# STAT 234 Lecture 17
# Confidence Intervals for one Population Mean
# Section 8.1-8.3

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# Outline

CIs for a population mean $\mu$:

- z-CI w/ known population SD: $\overline{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$
- z-CI w/ unknown population SD: $\overline{x} \pm z_{\alpha/2} \frac{s}{\sqrt{n}}$ (requires a large $n$)
- t-CI w/ unknown population SD: $\overline{x} \pm t_{\alpha/2,n-1} \frac{s}{\sqrt{n}}$ (requires a normal population distribution)

# t-Confidence Interval

In Lecture 16B, we introduced two CIs for a population mean $\mu$

- z-CI w/ known population SD: $\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$
- z-CI w/ unknown population SD: $\bar{x} \pm z_{\alpha/2} \frac{s}{\sqrt{n}}$ (requires a large sample size$n$)

What if $\sigma$ is *unknown* and the sample size $n$ is not so large?

By CLT, if $\overline{X}$ is the sample mean of i.i.d. $X_1, X_2, \ldots, X_n$ sampled from a population with **unknown mean** $\mu$ and SD $\sigma$, then

$$Z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}}$$

is nearly $N(0, 1)$ when the sample size $n$ is large.

# Student's $t$-Distributions

If $X_1, X_2, \ldots, X_n$ are i.i.d. from $N(\mu, \sigma^2)$, then

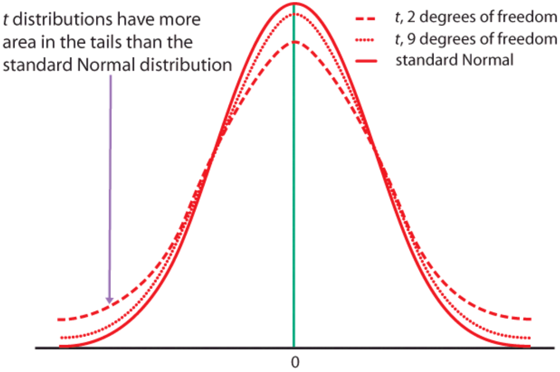
$$T = \frac{\overline{X} - \mu}{s/\sqrt{n}}$$

can be shown to have a *$t$ distribution* with *$n - 1$ degrees of freedom.*

What's is a $t$-distribution?

# Density Curves of $t$-Distributions

- Bell-shaped, symmetric about 0
- more spread out than normal — *heavier tails*
- Shape of the curves determined by the *degrees of freedom (df).* The larger the df, the lighter the tails, the closer the $t$-curve to the $N(0, 1)$ curve
- As df $= \infty$, $t$-curve = standard normal curve



# The Extra Variability of $t$-Distribution Makes Sense

$$T = \frac{\overline{X} - \mu}{s/\sqrt{n}} \sim t_{n-1} \quad \text{v.s.} \quad Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

- $T$ has greater variability than $Z$ because of the extra variability in the sample SD $s$.

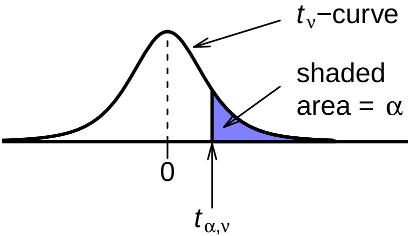
So the $t$-distribution has heavier tail than $N(0, 1)$.

- As the sample size increases, the sample SD $s$ becomes a more accurate estimate of the population SD $\sigma$.

So as df increases, $t$-curve approaches $N(0, 1)$ curve

## Notation $t_{\alpha,\nu}$ — t-critical value

The *t-critical value* $t_{\alpha,\nu}$ is the value for the $t$ distribution with $\nu$ degrees of freedom such that $P(T > t_{\alpha,\nu}) = \alpha$ if $T \sim t_\nu$

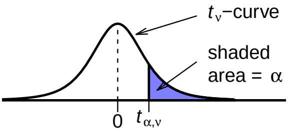



which can be found in R using the `qt()` command.

```
qt(alpha, df=nu, lower.tail=F)
```

# Finding t-Critical Values From a T-table (p.795 in MMSA)

e.g., $t_{4,0.01} = 3.747$

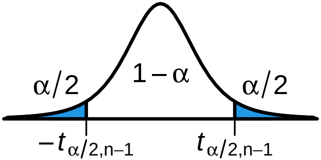



| ν \ α | 0.1 | 0.05 | 0.025 | 0.01 | 0.005 | 0.001 | 0.0005 |
|---|---|---|---|---|---|---|---|
| 1 | 3.078 | 6.314 | 12.706 | 31.821 | 63.657 | 318.309 | 636.619 |
| 2 | 1.886 | 2.920 | 4.303 | 6.965 | 9.925 | 22.327 | 31.599 |
| 3 | 1.638 | 2.353 | 3.182 | 4.541 | 5.841 | 10.215 | 12.924 |
| 4 | 1.533 | 2.132 | 2.776 | *3.747* | 4.604 | 7.173 | 8.610 |
| 5 | 1.476 | 2.015 | 2.571 | 3.365 | 4.032 | 5.893 | 6.869 |
| 6 | 1.440 | 1.943 | 2.447 | 3.143 | 3.707 | 5.208 | 5.959 |
| 7 | 1.415 | 1.895 | 2.365 | 2.998 | 3.499 | 4.785 | 5.408 |
| 8 | 1.397 | 1.860 | 2.306 | 2.896 | 3.355 | 4.501 | 5.041 |
| 9 | 1.383 | 1.833 | 2.262 | 2.821 | 3.250 | 4.297 | 4.781 |
| 10 | 1.372 | 1.812 | 2.228 | 2.764 | 3.169 | 4.144 | 4.587 |
| 11 | 1.363 | 1.796 | 2.201 | 2.718 | 3.106 | 4.025 | 4.437 |
| 12 | 1.356 | 1.782 | 2.179 | 2.681 | 3.055 | 3.930 | 4.318 |
| 13 | 1.350 | 1.771 | 2.160 | 2.650 | 3.012 | 3.852 | 4.221 |
| 14 | 1.345 | 1.761 | 2.145 | 2.624 | 2.977 | 3.787 | 4.140 |
| 15 | 1.341 | 1.753 | 2.131 | 2.602 | 2.947 | 3.733 | 4.073 |
| 16 | 1.337 | 1.746 | 2.120 | 2.583 | 2.921 | 3.686 | 4.015 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |

# t-Confidence Interval for a Population Mean

For i.i.d. $X_1, X_2, \ldots, X_n \sim N(\mu, \sigma^2)$, then

$$T = \frac{\overline{X} - \mu}{s/\sqrt{n}} \sim t_{n-1}$$




and hence

$$P\left(-t_{\alpha/2,n-1} < T = \frac{\overline{X} - \mu}{s/\sqrt{n}} < t_{\alpha/2,n-1}\right) = 1 - \alpha$$

or equivalently

$$P\left(\overline{X} - t_{\alpha/2,n-1}\frac{s}{\sqrt{n}} < \mu < \overline{X} + t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}\right) = 1 - \alpha$$

A $(1 - \alpha)100\%$ t-CI for $\mu$ is given by

$$\overline{X} \pm t_{\alpha/2,n-1} \times \frac{s}{\sqrt{n}}$$

# Example: Finding t-Critical Values for t-CIs (1)

Find the $t$-critical value for a 99% t-CI when $n = 10$

- df = $\nu = n - 1 = 10 - 1 = 9$

```
qt(0.01/2, df=10-1, lower.tail=FALSE)
[1] 3.249836
```

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Find the $t$-critical value for a 99% t-CI when $n = 10$

- df = $\nu = n - 1 = 10 - 1 = 9$
- $\alpha = 0.01$ for a 99% CI
- $t$-critical value = $t_{\alpha/2,n-1} = t_{0.01/2,10-1} = t_{0.005,9} = 3.250$

| | $\alpha$ | 0.1 | 0.05 | 0.025 | 0.01 | 0.005 | 0.001 | 0.0005 |
|---|---|---|---|---|---|---|---|---|
| $\nu$ | 6 | 1.440 | 1.943 | 2.447 | 3.143 | 3.707 | 5.208 | 5.959 |
| | 7 | 1.415 | 1.895 | 2.365 | 2.998 | 3.499 | 4.785 | 5.408 |
| | 8 | 1.397 | 1.860 | 2.306 | 2.896 | 3.355 | 4.501 | 5.041 |
| | 9 | 1.383 | 1.833 | 2.262 | 2.821 | *3.250* | 4.297 | 4.781 |
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```r
qt(0.01/2, df=10-1, lower.tail=FALSE)
[1] 3.249836
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| $\nu$ | 6 | 1.440 | 1.943 | 2.447 | 3.143 | 3.707 | 5.208 | 5.959 |
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- 99% CI is $\overline{X} \pm 3.250 \frac{s}{\sqrt{n}}$ when $n = 10$

```
qt(0.01/2, df=10-1, lower.tail=FALSE)
[1] 3.249836
```

# Example: Finding t-Critical Values for t-CIs (2)

Find the $t$-critical value for a 90% t-CI when $n = 12$

- df $= \nu = n - 1 = 12 - 1 = 11$

```
qt(0.1/2, df=12-1, lower.tail=FALSE)
[1] 1.795885
```

# Example: Finding t-Critical Values for t-CIs (2)

Find the $t$-critical value for a 90% t-CI when $n = 12$

- df = $\nu = n - 1 = 12 - 1 = 11$
- $\alpha = 0.1$ for a 90% CI

```
qt(0.1/2, df=12-1, lower.tail=FALSE)
[1] 1.795885
```

# Example: Finding t-Critical Values for t-CIs (2)

Find the $t$-critical value for a 90% t-CI when $n = 12$

- df = $\nu = n - 1 = 12 - 1 = 11$
- $\alpha = 0.1$ for a 90% CI
- $t$-critical value = $t_{\alpha/2,n-1} = t_{0.1/2,12-1} = t_{0.05,11} = 1.796$

| | $\alpha$ | 0.1 | 0.05 | 0.025 | 0.01 | 0.005 | 0.001 | 0.0005 |
|---|---|---|---|---|---|---|---|---|
| $\nu$ | 9 | 1.383 | 1.833 | 2.262 | 2.821 | 3.250 | 4.297 | 4.781 |
| | 10 | 1.372 | 1.812 | 2.228 | 2.764 | 3.169 | 4.144 | 4.587 |
| | 11 | 1.363 | *1.796* | 2.201 | 2.718 | 3.106 | 4.025 | 4.437 |
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| | $\alpha$ | 0.1 | 0.05 | 0.025 | 0.01 | 0.005 | 0.001 | 0.0005 |
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| $\nu$ | 9 | 1.383 | 1.833 | 2.262 | 2.821 | 3.250 | 4.297 | 4.781 |
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- 90% CI is $\overline{X} \pm 1.796 \frac{s}{\sqrt{n}}$ when $n = 12$

```
qt(0.1/2, df=12-1, lower.tail=FALSE)
[1] 1.795885
```

# Example: Thermal Conductivity of Glass

Thermal Conductivity is measured in terms of watts of heat power transmitted per square meter of surface per degree Celsius of temperature difference on the two sides of the material. In these units, glass has conductivity about 1.

The National Institute of Standards and Technology provides exact data on properties of materials. Here are measurements of the thermal conductivity of 11 randomly selected pieces of a particular type of glass:

```
1.11, 1.07, 1.11, 1.07, 1.12, 1.08, 1.08, 1.18, 1.18, 1.18, 1.12
```

The sample mean and sample SD are

$$\bar{x} \approx 1.1182, \quad \text{and} \quad s \approx 0.04378.$$

```
conduct = c(1.11,1.07,1.11,1.07,1.12,1.08,1.08,1.18,1.18,1.18,1.12)
mean(conduct)
[1] 1.118182
sd(conduct)
[1] 0.04377629
```

# 95% t-CI for Thermal Conductivity of Glass

For $n = 11$, df $= 11 - 1 = 10$, $t_{0.05/2,10} \approx 2.228$ can be found in table or in R.

| $\alpha$ | 0.1 | 0.05 | 0.025 | 0.01 | 0.005 | 0.001 | 0.0005 |
|---|---|---|---|---|---|---|---|
| $\nu$ 9 | 1.383 | 1.833 | 2.262 | 2.821 | 3.250 | 4.297 | 4.781 |
| 10 | 1.372 | 1.812 | *2.228* | 2.764 | 3.169 | 4.144 | 4.587 |
| 11 | 1.363 | 1.796 | 2.201 | 2.718 | 3.106 | 4.025 | 4.437 |
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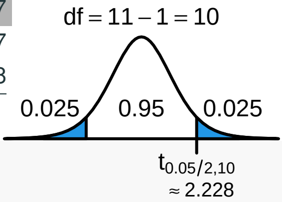


```
qt(0.05/2, df=10, lower.tail=F)
[1] 2.228139
```

95% CI for the mean conductivity of this type of glass is

$$\bar{x} \pm t_{0.05/2,10}\frac{s}{\sqrt{n}} = 1.1182 \pm 2.228 \times \frac{0.04378}{\sqrt{11}} = 1.1182 \pm 0.0294$$
$$= (1.0888, 1.1476)$$

# Finding T-Confidence Intervals in R

The `t.test()` command in R can construct t-CI. Observe the 95% CI (1.088773, 1.147591) given in the output agrees with our calculation.

```
conduct = c(1.11,1.07,1.11,1.07,1.12,1.08,1.08,1.18,1.18,1.18,1.12)
t.test(conduct, level=0.95)

    One Sample t-test

data:  conduct
t = 84.717, df = 10, p-value = 1.284e-15
alternative hypothesis: true mean is not equal to 0
95 percent confidence interval:
 1.088773 1.147591
sample estimates:
mean of x
 1.118182
```

# Which CIs to Use?

So far, we introduced 3 CIs for a population mean $\mu$:

1. z-CI w/ known population SD: $\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$
2. z-CI w/ unknown population SD: $\bar{x} \pm z_{\alpha/2}\frac{s}{\sqrt{n}}$ (needs a large $n$)
3. t-CI w/ unknown population SD: $\bar{x} \pm t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}$ (needs a normal population distribution)

Which one should we use?

- $\overline{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$ is out of question since the population SD $\sigma$ is almost never known

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- The t-CI $\overline{x} \pm t_{\alpha/2,n-1}\dfrac{s}{\sqrt{n}}$ always contains the z-CI $\overline{x} \pm z_{\alpha/2}\dfrac{s}{\sqrt{n}}$ since $t_{\alpha/2,n-1} > z_{\alpha/2}$ as t-distributions have a heavier tail than the standard normal distribution

- $\overline{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$ is out of question since the population SD $\sigma$ is almost never known
- The t-CI $\overline{x} \pm t_{\alpha/2,n-1} \frac{s}{\sqrt{n}}$ always contains the z-CI $\overline{x} \pm z_{\alpha/2} \frac{s}{\sqrt{n}}$ since $t_{\alpha/2,n-1} > z_{\alpha/2}$ as t-distributions have a heavier tail than the standard normal distribution
- *ALWAYS* use t-CIs!

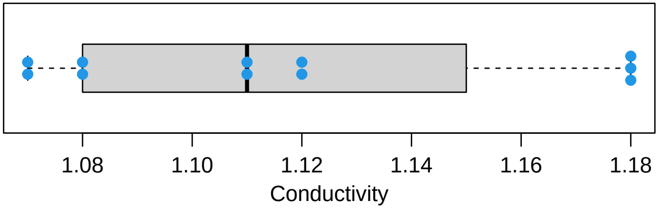
# Conditions to Use $t$-Confidence Intervals

Though $t$-confidence intervals doesn't require a big sample, they still require the following

- *Independence:* The observations should be independent
- *Normality*:
  - For $T = \frac{\overline{X}-\mu}{s/\sqrt{n}}$ to have a $t$-distribution, the population distribution has to be normal, which is rarely true.
  - In particular, it's inherently difficult to verify normality in small data sets.
  - Fortunately, $t$-CIs have some *robustness against non-normality* **except in the case of outliers and strong skewness**. However, their impact diminishes as the sample size gets larger.

# Checking Conditions for the Thermal Conductivity Example

- *Independence:* Suppose the observations are independent.
- *Normality*: The sample distribution does not appear to be extremely skewed, but it's very difficult to assess with such a small sample size. We might want to think about whether we would expect the population distribution to be skewed or not



# Example: Arsenic

Arsenic is toxic to humans and people can be exposed to it through contaminated drinking water, food, dust, and soil. Scientists have devised a non-invasive way to measure a person's level of arsenic poisoning: by examining toenail clippings. In a recent study, scientists measured the level of arsenic (in mg/kg) in toenail clippings of 8 people who lived near a former arsenic mine in Great Britain as follows:

```
0.8, 1.9, 2.7, 3.4, 3.9, 7.1, 11.9, 26.0
```

Suppose the 8 people examined were randomly sampled from residents near the former arsenic mine. Is it legitimate to construct a 95% CI for the mean level of arsenic (in mg/kg) in toenail clippings for residents near the former arsenic mine using a $t$-CI?

M. Button, G. R. T. Jenkin, C. F. Harrington and M. J. Watts, "Human toenails as a biomarker of exposure to elevated environment arsenic," *Journal of Environmental Monitoring*, 2009; 11(3):610-617. Data are reproduced from summary

# Example: Arsenic

Data: `0.8, 1.9, 2.7, 3.4, 3.9, 7.1, 11.9, 26.0`

Data Summary:

```
 min  Q1 median  Q3 max   mean       sd n
 0.8 2.5   3.65 8.3  26 7.2125 8.368041 8
```

At such a small sample size ($n = 8$), a $t$-CI can be used only if the population is fairly normal.

However, from the data summary we can see the sample is *extremely right-skewed* for

- min/Q1 being much closer to the median than max/Q3 is,
- *an extreme outlier 26.0* being over 3 IQRs above Q3.

It's hence not legitimate to use a $t$-CI.