STAT 234 Lecture 12
Condition Distributions
Section 5.3

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Conditional Distributions of Discrete Random Variables

Exercise 1 — Gas Station (p.242 in MMSA, Review)

A gas station has both **self-service** and **full-service** islands, each with a single regular unleaded pump with 2 hoses.

X = the # of hoses in use on the self-service island, and

Y = the # of hoses in use on the full-service island

The joint pmf of X and Y:

		Y (full-service)		
	$p(x, y)$	0	1	2
X	0	0.10	0.04	0.02
self-	1	0.08	0.20	0.06
service	2	0.06	0.14	0.30

What is $P(Y = 1 | X = 2)$?

	$p(x,y)$	Y 0	1	2	Row Sum $p_X(x)$
	0	0.10	0.04	0.02	0.16
X	1	0.08	0.20	0.06	0.34
	2	0.06	0.14	0.30	0.50

	$p(x,y)$	Y 0	1	2	Row Sum $p_X(x)$
	0	0.10	0.04	0.02	0.16
X	1	0.08	0.20	0.06	0.34
	2	0.06	0.14	0.30	0.50

By the definition of conditional probability,

$$P(Y = 1|X = 2) = \frac{P(X = 2, Y = 1)}{P(X = 2)} = \frac{p(2,1)}{p_X(2)} = \frac{0.14}{0.50} = 0.28.$$

$p(x,y)$		Y 0	1	2	Row Sum $p_X(x)$
	0	0.10	0.04	0.02	0.16
X	1	0.08	0.20	0.06	0.34
	2	0.06	0.14	0.30	0.50

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The conditional pmf of Y given $X = 2$ is

$$p_{Y|X}(y \mid x = 2) = \frac{P(X = 2, Y = y)}{P(X = 2)} = \frac{p(2,y)}{p_X(2)}$$

y	0	1	2
$p_{Y\|X}(y \mid x = 2)$	$\frac{0.06}{0.50} = 0.12$	$\frac{0.14}{0.50} = 0.28$	$\frac{0.30}{0.50} = 0.60$

$p(x, y)$		Y 0	1	2	$p_X(x)$
	0	0.10	0.04	0.02	0.16
X	1	0.08	0.20	0.06	0.34
	2	0.06	0.14	0.30	0.50

Similarly, the conditional pmf of Y given $X = 0$ is

$$p_{Y|X}(y \mid x = 0) = \frac{P(X = 0, Y = y)}{P(X = 0)} = \frac{p(0, y)}{p_X(0)}$$

y	0	1	2
$p_{Y\|X}(y \mid x = 0)$	$\frac{0.10}{0.16} = 0.625$	$\frac{0.04}{0.16} = 0.25$	$\frac{0.02}{0.16} = 0.125$

	$p(x,y)$	Y 0	1	2	$p_X(x)$
	0	0.10	0.04	0.02	0.16
X	1	0.08	0.20	0.06	0.34
	2	0.06	0.14	0.30	0.50

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y	0	1	2
$p_{Y\|X}(y \mid x = 0)$	$\frac{0.10}{0.16} = 0.625$	$\frac{0.04}{0.16} = 0.25$	$\frac{0.02}{0.16} = 0.125$

and the conditional pmf of Y given $X = 1$ is

$$p_{Y|X}(y \mid x = 1) = \frac{P(X = 1, Y = y)}{P(X = 1)} = \frac{p(1, y)}{p_X(1)}$$

y	0	1	2
$p_{Y\|X}(y \mid x = 1)$	$\frac{0.08}{0.34} \approx 0.235$	$\frac{0.20}{0.34} \approx 0.588$	$\frac{0.06}{0.34} \approx 0.176$

Conditional Distributions of Y given X

The **conditional pmf** of a discrete r.v. Y given another discrete r.v. $X = x$ is

$$p_{Y|X}(y \mid x) = \frac{P(X = x, Y = y)}{P(X = x)} = \frac{p(x, y)}{p_X(x)},$$

in which $p(x, y)$ is the joint pmf of X and Y, and $p_X(x)$ is the marginal pmf of X.

		Y			
conditional pmf	$p(y \mid x)$	0	1	2	Row Sum
	0	0.625	0.25	0.125	1
X	1	0.235	0.588	0.176	1
	2	0.12	0.18	0.60	1
marginal pmf	$p_Y(y)$	0.24	0.38	0.38	1

- Each row is a pmf for Y given some x value
- Observed the row sums of $p_{Y|X}(y \mid x)$ are all 1

Conditional Distributions of X given Y

The **conditional pmf** of a discrete r.v. X given another discrete r.v. $Y = y$ is

$$p_{X|Y}(x \mid y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{p(x, y)}{p_Y(y)},$$

in which $p(x, y)$ is the joint pmf of X and Y, and $p_Y(y)$ is the marginal pmf of Y.

	$p(x,y)$	Y = 0	1	2	$p_X(x)$
X	0	0.10	0.04	0.02	0.16
	1	0.08	0.20	0.06	0.34
	2	0.06	0.14	0.30	0.50
	$p_Y(y)$	0.24	0.38	0.38	

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	0	0.10	0.04	0.02	0.16
X	1	0.08	0.20	0.06	0.34
	2	0.06	0.14	0.30	0.50
	$p_Y(y)$	0.24	0.38	0.38	

	$p(x \mid y)$	Y = 0	1	2	marginal $p_X(x)$
	0	$\frac{0.10}{0.24} \approx 0.417$	$\frac{0.04}{0.38} \approx 0.105$	$\frac{0.02}{0.38} \approx 0.053$	0.16
X	1	$\frac{0.08}{0.24} \approx 0.333$	$\frac{0.20}{0.38} \approx 0.526$	$\frac{0.06}{0.38} \approx 0.158$	0.34
	2	$\frac{0.06}{0.24} = 0.25$	$\frac{0.14}{0.38} \approx 0.368$	$\frac{0.30}{0.38} \approx 0.790$	0.50
	column sum	1	1	1	1

- Each column is a pmf for X given some y value
- Observed the column sums $p_{X|Y}(x \mid y)$ are all 1

In summary,

A conditional pmf of Y given $X = x$ is $p_{Y|X}(y \mid x) = \dfrac{P(x, y)}{p_X(x)}$ which satisfies

$$0 \leq p_{Y|X}(y \mid x) \leq 1 \quad \text{and} \quad \sum_y p_{Y|X}(y \mid x) = 1, \quad \text{for all } x.$$

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$$0 \leq p_{X|Y}(x \mid y) \leq 1 \quad \text{and} \quad \sum_x p_{X|Y}(x \mid y) = 1, \quad \text{for all } y.$$

Conditional Distributions of Continuous Random Variables

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Given two continuous random variables with the joint pdf $f(x, y)$, the **conditional probability distribution of $\mathbf{X}$ given $\mathbf{Y} = \mathbf{y}$** is the function $f_{X|Y}$, and the **conditional probability distribution of $\mathbf{Y}$ given $\mathbf{X} = \mathbf{x}$** is the function $f_{Y|X}$ are defined as

$$f_{X|Y}(x \mid y) = \frac{f(x, y)}{f_Y(y)}, \quad f_{Y|X}(y \mid x) = \frac{f(x, y)}{f_X(x)}$$

- Recall the definition of conditional probability

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

- Similarly, given the joint pdf of two continuous r.v.'s, the conditional pdf is the ratio of the joint pdf and marginal pdf,

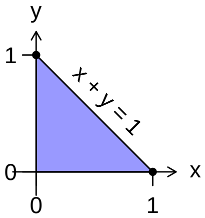
$$f_{X|Y}(x \mid y) = \frac{f(x, y)}{f_Y(y)}, \quad f_{Y|X}(y \mid x) = \frac{f(x, y)}{f_X(x)}$$

Example 5.21 (Mixed Nuts) on p.257 of MMSA

Recall in Lecture 9, the joint pdf for

$$X = \text{ the weight of almonds, and } Y = \text{ the weight of cashews}$$

in a can of mixed nuts is



$$f(x, y) = \begin{cases} 24xy & \text{if } 0 \leq x, y \leq 1, x + y < 1 \\ 0 & \text{otherwise} \end{cases}$$

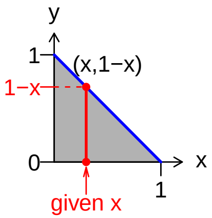
We calculated in L09 the marginal pdf's for X and for Y:

$$f_X(x) = 12x(1-x)^2, \quad f_Y(y) = 12y(1-y)^2, \text{ for } 0 \leq x, y \leq 1.$$

Conditional Distributions — Mixed Nuts

The conditional pdf $f_{X|Y}(x \mid y)$ of X (almond) given $Y = y$ (cashew) is

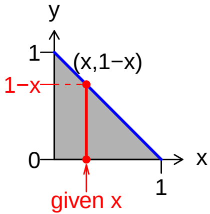
$$f_{X|Y}(x \mid y) = \frac{f(x,y)}{f_Y(y)}$$



Conditional Distributions — Mixed Nuts

The conditional pdf $f_{X|Y}(x \mid y)$ of X (almond) given $Y = y$ (cashew) is

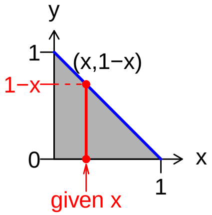
$$f_{X|Y}(x \mid y) = \frac{f(x,y)}{f_Y(y)} = \frac{24xy}{12y(1-y)^2}$$



Conditional Distributions — Mixed Nuts

The conditional pdf $f_{X|Y}(x \mid y)$ of X (almond) given $Y = y$ (cashew) is

$$
\begin{aligned}
f_{X|Y}(x \mid y) &= \frac{f(x,y)}{f_Y(y)} \\
&= \frac{24xy}{12y(1-y)^2} \\
&= \frac{2x}{(1-y)^2}, \qquad \text{for } 0 \le x \le 1-y\,.
\end{aligned}
$$



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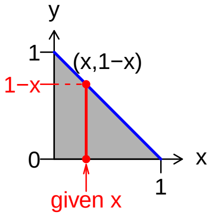
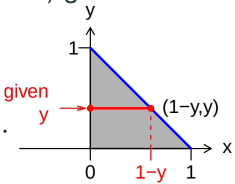


Similarly, the conditional pdf $f_{Y|X}(y \mid x)$ of Y (cashew) given $X = x$ (almond) is

$$f_{Y|X}(y \mid x) = \frac{f(x,y)}{f_X(x)} = \frac{2y}{(1-x)^2}, \quad \text{for } 0 \le y \le 1-x\,.$$



Example: Joint to Conditional

Recall on p.7 of L10A slides, for X and Y w/ the joint pdf

$$f(x,y) = 6xy^2, \quad \text{for } 0 \le x, y \le 1,$$

We found the marginal pdf's of X and of Y to be

$$f_X(x) = 2x,\ 0 < x < 1, \quad \text{and} \quad f_Y(y) = 3y^2,\ 0 < y < 1.$$

The conditional pdf of y given $X = x$ is

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{6xy^2}{2x} = 3y^2, \quad 0 < y < 1.$$

which is exactly the *marginal pdf* of Y.

Recall in L10A, we said X and Y are **independent** since $f(x,y) = 6xy^2 = (2x)(3y^2) = f_X(x)f_Y(y)$ for all $0 \le x, y \le 1$ and $f(x,y) = 0 = f_X(x)f_Y(y)$ elsewhere.

Conditional = Marginal, when Independent

What is the conditional distribution of Y given $X = x$ if X and Y are independent?

$$f_{Y|X}(y|X = x) = \frac{f_{X,Y}(x, y)}{f_X(x)} = \frac{f_X(x) f_Y(y)}{f_X(x)} = f_Y(y).$$

i.e., *conditional pdf $Y|X$ is the marginal pdf of Y*.

Conditional = Marginal, when Independent

What is the conditional distribution of Y given $X = x$ if X and Y are independent?

$$f_{Y|X}(y|X = x) = \frac{f_{X,Y}(x, y)}{f_X(x)} = \frac{f_X(x)f_Y(y)}{f_X(x)} = f_Y(y).$$

i.e., *conditional pdf $Y|X$ is the marginal pdf of Y*.

In fact, the following three are equivalent definitions of the independence of X and Y

- $f(x, y) = f_X(x)f_Y(y)$ . . . . (joint = product of marginal)
- $f_{Y|X}(y|X = x) = f_Y(y)$ . . . . (conditional $Y|X$ = marginal of Y)
- $f_{X|Y}(x|Y = y) = f_X(x)$ . . . . (conditional $X|Y$ = marginal of X)

Conditional = Marginal, when Independent

What is the conditional distribution of Y given $X = x$ if X and Y are independent?

$$f_{Y|X}(y|X = x) = \frac{f_{X,Y}(x, y)}{f_X(x)} = \frac{f_X(x)f_Y(y)}{f_X(x)} = f_Y(y).$$

i.e., *conditional pdf $Y|X$ is the marginal pdf of Y*.

In fact, the following three are equivalent definitions of the independence of X and Y

- $f(x, y) = f_X(x)f_Y(y)$ . . . . (joint = product of marginal)
- $f_{Y|X}(y|X = x) = f_Y(y)$ . . . . (conditional $Y|X$ = marginal of Y)
- $f_{X|Y}(x|Y = y) = f_X(x)$ . . . . (conditional $X|Y$ = marginal of X)

All the things above apply to joint/conditional/marginal **pmf** for discrete X, Y., too.

Conditional Expected Values

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For two random variables X and Y, the *conditional mean* or *conditional expected value of* Y given $X = x$ is

$$\mu_{Y|X=X} = \mathrm{E}(Y \mid X = x) = \begin{cases} \sum_y y\, p_{Y|X}(y \mid x) & \text{if } X, Y \text{ are discrete} \\ \int_{-\infty}^{\infty} y\, f_{Y|X}(y \mid x) dy & \text{if } X, Y \text{ are continuous} \end{cases}$$

where $p_{Y|X}(y|x)$ and $f_{Y|X}(y|x)$ are the conditional pmf/pdf of Y given X.

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For two random variables X and Y, the *conditional mean* or *conditional expected value of* Y given $X = x$ is

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where $p_{Y|X}(y|x)$ and $f_{Y|X}(y|x)$ are the conditional pmf/pdf of Y given X.

Note: The conditional mean of Y given $X = x$ $\mathrm{E}(Y \mid X = x)$ is NOT a single value but a function of the x value given.

Example (Gas Station) — Conditional Mean

Recall the conditional pmf of Y given $X = x$ is as follows.

	conditional pmf $p(y \mid x)$	Y 0	1	2	Row Sum
X	0	0.625	0.25	0.125	1
	1	0.235	0.588	0.176	1
	2	0.12	0.18	0.60	1

The conditional mean of Y given $X = x$ is

$$E(Y \mid X = x) = \begin{cases} 0 \cdot 0.625 + 1 \cdot 0.25 + 2 \cdot 0.125 = 0.5 & \text{if } x = 0 \\ 0 \cdot 0.235 + 1 \cdot 0.588 + 2 \cdot 0.176 = 0.94 & \text{if } x = 1 \\ 0 \cdot 0.12 + 1 \cdot 0.18 + 2 \cdot 0.6 = 1.38 & \text{if } x = 2 \end{cases}$$

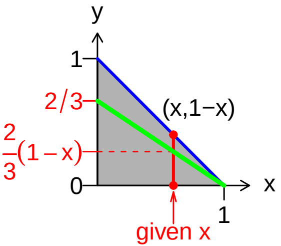
Example (Mixed Nuts) — Conditional Mean

Recall the conditional pdf $f_{Y|X}(y \mid x)$ of Y (cashew) given $X = x$ (almond) is

$$f_{Y|X}(y \mid x) = \frac{f(x,y)}{f_X(x)} = \frac{2y}{(1-x)^2}, \quad \text{for } 0 \le y \le 1-x\,.$$

The conditional expected weight of Y (cashew) in a can given there being $X = x$ lbs of almond in the can is

$$\begin{aligned} \mathrm{E}(Y \mid X = x) &= \int_{-\infty}^{\infty} y f_{Y|X}(y \mid x)dy \\ &= \int_0^{1-x} \frac{y \times 2y}{(1-x)^2} dy \\ &= \frac{2y^3}{3(1-x)^2}\bigg|_{y=0}^{y=1-x} = \frac{2}{3}(1-x). \end{aligned}$$



Bivariate Normal Distribution

Bivariate Normal Distribution

The joint pdf of the *Bivariate Normal Distribution* is

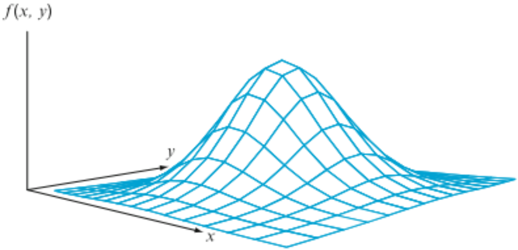
$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(\frac{-1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right\}\right)$$

denoted as

$$(X, Y) \sim BN(\mu_x, \mu_y, \sigma_x, \sigma_y, \rho)$$

One can show that

- The marginal distribution of X is $N(\mu_x, \sigma_x^2)$
- The marginal distribution of y is $N(\mu_y, \sigma_y^2)$
- The correlation of X and Y is ρ

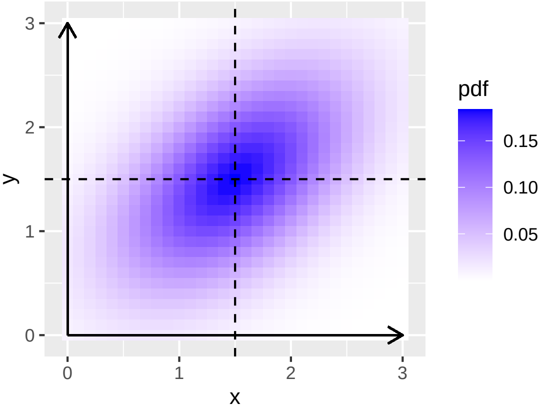


The 3D plot of the joint pdf

- looks like a mountain peaks at (μ_x, μ_y)
- The horizontal cross-sections are elliptical
- The vertical cross-sections are all proportional to normal pdf

Plot of Bivariate Normal joint pdf with

$$\mu_x = 1,5, \quad \mu_y = 0.5, \quad \sigma_x = 1, \quad \sigma_y = 1, \quad \rho = 0.5.$$



Conditional pdf of Y Given X for Bivariate Normal

As $X \sim N(\mu_x, \sigma_x^2)$ for if $(X, Y) \sim BN(\mu_x, \mu_y, \sigma_x, \sigma_y, \rho)$, the marginal pdf of X is

$$f(x) = \frac{1}{\sigma_x \sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right).$$

One can calculate the condition pdf of Y given $X = x$ by dividing the joint pdf by the marginal pdf. The algebra is messy but the result is simple. The conditional distribution of Y given $X = x$ is *Normal* with

$$\text{mean} = \mu_y + \rho\frac{\sigma_y}{\sigma_x}(x - \mu_x), \quad \text{and} \quad \text{variance} = \sigma_y^2(1 - \rho^2).$$