

STAT 25100 Lecture 5  
3.2 Conditional Probabilities, Multiplication Rule

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# Conditional Probability

## Example – Conditional Probability

A pair of dice is rolled. The sample space  $S$  is

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

▶ What is the probability of **doubles**  
= same number on both dice?

▶ If total is known to be 10+, what  
is the probability of getting a  
double?  $\frac{2}{6} = \frac{1}{3}$

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(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

- ▶ What is the probability of **doubles** = same number on both dice?

double =

$\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$

$$P(\text{double}) = \frac{\#(\text{double})}{\#(S)} = \frac{6}{36}.$$

- ▶ If total is known to be 10+, what is the probability of getting a double?  $\frac{2}{6} = \frac{1}{3}$

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- ▶ If total is known to be 10+, what is the probability of getting a double?  $\frac{2}{6} = \frac{1}{3}$

## Conditional Probabilities

The **conditional probability of  $A$  happens given that  $B$  has occurred** is denoted

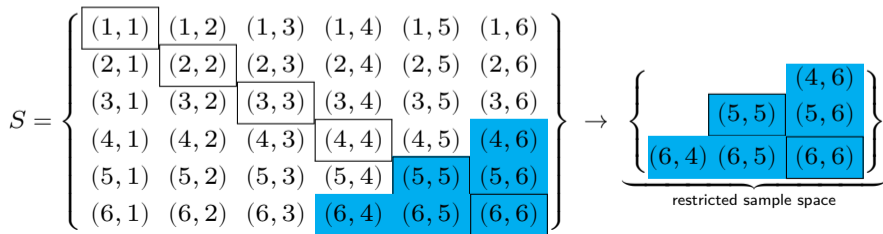
$$P(A | B),$$

and read as the probability of " *$A$  given  $B$* ."

For the example on the previous slide, let

$$\begin{cases} A = \text{getting a double,} \\ B = \text{total is } 10+, \end{cases} \quad \text{we have } P(A | B) = \frac{2}{6} \neq P(A) = \frac{6}{36}.$$

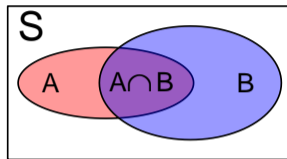
**The given info (total is 10+) changed (restricted) the sample space.**



## Definition of Conditional Probability

The **conditional probability**  $P(A | B)$  is defined as as

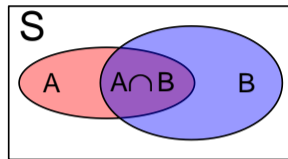
$$P(A | B) = \frac{P(A \cap B)}{P(B)} \quad \text{if } P(B) > 0.$$



## Definition of Conditional Probability

The **conditional probability**  $P(A | B)$  is defined as as

$$P(A | B) = \frac{P(A \cap B)}{P(B)} \quad \text{if } P(B) > 0.$$



**Example.**

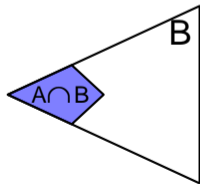
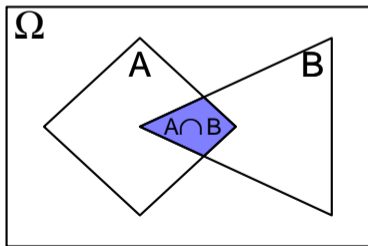
- ▶  $P(\text{total is } 10+) = 6/36$
- ▶  $P(\text{double} \cap \text{total is } 10+) = P(\{(5, 5) \text{ or } (6, 6)\}) = 2/36$

By definition of conditional probability,

$$P(\text{double} | \text{total is } 10+) = \frac{P(\text{double} \cap \text{total is } 10+)}{P(\text{total is } 10+)} = \frac{2/36}{6/36} = \frac{2}{6}.$$

## $P(A | B)$ v.s. $P(A \cap B)$

- ▶  $P(A \cap B)$  is the probability that  $A$  and  $B$  both occur (we are unsure whether  $B$  will occur)
- ▶  $P(A | B)$  is the probability that  $A$  occurs given that  $B$  has occurred
- ▶  $P(A \cap B) = \frac{P(A \cap B)}{P(S)} \rightarrow$  The sample space is  $S$
- ▶  $P(A | B) = \frac{P(A \cap B)}{P(B)} \rightarrow$  The sample space is  $B$ .



## Example 1 — King's Sibling

It's known that the King comes from a family of 2 children.  
Suppose any birth is equally likely to be either gender.

- a. What is the probability that the other child is his sister?

Sol:

- b. If the King is the elder child, what is the probability that the other child is his sister?

Sol:

## Example 1 — King's Sibling

It's known that the King comes from a family of 2 children.

Suppose any birth is equally likely to be either gender.

- a. What is the probability that the other child is his sister?

Sol: Possible genders of the two children: {BB, BG, GB, GG} ← equally likely

As one of the children is male (the King), the possible outcomes are restricted to {BB, BG, GB}, in two of which, the King has a sister.

$$P(\text{King's sibling is a sister} \mid \text{King is male}) = P(\{BG, GB\} \mid \{BB, BG, GB\}) = \frac{2}{3}.$$

- b. If the King is the elder child, what is the probability that the other child is his sister?

Sol: The possible outcomes are restricted to {BB, BG} if the elder child is male (the King). The King has a sister in one of the two cases.

$$P(\text{King's sibling is a sister} \mid \text{King is the elder child}) = P(\{BG\} \mid \{BB, BG\}) = \frac{1}{2}.$$

## Example 2 — Color on the Flip Side

You have 3 cards.

- ▶ Card 1 is Red on both sides, 

$R_1$	$R_2$
-------	-------

,
- ▶ Card 2 is Black on both sides, 

$B_1$	$B_2$
-------	-------

,
- ▶ Card 3 is Red on one side and Black on the other, 

$R_3$	$B_3$
-------	-------

,

After shuffling the cards behind your back, you select one of them at random and place it on your desk with your hand covering it. Upon lifting your hand, you observe that the face showing is red. Which of the following is the correct conditional probability

$P(\text{the other side is Red} \mid \text{the up side is Red})?$

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$P(\text{the other side is Red} \mid \text{the up side is Red})?$

1. Sample space = {Card 1, Card 2, Card 3}. Given the face is Red, it can only be Cards 1 or 3 and their flip sides are Red and Black,  $\Rightarrow$  Answer =  $1/2$ .

## Example 2 — Color on the Flip Side

You have 3 cards.

- ▶ Card 1 is Red on both sides,  $\boxed{R_1 \mid R_2}$ ,
- ▶ Card 2 is Black on both sides,  $\boxed{B_1 \mid B_2}$ ,
- ▶ Card 3 is Red on one side and Black on the other,  $\boxed{R_3 \mid B_3}$ ,

After shuffling the cards behind your back, you select one of them at random and place it on your desk with your hand covering it. Upon lifting your hand, you observe that the face showing is red. Which of the following is the correct conditional probability

$P(\text{the other side is Red} \mid \text{the up side is Red})?$

1. Sample space =  $\{\text{Card 1, Card 2, Card 3}\}$ . Given the face is Red, it can only be Cards 1 or 3 and their flip sides are Red and Black,  $\Rightarrow \text{Answer} = 1/2$ .
2. Sample space =  $\{R_1, R_2, B_1, B_2, R_3, B_3\}$ . Given the face is Red, it could be  $R_1$ ,  $R_2$ , or  $R_3$  and their flip sides are  $R_2$ ,  $R_1$  and  $B_3$ ,  $\Rightarrow \text{Answer} = 2/3$ .

## Example 2 — Color on the Flip Side (Cont'd)

By the definition of conditional probability

$$P(\text{the other side is Red} \mid \text{the up side is Red}) = \frac{P(\text{both sides are Red})}{P(\text{the up side is Red})}.$$

Which sample space allows us to compute  $P(\text{the up side is Red})$ ?

- ▶  $\{\text{Card 1, Card 2, Card 3}\}$
- ▶  $\{R_1, R_2, B_1, B_2, R_3, B_3\}$

## Example 2 — Color on the Flip Side (Cont'd)

By the definition of conditional probability

$$P(\text{the other side is Red} \mid \text{the up side is Red}) = \frac{P(\text{both sides are Red})}{P(\text{the up side is Red})}.$$

Which sample space allows us to compute  $P(\text{the up side is Red})$ ?

- ▶ {Card 1, Card 2, Card 3}
- ▶  $\{R_1, R_2, B_1, B_2, R_3, B_3\}$  ← Ans

## Example 3 — Unconditional Probability

- ▶ A deck of 52 poker cards is well-shuffled and two cards are drawn w/o replacement.

Find  $P(\text{second card is an Ace})$ , without knowing the first card.

Sol:

- ▶ A deck of 52 poker cards is well-shuffled and  $n$  cards are drawn w/o replacement. For  $1 \leq k \leq n$ , find  $P(k\text{th card is an Ace})$ , without knowing other cards.

Sol:

## Example 4

A friend randomly chooses two cards, w/o replacement, from an well-shuffled deck of 52 poker cards. In each of the following situations, determine the conditional probability that both cards are aces.

- Your friend tells you that the first card is an ace.
- Your friend tells you that the second card is an ace.
- Your friend tells you that at least one of the two cards is an ace.

Sol:

## Example 4

A friend randomly chooses two cards, w/o replacement, from an well-shuffled deck of 52 poker cards. In each of the following situations, determine the conditional probability that both cards are aces.

- Your friend tells you that the first card is an ace.  $3/51$
- Your friend tells you that the second card is an ace.  $3/51$
- Your friend tells you that at least one of the two cards is an ace.

Sol: Let  $A = \{\text{two Aces}\}$  and  $B = \{\text{at least one Ace}\}$ . Note  $A \cap B = A$  and  $P(B) = P(\text{at least one Ace}) = 1 - P(\text{neither is ace}) = 1 - \frac{48 \times 47}{52 \times 51}$ .

$$P(A | B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{\frac{4 \times 3}{52 \times 51}}{1 - \frac{48 \times 47}{52 \times 51}} = \frac{1}{33} \approx 0.0303.$$

d. Your friend tells you that one of the two cards is the Ace of Spade.

Sol:

d. Your friend tells you that one of the two cards is the Ace of Spade.

Sol: Let  $A = \{\text{two Aces}\}$  and  $C = \{\text{one of the two is } A\spadesuit\}$ .

Note  $C = C_1 \cup C_2$  where  $C_i = \{\text{ith card is } A\spadesuit\}$  and  $C_1, C_2$  are disjoint. Thus,

$$P(C) = P(C_1) + P(C_2) = \frac{1}{52} + \frac{1}{52} = \frac{2}{52}$$

$$P(AC) = P((AC_1) \cup (AC_2)) = P(AC_1) + P(AC_2) = \frac{1 \times 3}{52 \times 51} + \frac{3 \times 1}{52 \times 51}$$

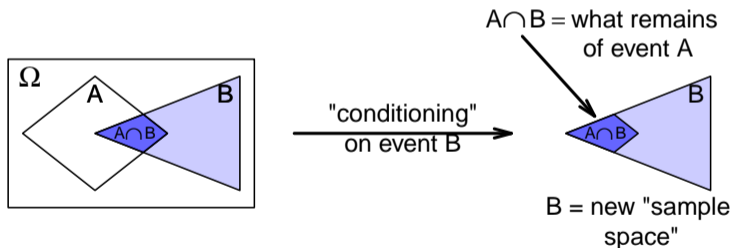
$$P(A | C) = \frac{P(AC)}{P(C)} = \frac{2 \times 1 \times 3 / (52 \times 51)}{(2/52)} = \frac{3}{51}.$$

## Calculation of Conditional Probabilities

Do **NOT** always calculate conditional probabilities by the definition.

$$P(A | B) = \frac{P(A \cap B)}{P(B)}$$

Sometimes, it's more straightforward to find  $P(A | B)$  by thinking about how  $B$  has changed the sample space instead of finding  $P(A \cap B)$ ,  $P(B)$  and their ratio.



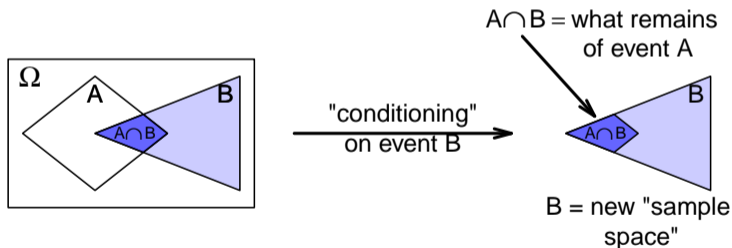
**Ex.** A deck of cards is well-shuffled and two cards are drawn w/o replacement. Find the probability that second card is a King given the first card is a King.

## Calculation of Conditional Probabilities

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**Ex.** A deck of cards is well-shuffled and two cards are drawn w/o replacement. Find the probability that second card is a King given the first card is a King. **3/51**

## Example: Bridge — Sol #1

In the card game bridge, the 52 cards are dealt out equally to 4 players — called **East**, **West**, **North**, and **South**. If **East has 8 ♠'s**, what is the probability that **West has 3 of the remaining 5 ♠'s**?

## Example: Bridge — Sol #1

In the card game bridge, the 52 cards are dealt out equally to 4 players — called **East**, **West**, **North**, and **South**. If **East has 8 ♠'s**, what is the probability that **West has 3 of the remaining 5 ♠'s**?

Sol 1: Given East has 8 spades in his 13 cards, there are 5 ♠'s in the remaining 39 cards, and 34 cards of other suits.

- ▶ The restricted sample space is the  $\binom{39}{13}$  possible hands of for West.
- ▶ Within the restricted sample space, the number of ways that West has 3 of the remaining 5 ♠'s is

$$\binom{5}{3} \binom{34}{10}$$

- ▶ The probability is thus

$$\frac{\binom{5}{3} \binom{34}{10}}{\binom{39}{13}} \approx 0.1614.$$

## Example 5: Bridge — Sol #2

Use the full sample space  $S$  of all possible hands for East and West.

$$\#(S) = \binom{52}{13} \binom{39}{13}.$$

Let  $E_8 = \{\text{East has 8 spades}\}$ ,  $W_3 = \{\text{West has 3 spades}\}$ .

▶  $\#(E_8) = \binom{13}{8} \binom{39}{5} \binom{39}{13}$

▶  $\#(E_8W_3) = \binom{13}{8} \binom{39}{5} \binom{5}{3} \binom{34}{10}$

▶ As  $P(E_8) = \#(E_8)/\#(S)$  and  $P(E_8W_3) = \#(E_8W_3)/\#(S)$ , we get

$$\begin{aligned} P(W_3 | E_8) &= \frac{P(E_8W_3)}{P(E_8)} = \frac{\#(E_8W_3)/\#(S)}{\#(E_8)/\#(S)} = \frac{\#(E_8W_3)}{\#(E_8)} \\ &= \frac{\binom{13}{8} \binom{39}{5} \binom{5}{3} \binom{34}{10}}{\binom{13}{8} \binom{39}{5} \binom{39}{13}} = \frac{\binom{5}{3} \binom{34}{10}}{\binom{39}{13}} \approx 0.1614. \end{aligned}$$

## Multiplication Rule

## Multiplication Rule

The definition of conditional probability

$$P(B | A) = \frac{P(A \cap B)}{P(A)}$$

can be used the other way around. Multiplying both sides by  $P(A)$ , we get the *Multiplication Law*:

$$P(A \cap B) = P(A) \times P(B | A)$$

If we want  $P(A \cap B)$ , and both  $P(A)$ ,  $P(B | A)$  are known or are easy to compute, we can use the Multiplication Law.

## Example 6: Multiplication Rule

A deck of cards is shuffled and the two top cards are placed face down on a table. What is the probability that both cards are Kings?

**Solution.** Let

$K_1 =$  1st card is a King,

$K_2 =$  2nd card is a King.

- ▶  $P(K_1) = P(\text{the 1st card is a King}) =$  .
- ▶ Given that the 1st card is a King, the conditional probability that the 2nd card is a King = ? .
- ▶ So the probability that both cards are Kings = ?

## Example 6: Multiplication Rule

A deck of cards is shuffled and the two top cards are placed face down on a table. What is the probability that both cards are Kings?

**Solution.** Let

$K_1 =$  1st card is a King,

$K_2 =$  2nd card is a King.

- ▶  $P(K_1) = P(\text{the 1st card is a King}) = 4/52.$
- ▶ Given that the 1st card is a King, the conditional probability that the 2nd card is a King = ? .
- ▶ So the probability that both cards are Kings = ?

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- ▶  $P(K_1) = P(\text{the 1st card is a King}) = 4/52.$
- ▶ Given that the 1st card is a King, the conditional probability that the 2nd card is a King = ?  $P(K_2 | K_1) = \frac{3}{51}.$
- ▶ So the probability that both cards are Kings = ?

## Example 6: Multiplication Rule

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**Solution.** Let

$K_1 =$  1st card is a King,

$K_2 =$  2nd card is a King.

▶  $P(K_1) = P(\text{the 1st card is a King}) = 4/52.$

▶ Given that the 1st card is a King, the conditional probability that the 2nd card is a King = ?  $P(K_2 | K_1) = \frac{3}{51}.$

▶ So the probability that both cards are Kings = ?

$$P(K_1 K_2) = P(K_1) \times P(K_2 | K_1) = \frac{4}{52} \times \frac{3}{51} = \frac{1}{221} \approx 0.0045.$$

## Multiplication Rule for Several Events

$$P(ABC) = P(A) \cdot P(B | A) \cdot P(C | AB)$$

$$P(ABCD) = P(A) \cdot P(B | A) \cdot P(C | AB) \cdot P(D | ABC)$$

$$P(ABCDE) = P(A) \cdot P(B | A) \cdot P(C | AB) \cdot P(D | ABC) \cdot P(E | ABCD)$$

and so on

## Example 7: Multiplication Rule for Several Events

Five (5) people are to be randomly chosen from 10 married couples (20 people in total) to form a committee. Find  $P(\text{the 5 chosen are unrelated (no married couples)})$ .

*Sol.* Let  $A_i = \{\text{the } i\text{th person chosen is not related to the first } i - 1 \text{ chosen}\}$ ,  $i = 2, 3, 4, 5$ . Note  $\{\text{All 5 are unrelated}\} = A_2A_3A_4A_5$ .

▶  $P(A_2) = 18/19$

▶  $P(A_3 | A_2) =$

▶  $P(A_4 | A_2A_3) =$

▶  $P(A_5 | A_2A_3A_4) =$

▶ By the Multiplication Rule,

$$\begin{aligned} P(A_2A_3A_4A_5) &= P(A_2)P(A_3 | A_2)P(A_4 | A_2A_3)P(A_5 | A_2A_3A_4) \\ &= \frac{18}{19} \times \frac{16}{18} \times \frac{14}{17} \times \frac{12}{16} \approx 0.52. \end{aligned}$$

## Example 8 — One Ace Each — Sol#1 (Counting)

An ordinary deck of 52 playing cards is randomly divided into 4 piles of 13 cards each. Compute  $P(\text{each pile has exactly 1 ace})$ .

Sol#1: Let the sample space  $S$  be the  $\binom{52}{13,13,13,13}$  possible ways to divide 52 cards into 4 piles for East, West, North and South.

The number of ways that each pile has exactly 1 ace is

$$\binom{4}{1,1,1,1} \binom{48}{12,12,12,12} = \frac{4!}{1!1!1!1!} \times \frac{48!}{12!12!12!12!} = 4! \times \frac{48!}{12!12!12!12!}$$

since

- ▶ there are  $\binom{4}{1,1,1,1}$  ways to divide the 4 aces into 4 labeled groups of size 1
- ▶ there are  $\binom{48}{12,12,12,12}$  ways to divide the 48 non-ace cards into 4 labeled groups of size 12. Thus

$$P(1 \text{ ace each}) = \frac{4! \times \frac{48!}{12!12!12!12!}}{\frac{52!}{13!13!13!13!}} = \frac{4! \times 13^4}{52 \cdot 51 \cdot 50 \cdot 49} \approx 0.1055.$$

## Example 8 — One Ace Each — Sol#2 (Conditional Prob.)

Sol#2: Define events  $E_i$ ,  $i = 1, 2, 3, 4$ , as follows:

$$E_1 = \{ \boxed{A\spadesuit} \text{ is in any piles} \}$$

$$E_2 = \{ \boxed{A\heartsuit} \text{ is not in the pile that include } \boxed{A\spadesuit} \}$$

$$E_3 = \{ \boxed{A\diamondsuit} \text{ is not in the two piles that include } \boxed{A\spadesuit} \boxed{A\heartsuit} \}$$

$$E_4 = \{ \text{all 4 aces are in different piles} \}$$

Clearly,  $P(E_1) = 1$ . Given  $E_1$ , the card  $\boxed{A\heartsuit}$  cannot be in the pile that includes  $\boxed{A\spadesuit}$  and hence must be one of the 39 cards for the other 3 piles. Thus

$$P(E_2 | E_1) = \frac{39}{51}.$$

Similarly, the card  $\boxed{A\diamondsuit}$  cannot be in the two piles that include  $\boxed{A\spadesuit}$  or  $\boxed{A\heartsuit}$ , and hence must be one of the 26 cards for the other 2 piles. Thus

$$P(E_3 | E_1 E_2) = \frac{26}{50}.$$

By the same logic, the last ace can only be one of the 13 cards of the last pile.

$$P(E_4 | E_1 E_2 E_3) = \frac{13}{49}.$$

$$\begin{aligned} P(E_1 E_2 E_3 E_4) &= P(E_1)P(E_2 | E_1)P(E_3 | E_1 E_2)P(E_4 | E_1 E_2 E_3) \\ &= 1 \times \frac{39}{51} \times \frac{26}{50} \times \frac{13}{49} \approx 0.1055. \end{aligned}$$

## Example 9A — Exactly 1 Match in Matching Problem

Suppose that each of  $N$  men at a party throws his hat into the center of the room. The hats are first mixed up, and then each man randomly selects a hat.

Find  $P(\text{exactly 1 of the } N \text{ men gets his own hat}) = P(\text{exactly 1 match in } N)$ .

Sol: Recall in Lecture 4 we showed that

$$P_N = P(\text{None of the } N \text{ men gets his own hat}) = \sum_{i=0}^N \frac{(-1)^i}{i!}.$$

Let  $E_i = \{i\text{th man gets his own hat and no other match}\}$ .

Then  $\{\text{exactly 1 match in } N\} = \cup_{i=1}^N E_i = E_1 \cup E_2 \cup \dots \cup E_N$ . Are  $E_i$ 's disjoint?

$$\begin{aligned} P(E_i) &= P(i\text{th man gets his hat}) \\ &\quad \times P(\text{none of the other } N-1 \text{ matches} \mid i\text{th man gets his own hat}) \\ &= \frac{1}{N} P_{N-1} \end{aligned}$$

$$\begin{aligned} \text{Thus, } P(\text{exactly 1 match in } N) &= P(\cup_{i=1}^N E_i) = \sum_{i=1}^N P(E_i) \\ &= \sum_{i=1}^N \frac{1}{N} P_{N-1} = N \cdot \frac{1}{N} P_{N-1} = P_{N-1}. \end{aligned}$$

## Example 9B — Exactly 2 Matches in Matching Problem

Find  $P(\text{exactly 2 of the } N \text{ men get their own hats}) = P(\text{Exactly 2 matches in } N)$ .

Sol: Let  $E_{i,j} = \{i\text{th and } j\text{th men get their own hats and no other matches}\}$ .

Then  $\{\text{exactly 2 matches in } N\} = \cup_{i,j=1}^N E_{i,j}$ . Are  $E_{i,j}$ 's disjoint?

$$\begin{aligned} P(E_{i,j}) &= P(\text{ith \& jth men get their hat}) \\ &\quad \times P(\text{none of the other } N - 2 \text{ matches} \mid \text{ith \& jth men gets their hat}) \\ &= \frac{1}{N(N-1)} P_{N-2} \end{aligned}$$

Thus,

$$\begin{aligned} P(\text{exactly 2 matches in } N) &= \sum_{i,j=1}^N P(E_{i,j}) = \sum_{i,j=1}^N \frac{1}{N(N-1)} P_{N-2} \\ &= \binom{N}{2} \frac{1}{N(N-1)} P_{N-2} = \frac{N(N-1)}{2!} \frac{1}{N(N-1)} P_{N-2} = \frac{1}{2!} P_{N-2} \end{aligned}$$

## Example 9C — Exactly $k$ Matches in Matching Problem

Find  $P(\text{exactly } k \text{ of the } N \text{ men get their own hats}) = P(\text{Exactly } 2 \text{ matches in } N)$ .

Sol: Let  $E_{i_1, i_2, \dots, i_k} = \{i_1 \text{th, } i_2 \text{th, } \dots, i_k \text{th men get their own hats and no other does}\}$ .

Then  $\{\text{exactly } k \text{ matches in } N\} = \cup_{i_1, i_2, \dots, i_k} E_{i_1, i_2, \dots, i_k}$ . Are  $E_{i_1, i_2, \dots, i_k}$ 's disjoint?

$$\begin{aligned} P(E_{i_1, i_2, \dots, i_k}) &= P(i_1^{th}, \dots, i_k^{th} \text{ men get their hats}) \\ &\quad \times P(\text{none of the other } N - k \text{ matches} \mid i_1^{th}, \dots, i_k^{th} \text{ men gets their hat}) \\ &= \frac{(N - k)!}{N!} P_{N-k} \end{aligned}$$

Thus,

$$\begin{aligned} P(\text{exactly } k \text{ matches in } N) &= \sum_{i_1, i_2, \dots, i_k} P(E_{i_1, i_2, \dots, i_k}) = \sum_{i_1, i_2, \dots, i_k} \frac{(N - k)!}{N!} P_{N-k} \\ &= \binom{N}{k} \frac{(N - k)!}{N!} P_{N-k} = \frac{N!}{k!(N - k)!} \frac{(N - k)!}{N!} P_{N-k} = \frac{1}{k!} P_{N-k} \end{aligned}$$