STAT253/317 Lecture 5: 4.4 Limiting Distribution II

Positive Recurrence and Null Recurrence

For a Markov chain, consider the return time to a recurrent state i

$$T_i = \min\{n > 0 : X_n = i | X_0 = i\}$$

We say a state i is

- **positive recurrent** if $\mathbb{E}[T_i] < \infty$.
- **null recurrent** if $\mathrm{P}(T_i < \infty) = 1$ but $\mathbb{E}[T_i] = \infty$.
- **transient** if $\mathrm{P}(T_i < \infty) < 1$

We say a state is **ergodic** if it is aperiodic and positive recurrent.

The Fundamental Limit Theorem of Markov Chains I

Consider a recurrent irreducible aperiodic Markov chain. Then

$$\lim_{n\to\infty} P_{ij}^{(n)} = \frac{1}{\mathbb{E}[T_j]}$$

Moreover, if a Markov chain is irreducible and ergodic,

$$\pi_j = \lim_{n\to\infty} P_{ij}^{(n)} = \frac{1}{\mathbb{E}[T_j]}$$

is uniquely determined by the set of equations

$$\pi_j \geq 0, \quad \sum_{j\in\mathfrak{X}} \pi_j = 1, \quad \pi_j = \sum_{i\in\mathfrak{X}} \pi_i P_{ij}$$

Proof. See Theorem 1.1, 1.2, 1.3 on p.81-86 in Karlin & Taylor (1975).

Why $\pi_i = 1/\mathbb{E}(T_i)$?

Consider a Markov chain started from state j. Let S_k be the time till the k-th visit to state i. Then

$$S_k = T_{ji} + T_{ii}(1) + \ldots + T_{ii}(k-1)$$

Here

- T_{ji} = the first time the process visits state i from state j, and
- $T_{ii}(m)$ = the time between the mth and $(m+1)$st visit to state i.

Observe that $T_{ii}(1)$, $T_{ii}(2), \ldots T_{ii}(k-1)$ are i.i.d. and have the same distribution as T_i.

For k large, the Law of Large Numbers tells us

$$\frac{1}{k}[T_{ji} + T_{ii}(1) + T_{ii}(2) + \cdots + T_{ii}(k-1)] \approx \mathbb{E}(T_i)$$

i.e., the chain visits state i about k times in $k\mathbb{E}(T_i)$ steps.

We have just seen that in n steps, we expect about $n\pi(i)$ visits to the state i. Hence setting $n = k\mathbb{E}(T_i)$, we get the relation

$$\pi_i = 1/\mathbb{E}(T_i).$$

Remark

From the result in the previous page, we can see that a state i is **null recurrent**, i.e., $\mathbb{E}(T_i) = \infty$, if and only if

$$\lim_{n \to \infty} P_{ji}^{(n)} = 0, \quad \text{for all } j \in \mathfrak{X}.$$

Proposition 4.5 Positive Recurrence is a Class Property

- From the Fundamental Limit Theorem of Markov Chains I

$$\pi_i = 1/\mathbb{E}[T_i]$$

  and that a state i is positive recurrent if and only if $\mathbb{E}[T_i] < \infty$ it follows that a state i is positive current if and only if $\pi_i > 0$

- If a state j communicate with a positive recurrent state i, then state j is also positive recurrent.

  Proof. Since $i \leftrightarrow j$, there exists n such that $P_{ij}^{(n)} > 0$. Along with the fact that i is positive recurrent, $\pi_i > 0$, we know $\pi_j = \sum_k \pi_k P_{kj}^{(n)} \geq \pi_i P_{ij}^{(n)} > 0$. So j is also positive recurrent.

Corollary: Null Recurrence is a Class Property

If state i is null recurrent and $i \leftrightarrow j$, then state j is also null recurrent.

Proof. Since recurrence is a class property, state j can only be positive or null recurrent as it communicates with a null recurrent state i. Suppose state j is positive recurrent. As positive recurrence is a class property, state i must also be positive recurrent not null recurrent if it communicates with state j. So state j can only be null recurrent.

Finite-State Markov Chains Have No Null Recurrent States

In a finite-state Markov chain all recurrent states are positive recurrent.

Proof.

It suffices to consider irreducible Markov chains only since a Markov chain restricted to one of its recurrent class is also a Markov chain.
Recall an irreducible Markov chain must be recurrent. Also recall that positive/null recurrence is a class property. Thus if one state is null recurrent, then all states are null recurrent. However, since $\sum_{j \in \mathfrak{X}} P_{ij}^{(n)} = 1$. As there are only finite number of states, it is impossible that $\lim_{n\to\infty} P_{ij}^{(n)} = 0$ for all $j \in \mathfrak{X}$. Thus no state can be null recurrent.

Remark. For a finite state Markov chain, a limiting distribution exists if it is irreducible and aperiodic

The Fundamental Limit Theorem of Markov Chain II ($\star\star\star\star\star$)

If a Markov chain is **irreducible**, then the Markov chain is **positive recurrent** if and only if there exists a solution to the set of equations:

$$\pi_i \geq 0, \quad \sum_{i \in \mathfrak{X}} \pi_i = 1, \quad \pi_j = \sum_{i \in \mathfrak{X}} \pi_i P_{ij}$$

If a solution exists then

- it will be unique, and

$$\pi_j = \begin{cases} \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P_{ij}^{(k)} & \text{if the chain is periodic} \\ \lim_{n\to\infty} P_{ij}^{(n)} & \text{if the chain is aperiodic} \end{cases}$$

Remark. When a Markov chain is periodic, though its limiting distribution $\lim_{n\to\infty} P_{ij}^{(n)}$ doesn't exist, another limit $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P_{ij}^{(k)}$ exists and is equal to the stationary distribution. The later limit can be interpret as the **long run proportion of time that the Markov chain is in state** j.

Example 1: One-Dimensional Random Walk

In Lecture 4, we have shown that 1-dim symmetric random walk has no stationary distribution.

- Conclusion: 1-dim symmetric random walk is null recurrent, i.e.

$$\mathbb{E}[T_i] = \infty \quad \text{for all state } i$$

In fact, in Lecture 3 we have shown that

$$P_{ii}^{(n)} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \binom{n}{n/2}\left(\frac{1}{2}\right)^n \approx \sqrt{\frac{2}{\pi n}} & \text{if } n \text{ is even} \end{cases}$$

Thus $\pi_i = \lim_{n\to\infty} P_{ii}^{(n)} = 0$, and hence $\mathbb{E}[T_i] = 1/\pi_i = \infty$.

Ex 2: 1-D Random Walk w/ Partially Reflective Boundary

$$
\begin{aligned}
P_{i,i+1} &= p \quad \text{for all } i = 0, 1, 2, \ldots \\
P_{i,i-1} &= 1 - p \quad \text{for all } i = 1, 2, \ldots \\
p_{00} &= 1 - p
\end{aligned}
$$

Try to solve $\pi_j = \sum_{i \in \mathfrak{X}} \pi_i P_{ij}$

$$
\begin{aligned}
\pi_0 &= \pi_0 P_{00} + \pi_1 P_{10} = (1-p)(\pi_0 + \pi_1) &&\Rightarrow \pi_1 = \tfrac{p}{1-p}\pi_0 \\
\pi_1 &= \pi_0 P_{01} + \pi_2 P_{21} = p\pi_0 + (1-p)\pi_2 &&\Rightarrow \pi_2 = \left(\tfrac{p}{1-p}\right)^2 \pi_0 \\
\pi_2 &= \pi_0 P_{12} + \pi_3 P_{32} = p\pi_1 + (1-p)\pi_3 &&\Rightarrow \pi_3 = \left(\tfrac{p}{1-p}\right)^3 \pi_0 \\
&\vdots \\
\pi_j &= p\pi_{j-1} + (1-p)\pi_{j+1} &&\Rightarrow \pi_{j+1} = \left(\tfrac{p}{1-p}\right)^{j+1} \pi_0
\end{aligned}
$$

Ex 2: 1-D Random Walk w/ Partially Reflective Boundary

$$\sum_{i=0}^{\infty} \pi_i = \pi_0 \sum_{i=0}^{\infty} \left(\frac{p}{1-p} \right)^i = \begin{cases} \pi_0 \left(\frac{1-p}{1-2p} \right) & \text{if } p < 1/2 \\ \infty & \text{if } p \geq 1/2 \end{cases}$$

Conclusion: The process is positive recurrent iff $p < 1/2$, in which case

$$\pi_i = \frac{1-2p}{1-p} \left(\frac{p}{1-p} \right)^i, \quad i = 0, 1, 2, \ldots$$

Ex 3: Ehrenfest Diffusion Model with N Balls

Recall that in Lecture 4, we show that Ehrenfest Diffusion Model is irreducible, has period $= 2$, and there exists a solution to the set of equations

$$\pi_i \geq 0, \quad \sum_{i \in \mathfrak{X}} \pi_i = 1, \quad \pi_j = \sum_{i \in \mathfrak{X}} \pi_i P_{ij}$$

which is

$$\pi_i = \binom{N}{i} \left(\frac{1}{2}\right)^N \quad \text{for } i = 0, 1, 2, \ldots, N$$

Though the limiting distribution $\lim_{n\to\infty} P_{ij}^{(n)}$ does not exist, we can show that

$$\lim_{n\to\infty} P_{ij}^{(2n)} = 2\binom{N}{j}(\frac{1}{2})^N, \quad \lim_{n\to\infty} P_{ij}^{(2n+1)} = 0 \quad \text{if } i+j \text{ is even}$$

$$\lim_{n\to\infty} P_{ij}^{(2n)} = 0, \quad \lim_{n\to\infty} P_{ij}^{(2n+1)} = 2\binom{N}{j}(\frac{1}{2})^N \quad \text{if } i+j \text{ is odd}$$

From the above, one can verify that $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P_{ij}^{(k)} = \binom{N}{j}(\frac{1}{2})^N = \pi_j$.

Exercise 4.50 on p.284

A Markov chain has transition probability matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0.2 & 0.4 & 0 & 0.3 & 0 & 0.1 \\ 0.1 & 0.3 & 0 & 0.4 & 0 & 0.2 \\ 0 & 0 & 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0.2 & 0.8 \end{pmatrix} \end{array}$$

Communicating classes:

$$\begin{array}{ccc} \{1,2\} & \{3,4\} & \{5,6\} \\ \uparrow & \uparrow & \uparrow \\ \text{transient} & \text{recurrent} & \text{recurrent} \end{array}$$

Find $\lim_{n\to\infty} P^{(n)}$.

Exercise 4.50 on p.284 (Cont'd)

Observe that $\lim_{n\to\infty} P_{ij}^{(n)} = 0$ if j is transient, hence,

$$\lim_{n\to\infty} P^{(n)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \end{pmatrix} \end{array}$$

Exercise 4.50 on p.284 (Cont'd)

Observe that $\lim_{n\to\infty} P_{ij}^{(n)} = 0$ if j is NOT accessible from i

$$\lim_{n\to\infty} P^{(n)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & 0 & 0 \\ 0 & 0 & ? & ? & 0 & 0 \\ 0 & 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & 0 & ? & ? \end{pmatrix} \end{array}$$

The two classes {3,4} and {5,6} do not communicate and hence the transition probabilities in between are all 0.

Exercise 4.50 on p.284 (Cont'd)

Recall we have shown that the limiting distribution of a two-state Markov chain with the transition matrix $\begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$ is $\left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$. As the Markov chain restricted to the closed class $\{3,4\}$ is also a Markov chain with the transition matrix

$$\begin{array}{c c} & \begin{array}{cc} 3 & 4 \end{array} \\ \begin{array}{c} 3 \\ 4 \end{array} & \begin{pmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{pmatrix} \end{array}.$$ Hence,

$$\lim_{n\to\infty} P^{(n)} = \begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{pmatrix} 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & 6/13 & 7/13 & 0 & 0 \\ 0 & 0 & 6/13 & 7/13 & 0 & 0 \\ 0 & 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & 0 & ? & ? \end{pmatrix} \end{array}$$

Exercise 4.50 on p.284 (Cont'd)

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0.2 & 0.4 & 0 & 0.3 & 0 & 0.1 \\ 0.1 & 0.3 & 0 & 0.4 & 0 & 0.2 \\ 0 & 0 & 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0.2 & 0.8 \end{pmatrix} \end{array}$$

For the same reason,

$$\lim_{n\to\infty} P^{(n)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & 6/13 & 7/13 & 0 & 0 \\ 0 & 0 & 6/13 & 7/13 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/7 & 5/7 \\ 0 & 0 & 0 & 0 & 2/7 & 5/7 \end{pmatrix} \end{array}$$