STAT 226 Lecture 20
Section 6.1 Baseline-Category Logit Models

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Chapter 6 Multicategory Logit Models

Response Y has $J > 2$ categories.

Extensions of logistic regression for nominal and ordinal Y assumes a multinomial distribution for Y.

- 6.1 Baseline-Categorical Logit Models for Nominal Responses
- 6.2 Cumulative Logit Models for Ordinal Responses

Review of Multinomial Distributions

If n trials are performed:

- in each trial there are $J > 2$ possible outcomes (categories)
- $\pi_j = \mathrm{P}(\text{category } j)$, for each trial, $\sum_{j=1}^{J} \pi_j = 1$
- trials are **independent**
- Y_j = number of trials fall in category j out of n trials

then the joint distribution of $(Y_1, Y_2, \ldots, Y_J)$ is said to have a *multinomial distribution*, with n trials and category probabilities $(\pi_1, \pi_2, \ldots, \pi_J)$, denoted as

$$(Y_1, Y_2, \ldots, Y_J) \sim \text{Multinom}(n; \pi_1, \pi_2, \ldots, \pi_J),$$

with probability function

$$\mathrm{P}(Y_1 = y_1, Y_2 = y_2, \ldots, Y_J = y_J) = \frac{n!}{y_1!\, y_2! \cdots y_J!} \pi_1^{y_1} \pi_2^{y_2} \cdots \pi_J^{y_J}$$

where $0 \le y_j \le n$ for all j and $\sum_j y_j = n$.

Odds for Multi-Category Response Variable

For a binary response variable, there is only one kind of odds that we may consider

$$\frac{\pi}{1-\pi}.$$

For a multi-category response variable with $J > 2$ categories and category probabilities $(\pi_1, \pi_2, \ldots, \pi_J)$, we may consider various kinds of odds, though some of them are more meaningful than others.

- odds between two categories: π_i/π_j.
- odds between a group of categories vs another group of categories, e.g.,

  $$\frac{\pi_1 + \pi_3}{\pi_2 + \pi_4 + \pi_5}.$$

  Note the two groups of categories should be non-overlapping.

Odds for Multi-Category Response Variable (Cont'd)

E.g., if $Y =$ choice of meat (in a broad sense) with 5 categories

beef, pork, chicken, turkey, fish

We may consider the odds of

- beef vs. chicken: $\pi_{\text{beef}}/\pi_{\text{chicken}}$
- red meat vs. white meat:

$$\frac{\pi_{\text{beef}} + \pi_{\text{pork}}}{\pi_{\text{chicken}} + \pi_{\text{turkey}} + \pi_{\text{fish}}}$$

- red meat vs. poultry:

$$\frac{\pi_{\text{beef}} + \pi_{\text{pork}}}{\pi_{\text{chicken}} + \pi_{\text{turkey}}}$$

Odds for Ordinal Variables

If Y is ordinal with ordered categories:

$$1 < 2 < \ldots < J$$

we may consider the odds of $Y \leq j$

$$\frac{P(Y \leq j)}{P(Y > j)} = \frac{\pi_1 + \pi_2 + \cdots + \pi_j}{\pi_{j+1} + \cdots + \pi_J}$$

e.g., Y = political ideology, with 5 levels

very liberal < slightly liberal < moderate

< slightly conservative < very conservative

we may consider the odds

$$\frac{P(\text{very or slightly liberal''})}{P(\text{moderate or conservative})} = \frac{\pi_{\text{vlib}} + \pi_{\text{slib}}}{\pi_{\text{mod}} + \pi_{\text{scon}} + \pi_{\text{vcon}}}$$

Odds Ratios for XY When Y is Multi-Category

For any sensible odds between two (groups of) categories of Y can be compared across two levels of X.

E.g., for Y = choice of meat, X = Country (Italy, Japan), we may consider

OR between Y (fish vs. beef) and X = IT or JP

$$= \frac{P(Y = \text{fish} \mid X = \text{IT}) / P(Y = \text{beef} \mid X = \text{IT})}{P(Y = \text{fish} | X = \text{JP}) / P(Y = \text{beef} \mid X = \text{JP})}$$

OR between Y (red meat vs. fish) and X = IT or JP

$$= \frac{P(Y = \text{beef or pork} \mid X = \text{IT}) / P(Y = \text{fish} \mid X = \text{IT})}{P(Y = \text{beef or pork} \mid X = \text{JP}) / P(Y = \text{fish} \mid X = \text{JP})}$$

- Again, ORs can be estimated from both prospective and retrospective studies.
- Usually we need more than 1 OR to describe XY associations completely.

6.1 Baseline-Category Logit Models for Nominal Responses

Let $\pi_j = \mathrm{P}(Y = j),\ j = 1, 2, \ldots, J$.

Baseline-category logits are

$$\log\left(\frac{\pi_j}{\pi_J}\right), \quad j = 1, 2, \ldots, J - 1.$$

Baseline-category logit model has form

$$\log\left(\frac{\pi_j}{\pi_J}\right) = \alpha_j + \beta_j x, \quad j = 1, 2, \ldots, J - 1.$$

or equivalently,

$$\pi_j = \pi_J \exp(\alpha_j + \beta_j x) \quad j = 1, 2, \ldots, J - 1.$$

- Separate set of parameters (α_j, β_j) for each logit.
- Equation for π_J is not needed since $\log(\pi_J/\pi_J) = 0$

Choice of the Baseline-Category Is Arbitrary

Equation for other pair of categories, say, categories a and b can then be determined as

$$\begin{aligned} \log\left(\frac{\pi_a}{\pi_b}\right) &= \log\left(\frac{\pi_a/\pi_J}{\pi_b/\pi_J}\right) = \log\left(\frac{\pi_a}{\pi_J}\right) - \log\left(\frac{\pi_b}{\pi_J}\right) \\ &= (\alpha_a + \beta_a x) - (\alpha_b + \beta_b x) \\ &= (\alpha_a - \alpha_b) + (\beta_a - \beta_b)x \end{aligned}$$

Choice of the Baseline-Category Is Arbitrary

Equation for other pair of categories, say, categories a and b can then be determined as

$$\log\left(\frac{\pi_a}{\pi_b}\right) = \log\left(\frac{\pi_a/\pi_J}{\pi_b/\pi_J}\right) = \log\left(\frac{\pi_a}{\pi_J}\right) - \log\left(\frac{\pi_b}{\pi_J}\right)$$
$$= (\alpha_a + \beta_a x) - (\alpha_b + \beta_b x)$$
$$= (\alpha_a - \alpha_b) + (\beta_a - \beta_b)x$$

Any of the categories can be chosen to be the baseline

- The model will fit equally well, achieving the same likelihood and producing the same fitted values.
- The coefficients α_j, β_j's will change, but their differences

$$\alpha_a - \alpha_b \quad \text{and} \quad \beta_a - \beta_b$$

between any two categories a and b will stay the same.

Baseline-Category Logit Models w/ Ordinal Response

Could also use this model with ordinal response variables, but this would ignore ordinal information.

Exercise 2.21 (Job Satisfaction and Income, ICDA, p.61)

Data from General Social Survey (1991)

Income (x)	Job Satisfaction (Y) Dissat	Little	Moderate	Very
0-5K	2	4	13	3
5-15K	2	6	22	4
15-25K	0	1	15	8
$>$25K	0	3	13	8

Goal: to know if one's job satisfaction changes with income

From the table above, there seems to be higher percentages of people in the more satisfied categories in the higher income groups.

How to we test if the tendency is significant?

Income	Job Satisfaction (Y)			
(x)	Dissat	Little	Moderate	Very
0-5K	2	4	13	3
5-15K	2	6	22	4
15-25K	0	1	15	8
>25K	0	3	13	8

Note X = Income is ordinal w/ 4 categories.

To utilize the ordinal info of X, instead of creating dummy variables for the categories of X as if X is nominal, we convert the categories to

$$X = \text{income scores (3K, 10K, 20K, 35K)},$$

and fit the baseline-category logit model

$$\log\left(\frac{\pi_j}{\pi_J}\right) = \alpha_j + \beta_j x, \quad j = 1, 2, 3.$$

for $J = 4$ job satisfaction categories.

VGAM Library in R

ML estimates for coefficients (α_j, β_j) in logit model can be found via the R function vglm() in the package VGAM w/ multinomial family.

You will have to install the VGAM library first, by the following command. You only need to install ONCE!

```
install.packages("VGAM")    # JUST RUN THIS ONCE!
```

Once installed, must load VGAM at every R session before it can be used.

```
library(VGAM)
```

Fitting Baseline-Category Logit Models in R

Income	Job Satisfaction (Y)			
(X)	Dissat	Little	Moderate	Very
0-5K	2	4	13	3
5-15K	2	6	22	4
15-25K	0	1	15	8
$>$25K	0	3	13	8

Recall we use X = income score (3K, 10K, 20K, 35K) as the predictor.

```
Income = c(3,10,20,35)
Diss = c(2,2,0,0)
Little = c(4,6,1,3)
Mod = c(13,22,15,13)
Very = c(3,4,8,8)
jobsat.fit1 = vglm(cbind(Diss,Little,Mod,Very) ~ Income,
                   family=multinomial)
```

```
coef(jobsat.fit1, matrix=TRUE)
            log(mu[,1]/mu[,4]) log(mu[,2]/mu[,4]) log(mu[,3]/mu[,4])
(Intercept)             0.4298            0.45627            1.70393
Income                 -0.1854           -0.05441           -0.03739
```

The fitted model is

$$\log\left(\frac{\widehat{\pi}_1}{\widehat{\pi}_4}\right) = \widehat{\alpha}_1 + \widehat{\beta}_1 x = 0.430 - 0.185x \qquad \text{(Dissat. v.s. Very Sat.)}$$

$$\log\left(\frac{\widehat{\pi}_2}{\widehat{\pi}_4}\right) = \widehat{\alpha}_2 + \widehat{\beta}_2 x = 0.456 - 0.054x \qquad \text{(Little v.s. Very Sat.)}$$

$$\log\left(\frac{\widehat{\pi}_3}{\widehat{\pi}_4}\right) = \widehat{\alpha}_3 + \widehat{\beta}_3 x = 1.704 - 0.037x \qquad \text{(Moderate v.s. Very Sat.)}$$

As $\widehat{\beta}_j < 0$ for $j = 1, 2, 3$, for each logit, estimated odds of being in less satisfied category (instead of very satisfied) decrease as $x =$ income increases.

Interpretation of Coefficients

Interpretation of β_i in the model $\log(\widehat{\pi_i/\pi_J}) = \widehat{\alpha}_i + \widehat{\beta}_i x$:
For every 1-unit increase in x, the odds of Y being in category i rather than category J become e^{β_i} times as large.

Example (Job Satisfaction)

$$\log(\widehat{\pi_1/\pi_4}) = 0.430 - 0.185x \qquad \text{(Dissat. v.s. Very Sat.)}$$

$$\log(\widehat{\pi_2/\pi_4}) = 0.456 - 0.054x \qquad \text{(Little v.s. Very Sat.)}$$

$$\log(\widehat{\pi_3/\pi_4}) = 1.704 - 0.037x \qquad \text{(Moderate v.s. Very Sat.)}$$

Estimated odds of being

"dissatisfied"		$e^{-0.185} \approx 0.83$
"little satisfied"	rather than "very satisfied" become	$e^{-0.054} \approx 0.95$
"moderately satisfied"		$e^{-0.037} \approx 0.96$

times as large for each 1K increase in income.

Interpretation of Coefficients

Example (Job Satisfaction)

$$\log(\widehat{\pi}_1/\widehat{\pi}_4) = 0.430 - 0.185x \qquad \text{(Dissat. v.s. Very Sat.)}$$

$$\log(\widehat{\pi}_2/\widehat{\pi}_4) = 0.456 - 0.054x \qquad \text{(Little v.s. Very Sat.)}$$

$$\log(\widehat{\pi}_3/\widehat{\pi}_4) = 1.704 - 0.037x \qquad \text{(Moderate v.s. Very Sat.)}$$

The estimated odds of being “little satisfied” rather than “dissatisfied” (neither is the baseline category) become

$$e^{-0.054-(-0.185)} \approx 1.14$$

times as large for each 1K increase in income.

Probabilities of Categories

Baseline-Category Logit Model:

$$\log\left(\frac{\pi_j}{\pi_J}\right) = \alpha_j + \beta_j x \iff \pi_j = \pi_J e^{\alpha_j + \beta_j x} \quad j = 1, 2, \ldots, J-1.$$

The probability π_J for the baseline category can be determined from $\sum_{j=1}^{J} \pi_j = 1$ as follows:

$$1 = \sum_{j=1}^{J} \pi_j = \pi_J + \sum_{j=1}^{J-1} \pi_J e^{\alpha_j + \beta_j x} = \pi_J \left(1 + \sum_{j=1}^{J-1} e^{\alpha_j + \beta_j x}\right)$$

So $\pi_J = \dfrac{1}{1 + \sum_{k=1}^{J-1} e^{\alpha_k + \beta_k x}}$. The probabilities π_j for other categories can be obtained from $\pi_j = \pi_J e^{\alpha_j + \beta_j x}$ to be

$$\pi_j = \frac{e^{\alpha_j + \beta_j x}}{1 + \sum_{k=1}^{J-1} e^{\alpha_k + \beta_k x}}, \text{ for } j = 1, 2, \ldots, J-1$$

Probabilities of Categories (Job Satisfaction)

$$\widehat{\pi}_1 = \frac{e^{0.430-0.185x}}{1 + e^{0.430-0.185x} + e^{0.456-0.054x} + e^{1.704-0.037x}}$$

$$\widehat{\pi}_2 = \frac{e^{0.456-0.054x}}{1 + e^{0.430-0.185x} + e^{0.456-0.054x} + e^{1.704-0.037x}}$$

$$\widehat{\pi}_3 = \frac{e^{1.704-0.037x}}{1 + e^{0.430-0.185x} + e^{0.456-0.054x} + e^{1.704-0.037x}}$$

$$\widehat{\pi}_4 = \frac{1}{1 + e^{0.430-0.185x} + e^{0.456-0.054x} + e^{1.704-0.037x}}$$

E.g., at $x = 20$ (K), estimated prob. of being “dissatisfied” and “very satisfied” are respectively,

$$\widehat{\pi}_1 = \frac{e^{0.430-0.185(20)}}{1 + e^{0.430-0.185(20)} + e^{0.456-0.054(20)} + e^{1.704-0.037(20)}} \approx 0.009$$

$$\widehat{\pi}_4 = \frac{1}{1 + e^{0.430-0.185(20)} + e^{0.456-0.054(20)} + e^{1.704-0.037(20)}} \approx 0.240$$

Obtaining Probabilities of Categories in R

In R, we can obtain the prob. of being "diss", "little", "moderate", or "very satisfied" when income is 20K using `predict()`.

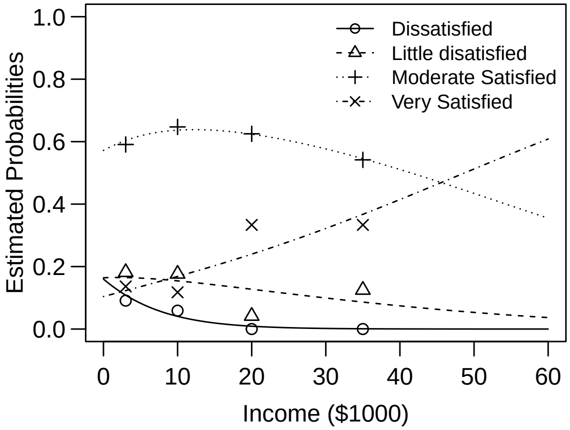
```
predict(jobsat.fit1, data.frame(Income=20), type="response")
      Diss Little    Mod   Very
1 0.009043 0.1274 0.6238 0.2397
```

We see $\widehat{\pi}_1 \approx 0.009, \widehat{\pi}_2 \approx 0.127, \widehat{\pi}_3 \approx 0.624$, and $\pi_4 \approx 0.240$. Observe that $\widehat{\pi}_1 + \widehat{\pi}_2 + \widehat{\pi}_3 + \widehat{\pi}_4 = 1$.

Caution: without specifying `type="response"`, `predict()` would return the values of the logits $\log(\widehat{\pi}_i/\widehat{\pi}_J) = \alpha_i + \beta_i x$, not the probabilities $\widehat{\pi}_i$.

```
predict(jobsat.fit1, data.frame(Income=20))
  log(mu[,1]/mu[,4]) log(mu[,2]/mu[,4]) log(mu[,3]/mu[,4])
1             -3.278             -0.632             0.9562
```

Plot of sample proportions and estimated probabilities of Job Satisfaction as a function of Income



Observe that though π_j/π_J is a monotone function of x, π_j may NOT be **monotone** in x.

Deviance and Goodness of Fit

For grouped multinomial response data,

	conditions of trial (explanatory variables)				number of trials	multinomial counts			
Condition 1	x_{11}	x_{12}	$\ldots$	x_{1p}	n_1	y_{11}	y_{12}	$\ldots$	y_{1J}
Condition 2	x_{21}	x_{22}	$\ldots$	x_{2p}	n_2	y_{21}	y_{22}	$\ldots$	y_{2J}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
Condition N	x_{N1}	x_{N2}	$\ldots$	x_{Np}	n_N	y_{N1}	y_{N2}	$\ldots$	y_{NJ}

(Residual) Deviance for a Model M is defined as

$$\text{Deviance} = -2(L_M - L_S) = 2\sum_{ij} y_{ij}\log\left(\frac{y_{ij}}{n_i\widehat{\pi}_j(\mathbf{x}_i)}\right)$$

$$= 2\sum_{ij}(\text{observed})\log\left(\frac{\text{observed}}{\text{fitted}}\right)$$

where $\widehat{\pi}_j(\mathbf{x}_i)$ = estimated prob. based on Model M

L_M = max. log-likelihood for Model M

L_S = max. log-likelihood for the saturated model

DF of Deviance

df for deviance of Model M is

$$N(J-1) - (\text{\# of parameters in the model}).$$

where N = # of rows in the data, J = # of levels of the response

If the model has p explanatory variables,

$$\log\left(\frac{\pi_j}{\pi_J}\right) = \alpha_j + \beta_{1j}x_1 + \cdots + \beta_{pj}x_p, \quad j = 1, 2, \ldots, J-1.$$

there are $p+1$ coefficients per equation, hence $(J-1)(p+1)$ coefficients in total.

df for deviance $= N(J-1) - (J-1)(p+1) = (J-1)(N-p-1)$.

```
deviance(jobsat.fit1)
[1] 4.658
df.residual(jobsat.fit1)
[1] 6
```

Goodness of Fit Test (GOF test)

If the estimated expected counts $n_i\widehat{\pi}_j(\mathbf{x}_i)$ are large enough (≥ 5), the deviance has a large sample chi-squared distribution with df = df of deviance.

We can use deviance to conduct Goodness of Fit test

- H_0: Model M is correct (fits the data as well as the saturated model)
- H_A: Saturated model is correct

When H_0 is rejected, it means that Model M doesn't fit as well as the saturated model.

Example (Job Satisfaction): the P-value for the GOF test is 58.8%, no evidence of lack of fit. However, this P-value is not reliable because most of the cell counts are small.

```
deviance(jobsat.fit1)
[1] 4.658
df.residual(jobsat.fit1)
[1] 6
pchisq(4.657999, df=6, lower.tail=F)
[1] 0.5884
```

Wald CIs and Wald Tests for Coefficients

- Wald CI for β_j is $\widehat{\beta}_j \pm z_{\alpha/2}\text{SE}(\widehat{\beta}_j)$.
- Wald test of H_0: $\beta_j = 0$ uses $z = \dfrac{\widehat{\beta}_j}{\text{SE}(\widehat{\beta}_j)} \sim N(0,1)$

Example (Job Satisfaction):

```
# the 4 to 6th coefficients, the 1st to 3rd are intercepts
coef(summary(jobsat.fit1))[4:6,]
         Estimate Std. Error z value Pr(>|z|)
Income:1 -0.18537    0.10251  -1.808  0.07057
Income:2 -0.05441    0.03112  -1.748  0.08038
Income:3 -0.03739    0.02088  -1.790  0.07340
```

- 95% for β_1: $-0.185 \pm 1.96 \times 0.1025 \approx (-0.386, 0.016)$
- 95% for e^{β_1}: $(e^{-0.386}, e^{0.016}) \approx (0.680, 1.016)$

Interpretation: Estimated odds of being “dissatisfied” rather than “very satisfied” become 0.680 to 1.016 times as large for each 1K increase in income w/ 95% confidence.

Likelihood Ratio Tests

Example (Job Satisfaction): Overall test of income effect

$$\mathrm{H}_0 : \beta_1 = \beta_2 = \beta_3 = 0$$

is equivalent of the comparison of the two models

$$\mathrm{H}_0 : \log\left(\pi_j/\pi_4\right) = \alpha_j, \quad j = 1, 2, 3$$
$$\mathrm{H}_1 : \log\left(\pi_j/\pi_4\right) = \alpha_j + \beta_j x, \quad j = 1, 2, 3.$$

$$\begin{aligned} \text{LRT} &= -2(L_0 - L_1) = -2(-21.358 - (-16.954)) = 8.808 \\ &= \text{diff in deviances} = 13.467 - 4.658 = 8.809 \\ Df &= \text{diff. in number of parameters} = 6 - 3 = 3 \\ &= \text{diff. in residual df} = 9 - 6 = 3 \end{aligned}$$

P-value $= \mathrm{P}(\chi^2_3 > 8.809) \approx 0.03194.$

```
lrtest(jobsat.fit1)
Likelihood ratio test

Model 1: cbind(Diss, Little, Mod, Very) ~ Income
Model 2: cbind(Diss, Little, Mod, Very) ~ 1
  #Df LogLik Df Chisq Pr(>Chisq)
1   6  -16.9
2   9  -21.4  3  8.81      0.032
jobsat.fit2 = vglm(cbind(Diss,Little,Mod,Very) ~ 1,
                     family=multinomial)
logLik(jobsat.fit2)
[1] -21.36
logLik(jobsat.fit1)
[1] -16.95
```

Note that H_0 implies job satisfaction is independent of income. We got some evidence (P-value $= 0.032$) of dependence between job satisfaction and income.

Note we get a different conclusion if we conduct Pearson's Chi-square test of independence:

$$X^2 = 11.5, \quad df = (4-1)(4-1) = 9 \quad , P\text{-value} = 0.2415$$

```
jobsat = matrix(c(2,2,0,0,4,6,1,3,13,22,15,13,3,4,8,8), nrow=4)
options(digits=6)
chisq.test(jobsat)
Warning in chisq.test(jobsat): Chi-squared approximation may be incorre


    Pearson's Chi-squared test


data:  jobsat
X-squared = 11.52, df = 9, p-value = 0.241
```

LR test of independence gives similar conclusion ($G^2 = 13.47$, $df = 9$, P-value $= 0.1426$)

Why the Baseline Category Logit model give different conclusion from Pearson's test of independence?