STAT253/317 Lecture 3: 4.3 Classification of States

Definition. Consider a Markov chain $\{X_n, n \geq 0\}$ with state space $\mathfrak{X}$. For two states $i, j \in \mathfrak{X}$, we say state j is **accessible** from state i if $P_{ij}^{(n)} > 0$ for some n, and we denote it as

$$i \to j.$$

Note that **accessibility is transitive**: for $i, j, k \in \mathfrak{X}$,

$$\text{if } i \to j \text{ and } j \to k, \text{ then } i \to k.$$

Proof.

$$\begin{aligned} i \to j \quad &\Rightarrow \quad P_{ij}^{(m)} > 0 \text{ for some } m \\ j \to k \quad &\Rightarrow \quad P_{jk}^{(n)} > 0 \text{ for some } n \end{aligned}$$

By Chapman-Kolmogorov Equation:

$$P_{ik}^{(m+n)} = \sum_{l \in \mathfrak{X}} P_{il}^{(m)} P_{lk}^{(n)} \geq P_{ij}^{(m)} P_{jk}^{(n)} > 0,$$

which shows $i \to k$.

Communicability

Definition. Consider a Markov chain $\{X_n, n \geq 0\}$ chain with state space $\mathfrak{X}$. Two states $i, j \in \mathfrak{X}$ are said to **communicate** if $i \to j$, and $j \to i$. We denote it as

$$i \longleftrightarrow j.$$

Fact. Communicability is also transitive, meaning that

$$\text{if } i \longleftrightarrow j \text{ and } j \longleftrightarrow k, \text{ then } i \longleftrightarrow k.$$

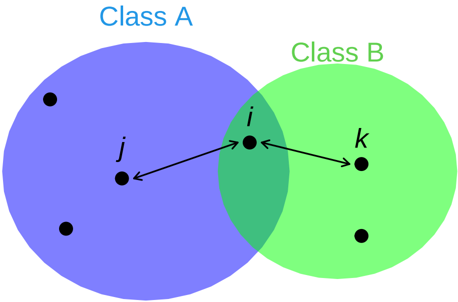
The proof is straight forward from the transitivity of accessibility.

Communicative Class

Definition. Two states that communicate with each other are in the same **class**. A state that communicates with no other states itself is a class.

Fact. Two classes are either identical or disjoint.

Proof. If two classes A and B have one state i in common, then all states in A communicate with i and all states in B do too. Consequently, all states with A can communicate with states in B (through state i). Class A and Class B must be identical.



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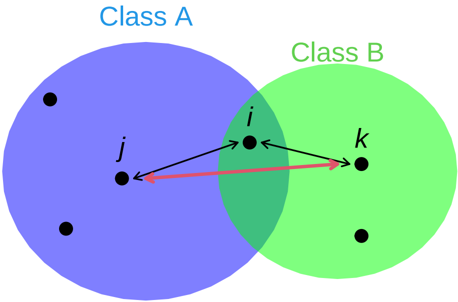

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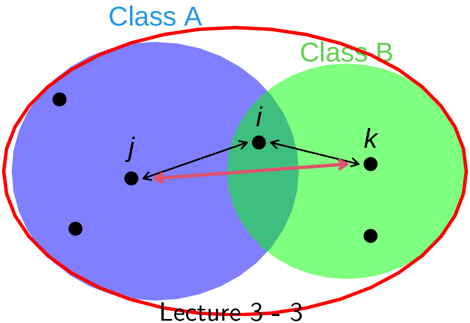

Example 1. Specify the classes of the following Markov chains.

$$\mathbb{P}_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.3 & 0.6 & 0.1 & 0 \\ 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0.9 & 0.1 \end{pmatrix} \end{array} \quad \mathbb{P}_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Example 2. How many classes does the Ehrenfest diffusion model with K balls have?

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For $\mathbb{P}_1$, $1 \leftrightarrow 2 \rightarrow 3 \leftrightarrow 4$. Classes: $\{1,2\}$, $\{3,4\}$.

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For $\mathbb{P}_2$, $\begin{array}{ccc} 1 & \longleftrightarrow & 2 \\ & \nwarrow & \uparrow \\ 4 & \leftarrow & 3 \\ \circlearrowleft & & \circlearrowleft \end{array}$. Classes: $\{1,2\}$, $\{3\}$, $\{4\}$.

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Example 2. How many classes does the Ehrenfest diffusion model with K balls have?

All states communicate. Only one class.

Closed Classes

Definition. A class C is said to be closed if

$$P_{ij} = 0 \quad \text{for all } i \text{ in } C \text{ and } j \text{ not in } C.$$

Once the process gets into a closed class. It will never leave the class since the outgoing probabilities from the class are all 0.

Examples.

- For $\mathbb{P}_1$ in the previous slide, the class {1,2} is not closed because it has a non-zero outgoing probability $P_{23} = 0.1 > 0$. The class $\{3, 4\}$ is closed.
- For $\mathbb{P}_2$ in the previous slide, the classes $\{1, 2\}$ and $\{4\}$ are closed, and $\{3\}$ is not closed.

A Markov Chain Restricted to a Closed Class is Also a Markov Chain

Example.

$$\mathbb{P}_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.3 & 0.6 & 0.1 & 0 \\ 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0.9 & 0.1 \end{pmatrix} \end{array}$$

- For $\mathbb{P}_1$ above, the Markov chain restricted to the class {3,4} is also a Markov chain, with transition matrix

$$\begin{array}{c} \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cc} 3 & 4 \end{array} \\ \begin{pmatrix} 0.2 & 0.8 \\ 0.9 & 0.1 \end{pmatrix} \end{array}$$

- The Markov chain for $\mathbb{P}_1$ can not be restricted to {1,2} as it may transit out of the state space from 2 to 3.

Irreducibility

A Markov chain is said to be **irreducible** if it has only 1 class.

Recurrence & Transience

Consider a Markov chain $\{X_n, n \geq 0\}$ chain with state space $\mathfrak{X}$.
For $i \in \mathfrak{X}$, define

$$f_i = \mathrm{P}(X_n = i \text{ for some n} > 0 | X_0 = i)$$

If $f_i = 1$, we say state i is **recurrent**
If $f_i < 1$, we say state i is **transient**

- It's generally difficult to compute f_i directly.
  We need other tools to determine whether a state is recurrent or transient.

States in a Non-Closed Class Are Always Transient

For a class A that is NOT closed, there must exists some state k not in A such that

$$P_{i_0,k} > 0, \quad \text{for some state } i_0 \text{ in class } A$$

but

$$P_{ki}^{(n)} = 0 \quad \text{for all state } i \text{ in class } A \text{ and for all } n.$$

Otherwise, state i would be accessible from state k ($k \longrightarrow i$). As i and i_0 are in the same class, we know $i \longleftrightarrow i_0$. Combining all the above, we have

$$k \longrightarrow i \longleftrightarrow i_0 \xrightarrow{P_{i_0,k}>0} k.$$

and hence k would communicate with i_0, contradicting to the assumption that k is not in A.

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Starting from a state j in a non-closed class A, there is a positive probability that the Markov chain will move to state k and never comes back to the class. Hence state j must be transient.

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Are states in an closed class always recurrent?

Fact 1 If state i is recurrent, then starting from state i, the process will revisit state i infinitely often.

Fact 2 If state i is transient, then starting from state i, the number of times the process revisits state i is finite, with expected value $1/(1-f_i)$.

Reason: Let N_i be the number of times the process revisits state i after starting from i. Observe that

$$\begin{aligned} \mathrm{P}(N_i = k) &= \mathrm{P}(\text{returns to } i \text{ after 1st departure}) \\ &\quad \times \cdots \times \mathrm{P}(\text{returns to } i \text{ after } k\text{th departure}) \\ &\quad \times \mathrm{P}(\text{never returns to } i \text{ after k+1st departure}) \\ &= f_i^k(1-f_i), \quad k = 0, 1, 2, \ldots \end{aligned}$$

i.e., N_i has a geometric distribution with mean $1/(1-f_i)$.

Claim:

$$\mathbb{E}(\# \text{ of visit to state } i | X_0 = i) = \sum_{n=1}^{\infty} P_{ii}^{(n)}$$

Proof. Define

$$I_{ni} = \begin{cases} 1 & \text{if } X_n = i \\ 0 & \text{if } X_n \neq i \end{cases}, \quad n \geq 0,\ i \in \mathfrak{X}.$$

Observe that $\sum_{n=1}^{\infty} I_{ni}$ is the number of visits to state i.

$$\begin{aligned} \mathbb{E}\left[\sum_{n=1}^{\infty} I_{ni} \middle| X_0 = i\right] &= \sum_{n=1}^{\infty} \mathbb{E}[I_{ni} | X_0 = i] \\ &= \sum_{n=1}^{\infty} \mathrm{P}(X_n = i | X_0 = i) \\ &= \sum_{n=1}^{\infty} P_{ii}^{(n)} \end{aligned}$$

Conclusion from Fact 1, Fact 2 and the Claim Above:

$$\begin{aligned} \text{State } i \text{ is recurrent} &\iff \mathbb{E}(\# \text{ of visit to state } i | X_0 = i) = \infty. \\ &\iff \sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty \end{aligned}$$

Proposition 4.1

$$\text{State } i \text{ is } \begin{cases} \text{recurrent if} & \sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty \\ \text{transient if} & \sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty \end{cases}$$

Implication of Proposition 4.1:

States in a finite-state Markov chain CANNOT be all transient.

Reason. Observe that $\sum_{i \in X} I_{ni} = 1$ for all n since X_n must be in one of the states. Thus

$$\sum_{n=1}^{\infty} \sum_{i \in X} I_{ni} = \sum_{n=1}^{\infty} 1 = \infty.$$

Since $\mathfrak{X}$ is finite, there exists at least one state i such that

$$\sum_{n=1}^{\infty} I_{ni} = \infty.$$

Such states are recurrent. Otherwise $\sum_{n=1}^{\infty} \sum_{i \in X} I_{ni}$ will be $< \infty$.

Corollary 4.2

If $i \longleftrightarrow j$, and i is recurrent, then j is also recurrent.

Proof.

$$
\begin{aligned}
i \to j \quad &\Rightarrow \quad P_{ij}^{(k)} > 0 \text{ for some } k \\
j \to i \quad &\Rightarrow \quad P_{ji}^{(l)} > 0 \text{ for some } l
\end{aligned}
$$

By Chapman-Kolmogorov Equation:

$$P_{jj}^{(l+n+k)} \geq P_{ji}^{(l)} P_{ii}^{(n)} P_{ij}^{(k)}, \text{ for all } k = 0, 1, 2, \ldots$$

Thus

$$\sum_{n=1}^{\infty} P_{jj}^{(n)} \geq \sum_{n=1}^{\infty} P_{jj}^{(l+n+k)} \geq \underbrace{P_{ji}^{(l)}}_{>0} \underbrace{\sum_{n=1}^{\infty} P_{ii}^{(n)}}_{=\infty} \underbrace{P_{ij}^{(k)}}_{>0} = \infty$$

Corollary 4.2 implies that all states of a finite irreducible Markov chain are recurrent.

Example: One-Dimensional Random Walk

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with prob. } p \\ X_n - 1 & \text{with prob. } 1 - p \end{cases}$$

- State space $\{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$
- All states communicate

$$\cdots \longleftrightarrow -2 \longleftrightarrow -1 \longleftrightarrow 0 \longleftrightarrow 1 \longleftrightarrow 2 \longleftrightarrow \cdots$$

Only one class $\Rightarrow$ Irreducible

$\Rightarrow$ States are all transient or all recurrent.

It suffices to check whether 0 is recurrent or transient, i.e., whether

$$\sum_{n=1}^{\infty} P_{00}^{(n)} = \infty \text{ or } < \infty$$

Example: One-Dimensional Random Walk (Cont'd)

$$
\begin{aligned}
P_{00}^{(2n+1)} &= 0 \quad \text{(Why?)} \\
P_{00}^{(2n)} &= \binom{2n}{n} p^n (1-p)^n \\
&= \frac{(2n)!}{n!\,n!} p^n (1-p)^n \\
&\approx \frac{(2n)^{2n+0.5} e^{-2n} \sqrt{2\pi}}{(n^{n+0.5} e^{-n} \sqrt{2\pi})^2} p^n (1-p)^n \\
&= \frac{1}{\sqrt{\pi n}} [4p(1-p)]^n
\end{aligned}
$$

Stirlin's Formula: $n! \approx n^{n+0.5} e^{-n} \sqrt{2\pi}$

Thus

$$
\sum_{n=1}^{\infty} P_{ii}^{2n} \approx \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi n}} [4p(1-p)]^n \begin{cases} < \infty & \text{if } p \neq 1/2 \\ = \infty & \text{if } p = 1/2 \end{cases}
$$

Conclusion: One-dimensional random walk is recurrent if $p = 1/2$, and transient otherwise.

Example: Two-Dimensional Symmetric Random Walk

Irreducible. Just check if 0 is recurrent.

$$
\begin{aligned}
P_{00}^{(2n)} &= \sum_{i=0}^{n} \frac{(2n)!}{i!i!(n-i)!(n-i)!} \left(\frac{1}{4}\right)^{2n} \\
&= \binom{2n}{n} \underbrace{\sum_{i=0}^{n} \binom{n}{i} \binom{n}{n-i}}_{=\binom{2n}{n}} \left(\frac{1}{4}\right)^{2n} \\
&= \binom{2n}{n}^2 \left(\frac{1}{4}\right)^{2n} \approx \frac{1}{\pi n} \quad \text{by Stirlin's Formula}
\end{aligned}
$$

Thus $\sum_{n=1}^{\infty} P_{00}^{(2n)} = \infty$.

Two-dimensional symmetric random walk is **recurrent**.

Example: d-Dimensional Symmetric Random Walk

In general, for a d-dimensional symmetric random walk, it can be shown that

$$P_{00}^{(2n)} \approx (1/2)^{d-1} \left(\frac{d}{n\pi} \right)^{d/2}$$

Thus

$$\sum_{n=1}^{\infty} P_{00}^{(2n)} \begin{cases} = \infty & \text{for } d = 1 \text{ or } 2 \\ < \infty & \text{for } d \geq 3 \end{cases}.$$

"A drunken man will find his way home.
A drunken bird might be lost forever."