# STAT 222 Lecture 25
# Confounded Two-Level Factorial Designs
# In 4 and 8 Blocks

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# Coverage

Section 13.1-13.3, 13.5 of Dean & Voss

# Confounded $2^k$ Factorial in More Blocks

In Lecture 24, we introduced a way to divide treatments in a single replicate $2^k$ factorial design in *two blocks* of size $2^{k-1}$ such that all but one parameter in full $k$-way model are estimable.

- A block of size $2^{k-1}$ can be too big (if $k \geq 5$) to be homogeneous

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- A block of size $2^{k-1}$ can be too big (if $k \geq 5$) to be homogeneous
- Better if one can divide the $2^k$ in 4 block, 8 blocks, or in $2^s$ blocks

# Confounded $2^4$ Factorial in 4 Blocks

| TC | $ABC$ | $ABCD$ | Block *I* | *II* | *III* | *IV* |
|---|---|---|---|---|---|---|
| 0000 | $-1$ | 1 | | ✓ | | |
| 0001 | $-1$ | $-1$ | ✓ | | | |
| 0010 | 1 | $-1$ | | | ✓ | |
| 0011 | 1 | 1 | | | | ✓ |
| 0100 | 1 | $-1$ | | | ✓ | |
| 0101 | 1 | 1 | | | | ✓ |
| 0110 | $-1$ | 1 | | ✓ | | |
| 0111 | $-1$ | $-1$ | ✓ | | | |
| 1000 | 1 | $-1$ | | | ✓ | |
| 1001 | 1 | 1 | | | | ✓ |
| 1010 | $-1$ | 1 | | ✓ | | |
| 1011 | $-1$ | $-1$ | ✓ | | | |
| 1100 | $-1$ | 1 | | ✓ | | |
| 1101 | $-1$ | $-1$ | ✓ | | | |
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For example, one can divide according to the coefficients of the two contrasts $(ABC, ABCD)$:

- Block I if $(c_{ijk\ell}^{ABC}, c_{ijk\ell}^{ABCD}) = (-1, -1)$
- Block II if $(c_{ijk\ell}^{ABC}, c_{ijk\ell}^{ABCD}) = (-1, 1)$
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- Only $15 - 3 = 12$ degrees of freedom left for treatment. That is, only 12 parameters are estimable.

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- Only $15 - 3 = 12$ degrees of freedom left for treatment. That is, only 12 parameters are estimable.
- There is one more contrast that is confounded, in addition to ABC and ABCD

Recall that coefficients of $ABC$ and $ABCD$ are

$$c_{ijk\ell}^{ABC} = c_{ijk\ell}^{A} c_{ijk\ell}^{B} c_{ijk\ell}^{C}, \quad c_{ijk\ell}^{ABCD} = c_{ijk\ell}^{A} c_{ijk\ell}^{B} c_{ijk\ell}^{C} c_{ijk\ell}^{D}$$

where

$$c_{ijk\ell}^{A} = \begin{cases} -1 & \text{if } i = 0 \\ 1 & \text{if } i = 1 \end{cases}, \; c_{ijk\ell}^{B} = \begin{cases} -1 & \text{if } j = 0 \\ 1 & \text{if } j = 1 \end{cases},$$

$$c_{ijk\ell}^{C} = \begin{cases} -1 & \text{if } k = 0 \\ 1 & \text{if } k = 1 \end{cases}, \; c_{ijk\ell}^{D} = \begin{cases} -1 & \text{if } \ell = 0 \\ 1 & \text{if } \ell = 1 \end{cases}.$$

The product of coefficients of ABC and ABCD is

$$\begin{aligned} c_{ijk\ell}^{ABC} c_{ijk\ell}^{ABCD} &= c_{ijk\ell}^{A} c_{ijk\ell}^{B} c_{ijk\ell}^{C} c_{ijk\ell}^{A} c_{ijk\ell}^{B} c_{ijk\ell}^{C} c_{ijk\ell}^{D} \\ &= \underbrace{(c_{ijk\ell}^{A})^2}_{=1} \underbrace{(c_{ijk\ell}^{B})^2}_{=1} \underbrace{(c_{ijk\ell}^{C})^2}_{=1} c_{ijk\ell}^{D} = c_{ijk\ell}^{D} \end{aligned}$$

- In Blocks I and IV, $c_{ijk\ell}^{D}$ is always $+1$
- In Blocks II and III, $c_{ijk\ell}^{D}$ is always $-1$

Main effect D is also confounded with block and hence is not estimable.

We wish all the main effects can be estimated. Hence, it's not a good idea to divide blocks using the contrasts ABC and ABCD.

In general, when one divides a $2^k$ design into 4 blocks by two contrasts,

- not only the two contrasts are confounded with blocks
- the "product" of the two contrasts is also confounded with blocks

In the "*product*" of contrasts, *any letter with exponent 2 can be ignored*. For example,

- If ABCD and ACD are confounded, $(ABCD)(ACD) = A^2BC^2D^2 = B$ is also confounded
- If ABC and BCD are confounded, $(ABC)(BCD) = AB^2C^2D = AD$ is also confounded

# $2^4$ Design in 4 Blocks

- We wish to estimate all main effects and 2-way interactions
- It's a bad idea to divide $2^4$ Design into 4 blocks using the 4-way interaction ABCD and a 3-way interaction. This way, one of the main effects would be confounded
- Better to divide using two 3-way contrasts like ABC and ABD. Their product is a two way interaction. This way, all main effects and all but one of the two-way interactions can be estimated
- It's impossible to divide a $2^4$ design into 4 blocks in a way that all main effects and two-way interactions are estimable.

## How to Divide a $2^5$ Design in 4 Blocks

Consider a $2^5$ design with 5 factors A, B, C, D, E.

- Again, we wish to estimate all main effects and 2-way interactions
- It's a bad idea to divide using the 5-way contrasts ABCDE, since, for example

$$(ABCD)(ABCDE) = E \quad \text{and} \quad (ABC)(ABCDE) = DE.$$

  Some main effect or two-way interaction will be confounded.
- Better using a 3-way interaction and a 4-way interaction with as few letters in common as possible, like

$$(ABCD)(CDE) = ABE.$$

# Example: Mangold Experiment (Section 13.5)

A study to investigate the effects of 5 different fertilizers on the growth of mangold roots. The five factors were

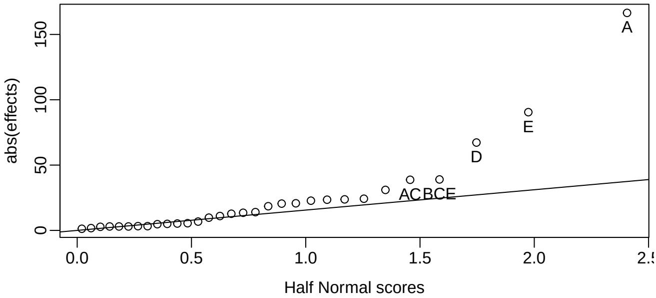
- A: Sulphate of ammonia (0 or 0.6 cwt per acre),
- B: Superphosphate (0 or 0.5 cwt per acre),
- C: Muriate of potash (0 or 1.0 cwt per acre),
- D: Agricultural salt (0 or 5 cwt per acre), and
- E: Dung (0 or 10 tons per acre).

The $2^5 = 32$ treatments are divided into 4 blocks of size 8, by the contrasts ABD and BCE. Their product $(ABD)(BCE) = ACDE$ is also confounded.

```
mangold = read.table(
  "https://www.stat.uchicago.edu/~yibi/s222/mangold.txt",h=T)
mangold$A = as.factor(mangold$A)
mangold$B = as.factor(mangold$B)
mangold$C = as.factor(mangold$C)
mangold$D = as.factor(mangold$D)
mangold$E = as.factor(mangold$E)
contrasts(mangold$A) = contr.sum(2)
contrasts(mangold$B) = contr.sum(2)
contrasts(mangold$C) = contr.sum(2)
contrasts(mangold$D) = contr.sum(2)
contrasts(mangold$E) = contr.sum(2)
lm1 = lm(y ~ A*B*C*D*E, data=mangold)
lm1$coef
   (Intercept)             A1             B1             C1
       1098.50        -166.50           9.75          -4.75
            D1             E1          A1:B1          A1:C1
        -67.25         -90.50           3.25          38.75
         B1:C1          A1:D1          B1:D1          C1:D1
        -14.00          22.75          13.50          31.00
         A1:E1          B1:E1          C1:E1          D1:E1
         -3.00          -1.75          -5.25         -20.75
      A1:B1:C1       A1:B1:D1       A1:C1:D1       B1:C1:D1
          3.00           3.00          18.50           1.25
```

# Half-Normal Probablity Plot — Mangold Experiment

```
library(daewr)
halfnorm(lm1$coef[-1])
```

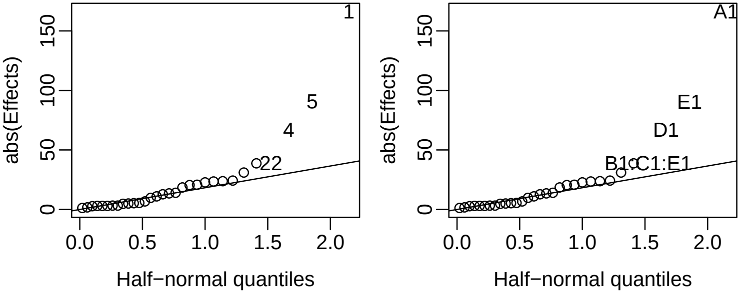



```
zscore= 0.02022 0.06068 0.1012 0.142 0.1829 0.2242 0.2659 0.308 0.3507
```

- A, D, E main effects are the most prominent ones
- Note BCE is confounded and hence it might just stands out because of the block effect
- There might be a bit AC interactions
- All other effects are small

If cannot install the `daewr` library, one can use the `halfnorm` plot in the `faraway` library.

```
library(faraway)
par(mai=c(.6,.6,.05,.05),mgp=c(2,.5,0))
halfnorm(lm1$coef[-1], nlab=4, ylab= "abs(Effects)")
qqline(c(-abs(lm1$coef[-1]), abs(lm1$coef[-1])))
halfnorm(lm1$coef[-1], nlab=4, labs= names(lm1$coef[-1]),
         ylab= "abs(Effects)")
qqline(c(-abs(lm1$coef[-1]),abs(lm1$coef[-1])))
```




One can specify the number of labelled effects to 4 by adding `nlab=4`.

# Analysis By Pooling Terms Into Errors

If the 3-, 4-, and 5-way interactions are assumed to be negligible, their SS, apart from ABD, BCE, and ACDE, which are confounded with blocks, can be pooled into error.

```
mangold$block = as.factor(mangold$block)
lm2 = lm(y ~ block + (A+B+C+D+E)^2, data=mangold)
```

See the ANOVA table on the next page.

- A, D, E main effects are highly significant
- AC interaction is a bit significant

```
anova(lm2)
Analysis of Variance Table

Response: y
          Df Sum Sq Mean Sq F value      Pr(>F)
block      3  52832   17611    2.59     0.09724
A          1 887112  887112  130.63 0.000000037
B          1   3042    3042    0.45     0.51503
C          1    722     722    0.11     0.74957
D          1 144722  144722   21.31     0.00048
E          1 262088  262088   38.59 0.000031564
A:B        1    338     338    0.05     0.82693
A:C        1  48050   48050    7.08     0.01964
A:D        1  16562   16562    2.44     0.14238
A:E        1    288     288    0.04     0.84003
B:C        1   6272    6272    0.92     0.35408
B:D        1   5832    5832    0.86     0.37097
B:E        1     98      98    0.01     0.90622
C:D        1  30752   30752    4.53     0.05305
C:E        1    882     882    0.13     0.72435
D:E        1  13778   13778    2.03     0.17790
Residuals 13  88286    6791
```

# 95% Simultaneous CIs for the 5 Main Effects

We can construct CIs based on the model `lm2`, with MSE $=$ SSE/dfE $= 88286/13 \approx 6791.23$

For all 5 main effects, their SE's all equals

$$\mathrm{SE} = \sqrt{\mathsf{MSE}\left(\frac{1}{r}+\frac{1}{r}\right)} = \sqrt{6791.23\left(\frac{1}{16}+\frac{1}{16}\right)} \approx 29.136$$

since the means of the two levels of a factor are both an average of 16 numbers.

Bonferroni's critical value for 5 CIs is about 3.012

```
qt(0.05/2/5, df=13, lower.tail=FALSE)
[1] 3.012
```

```r
library(mosaic)
mean(y ~ A, data=mangold)
   0    1
 932 1265
```

The 99% CI for A main effect $\alpha_1 - \alpha_0$ is

$$(1265 - 932) \pm 3.012 \times 29.136 \approx 333 \pm 87.77 = (245.23, 420.77)$$

Adding Sulphate of ammonia increase the yield by 245.23 to 420.77.

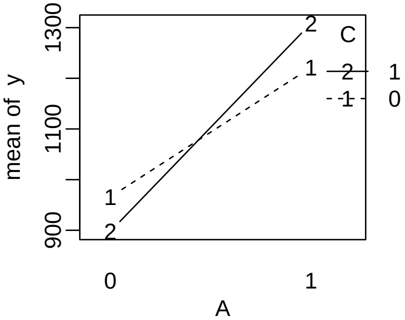
```
mean(y ~ B, data=mangold)
   0    1
1108 1089
mean(y ~ C, data=mangold)
   0    1
1094 1103
mean(y ~ D, data=mangold)
   0    1
1031 1166
mean(y ~ E, data=mangold)
   0    1
1008 1189
```

The simultaneous CIs for B, C, D, E are

$$\begin{aligned}
\text{For B} &: (1088.75 - 1108.25) \pm 87.77 = -19.5 \pm 87.77 = (-107.27, 68.27) \\
\text{For C} &: (1103.25 - 1093.75) \pm 87.77 = 9.5 \pm 87.77 = (-78.27, 97.27) \\
\text{For D} &: (1165.75 - 1031.25) \pm 87.77 = 134.5 \pm 87.77 = (46.73, 222.27) \\
\text{For E} &: (1189.00 - 1008.00) \pm 87.77 = 181 \pm 87.77 = (93.23268.77)
\end{aligned}$$

# AC Interaction Plots

```
par(mai=c(.6,.6,.15,.3),mgp=c(2,.5,0))
with(mangold, interaction.plot(A, C, y, type="b"))
```




- Slight AC interaction
- To maximize the yield, A should be set at high level
- When A is high, C should also be set at the high level to maximize the yield.

# $2^k$ Designs in 8 Blocks

- Can use 3 contrasts to divide into 8 blocks
- All possible products of subsets the 3 contrasts are also confounded.
- In total, $8-1$ contrasts are confounded
- E.g., in a $2^6$ design, if one divides using the BCD, ABE, and ADF contrasts, then

$$\begin{aligned}(BCD)(ABE) &= ACDE,\\ (BCD)(ADF) &= ABCF,\\ (ABE)(ADF) &= BDEF,\\ (BCD)(ABE)(ADF) &= CEF\end{aligned}$$

are also confounded.