STAT253/317 Lecture 16

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7.3 Limit Theorems

Revision of Example 7.7

A coin with probability p to land heads (and $q = 1 - p$ to land tails) is tossed continually.

- What is the probability to get k **heads in a row** before getting k **tails in a row**?
- How many tosses is expected to get k heads in a row or k tails in a row?

Solution.

- Suppose the coin is tossed at every integer time points $t = 1, 2, 3 \ldots.$
- An *event* occurs whenever getting k heads in a row. Here we required the rows of heads for different events must be non-overlapping.
- Define $N_H(t) = \#$ of events occurred at or before time t. $\{N_H(t),\ t \geq 0\}$ is a renewal processes (why?).
- What is the mean length of the interarrival times for $\{N_H(t)\ t \geq 0\}$?

Revision of Example 7.7 (Cont'd)

Let T_k = the # of tosses required to get k heads in a row.
To get k consecutive heads, one must first get $k-1$ consecutive heads, which takes T_{k-1} steps. So

$$T_k = \begin{cases} T_{k-1}+1 & \text{w/ prob. } p, \quad \text{(if heads in the next toss)} \\ T_{k-1}+1+T'_k & \text{w/ prob. } 1-p \text{ (if tails in the next toss).} \end{cases}$$

Here $T'_k \sim T_k$ and T'_k is independent of the past, and hence is independent of T_{k-1}. So

$$\begin{aligned} \mathbb{E}[T_k|T_{k-1}] &= T_{k-1}+1+(1-p)\mathbb{E}[T'_k|T_{k-1}] \\ &= T_{k-1}+1+(1-p)\mathbb{E}[T'_k] \quad \text{(since } T'_k, T_{k-1} \text{ are indep.)} \\ &= T_{k-1}+1+(1-p)\mathbb{E}[T_k] \quad \text{(since } T'_k \sim T_k\text{)} \end{aligned}$$

and hence

$$\mathbb{E}[T_k] = \mathbb{E}[\mathbb{E}[T_k|T_{k-1}]] = \mathbb{E}[T_{k-1}]+1+(1-p)\mathbb{E}[T_k]$$

Subtracting $(1-p)\mathbb{E}[T_k]$ from both sides we get

$$p\mathbb{E}[T_k] = \mathbb{E}[T_{k-1}]+1, \quad k = 2, 3, 4, \ldots.$$

Revision of Example 7.7 (Cont'd)

Observe that T_1 has a geometric distribution and hence$\mathbb{E}[T_1] = 1/p$. Using the iterative relation, we get that

$$\begin{aligned}
\mathbb{E}[T_2] &= (\mathbb{E}[T_1] + 1)/p = 1/p^2 + 1/p \\
\mathbb{E}[T_3] &= (\mathbb{E}[T_2] + 1)/p = 1/p^3 + 1/p^2 + 1/p \\
&\vdots \\
\mathbb{E}[T_k] &= (\mathbb{E}[T_{k-1}] + 1)/p = 1/p^k + \ldots + 1/p^2 + 1/p = \frac{1-p^k}{p^k(1-p)}
\end{aligned}$$

By Proposition 7.1, we know

$$\lim_{t\to\infty} \frac{N_H(t)}{t} = \frac{1}{\mathbb{E}[T_k]} = \frac{p^k(1-p)}{1-p^k} = \frac{p^k q}{1-p^k}.$$

Similarly, considering the renewal process

$$N_T(t) = \# \text{ of times to see } k \text{ tails in a row by time } t.$$

By Proposition 7.1, we also have $\lim_{t\to\infty} \frac{N_T(t)}{t} = \frac{q^k p}{1-q^k}$.

Example 7.7 Solution (Cont'd)

Consider the counting process $N(t) = N_H(t) + N_T(t)$.

- $\{N(t) : t \geq 0\}$ is also a renewal process (Why?)
  Here an event occurs whenever one gets k heads in a row or k tails in a row.
- As $t \to \infty$,

$$\frac{N(t)}{t} = \frac{N_H(t)}{t} + \frac{N_T(t)}{t} \to \frac{p^k q}{1-p^k} + \frac{q^k p}{1-q^k}.$$

  which is the reciprocal of mean length $\mathbb{E}[T]$ of interarrival times for $\{N(t) : t \geq 0\}$.
- From the above we can answer the 2nd question: the expected number of tosses required to get k heads in a row or k tails in a row is

$$\mathbb{E}[T] = \frac{1}{p^k q/(1-p^k) + q^k p/(1-q^k)}.$$

Example 7.7 Solution (Cont'd)

With probability 1, the long run proportion of events that are k-heads is

$$\lim_{t\to\infty}\frac{N_H(t)}{N(t)} = \lim_{t\to\infty}\frac{N_H(t)}{N_H(t)+N_T(t)} = \lim_{t\to\infty}\frac{N_H(t)/t}{N_H(t)/t+N_T(t)/t}$$
$$= \frac{p^k q/(1-p^k)}{p^k q/(1-p^k)+q^k p/(1-q^k)}$$

The probability of getting k-heads before k-tails is, by SLLN, equal to the long-run proportion of events that are k-heads.

$$\mathrm{P}(k\text{-heads before }k\text{-tails}) = \frac{p^k q/(1-p^k)}{p^k q/(1-p^k)+q^k p/(1-q^k)}$$

Theorem 7.2 CLT for Renewal Processes

Suppose that μ and σ^2 are, respectively, the mean and variance of the interarrival times of a renewal process $\{N(t), t \geq 0\}$. Then

$$\lim_{t\to\infty} \frac{\mathrm{Var}(N(t))}{t} = \frac{\sigma^2}{\mu^3},$$

and $N(t)$ is asymptotically $N(t/\mu, \sigma^2 t/\mu^3)$, i.e.,

$$\lim_{t\to\infty} \mathrm{P}\left(\frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}} < x\right) = \Phi(x)$$

where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{\pi}} e^{-z^2/2} dz$ is the CDF of the standard normal distribution.

Proof of Theorem 7.2

$$
\begin{aligned}
& \mathrm{P}\left(\frac{N(t)-t/\mu}{\sqrt{t\sigma^2/\mu^3}} < x\right) \\
= & \mathrm{P}\left(N(t) < \frac{t}{\mu} + \frac{\sigma}{\mu}\sqrt{\frac{t}{\mu}}x\right) \\
= & \mathrm{P}(N(t) \le n) && \left(\text{Let } n = \left\lfloor \frac{t}{\mu} + \frac{\sigma}{\mu}\sqrt{\frac{t}{\mu}}x \right\rfloor\right) \\
= & \mathrm{P}(S_n \ge t) && (\text{Recall } N(t) \le n \Leftrightarrow S_n \ge t) \\
= & \mathrm{P}\left(\frac{S_n - n\mu}{\sqrt{n}\sigma} \ge \frac{t-n\mu}{\sqrt{n}\sigma}\right) \\
\longrightarrow & 1 - \Phi\left(\frac{t-n\mu}{\sqrt{n}\sigma}\right) && \text{as } n \to \infty \text{ by the CLT for } S_n \\
= & \Phi\left(-\frac{t-n\mu}{\sqrt{n}\sigma}\right) && (\text{since } 1-\Phi(z) = \Phi(-z))
\end{aligned}
$$

Here $\lfloor y \rfloor$ means the greatest integer less or equal to y

Proof of Theorem 7.2 (Cont'd)

It remains to show that

$$-\frac{t - n\mu}{\sqrt{n}\sigma} \longrightarrow x \quad \text{as } t \to \infty.$$

Since $n \le \frac{t}{\mu} + \frac{\sigma}{\mu}\sqrt{\frac{t}{\mu}}x$, as $t \to \infty$, we have

$$\frac{t - n\mu}{\sqrt{n}\,\sigma} \ge \frac{t - (\frac{t}{\mu} + \frac{\sigma}{\mu}\sqrt{\frac{t}{\mu}}x)\mu}{\sigma\sqrt{\frac{t}{\mu} + \frac{\sigma}{\mu}\sqrt{\frac{t}{\mu}}x}} = \frac{-\sigma x\sqrt{t/\mu}}{\sigma\sqrt{\frac{t}{\mu} + \frac{\sigma}{\mu}\sqrt{\frac{t}{\mu}}x}} \longrightarrow -x.$$

Similarly because $n \ge \frac{t}{\mu} + \frac{\sigma}{\mu}\sqrt{\frac{t}{\mu}} - 1$, we have

$$\frac{t - n\mu}{\sqrt{n}\,\sigma} \le \frac{t - (\frac{t}{\mu} + \frac{\sigma}{\mu}\sqrt{\frac{t}{\mu}}x - 1)\mu}{\sigma\sqrt{\frac{t}{\mu} + \frac{\sigma}{\mu}\sqrt{\frac{t}{\mu}}x}} = \frac{-\sigma x\sqrt{t/\mu} - \mu}{\sigma\sqrt{\frac{t}{\mu} + \frac{\sigma}{\mu}\sqrt{\frac{t}{\mu}}x - 1}} \longrightarrow -x$$

as $t \to \infty$.