STAT 234 Lecture 13
Central Limit Theorem
Section 6.1-6.2

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- the average of random sample $\{X_1, \ldots, X_n\}$, $\overline{X} = \frac{1}{n}(X_1 + \cdots + X_n)$ is called the *sample mean*
- Observe that the sample mean $\overline{X}$ is also a random variable, which has a probability distribution, called the *sampling distribution of the (sample) mean*.

Weak Law of Large Number

In Lectured 11, we showed if $X_1, \ldots, X_n$ are **i.i.d.** random variables with *mean* μ and *variance* σ^2, then

$$\mathrm{E}(\overline{X}) = \mu, \quad \mathrm{Var}(\overline{X}) = \frac{\sigma^2}{n}$$

from which we can prove the *Weak Law of Large Numbers*:

$$\text{as } n \to \infty, \quad \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \to \mu.$$

Intuitively, this is clear from the mean and the variance of $\overline{X}$; the "center" of the distribution $\overline{X}$ is μ, and the "spread" around it becomes smaller and smaller as n grows.

Sampling Distribution of the (Sample) Mean

Note that the sample mean $\overline{X}$ itself is a random variable, and hence it has a probability distribution, called the *sampling distribution of the (sample) mean*.

The sampling distribution of $\overline{X}$ depends on the **population distribution**. Here are some examples.

- If $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$, then $\overline{X} \sim N(\mu, \sigma^2/n)$.
- If $\overline{X}$ is the average of n Bernoulli random variables $X_1, \ldots, X_n \sim$ Bernoulli(p), then $n\overline{X} \sim Bin(n, p)$, i.e.,

$$P\left(\overline{X} = \frac{k}{n}\right) = \binom{n}{k} p^k (1-p)^{n-k}, \quad 0 \leq k \leq n.$$

and so on.

Central Limit Theorem (CLT)

Let $X_1, X_2, \ldots$ be **i.i.d.** random variables with *mean* μ and *variance* σ^2. CLT asserts that, when n is large,

- the distribution of the *sample mean* $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is approximately

$$N\left(\mu_{\overline{x}} = \mu,\ \sigma^2_{\overline{x}} = \frac{\sigma^2}{n}\right).$$

- the distribution of the *total* $T = \sum_{i=1}^{n} X_i$ is approximately

$$N\left(\mu_T = n\mu,\ \sigma^2_T = n\sigma^2\right).$$

Example 1: Card Game

Recall the card game in Lecture 4, draw ONE card from a well-shuffled deck of cards and get a reward based on the card drawn as follows.

Event	reward X	$p(x)$
Heart (not ace)	\$1	$12/52$
Ace	\$5	$4/52$
King of spades	\$10	$1/52$
All else	\$0	$35/52$
Total		1

- The card drawn is placed back to the deck before he draws the card for the next game.
- Let X_i be the reward he get in the ith game, then X_i's are i.i.d. and his total reward from the 300 games is

$$X_1 + X_2 + \cdots + X_{300}$$

Example 1: Card Game

Recall the pmf for the reward X_i from one game is

x	0	1	5	10
$p_X(x)$	$35/52$	$12/52$	$4/52$	$1/52$

The expected reward from one game and the variance are

$$\mu = \mathrm{E}(X) = 0 \cdot \frac{35}{52} + 1 \cdot \frac{12}{52} + 5 \cdot \frac{4}{52} + 10 \cdot \frac{1}{52} = \frac{21}{26}$$

$$\mathrm{E}(X^2) = 0^2 \cdot \frac{35}{52} + 1^2 \cdot \frac{12}{52} + 5^2 \cdot \frac{4}{52} + 10^2 \cdot \frac{1}{52} = \frac{53}{13}$$

$$\sigma^2 = \mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2 = \frac{53}{13} - \left(\frac{21}{26}\right)^2 = \frac{2315}{26^2}$$

So if a gambler played the game 300 times, his expected value, variance of his total reward is

$$\mathrm{E}(X_1 + \cdots + X_{300}) = 300\mu = 300 \times \frac{21}{26} \approx 243.308$$

$$\mathrm{Var}(X_1 + \cdots + X_{300}) = 300\sigma^2 = 300 \times \frac{2315}{26^2}$$

$$\mathrm{SD}(X_1 + \cdots + X_{300}) = \sqrt{300 \times \frac{2315}{26^2}} = 32.052$$

The gambler is expected to get \$243.308 from the 300 games, with a standard deviation \$32.052.

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What is the probability that the gambler can earn \$250 or more from the 300 games?

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Solution: By CLT, as $n = 300$ is large, the distribution of the total rewards $T = \sum_{i=1}^{300} X_i$ is approx. normal w/

$$\mu_T = n\mu = 300\mu = 243.308, \quad \sigma_T = \sqrt{300}\sigma = 32.052.$$

Thus

$$P(\text{total reward} > \$250) = P\left(Z > \frac{250 - 243.308}{32.052}\right)$$
$$\approx P(Z > 0.21) \approx 1 - 0.5832 \approx 0.417$$

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1- pnorm(250, m = 243.308, s = 32.052)
[1] 0.4173
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Example 2: Shipping Packages

Suppose a company ships packages that vary in weight:

- Packages have mean 15 lb and standard deviation 10 lb.
- Packages weights are independent from each other

Q: What is the probability that the average weight of 100 packages exceeds 17 lb?

Example 2: Shipping Packages — Solutions

Let W_i be the weight of the ith package and the total weights of 100 packages is

$$\overline{W} = \frac{1}{100}\sum_{i=1}^{100} W_i,$$

where W_i's are i.i.d. with mean $\mu_W = 15$ and SD $\sigma_W = 10$.

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By CLT, $\overline{W}$ is approx. $N(\mu_{\overline{W}} = 15, \sigma^2_{\overline{W}} = 1^2)$,

$$P(\overline{W} > 17) = P\left(\frac{\overline{W} - \mu_{\overline{W}}}{\sigma_{\overline{W}}} > \frac{17 - \mu_{\overline{W}}}{\sigma_{\overline{W}}}\right)$$

$$= P\left(Z > \frac{17 - 15}{1}\right) \approx 1 - \Phi(2) \approx 0.023$$

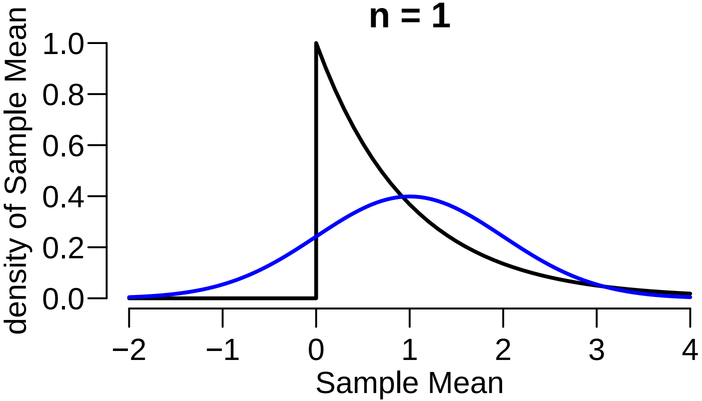
```
1- pnorm(2)
[1] 0.02275
```

If the population distribution is exponential with density

$$f(x) = e^{-x}, \quad \text{for } x > 0, \quad \mu = 1,\ \sigma^2 = 1$$

black curve: the exact sampling distribution of $\overline{X}$,

blue curve: the normal approximation



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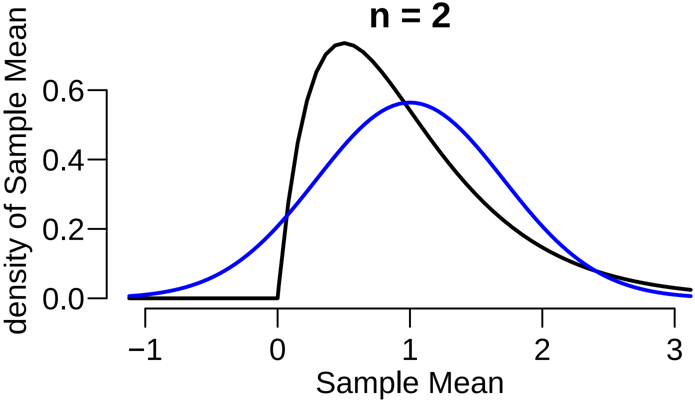

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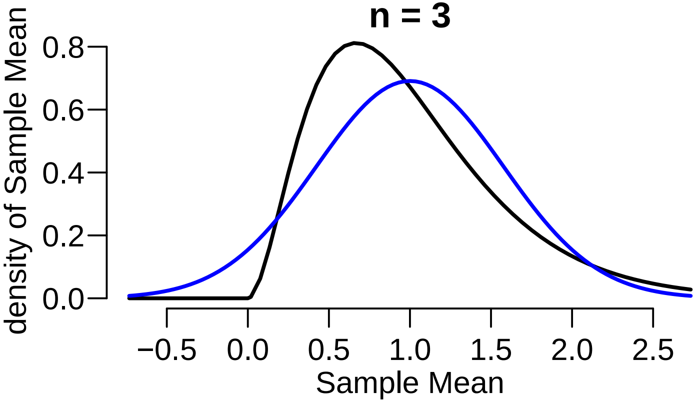

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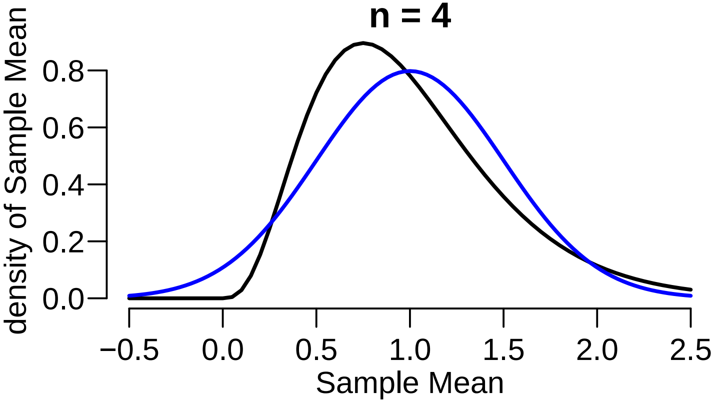

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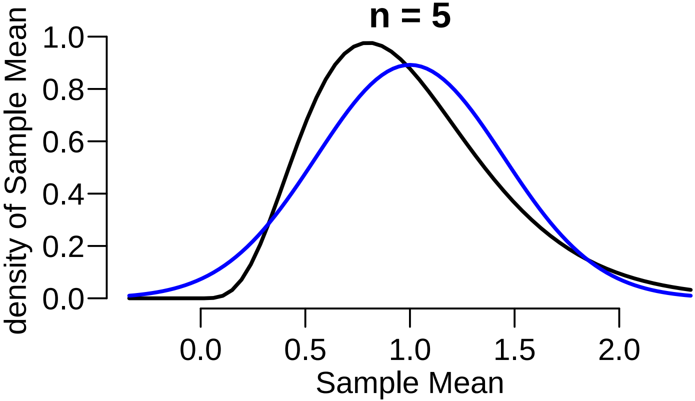

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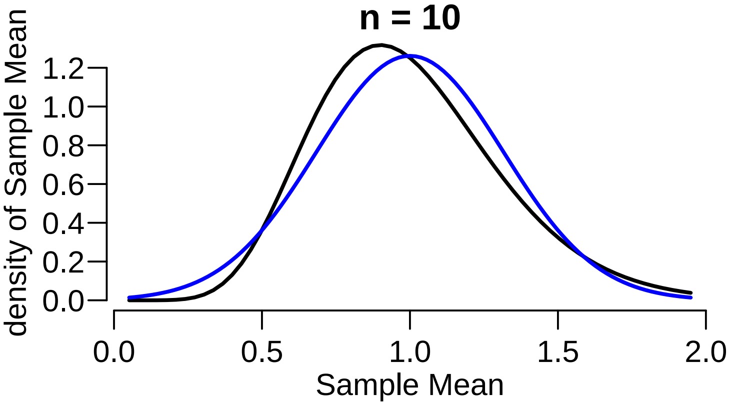

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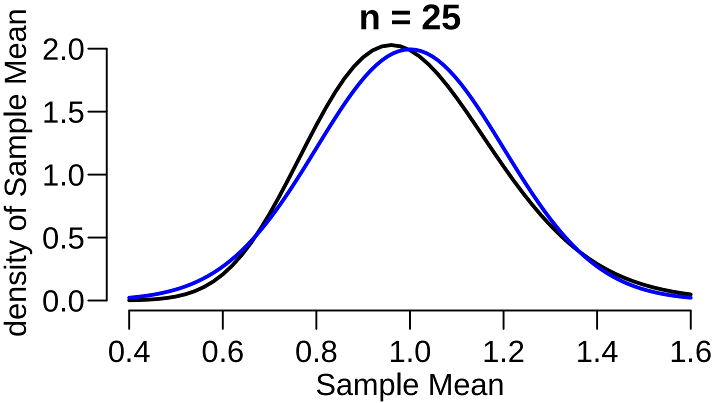

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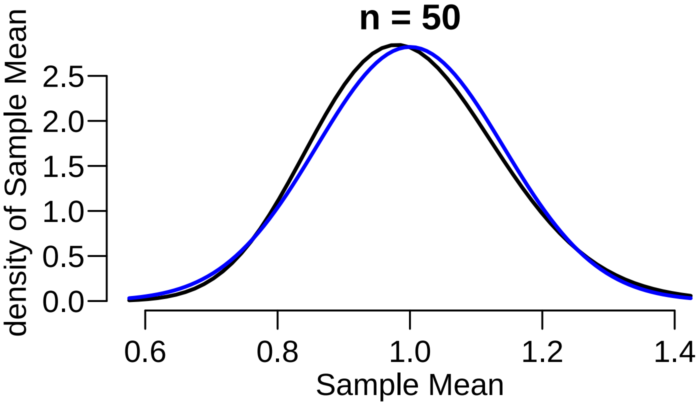

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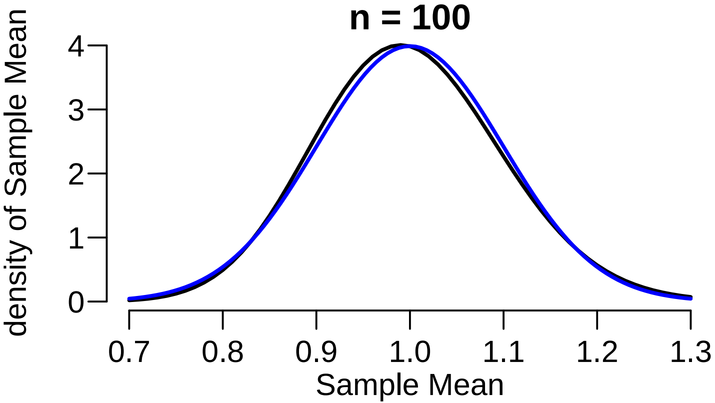

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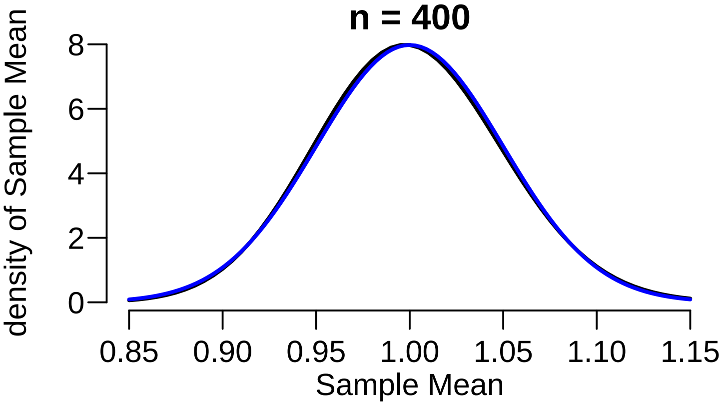

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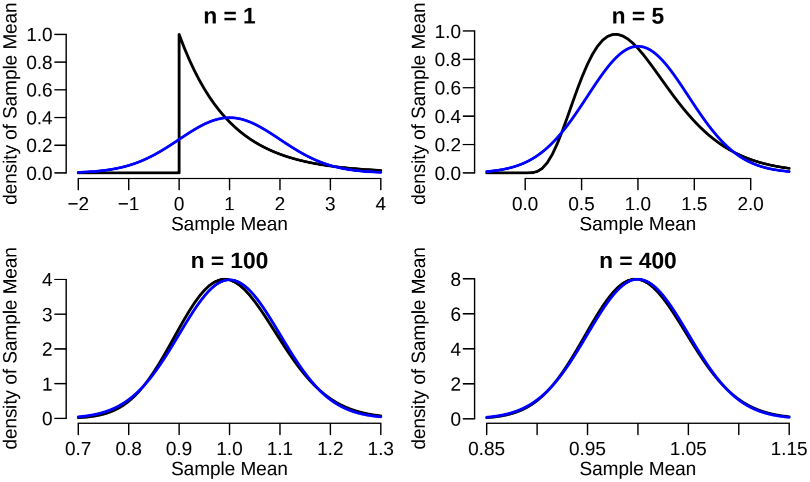

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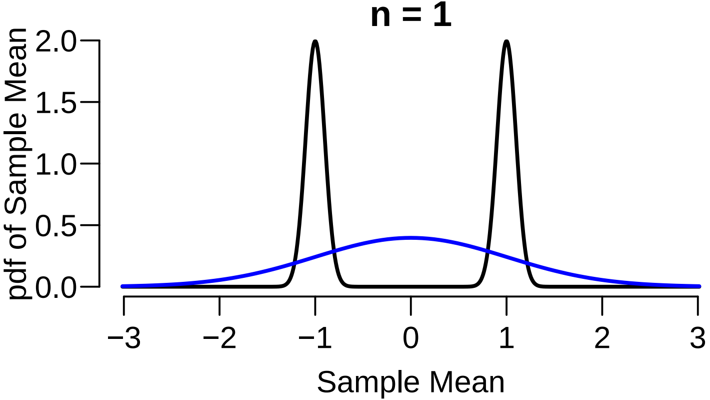

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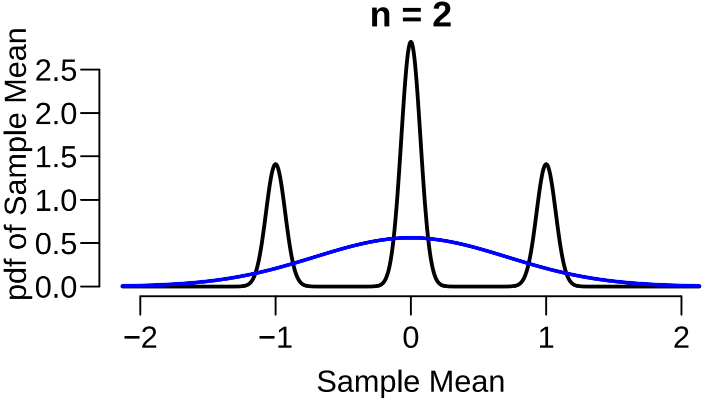

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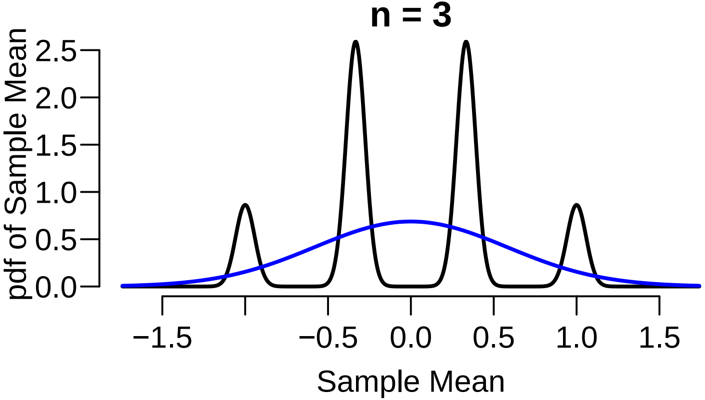

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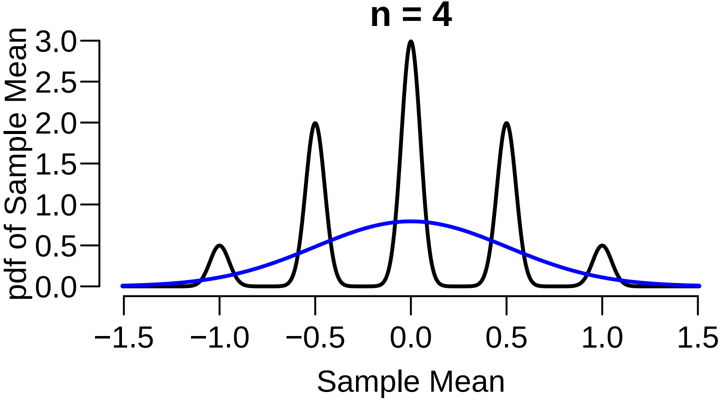

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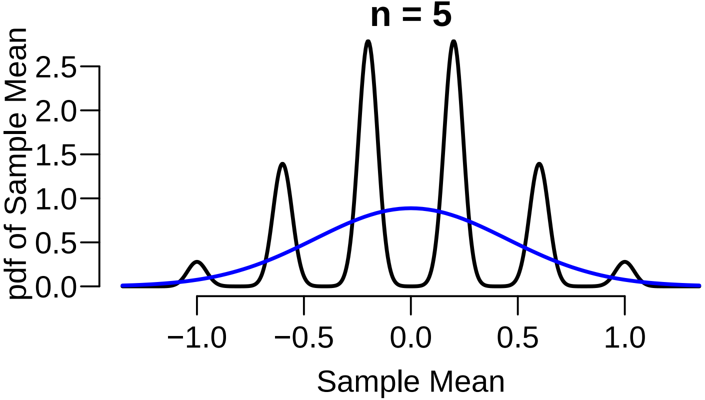

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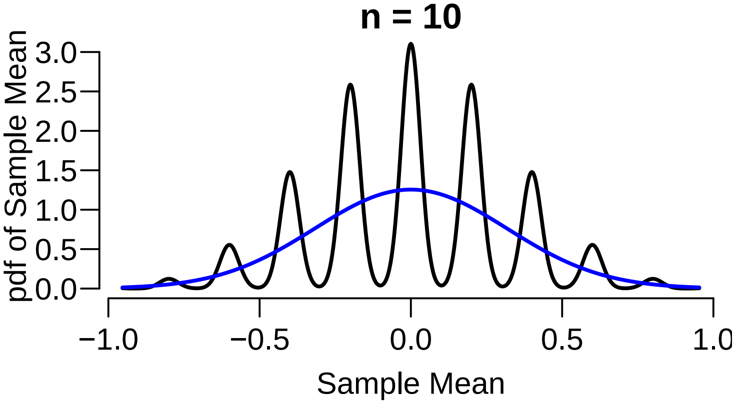

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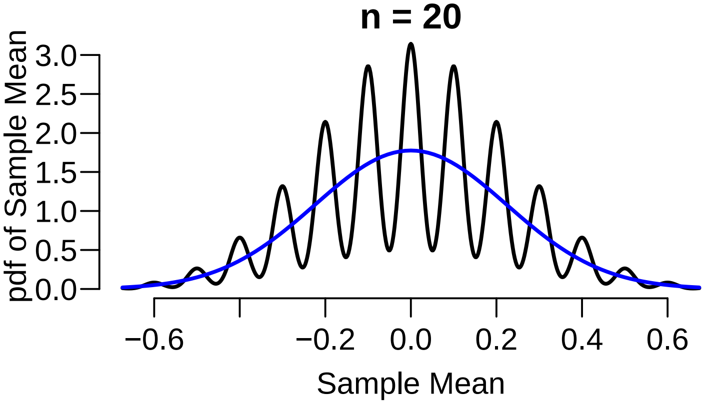

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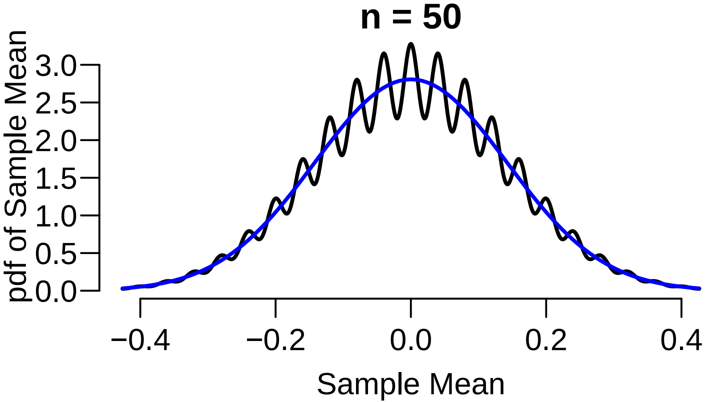

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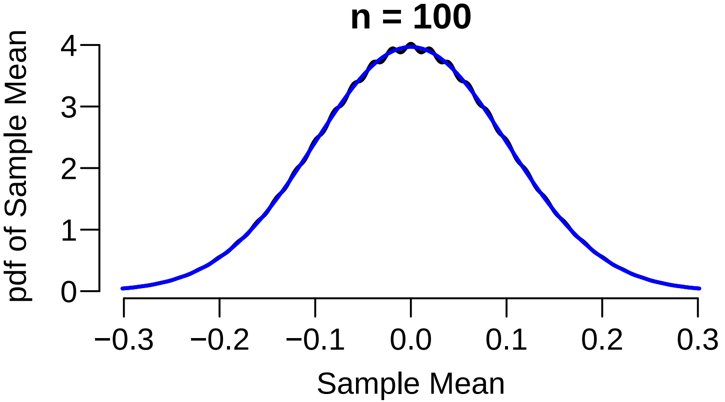

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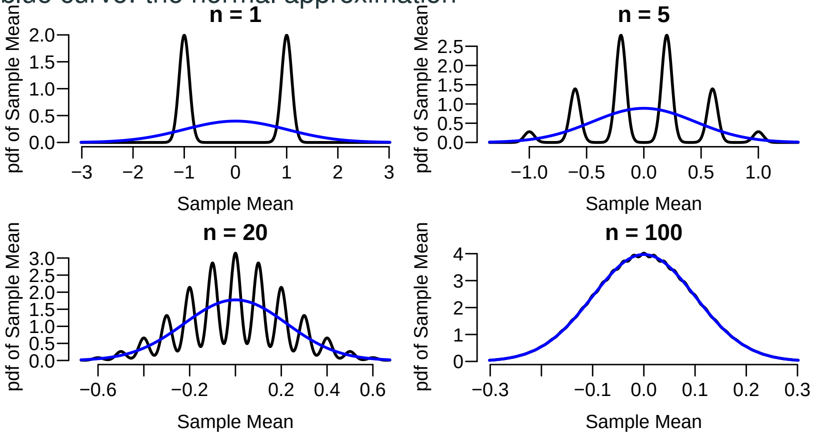

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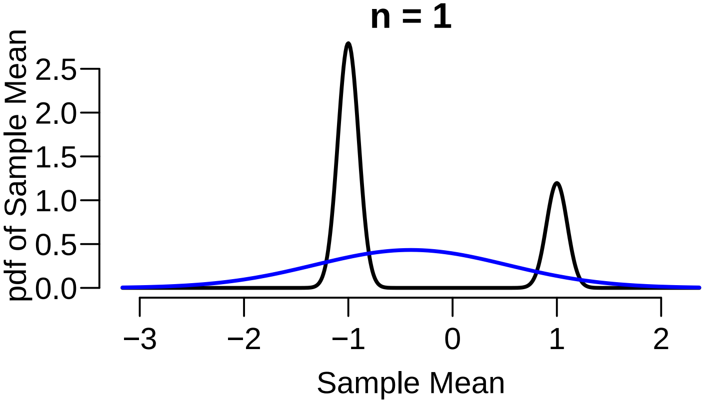

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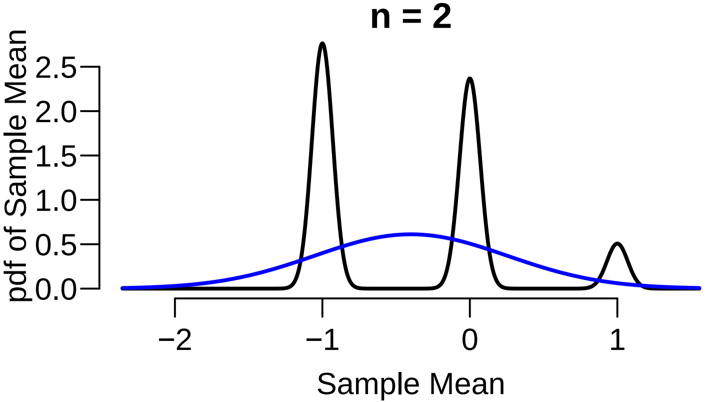

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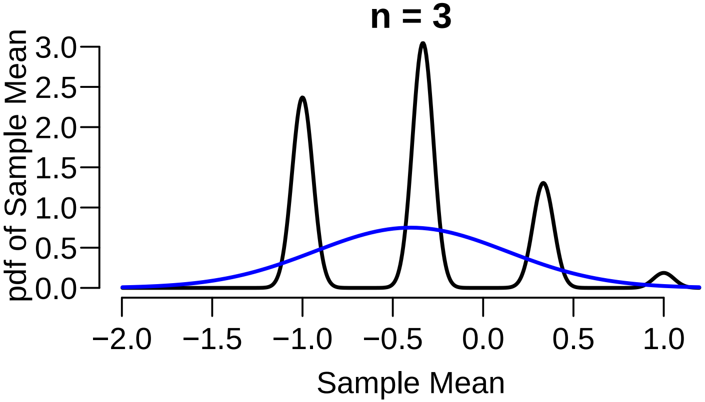

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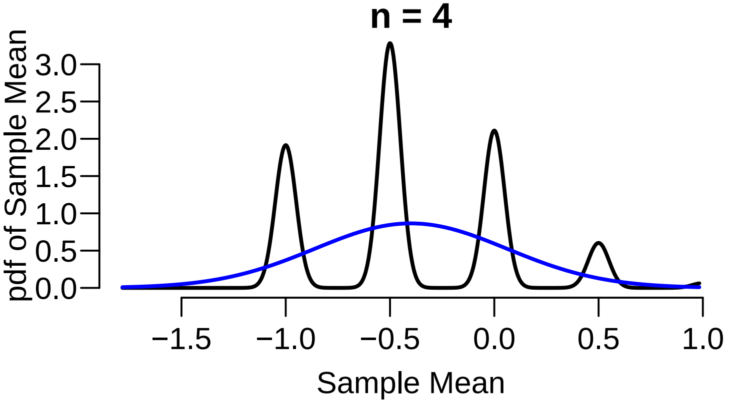

If the population distribution is Bimodal with density

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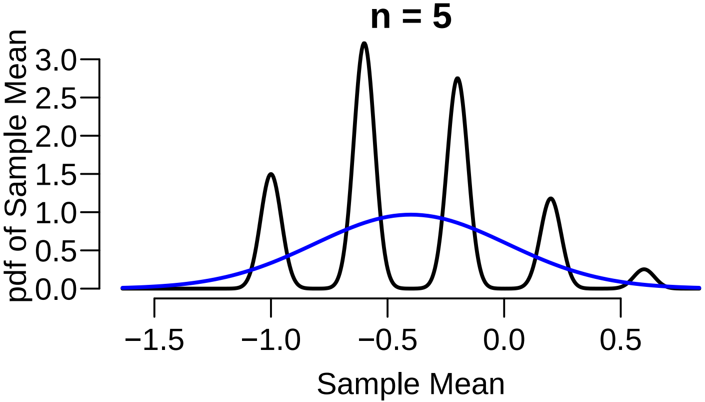

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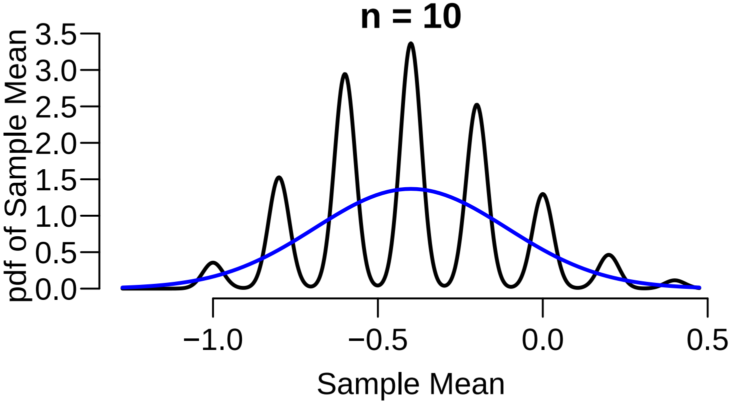

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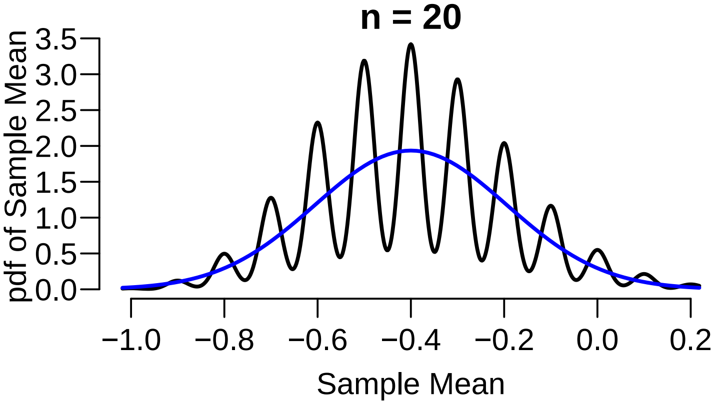

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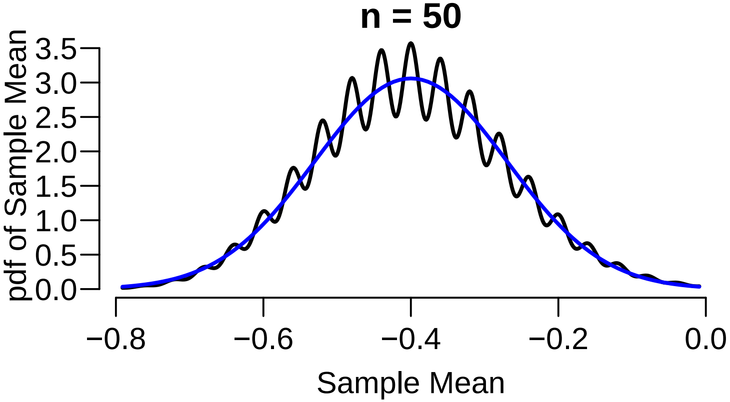

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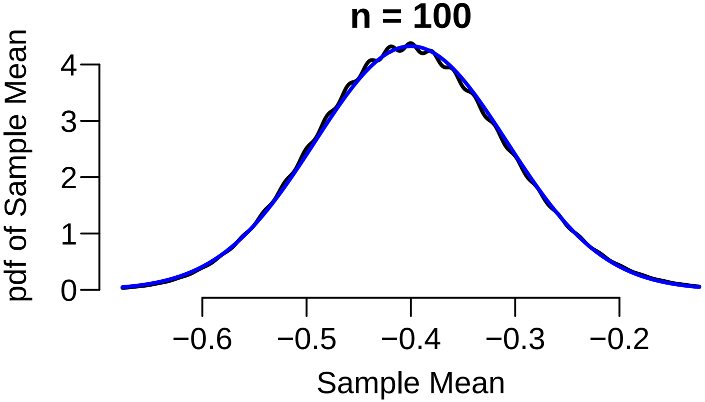

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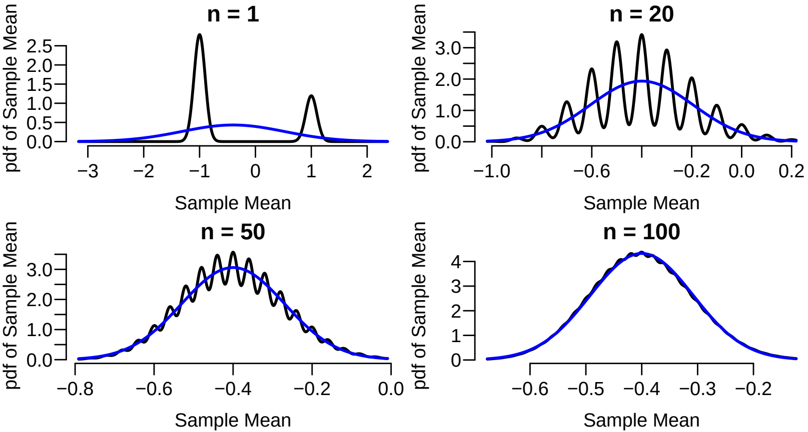

Normal Approximation to Binomial Distribution

Normal approximation to the Binomial distributions is a special case of CLT:

$$X = \sum_{i=1}^{n} X_i \sim Bin(n, p),$$

where $X_1, X_2, ..., X_n$ are n *independent Bernoulli* random variables with success probability p.

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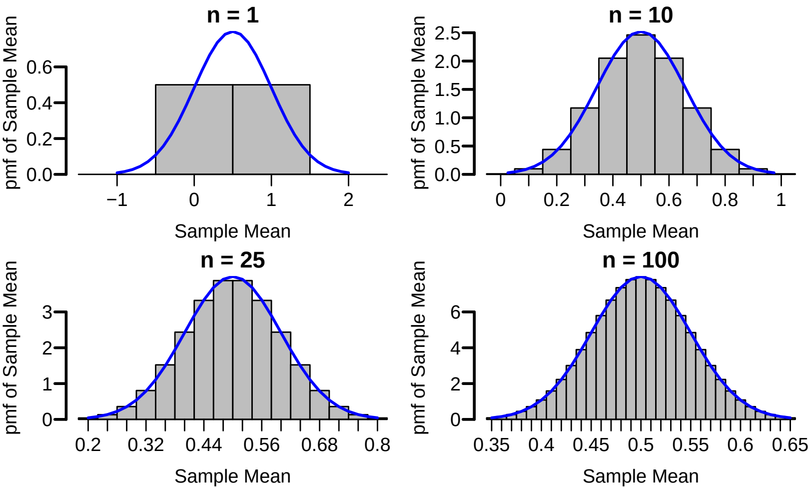
By CLT, for large n, $Y \sim \mathrm{Bin}(n, p)$ is approximately distributed as

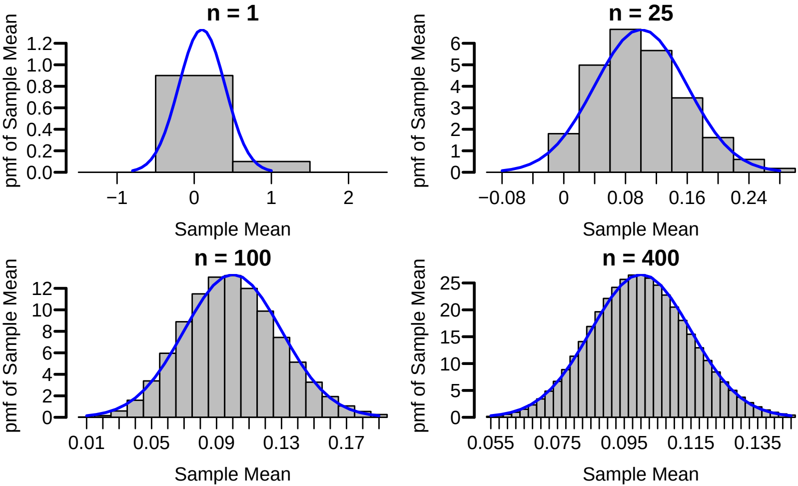
$$N(\mu_Y = np,\ \sigma_Y^2 = np(1-p)).$$

Normal Approximation to $Bin(n, p = 0.5)$

When $X_1, \ldots, X_n \sim$ Bernoulli$(p = 0.5)$, the sampling distribution of $\overline{X}$ is



For $X_1, \ldots, X_n \sim \text{Bernoulli}(p = 0.1)$, the sampling distribution of $\overline{X}$ is



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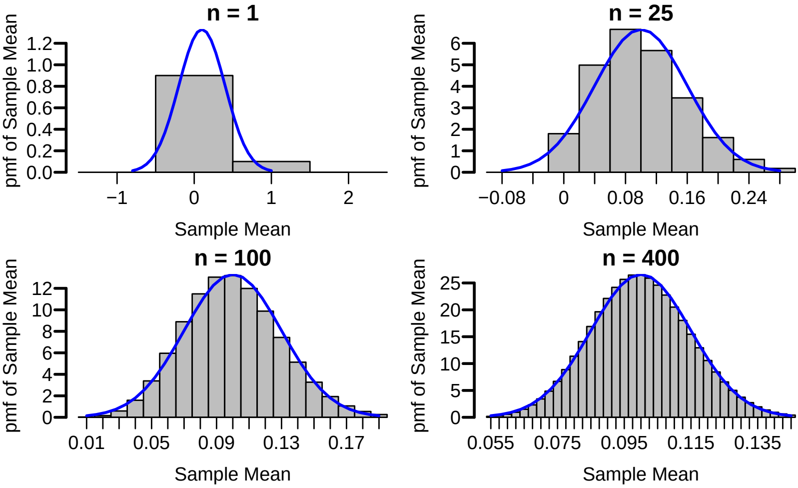

If the population distribution is skewed, so is the sampling distribution of the sample mean, though the skewness diminishes as the number of draws goes up.

Example 3: Roulette Calibration

With a perfectly balanced roulette wheel, red numbers should turn up 18 in 38 of the time. To test its wheel, one casino records the results of 3800 plays. Let X be the number of reds the casino got.

Q1: If the roulette wheel is perfectly balanced, what is the chance that $X \geq 1890$?

Q2 If the casino gets 1890 reds, do you think the roulette wheel should be calibrated?

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Thus

$$E(X) = np = 3800(18/38) = 1800$$

$$SD(X) = \sqrt{np(1-p)} = \sqrt{3800(18/38)(20/38)} \approx 30.78$$

By CLT, X is approximately $N(\mu = 1800, \sigma^2 = (30.78)^2)$. Thus,

$$P(X \geq 1890) \approx P\left(\frac{X - 1800}{30.78} \geq \frac{1890 - 1800}{30.78}\right) \approx P(Z \geq 2.92) \approx 0.00173$$

```
1-pnorm(1890, m = 1800, s = sqrt(3800*(18/38)*(20/38)))
[1] 0.001728
```

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As $X \sim \text{Bin}(n = 3800,\ p = 18/38)$, the exact probability of $X \geq 1890$ is

$$P(X \geq 1890) = \sum_{k=1890}^{3800} \binom{3800}{k} \left(\frac{18}{38}\right)^k \left(\frac{20}{38}\right)^{3800-k} \approx 0.00183$$

found using R as follows.

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Q2 If the casino gets 1890 reds, do you think the roulette wheel should be calibrated? Yes. $X \geq 1890$ is very unlikely to happen.

How Large n Has to Be to Use CLT?

- If the population is normal, then any n will do.
- If the population distribution is symmetric, then n should be at least 30 or so.
- The more skew or irregular the population, the larger n has to be
- For the Binomial distribution, a rule of thumb is that n should be such that

$$np \geq 10 \quad \text{and} \quad n(1-p) \geq 10.$$