Why Consider Two or More Random Variables?

- Our focus so far has been on the distribution of a single random variable.
- In many situations, there are two or more variables of interest, and we want to know how they are related. For example, I am interested to know
  - \( X_1 \): the number of hours spent on studying per week
  - \( X_2 \): final grade of stat234.
- Since the *relationship* is important, we cannot study them separately and need to consider them *jointly*. 
Joint Probability Distribution for Discrete R.V.
The joint probability mass function (joint pmf), or, simply the joint distribution, of two discrete r.v. $X$ and $Y$ is defined as

$$p(x, y) = P(X = x, Y = y) = P(\{X = x\} \cap \{Y = y\}).$$
The *joint probability mass function (joint pmf)*, or, simply the *joint distribution*, of two discrete r.v. $X$ and $Y$ is defined as

$$p(x, y) = P(X = x, Y = y) = P\left(\{X = x\} \cap \{Y = y\}\right).$$

**Properties of the joint probability distribution:**

1. $p(x, y) \geq 0$.
2. Define the probability for an event $A$ as,

   $$P(A) = P((x, y) \in A) = \sum_{(x,y)\in A} p(x, y).$$

3. If we set $A = S$ in (2), then

   $$P(S) = \sum_x \sum_y p(x, y) = 1.$$
Exercise 1 — Gas Station (p.242 in MMSA)

A gas station has both self-service and full-service islands, each with a single regular unleaded pump with 2 hoses.

\[ X = \text{the # of hoses in use on the self-service island, and} \]
\[ Y = \text{the # of hoses in use on the full-service island} \]

The joint pmf of \( X \) and \( Y \):

<table>
<thead>
<tr>
<th>( p(x, y) )</th>
<th>( Y ) (full-service)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X )</td>
<td>0</td>
</tr>
<tr>
<td>self-service</td>
<td>0</td>
</tr>
<tr>
<td>service</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0.06</td>
</tr>
</tbody>
</table>

What is \( P(X = 2 \text{ and } Y = 1) \)?
A gas station has both **self-service** and **full-service** islands, each with a single regular unleaded pump with 2 hoses.

\[ X = \text{the \# of hoses in use on the self-service island, and} \]

\[ Y = \text{the \# of hoses in use on the full-service island} \]

The joint pmf of \( X \) and \( Y \):

\[
\begin{array}{c|ccc}
X & Y (\text{full-service}) \\
0 & 0.10 & 0.04 & 0.02 \\
1 & 0.08 & 0.20 & 0.06 \\
2 & 0.06 & 0.14 & 0.30 \\
\end{array}
\]

What is \( P(X = 2 \text{ and } Y = 1) \)? \( p(2, 1) = 0.14 \)
Exercise 1 — Gas Station (2)

<table>
<thead>
<tr>
<th></th>
<th>$p(x, y)$</th>
<th>$Y$ (full-service)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
<td>0</td>
<td>0.10 0.04 0.02</td>
</tr>
<tr>
<td>self-</td>
<td>1</td>
<td>0.08 0.20 0.06</td>
</tr>
<tr>
<td>service</td>
<td>2</td>
<td>0.06 0.14 0.30</td>
</tr>
</tbody>
</table>

What is $P(X \leq 1$ and $Y \leq 1$)?
Exercise 1 — Gas Station (2)

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(x, y)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>X</td>
<td>0.10</td>
<td>0.04</td>
<td>0.02</td>
</tr>
<tr>
<td>self- service</td>
<td>0.08</td>
<td>0.20</td>
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</tr>
<tr>
<td>service</td>
<td>0.06</td>
<td>0.14</td>
<td>0.30</td>
</tr>
</tbody>
</table>

What is $P(X \leq 1$ and $Y \leq 1$)?

\[
P(X = 0, Y = 0) + P(X = 0, Y = 1) + P(X = 1, Y = 0) + P(X = 1, Y = 1) \\
= p(0, 0) + p(0, 1) + p(1, 0) + p(1, 1) \\
= 0.10 + 0.04 + 0.08 + 0.20 = 0.42
\]
What is the probability that more self-service hoses in use than full service hoses $P(X > Y)$?

<table>
<thead>
<tr>
<th>$p(x, y)$</th>
<th>$Y$ (full-service)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>$X$</td>
<td>0.10</td>
</tr>
<tr>
<td>self-</td>
<td>0.08</td>
</tr>
<tr>
<td>service</td>
<td>0.06</td>
</tr>
</tbody>
</table>
What is the probability that more self-service hoses in use than full service hoses $P(X > Y)$?

\[
P(X = 1, Y = 0) + P(X = 2, Y = 1) + P(X = 2, Y = 0) \\
= p(1, 0) + p(2, 1) + p(2, 1) \\
= 0.08 + 0.06 + 0.14 = 0.28
\]
Marginal Distribution
Obtaining pmf of $X$ From the Joint Distribution of $(X, Y)$

<table>
<thead>
<tr>
<th>$p(x, y)$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>Row Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.10</td>
<td>0.04</td>
<td>0.02</td>
<td></td>
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<td>2</td>
<td>0.06</td>
<td>0.14</td>
<td>0.30</td>
<td></td>
</tr>
</tbody>
</table>

$P(X = 0) =$
Obtaining pmf of $X$ From the Joint Distribution of $(X, Y)$

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>Row Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.10</td>
<td>0.04</td>
<td>0.02</td>
<td>0.16</td>
</tr>
<tr>
<td>1</td>
<td>0.08</td>
<td>0.20</td>
<td>0.06</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.06</td>
<td>0.14</td>
<td>0.30</td>
<td></td>
</tr>
</tbody>
</table>

$P(X = 0) = P(X = 0, Y = 0) + P(X = 0, Y = 1) + P(X = 0, Y = 2)$

$= 0.10 + 0.04 + 0.02 = 0.16$
Obtaining pmf of $X$ From the Joint Distribution of $(X, Y)$

<table>
<thead>
<tr>
<th>$p(x, y)$</th>
<th>$Y$</th>
<th>Row Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0.10</td>
<td>0.04</td>
</tr>
<tr>
<td>$X$</td>
<td>1</td>
<td>0.08</td>
</tr>
<tr>
<td>2</td>
<td>0.06</td>
<td>0.14</td>
</tr>
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$P(X = 0) = P(X = 0, Y = 0) + P(X = 0, Y = 1) + P(X = 0, Y = 2)$

$= 0.10 + 0.04 + 0.02 = 0.16$

Likewise,

$P(X = 1) = 0.08 + 0.20 + 0.06 = 0.34$
Obtaining pmf of $X$ From the Joint Distribution of $(X, Y)$

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<th>$p(x, y)$</th>
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<tbody>
<tr>
<td></td>
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<td>1</td>
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<td>0.10</td>
<td>0.04</td>
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<tr>
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<td>0.20</td>
</tr>
<tr>
<td>2</td>
<td>0.06</td>
<td>0.14</td>
</tr>
</tbody>
</table>

$P(X = 0) = P(X = 0, Y = 0) + P(X = 0, Y = 1) + P(X = 0, Y = 2)$

$= 0.10 + 0.04 + 0.02 = 0.16$

Likewise,

$P(X = 1) = 0.08 + 0.20 + 0.06 = 0.34$

$P(X = 2) = 0.06 + 0.14 + 0.30 = 0.50$
Obtaining pmf of $X$ From the Joint Distribution of $(X, Y)$

<table>
<thead>
<tr>
<th>$p(x, y)$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>Row Sum $p_X(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.10</td>
<td>0.04</td>
<td>0.02</td>
<td>0.16</td>
</tr>
<tr>
<td>$X$</td>
<td></td>
<td></td>
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$$P(X = 0) = P(X = 0, Y = 0) + P(X = 0, Y = 1) + P(X = 0, Y = 2)$$
$$= 0.10 + 0.04 + 0.02 = 0.16$$

Likewise,

$$P(X = 1) = 0.08 + 0.20 + 0.06 = 0.34$$

$$P(X = 2) = 0.06 + 0.14 + 0.30 = 0.50$$

The pmf $p_X(x)$ of $X$ is thus

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_X(x)$</td>
<td>0.16</td>
<td>0.34</td>
<td>0.50</td>
</tr>
</tbody>
</table>
Obtaining pmf of $Y$ From the Joint Distribution of $(X, Y)$

<table>
<thead>
<tr>
<th>$p(x, y)$</th>
<th>$Y$</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
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<td>0.14</td>
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<td></td>
</tr>
<tr>
<td><strong>Column sum</strong></td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

$P(Y = 0) =$
Obtaining pmf of $Y$ From the Joint Distribution of $(X, Y)$

<table>
<thead>
<tr>
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</tr>
</tbody>
</table>

Column sum: 0.24

$P(Y = 0) = P(X = 0, Y = 0) + P(X = 1, Y = 0) + P(X = 2, Y = 0)$

$= 0.10 + 0.08 + 0.06 = 0.24$
Obtaining pmf of $Y$ From the Joint Distribution of $(X, Y)$

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<td>0.14</td>
<td>0.30</td>
</tr>
</tbody>
</table>

| Column sum | 0.24 | 0.38 |

$P(Y = 0) = P(X = 0, Y = 0) + P(X = 1, Y = 0) + P(X = 2, Y = 0)$

$= 0.10 + 0.08 + 0.06 = 0.24$

Likewise, $P(Y = 1) = 0.04 + 0.20 + 0.14 = 0.38$
Obtaining pmf of \( Y \) From the Joint Distribution of \((X, Y)\)

<table>
<thead>
<tr>
<th>( p(x, y) )</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
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<td>0.30</td>
</tr>
</tbody>
</table>

Column sum

|          | 0.24 | 0.38 | 0.38 |

\[
P(Y = 0) = P(X = 0, Y = 0) + P(X = 1, Y = 0) + P(X = 2, Y = 0)
= 0.10 + 0.08 + 0.06 = 0.24
\]

Likewise,

\[
P(Y = 1) = 0.04 + 0.20 + 0.14 = 0.38
\]

\[
P(Y = 2) = 0.02 + 0.06 + 0.30 = 0.38
\]
Obtaining pmf of $Y$ From the Joint Distribution of $(X, Y)$

$Y$

<table>
<thead>
<tr>
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</tr>
<tr>
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</tr>
</tbody>
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$P(Y = 0) = P(X = 0, Y = 0) + P(X = 1, Y = 0) + P(X = 2, Y = 0)$

$= 0.10 + 0.08 + 0.06 = 0.24$

Likewise,

$P(Y = 1) = 0.04 + 0.20 + 0.14 = 0.38$

$P(Y = 2) = 0.02 + 0.06 + 0.30 = 0.38$

The pmf $p_Y(y)$ of $Y$ is thus

<table>
<thead>
<tr>
<th>$y$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_Y(y)$</td>
<td>0.24</td>
<td>0.38</td>
<td>0.38</td>
</tr>
</tbody>
</table>
The marginal probability mass functions (marginal pmf’s) of $X$ and of $Y$ are obtained by summing $p(x, y)$ over values of the other variable.

$$p_X(x) = \sum_y p(x, y), \quad p_Y(y) = \sum_x p(x, y).$$

**Example:** Gas Station

<table>
<thead>
<tr>
<th>$p(x, y)$</th>
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<th>1</th>
<th>2</th>
<th>Row Sum $p_X(x)$</th>
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</thead>
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<td>0.30</td>
<td>0.50</td>
</tr>
</tbody>
</table>

We call them **marginal distributions** because they show up at the table margins when the joint distribution is written in a tabular form.
Joint Distribution of Continuous Random Variables
Joint Distribution of Two Continuous Random Variables

Let $X$ and $Y$ be continuous rv. Then $f(x, y)$ is their **joint probability density function** or **joint pdf** for $X$ and $Y$ if for any two-dimensional set $A$

$$P[(X, Y) \in A] = \int_{A} \int f(x, y) \, dx \, dy$$

In particular, if $A$ is the two-dimensional rectangle \{a \leq x \leq b, c \leq y \leq d\}, then

$$P[(X, Y) \in A] = P(a \leq X \leq b, c \leq Y \leq d) = \int_{c}^{d} \int_{a}^{b} f(x, y) \, dx \, dy$$

Conditions for a joint pdf

- It must be nonnegative: $f(x, y) \geq 0$ for all $x$ and $y$
- $\int \int f(x, y) \, dx \, dy = 1$
• Each can of mixed nuts contains *almonds, cashews, and peanuts*
Example 5.5 on p.237-238 of MMSA

- Each can of mixed nuts contains *almonds*, *cashews*, and *peanuts*
- Weights of the 3 types of nuts in a can are random but the total is exactly 1 lb
Each can of mixed nuts contains *almonds*, *cashews*, and *peanuts*

Weights of the 3 types of nuts in a can are random but the total is exactly 1 lb

In a randomly selected can, let

\[ X = \text{the weight of almonds}, \quad Y = \text{the weight of cashews}. \]

The weight of peanuts in the can is thus \((1 - X - Y)\)
Each can of mixed nuts contains *almonds, cashews, and peanuts*. 

Weights of the 3 types of nuts in a can are random but the total is **exactly 1 lb**

In a randomly selected can, let 

\[ X = \text{the weight of almonds, and } Y = \text{the weight of cashews.} \]

The weight of peanuts in the can is thus 

\[ (1 - X - Y) \]

Natural constraints on \( X \) & \( Y \):

\[ 0 \leq X \leq 1, \ 0 \leq Y \leq 1, \ X + Y < 1 \]

Joint pdf of \( X \) & \( Y \):

\[
 f(x, y) = \begin{cases} 
 24xy & \text{if } 0 \leq x \leq 1, 0 \leq y \leq 1, x + y < 1 \\
 0 & \text{otherwise} 
\end{cases}
\]
Clearly, \( f(x, y) \geq 0 \). It remains to check \( \iint f(x, y) \, dx \, dy = 1 \).

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = \int_{0}^{1} \int_{0}^{1-y} 24xy \, dx \, dy
\]

To compute the double integral above,
1. hold one variable fixed (e.g., \( y \))
2. integrate the other variable \( x \) along the line of the fixed \( y \)
3. integrate the variable \( y \) that is fixed in the prior steps

\[
\int_{0}^{1} \int_{0}^{1-y} 24xy \, dx \, dy = \int_{0}^{1} \left[ 12(1-y)^2 \right] y \, dy = \left[ 6y^2 - 8y^3 + 3y^4 \right]_{0}^{1} = 1
\]
Clearly, \( f(x, y) \geq 0 \). It remains to check \( \iint f(x, y) \, dx \, dy = 1 \).

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = \int_{0}^{1} \int_{0}^{1-y} 24xy \, dx \, dy
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To compute the double integral above,

1. hold one variable fixed (e.g., \( y \))
Checking Conditions on a Joint PDF

Clearly, \( f(x, y) \geq 0 \). It remains to check \( \iint f(x, y) \, dx \, dy = 1 \).

\[
\iint_{-\infty}^{\infty} f(x, y) \, dx \, dy = \int_0^1 \int_0^{1-y} 24xy \, dx \, dy
\]

To compute the double integral above,

1. hold one variable fixed (e.g., \( y \))
2. integrate the other variable \( x \) along the line of the fixed \( y \)
   - key: express the end points of the line in terms of the fixed \( y \), which will be the upper and lower limits for the integral over \( x \)
   
   \[
   \int_0^{1-y} 24xy \, dx = 12x^2y \bigg|_{x=0}^{x=1-y} = 12(1 - y)^2y
   \]
3. integrate the variable \( y \) that is fixed in the prior steps

\[
\int_0^1 \int_0^{1-y} 24xy \, dx \, dy = \int_0^1 12(1 - y)^2y \, dy = 6y^2 - 8y^3 + 3y^4 \bigg|_0^1 = 1.
\]
What is $P(X > 0.3) = P(\text{at least 30\% almonds in a can})$?
What is $P(X > 0.3) = P(\text{at least 30\% almonds in a can})$?

$$P(X > 0.3) = \int_{x>0.3} \int f(x,y) \, dx \, dy$$

$$= \int_0^{0.7} \int_{0.3}^{1-y} 24xy \, dx \, dy$$
What is $P(X > 0.3) = P(\text{at least 30\% almonds in a can})$?

$$P(X > 0.3) = \int_{x>0.3} \int f(x, y) \, dx \, dy$$

$$= \int_0^{0.7} \int_0^{1-y} 24xy \, dx \, dy$$

where

$$\int_0^{1-y} 24xy \, dx = 12x^2y \bigg|_{x=0.3}^{x=1-y}$$

$$= 12((1 - y)^2 - 0.3^2)y = 12(0.91y - 2y^2 + y^3).$$

Putting it back to the double integral, we get

$$\int_{0.3}^{1-y} \int_0^{1-y} 24xy \, dx \, dy = \int_{0.3}^{0.7} 12(0.91y - 2y^2 + y^3) \, dy$$

$$= 5.46y^2 - 8y^3 + 3y^4 \bigg|_{0.3}^{0.7}$$

$$= 0.6517.$$
Finding Probabilities From the Joint PDF $P(X > 0.3)$

What is $P(X > 0.3) = P(\text{at least 30\% almonds in a can})$?

$$P(X > 0.3) = \int_{x>0.3} \int f(x, y) \, dx \, dy$$

where

$$\int_{0.3}^{1-y} 24xy \, dx = 12x^2y \bigg|_{x=0.3}^{x=1-y}$$

$$= 12((1 - y)^2 - 0.3^2)y = 12(0.91y - 2y^2 + y^3).$$

Putting it back to the double integral, we get

$$\int_{0}^{0.7} \int_{0.3}^{1-y} 24xy \, dx \, dy = \int_{0}^{0.7} 12(0.91y - 2y^2 + y^3) \, dy$$

$$= 5.46y^2 - 8y^3 + 3y^4 \bigg|_{0}^{0.7} = 0.6517.$$
What is the probability that less than 30% are peanuts in a randomly selected can?

\[ P(\text{less than 30\% are Peanuts}) \]

\[ = \]

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What is the probability that less than 30% are peanuts in a randomly selected can?

\[
P(\text{less than 30% are Peanuts})
\]

\[
= P(\text{at least 70% are almonds or cashews})
\]

\[
= \]

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\[
by \text{ Complement Rule}
\]

\[
P(X + Y > 0.7) = \text{integral of } f(x, y) \text{ over the gray region}
\]

\[
P(X + Y < 0.7) = \text{integral of } f(x, y) \text{ over the green region}
\]
What is the probability that less than 30% are peanuts in a randomly selected can?

\[
P(\text{less than 30\% are Peanuts})
\]
\[
= P(\text{at least 70\% are almonds or cashews})
\]
\[
= P(X + Y > 0.7)
\]
\[
= 
\]
Finding Probabilities From the Joint PDF $P(X + Y > 0.7)$

What is the probability that less than 30% are peanuts in a randomly selected can?

\[
P(\text{less than 30\% are Peanuts})
\]

\[
= P(\text{at least 70\% are almonds or cashews})
\]

\[
= P(X + Y > 0.7)
\]

\[
= 1 - P(X + Y \leq 0.7) \quad \text{by } \textit{Complement Rule}
\]
Finding Probabilities From the Joint PDF $P(X + Y > 0.7)$

What is the probability that less than 30% are peanuts in a randomly selected can?

$$P(\text{less than 30% are Peanuts})$$

$$= P(\text{at least 70% are almonds or cashews})$$

$$= P(X + Y > 0.7)$$

$$= 1 - P(X + Y \leq 0.7) \text{ by Complement Rule}$$

where

$$P(X + Y > 0.7) = \text{integral of } f(x, y) \text{ over the gray region}$$

$$P(X + Y < 0.7) = \text{integral of } f(x, y) \text{ over the green region}$$
Finding Probabilities From the Joint PDF $P(X + Y > 0.7)$ (Cont’d)

\[ P(X + Y < 0.7) = \int_{x+y<0.7} \int f(x, y) \, dx \, dy \]

\[ = \int_{0}^{0.7} \int_{0}^{0.7-y} 24xy \, dx \, dy \]

where

\[ \int_{0}^{0.7-y} 24xy \, dx = 12x^2y \bigg|_{x=0}^{x=0.7-y} = 12(0.7 - y)^2y \]

Putting it back to the double integral, we get

\[ Z_{0.7} - y \]

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Finding Probabilities From the Joint PDF $P(X + Y > 0.7)$ (Cont’d)

$$P(X + Y < 0.7) = \int_{x+y<0.7} \int f(x, y) \, dx \, dy$$

$$= \int_0^{0.7} \int_0^{0.7-y} 24xy \, dx \, dy$$

Where

$$\int_0^{0.7-y} 24xy \, dx = 12x^2y \bigg|_{x=0}^{x=0.7-y} = 12(0.7 - y)^2y$$

Putting it back to the double integral, we get

$$\int_0^{0.7} \int_0^{0.7-y} 24xy \, dx \, dy = \int_0^{0.7} 12(0.7 - y)^2y \, dy = \int_0^{0.7} (-4y)d(0.7 - y)^3$$

$$= -4y(0.7 - y)^3 \bigg|_0^{0.7} + \int_0^{0.7} 4(0.7 - y)^3 \, dy$$

$$= 0 - (0.7 - y)^4 \bigg|_0^{0.7} = (0.7)^4 = 0.2401.$$  

Hence, $P(\text{less than 30% peanut}) = 1 - 0.2401 = 0.7599$. 
Given the joint pdf \( f(x, y) \) of two continuous random variables, the *marginal probability density function* (\( p \)), or simply the *marginal density*, of \( X \) and \( Y \), can be obtained by *integrating the joint pdf over the other variable*.

\[
\begin{align*}
    f_X(x) &= \int_{-\infty}^{\infty} f(x, y) \, dy, \quad \text{for } -\infty < x < \infty, \\
    f_Y(y) &= \int_{-\infty}^{\infty} f(x, y) \, dx, \quad \text{for } -\infty < y < \infty.
\end{align*}
\]

Recall the *marginal pmf’s* of discrete random variables are obtained by *summing the joint pmf over values of the other variable*.

\[
\begin{align*}
    p_X(x) &= \sum_y p(x, y), \quad p_Y(y) = \sum_x p(x, y).
\end{align*}
\]
The marginal pdfs of $X$ (almond) is

\[
f_X(x) = \int_{-\infty}^{\infty} f(x, y) \, dy
= \int_{0}^{1-x} 24xy \, dy = 12xy^2 \bigg|_{y=0}^{y=1-x}
= 12x(1 - x)^2, \text{ for } 0 \leq x \leq 1.
\]
The marginal pdfs of $X$ (almond) is

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_{0}^{1-x} 24xydy = 12xy^2 \bigg|_{y=0}^{y=1-x} = 12x(1 - x)^2, \text{ for } 0 \leq x \leq 1.$$ 

The marginal pdfs of $Y$ (cashew) is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx = \int_{0}^{1-y} 24xydx = 12x^2y \bigg|_{x=0}^{x=1-y} = 12y(1 - y)^2, \text{ for } 0 \leq y \leq 1.$$
Independent Random Variables
**Independent Random Variables**

- Recall that two events $A$ and $B$ are *independent* if
  \[ P(A \cap B) = P(A)P(B) \]
- Two random variables $X$ and $Y$ are *independent* if
  \[ P(X \in A, Y \in B) = P(X \in A) \, P(Y \in B) \]
  for any sets $A$ and $B$.
- It can be shown that two random variables $X$ and $Y$ are *independent* if and only if
  \[
  \begin{align*}
  p(x, y) &= p_X(x)p_Y(y) & \text{if } X \text{ and } Y \text{ are discrete} \\
  f(x, y) &= f_X(x)f_Y(y) & \text{if } X \text{ and } Y \text{ are continuous}
  \end{align*}
  \]
  for all $x$ and $y$, i.e., the joint distribution of $X$ and $Y$ is the product of their marginal distributions.
Are $X$ and $Y$ Independent?

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<thead>
<tr>
<th>$f(x, y)$</th>
<th>$y$</th>
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<tbody>
<tr>
<td></td>
<td>1</td>
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<td>1</td>
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<td>2</td>
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<tr>
<td>3</td>
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</table>

1. Find the marginal distributions
2. Check whether $p(x, y) = p_X(x)p_Y(y)$ for all possible $x, y$ pairs.

- $p(1, 1) = 0.05 \times 0.2 = p_X(1)p_Y(1)$.

- $X$ and $Y$ are NOT independent.
Are $X$ and $Y$ Independent?

<table>
<thead>
<tr>
<th>$f(x, y)$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>$f_X(x)$</th>
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<tr>
<td>1</td>
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<td>0.10</td>
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| $f_Y(y)$ | 0.20 | 0.60 | 0.20 |

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$$p(x, y) = p_X(x)p_Y(y)$$
**Are X and Y Independent?**

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1. Find the marginal distributions
2. Check whether

\[ p(x, y) = p_X(x)p_Y(y) \]

for all possible \( x, y \) pairs.

- \( p(1, 1) = 0.05 \neq 0.2 \times 0.2 = p_X(1)p_Y(1) \).
- \( X \) and \( Y \) are NOT independent.
Given the marginal pmfs of two independent r.v.'s, \( X \) and \( Y \), find their joint pmf.

Since \( X \) and \( Y \) are independent,

1. \( p(1, 1) = p_X(1)p_Y(1) = 0.2 \times 0.2 = 0.04 \)
2. also \( p(1, 2) = p_X(1)p_Y(2) = 0.2 \times 0.6 = 0.12 \).
3. Repeat filling the blank for \( p(x, y) \) by \( p_X(x)p_Y(y) \) for all \( x, y \) pairs.
Given the marginal pmfs of two independent r.v.’s, $X$ and $Y$, find their joint pmf.

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Given the marginal pmfs of two *independent* r.v.’s, $X$ and $Y$, find their joint pmf.

\[
\begin{array}{|c|c|c|c|c|}
\hline
p(x, y) & 1 & 2 & 3 & f_X(x) \\
\hline
1 & 0.04 & 0.12 & 0.04 & 0.2 \\
x & 2 & 0.12 & 0.36 & 0.12 & 0.6 \\
3 & 0.04 & 0.12 & 0.04 & 0.2 \\
\hline
f_Y(y) & 0.2 & 0.6 & 0.2 & \\
\hline
\end{array}
\]

Since $X$ and $Y$ are *independent*,

1. $p(1, 1) = p_X(1)p_Y(1) = 0.2 \times 0.2 = 0.04$
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3. Repeat filling the blank for $p(x, y)$ by $p_X(x)p_Y(y)$ for all $x, y$ pairs.
If $X$ and $Y$ are independent with marginal pdfs

$$f_X(x) = e^{-x} \quad \text{and} \quad f_Y(y) = 2e^{-2y},$$

for $0 < x, y < \infty$, then their joint pdf is

$$f(x, y) = f_X(x)f_Y(y) = 2e^{-(x+2y)}, \quad 0 < x, y < \infty.$$