STAT253/317 Winter 2020 Lecture 1

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4.1 Introduction to Markov Chains

Stochastic Processes

A stochastic process is a family of random variables $\{X_t : t \in \mathcal{T}\}$ such that

- For each $t \in \mathcal{T}$, X_t is a random variable
- The index set $\mathcal{T}$ can be discrete or continuous
  - $\mathcal{T} = \{0, 1, 2, 3, 4\}$
  - $\mathcal{T} = \mathbb{R}, \mathbb{R}^+, \mathbb{R}^2, \mathbb{R}^3$

Examples:

- Discrete Time Markov Chains .... Chapter 4
- Poisson Processes, Counting Processes .... Chapter 5
- Continuous Time Markov Chain .... Chapter 6
- Renewal Theory .... Chapter 7
- Queuing Theory .... Chapter 8
- Brownian Motion .... Chapter 10

4.1 Introduction to Markov Chain

Consider a stochastic process $\{X_n : \ n = 0, 1, 2, \ldots\}$ taking values in a finite or countable set $\mathfrak{X}$.

- $\mathfrak{X}$ is called the **state space**
- If $X_n = i$, $i \in \mathfrak{X}$, we say the process is in state i at time n
- Since $\mathfrak{X}$ is countable, there is a 1-1 map from $\mathfrak{X}$ to the set of non-negative integers $\{0, 1, 2, 3, \ldots\}$
  From now on, we assume $\mathfrak{X} = \{0, 1, 2, 3, \ldots\}$

Definition

A stochastic process $\{X_n : \ n = 0, 1, 2, \ldots\}$ is called a **Markov chain** if it has the following property:

$$P(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_2 = i_2, X_1 = i_1, X_0 = i_0)$$
$$= P(X_{n+1} = j | X_n = i)$$

for all states $i_0,\ i_1,\ i_2, \ldots, i_{n-1},\ i,\ j \in \mathfrak{X}$ and $n \geq 0$.

Transition Probability Matrix

If $P(X_{n+1} = j | X_n = i) = P_{ij}$ does not depend on n, then the process $\{X_n : \ n = 0, 1, 2, \ldots\}$ is called a **stationary Markov chain**. From now on, we consider stationary Markov chain only.

$\{P_{ij}\}$ is called the **transition probabilities**.
The matrix

$$\mathbb{P} = \begin{pmatrix} P_{00} & P_{01} & P_{02} & \cdots & P_{0j} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots & P_{1j} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ P_{i0} & P_{i1} & P_{i2} & \cdots & P_{ij} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}$$

is called the **transition probability matrix**.
Naturally, the transition probabilities $\{P_{ij}\}$ satisfies the following

- $P_{ij} \geq 0$ for all i, j
- Rows sums are 1: $\sum_j P_{ij} = 1$ for all i.

In other words, $\mathbb{P}\mathbf{1} = \mathbf{1}$, where $\mathbf{1} = (1, 1, \cdots, 1, \cdots)^T$

Example 1: Busy Phone Line

Consider the status of a phone line on discrete time intervals: 1, 2, ...

- Suppose calls come in independently at constant rate: for each time interval, there is a probability α that one call comes in that interval. Assume there is at most one call per interval.
- An incoming call can go through only if the line is free when the call comes in, and will occupied the line starting the next time interval until the call ends.
- All unanswered calls are missed. (Cannot "stay on the line")
- Suppose the length of calls is fixed at 2 (time intervals).

Can you find a Markov chain in this model?

Example 1: Busy Phone Line (Cont'd)

Let X_n be the status (busy or free) of the phone line in the nth time interval. Is $\{X_n\}$ a Markov chain?

Answer: No. Observe that

$$P(X_3 = \text{free}|X_2 = \text{busy}, X_1 = \text{free}) = 0$$
$$P(X_3 = \text{free}|X_2 = \text{busy}, X_1 = \text{busy}) > 0$$

The distribution of X_3 depends on not just X_2 but also the X_1 and hence $\{X_n\}$ is NOT a Markov Chain.

Example 1: Busy Phone Line (Cont'd)

Let Y_n be the remaining time of the call in the nth time interval if the line is busy, and $Y_n = 0$ if the line is free in the nth time interval. Is $\{Y_n\}$ a Markov chain?

$$Y_{n+1} = \begin{cases} Y_n - 1 & \text{if } Y_n > 0 \\ 2 & \text{with prob } \alpha \text{ if } Y_n = 0 \\ 0 & \text{with prob } 1-\alpha \text{ if } Y_n = 0 \end{cases}$$

The transition matrix is

$$\mathbb{P} = \begin{matrix} & 0 & 1 & 2 \\ 0 & 1-\alpha & 0 & \alpha \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{matrix}$$

Example 1: Busy Phone Line (Cont'd)

Is $\{Y_n\}$ still a Markov chain if the lengths of calls are random: 50% of the calls are of length 1, 30% of length 2 and 20% of length 3? Assume the lengths of calls are independent of each other.

$$Y_{n+1} = \begin{cases} Y_n - 1 & \text{if } Y_n > 0 \\ 1 & \text{with prob } 0.5\alpha \text{ if } Y_n = 0 \\ 2 & \text{with prob } 0.3\alpha \text{ if } Y_n = 0 \\ 3 & \text{with prob } 0.2\alpha \text{ if } Y_n = 0 \\ 0 & \text{with prob } 1-\alpha \text{ if } Y_n = 0 \end{cases}$$

The transition matrix is

$$\mathbb{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1-\alpha & 0.5\alpha & 0.3\alpha & 0.2\alpha \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Example 2: Random Walk

Consider the following random walk on integers

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with prob } p \\ X_n - 1 & \text{with prob } 1 - p \end{cases}$$

This is a Markov chain because given $X_n, X_{n-1}, X_{n-2}, \ldots$, the distribution of X_{n+1} depends only on X_n but not $X_{n-1}, X_{n-2}, \ldots$. The state space is

$$\mathfrak{X} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\} = \mathbb{Z} = \text{all integers}$$

The transition probability is

$$P_{ij} = \begin{cases} p & \text{if } j = i + 1 \\ 1 - p & \text{if } j = i - 1 \\ 0 & \text{otherwise} \end{cases}$$

Example 2: Random Walk (Cont'd)

$$\mathbb{P} = \begin{array}{c} \\ \vdots \\ -3 \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccccccc} \cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \cdots \end{array} \\ \left(\begin{array}{ccccccccc} \ddots & \ddots & & & & & & & \\ \ddots & 0 & p & & & & & & \\ & 1-p & 0 & p & & & & & \\ & & 1-p & 0 & p & & & & \\ & & & 1-p & 0 & p & & & \\ & & & & 1-p & 0 & p & & \\ & & & & & 1-p & 0 & p & \\ & & & & & & 1-p & 0 & \ddots \\ & & & & & & & \ddots & \ddots \end{array} \right) \end{array}$$

Example 3: Ehrenfest Diffusion Model

Two containers A and B, containing a sum of K balls. At each stage, a ball is selected at random from the totality of K balls, and move to the other container. Let

$X_0 = \#$ of balls in container A in the beginning

$X_n = \#$ of balls in container A after n movements, $n = 1, 2, 3, \ldots$

$$\mathfrak{X} = \{0, 1, 2, \ldots, K\}$$

$$P_{ij} = \begin{cases} \dfrac{i}{K} & \text{if } j = i - 1 \\ \dfrac{K - i}{K} & \text{if } j = i + 1 \\ 0 & \text{otherwise} \end{cases}$$

Example 4: Discrete Queuing Process

A line of customers await in front of 1 server.

- It takes one unit of time to serve 1 customer
- During each period of time, only 1 customer is served.
- If no customer awaits, the server idles

Let $\xi_n = \#$ of customers arriving in the n-th period. Suppose $\{\xi_n, n = 0, 1, 2, \cdots\}$ are i.i.d. with

$$\mathrm{P}(\xi_n = k) = a_k, \quad k = 0, 1, 2, \dots$$

$$a_k \geq 0 \text{ for all } k, \quad \text{and } \sum_{k=0}^{\infty} a_k = 1$$

Let $X_n = \#$ of customers await during the n-th period, including the one being served. Then

$$X_{n+1} = \begin{cases} X_n - 1 + \xi_n & \text{if } X_n \geq 1 \\ \xi_n & \text{if } X_n = 0 \end{cases}$$

Example 3: Discrete Queuing Process (Cont'd)

First, observe that $P_{0k} = P(X_{n+1} = k | X_n = 0) = P(\xi_n = k) = a_k$ since $X_{n+1} = \xi_n$ if $X_n = 0$.

Second, as $X_{n+1} = X_n - 1 + \xi_n = i - 1 + \xi_n$ if $X_n = i > 0$, $X_{n+1} = k$ implies $\xi_n = k - i + 1$ and hence,

$$P_{ik} = P(X_{n+1} = k | X_n = i) = P(\xi_n = k - i + 1) = a_{k-i+1}.$$

The transition probability matrix is then

$$\mathbb{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\ \left(\begin{array}{cccccc} a_0 & a_1 & a_2 & a_3 & a_4 & \cdots \\ a_0 & a_1 & a_2 & a_3 & a_4 & \cdots \\ 0 & a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & 0 & a_0 & a_1 & a_2 & \cdots \\ 0 & 0 & 0 & a_0 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right) \end{array}$$