STAT253/317 Lecture 17

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7.4 Renewal Reward Processes

Let $\{N(t), t \geq 0\}$ be a renewal process with i.i.d. interarrival times $\{X_i, i \geq 1\}$. Let R_i, $i = 1, 2, \ldots$ be i.i.d random variables. R_i may depend on the ith interarrival time X_i, but (X_i, R_i) are i.i.d. random variable pairs. The compound process

$$R(t) = \sum_{i=1}^{N(t)} R_i$$

is called a *renewal reward process*. R_i may be considered as *reward* earned during the ith cycle, and $R(t)$ represents the total reward earned up to time t.

Proposition 7.3 If $\mathbb{E}[R_1] < \infty$ and $\mathbb{E}[X_1] < \infty$, then

(a) $\displaystyle\lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}[R_1]}{\mathbb{E}[X_1]}$ with probability 1

(b) $\displaystyle\lim_{t\to\infty} \frac{\mathbb{E}[R(t)]}{t} = \frac{\mathbb{E}[R_1]}{\mathbb{E}[X_1]}$

Proof of Proposition 7.3(a)

We give the proof for (a) only. To prove this, write

$$\frac{R(t)}{t} = \frac{\sum_{i=1}^{N(t)} R_i}{t} = \frac{\sum_{i=1}^{N(t)} R_i}{N(t)} \frac{N(t)}{t}$$

By the Strong Law of Large Numbers (SLLN) and that $\lim_{t\to\infty} N(t) = \infty$ w/ prob. 1, we know

$$\frac{\sum_{i=1}^{N(t)} R_i}{N(t)} \to \mathbb{E}[R_1] \quad \text{as } t \to \infty \quad \text{w/ prob. 1.}$$

By Proposition 7.1

$$\frac{N(t)}{t} \to \frac{1}{\mathbb{E}[X_1]} \quad \text{as } t \to \infty.$$

The result thus follows.

Example 7.12 (A Car Buying Model)

- Mr. Brown buys a new car whenever his old one breaks down or reaches the age of T years
- Let Y_i be the lifetime of his ith car. Suppose Y_i's are i.i.d with CDF

$$H(y) = P(Y \leq y), \quad \text{and density } h(y) = H'(y).$$

- Cost to by a new car $= C_1$;
- If the car breaks down, an additional cost of C_2 is incurred.
- What is Mr. Brown's long run average cost (per unit of time, not per car)?

Example 7.12 (A Car Buying Model) Solutions

- An event occurs whenever Mr. Brown buys a new car
- Interarrival times: $X_i = \min(Y_i, T)$
- Cost incurred in the ith cycle: $R_i = C_1 + C_2 \mathbf{1}_{\{Y_i \le T\}}$
- Are (X_i, R_i), $i = 1, 2, \ldots$ i.i.d?
- Total cost up to time t: $R(t) = \sum_{i=1}^{N(t)} R_i$

$$\mathbb{E}[X_i] = \int_0^\infty \min(y, T) h(y) dy = \int_0^T y h(y) dy + T(1 - H(T))$$
$$\mathbb{E}[R_i] = C_1 + C_2 \mathrm{P}(Y_i \le T) = C_1 + C_2 H(T)$$

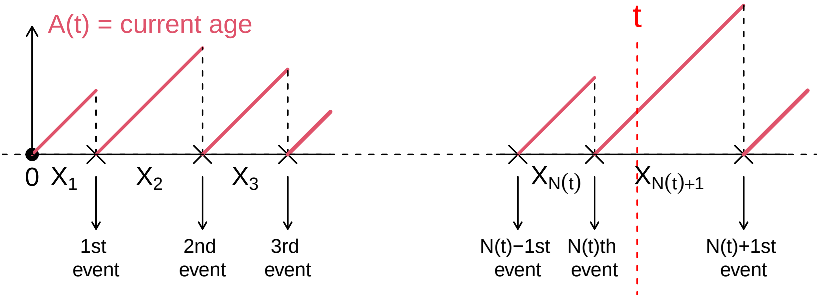
- average cost per car $= \mathbb{E}[R_i] = C_1 + C_2 H(T)$
- long-run average cost (per unit of time)

$$\lim_{t \to \infty} \frac{R(t)}{t} = \frac{C_1 + C_2 H(T)}{\int_0^T y h(y) dy + T(1 - H(T))}$$

Example 7.18 Current Age

Let $\{N(t), t \geq 0\}$ be a renewal process with i.i.d. interarrival times $\{X_i, i \geq 1\}$. Consider the **current age** of the item in use at time t

$$A(t) = t - S_{N(t)}.$$

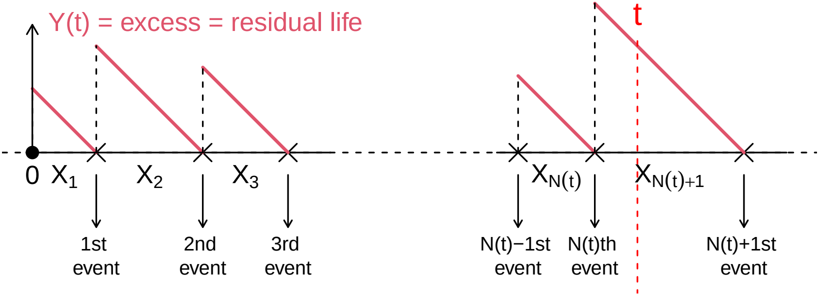


What is the long-run average of age $\lim_{t\to\infty} \frac{\int_0^t A(s)ds}{t}$?

Example 7.19 Residual Life of a Renewal Process

Consider the **residual life** or **excess** of the item in use at time t

$$Y(t) = S_{N(t)+1} - t.$$

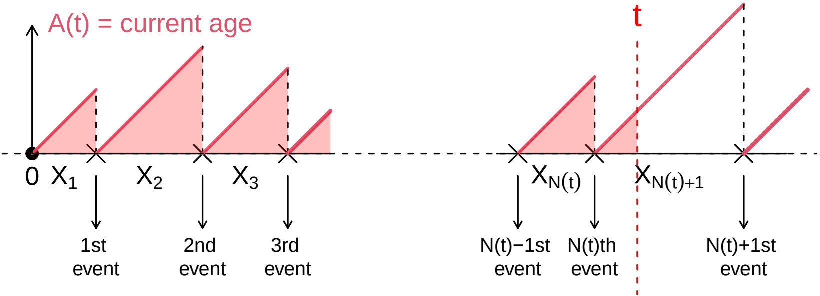


What is the long-run average of residual life

$$\lim_{t\to\infty} \frac{\int_0^t Y(s)ds}{t}?$$

Example 7.18 Age of a Reward Renewal Process (Cont'd)

Observe that $\int_0^t A(s)ds$ is the area of the shaded regions below.



$$\sum_{i=1}^{N(t)} \frac{X_i^2}{2} \leq \int_0^t A(s)ds < \sum_{i=1}^{N(t)+1} \frac{X_i^2}{2}$$

Observe that $\sum_{i=1}^{N(t)} \frac{X_i^2}{2}$ is a renewal reward process $R(t) = \sum_{i=1}^{N(t)} R_i$ with reward $R_i = X_i^2/2$.

Example 7.18 Current Age (Cont'd)

Since

$$R(t) \le \int_0^t A(s)ds < R(t) + \frac{X_{N(t)+1}^2}{2},$$

and

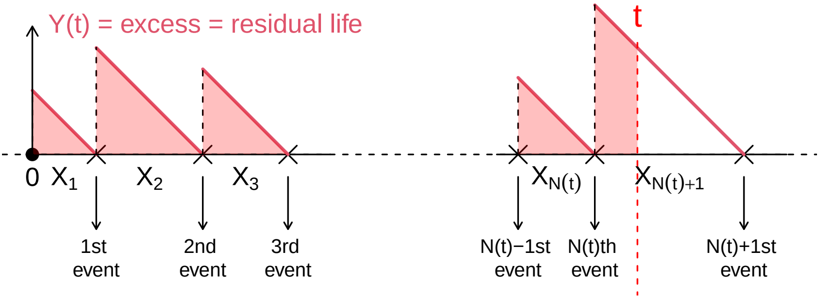
$$\frac{X_{N(t)+1}^2}{2t} \to 0 \quad \text{as } t \to \infty.$$

by Proposition 7.3, the long-run average age of the item in use is

$$\frac{\int_0^t A(s)ds}{t} = \lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}[R_1]}{\mathbb{E}[X_1]} = \frac{\mathbb{E}[X_1^2]}{2\mathbb{E}[X_1]}.$$

Example 7.19 Residual Life (Cont'd)

Similarly, for the residual life, $\int_0^t Y(s)ds$ is the area of the shaded regions below.



$$\sum_{i=1}^{N(t)} \frac{X_i^2}{2} \leq \int_0^t Y(s)ds < \sum_{i=1}^{N(t)+1} \frac{X_i^2}{2}$$

By the same argument, the long-run average of residual life of the item in use is

$$\frac{\int_0^t Y(s)ds}{t} = \lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}[R_1]}{\mathbb{E}[X_1]} = \frac{\mathbb{E}[X_1^2]}{2\mathbb{E}[X_1]}.$$

7.5.1 Alternating Renewal Processes

Considers a system that can be in one of two states: **ON** or **OFF**. Initially it is ON, and remains ON for a time Z_1; it then goes OFF and remains OFF for a time Y_1. It then goes ON for a time Z_2; then OFF for a time Y_2; then on, and so on. Suppose

- (Z_k, Y_k) are i.i.d random vectors, though Z_k and Y_k might depend on each other
- Y_k, Z_k are non-negative with finite means.

Then a renewal process $\{N(t), t \geq 0\}$ with interarrival times

$$X_k = Z_k + Y_k, \qquad k \geq 1$$

is called an *alternating renewal process*. Let

$$U(t) = \begin{cases} 1 & \text{if the system is ON at time } t \\ 0 & \text{otherwise} \end{cases}$$

Q: What is the long-run proportion of time that the system is ON?

$$\lim_{t\to\infty} \frac{\int_0^t U(s)ds}{t}?$$

Alternating Renewal Processes (Cont'd)

An alternating renewal process can be regarded as a renewal reward process with reward $R_i = Z_i$,

$$R(t) = \sum_{i=1}^{N(t)} Z_i$$

Then

$$R(t) \leq \int_0^t U(s)ds < R(t) + Z_{N(t)+1}$$

By Proposition 7.3, with probability 1,

$$\lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}[Z_1]}{E[X_1]} = \frac{\mathbb{E}[Z_1]}{\mathbb{E}[Z_1] + \mathbb{E}[Y_1]}$$

and hence

$$\lim_{t\to\infty} \frac{\int_0^t U(s)ds}{t} = \lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}[Z_1]}{\mathbb{E}[Z_1] + \mathbb{E}[Y_1]} = \frac{\mathbb{E}[\mathrm{ON}]}{\mathbb{E}[\mathrm{ON}] + \mathbb{E}[\mathrm{OFF}]}.$$

Definition: Lattice Distribution

A random variable X is said to have a **lattice** distribution if there is an $h > 0$ for which

$$\sum_{k=-\infty}^{\infty} P(X = kh) = 1,$$

i.e., X is lattice if it only takes on integral multiples of some nonnegative number h. The largest h having this property is called the *period* of X.

Examples.

- Continuous distributions, mixtures of discrete and continuous distributions are both non-lattice.
- Integer-valued random variables are lattice, e.g., Poisson, binomial
- A lattice distribution must be discrete, but a discrete distribution may not be lattice, e.g., if

  $$\mathrm{P}(X = 1/n) = 1/2^n, \quad n = 1, 2, 3, \ldots$$

  then X is discrete but non-lattice because we cannot find an $h > 0$ such that all $1/n$'s are all multiples of h.

Theorem

If the interarrival distribution is non-lattice, then

$$\lim_{t\to\infty} \mathrm{P}(\text{ON at time } t) = \lim_{t\to\infty} \mathrm{P}(U(t) = 1) = \frac{\mathbb{E}[Z_1]}{\mathbb{E}[Z_1] + \mathbb{E}[Y_1]}$$

Remark. If interarrival distribution is lattice, $\lim_{t\to\infty} \mathrm{P}(U(t) = 1)$ may not exist.

Exercise 7.39

- Two machines work independently, each functions for an exponential time with rate λ and then fails
- A single repairmen. All repair times are independent with distribution function G
- If the repairmen is free when a machine fails, he will begin repairing that machine immediately; Otherwise, then that machine must wait until the other machine has been repaired.
- Once repaired, a machine is as good as a new one.
- What proportion of time is the repairmen idle?

Solution.

- ON when the repairmen is idle, OFF when busy
- length of ON (idle) time: $Z \sim Exp(2\lambda)$, $\mathbb{E}[Z] = 1/(2\lambda)$
- length of OFF (busy) time Y; want to find $\mathbb{E}[Y]$

Exercise 7.39 Solutions

- T = length of time to repair the first failing machine $\sim G$
- U = the time the working machine can function after the first machine failed. By the memoryless property, $U \sim Exp(\lambda)$
- Note that

$$Y = \begin{cases} T & \text{if } U > T \\ T + Y' & \text{if } U < T \end{cases}$$
$$= T + Y'\mathbf{1}_{\{U<T\}}$$

  where Y' is the time the repairmen remains busy after the first failing machine is fixed. Note Y' is independent of T and U, and has the same distribution as Y. Thus

$$\mathbb{E}[Y] = \mathbb{E}[T] + \mathbb{E}[Y]\mathrm{P}(T > U) \;\Rightarrow\; \mathbb{E}[Y] = \frac{\mathbb{E}[T]}{\mathrm{P}(T < U)}$$

- long-run proportion of ON (idle) time

$$\frac{\mathbb{E}[Z]}{\mathbb{E}[Z] + \mathbb{E}[Y]} = \frac{1/(2\lambda)}{1/(2\lambda) + \mathbb{E}[Y]}$$

Example 7.23 & 7.24

Let $\{N(t),\ t = 0\}$ be a renewal process with i.i.d. interarrival times X_i, $i = 1, 2, \ldots$, where $\mu = \mathbb{E}[X_i]$ and $F(x) = P(X_i \leq x)$. Consider the **current age** of the item in use at time t

$$A(t) = t - S_{N(t)},$$

and the **residual life** of the item in use at time t

$$Y(t) = S_{N(t)+1} - t.$$

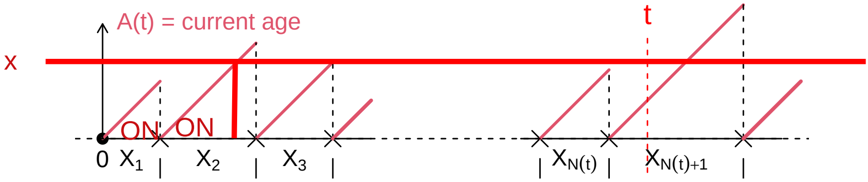
Proposition. The long-run proportion of time that $A(t) \leq x$ is the same as the long-run proportion of time that $Y(t) \leq x$, and is equal to

$$F_e(x) = \frac{1}{\mu} \int_0^x (1 - F(u)) du.$$

Furthermore, if F is non-lattice, then

$$\lim_{t \to \infty} \mathrm{P}(A(t) \leq x) = \lim_{t \to \infty} \mathrm{P}(Y(t) \leq x) = F_e(x).$$

Example 7.23 Current Age(Con'd)



- let's say the system is ON at time t if $A(t) \le x$
- length of ON time $Y_i = \min(X_i, x)$

$$\mathbb{E}[Y_i] = \mathbb{E}[\min(X_i, x)] = \int_0^{\infty} \mathrm{P}(\min(X_i, x) > u)du$$

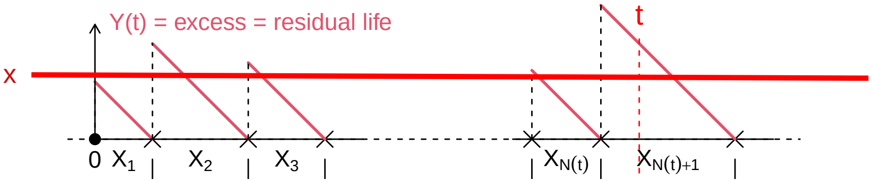
=\int_0^{\infty} min(u,x)f(u)du

$$= \int_0^{x} (1 - F(u))du$$

- length of a cycle = X_i, $\mathbb{E}[\text{ON}] + \mathbb{E}[\text{OFF}] = \mathbb{E}[X_i] = \mu$
- long-run proportion of time that $A(t) \le x$ is

$$\frac{\mathbb{E}[\text{ON}]}{\mathbb{E}[\text{ON}] + \mathbb{E}[\text{OFF}]} = \frac{1}{\mu} \int_0^{x} (1 - F(u))du$$

Example 7.24 Residual Life (Con'd)



- let's say the system is OFF at time t if $Y(t) \leq x$
- length of OFF time $Z_i = \min(X_i, x)$

$$\mathbb{E}[Z_i] = \mathbb{E}[\min(X_i, x)] = \int_0^x (1 - F(u))du$$

- length of a cycle $= X_i$, $\mathbb{E}[\text{ON}] + \mathbb{E}[\text{OFF}] = \mathbb{E}[X_i] = \mu$
- long-run proportion of time that $Y(t) \leq x$ is

$$\frac{\mathbb{E}[\text{OFF}]}{\mathbb{E}[\text{ON}] + \mathbb{E}[\text{OFF}]} = \frac{1}{\mu} \int_0^x (1 - F(u))du$$

Remark: The ON time in Example 7.23 is not the same as the ON time in Example 7.24

The density and kth moment of the distribution F_e is

$$f_e(x) = \frac{1}{\mu}(1 - F(x)), \quad \text{and} \quad \int_0^\infty x^k f_e(x) dx = \frac{\mathbb{E}[X^{k+1}]}{(k+1)\mathbb{E}[X]}$$

where X is an interarrival time.

Recall that

$$\frac{m(t)}{t} = \frac{1}{\mu} - \frac{1}{t} + \frac{\mathbb{E}[Y(t)]}{t\mu}$$

If F is non-lattice, since the limiting distribution of $Y(t)$ is F_e, we have

$$\lim_{t\to\infty} = \mathbb{E}[Y(t)] = \frac{\mu^2 + \sigma^2}{2\mu}$$

Thus

$$m(t) = \frac{t}{\mu} - 1 + \frac{\mu^2 + \sigma^2}{2\mu^2} + o(t) = \frac{t}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2} + o(t)$$