STAT 222 Lecture 18-19
Complete Block Designs

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Coverage

- Chapter 10 Randomized Complete Block Designs (RCBD)
- Chapter 12 Latin Square Designs (except Section 12.2.3 and 12.5)

Randomized Complete Block Designs (RCBD)

Exercise 12.8: Air Freshener Sale (p. 427 Dean & Voss)

- Goal: comparing 4 price+display treatments on the sales of a brand of air fresheners.
  - Treatment A = high price + extra display
  - Treatment B = middle price + extra display
  - Treatment C = low price + extra display
  - Treatment D = middle price + no extra display
- conducted at 8 stores for 4 weeks and each treatment lasts for one week in each store
- Response: unit sales in a one-week period

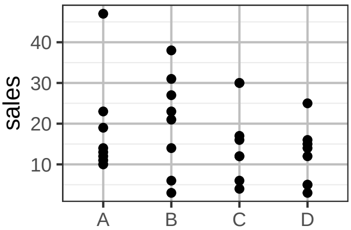
Week	Store							
	1	2	3	4	5	6	7	8
1	B 31	A 23	C 12	D 3	A 10	C 30	B 23	D 14
2	A 19	D 16	B 14	C 4	B 21	D 25	C 17	A 14
3	D 15	C 30	A 12	B 6	C 12	A 47	D 5	B 3
4	C 16	B 27	D 5	A 11	D 12	B 38	A 13	C 6

One-Way Analysis of Air Freshener Sales Data

```
freshener = read.table(
  "http://www.stat.uchicago.edu/~yibi/s222/air.freshener.txt",h=T)
freshener$trtmt = factor(freshener$trtmt, labels=LETTERS[1:4])
```

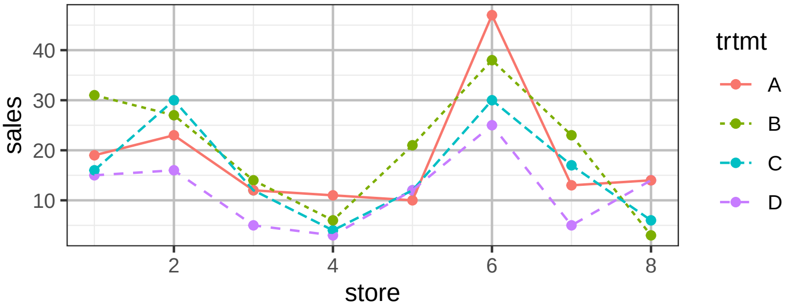
Treatments appear insignificant if analyzed like data from a one-way Completely Randomized Design (CRD) w/ 4 treatments.

$$y_{ij} = \mu_i + \varepsilon_{ij},$$



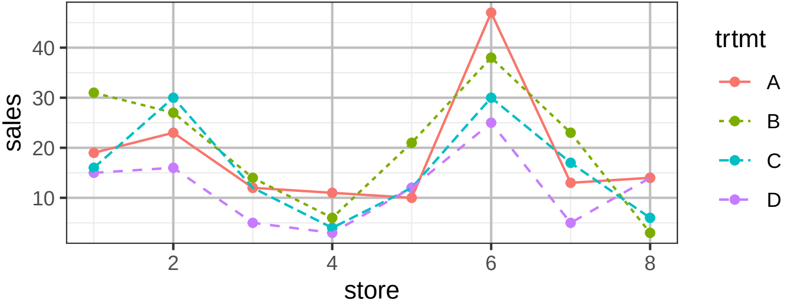

```
lm0 = lm(sales ~ trtmt, data=freshener)
anova(lm0)
Analysis of Variance Table

Response: sales
          Df Sum Sq Mean Sq F value Pr(>F)
trtmt      3    329     110    0.98   0.41
Residuals 28   3123     112
```

Air Freshener Sales — Why Account For `Store` Effect?



- Substantial variation in sales between stores
- Within each store, Treatment D was almost always the worst
  - evidence of treatment effects

Air Freshener Sales — Why Account For `Store` Effect?



- Substantial variation in sales between stores
- Within each store, Treatment D was almost always the worst
  - evidence of treatment effects
- Better take `store` effect into account

Block Designs

- A *block* is a set of experimental units that are homogeneous in some sense. Hopefully, units in the same block will have similar responses (if applied with the same treatment.)
- **Block designs**: randomize the units within each block to the treatments.

Randomized Complete Block Designs (RCBD)

g treatments to compare, b blocks of units available, each block contains $k = rg$ units.

- Within each block, the $k = rg$ units are **randomized** to the g treatments, r units each.
- "Complete" means each of the g treatments appears the same number of times (r) in every block.
- Mostly, block size $k = \#$ of treatments g, i.e., $r = 1$.
- Matched-pair design is one kind of RCBD with block size $k = 2$.

	Block 1	Block 2	$\cdots$	Block b
Treatment 1	y_{11}	y_{12}	$\cdots$	y_{1b}
Treatment 2	y_{21}	y_{22}	$\cdots$	y_{2b}
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$
Treatment g	y_{g1}	y_{g2}	$\cdots$	y_{gb}

Normally, data are shown arranged by block and treatment.
Cannot tell from the data what was/was not randomized.

Things That Can Be Blocked On

- Block when you can identify a source of variation (e.g., age, gender, medical history, etc)
- Block on machine/operator/batch (e.g., milk produced in a day)
- Block spatially
- Block on time
- Block on ...

Advantages of Blocking

- Blocking is the second basic principle of experimental design after randomization.

  > "Block what you can, randomize everything else."

- If units are highly variable, grouping them into more similar blocks can lead to a large reduction in noise (more power to detect difference in treatment effects).
- The choice of blocks is crucial

Example 2: Auditor Training

An accounting firm tested 3 training methods in statistical sampling for auditing,

1. study at home with programmed training materials,
2. training sessions at local offices conducted by local staff, and
3. training sessions in Chicago conducted by national staff.

- 30 auditors grouped into 10 blocks of 3, according to time elapsed since college graduation (new graduates in block 1, those graduated most distantly in block 10)
- Auditors in each block were randomly assigned to the 3 training methods
- Each auditor is tested and scored at the end of the training

Training	Block									
Method	1	2	3	4	5	6	7	8	9	10
1	73	76	75	74	76	73	68	64	65	62
2	81	78	76	77	71	75	72	74	73	69
3	92	89	87	90	88	86	88	82	81	78

Models for RCBDs

$$y_{ij} = \mu + \underset{\text{(trt)}}{\alpha_i} + \underset{\text{(block)}}{\beta_j} + \varepsilon_{ij}$$

in which

- y_{ij} = response of the unit receiving treatment i in block j
- μ = the grand mean
- α_i = the treatment effects
- β_j = the block effects
- ε_{ij} = measurement errors, i.i.d. $\sim N(0, \sigma^2)$

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Q: In an RCBD model, are we more interested in α_i's or β_j's?

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Q: In an RCBD model, are we more interested in α_i's or β_j's?

Just like models for factorial data, the model above is over-parameterized. Need to impose constraints on parameters, like the zero-sum constraints

$$\sum_{i=1}^{g} \alpha_i = 0 \quad \text{and} \quad \sum_{j=1}^{b} \beta_j = 0.$$

Parameter Estimates for RCBD Models

The model for a RCBD

$$y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}, \quad \varepsilon_{ij}\text{'s are i.i.d. } N(0, \sigma^2).$$

has the same format as the additive model for a balanced 2-way factorial design,
$\Rightarrow$ identical formulas for the parameter estimates

$$\begin{aligned}
\widehat{\mu} &= \bar{y}_{\bullet\bullet} \\
\widehat{\alpha}_i &= \bar{y}_{i\bullet} - \bar{y}_{\bullet\bullet} && \text{for } i = 1, \ldots, g \\
\widehat{\beta}_j &= \bar{y}_{\bullet j} - \bar{y}_{\bullet\bullet} && \text{for } j = 1, \ldots, b
\end{aligned}$$

Parameter Estimates for RCBD Models

The model for a RCBD

$$y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}, \quad \varepsilon_{ij}\text{'s are i.i.d. } N(0, \sigma^2).$$

has the same format as the additive model for a balanced 2-way factorial design,
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$$\begin{aligned} \widehat{\mu} &= \bar{y}_{\bullet\bullet} \\ \widehat{\alpha}_i &= \bar{y}_{i\bullet} - \bar{y}_{\bullet\bullet} && \text{for } i = 1, \ldots, g \\ \widehat{\beta}_j &= \bar{y}_{\bullet j} - \bar{y}_{\bullet\bullet} && \text{for } j = 1, \ldots, b \end{aligned}$$

Questions: Why not include treatment-block interactions?

Sum of Squares and Degrees of Freedom

The sum of squares and degrees of freedom for RCBD are just like those for additive models:

$$SST = SS_{trt} + SS_{block} + SSE$$

where

$$SST = \sum_{i=1}^{g}\sum_{j=1}^{b}(y_{ij} - \bar{y}_{\bullet\bullet})^2$$

$$SS_{trt} = \sum_{i=1}^{g}\sum_{j=1}^{b}(\bar{y}_{i\bullet} - \bar{y}_{\bullet\bullet})^2 = b\sum_{i=1}^{g}(\bar{y}_{i\bullet} - \bar{y}_{\bullet\bullet})^2$$

$$SS_{block} = \sum_{i=1}^{g}\sum_{j=1}^{b}(\bar{y}_{\bullet j} - \bar{y}_{\bullet\bullet})^2 = g\sum_{j=1}^{b}(\bar{y}_{\bullet j} - \bar{y}_{\bullet\bullet})^2$$

$$SSE = \sum_{i=1}^{g}\sum_{j=1}^{b}(y_{ij} - \bar{y}_{i\bullet} - \bar{y}_{\bullet j} + \bar{y}_{\bullet\bullet})^2.$$

Total	Treatment	Block	Error
$dfT = bg - 1$	$df_{trt} = g - 1$	$df_{block} = b - 1$	$dfE = (g-1)(b-1)$

Expected Values for the Mean Squares

Just like CRD, the mean squares for RCBD is the sum of squares divided by the corresponding d.f.

$$\mathrm{MS}_{trt} = \frac{\mathrm{SS}_{trt}}{g-1}, \quad \mathrm{MS}_{block} = \frac{\mathrm{SS}_{block}}{b-1}, \quad \mathrm{MSE} = \frac{\mathrm{SSE}}{(g-1)(b-1)}.$$

Under the model for RCBD,

$$y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij} \quad \varepsilon_{ij}\text{'s are i.i.d. } N(0, \sigma^2),$$

with the zero-sum constraints $\sum_{i=1}^{g} \alpha_i = \sum_{j=1}^{b} \beta_j = 0$, one can show that

$$\mathrm{E}(\mathrm{MS}_{trt}) = \sigma^2 + \frac{b}{g-1} \sum_{i=1}^{g} \alpha_i^2$$
$$\mathrm{E}(\mathrm{MS}_{block}) = \sigma^2 + \frac{g}{b-1} \sum_{j=1}^{b} \beta_j^2$$
$$\mathrm{E}(\mathrm{MSE}) = \sigma^2$$

MSE is again an unbiased estimator for σ^2.

ANOVA F-Test for Treatment Effect

A test of equal treatment effects

$$H_0: \ \alpha_1 = \alpha_2 = \cdots = \alpha_g \quad \text{v.s.} \quad H_a: \text{not all } \alpha_i\text{'s are equal}$$

is equivalent to a test of whether all α_i's are zero

$$H_0: \ \alpha_1 = \alpha_2 = \cdots = \alpha_g = 0 \quad \text{v.s.} \quad H_a: \text{not all } \alpha_i\text{'s are zero}$$

as the constraint $\sum_{i=1}^{g} \alpha_i = 0$. The test statistic is

$$F_{trt} = \frac{\text{MS}_{trt}}{\text{MSE}} \sim F_{g-1,\,(g-1)(b-1)} \quad \text{under H}_0.$$

The ANOVA table is given in the next page.

ANOVA Table for RCBD

Source	d.f.	SS	MS	F
Block	$b-1$	SS_{block}	MS_{block}	$(F_{block} = \frac{\text{MS}_{block}}{\text{MSE}})$
Treatment	$g-1$	SS_{trt}	MS_{trt}	$F_{trt} = \frac{\text{MS}_{trt}}{\text{MSE}}$
Error	$(b-1)(g-1)$	SSE	MSE	
Total	$bg-1$	SST		

The F statistic F_{block} for testing the block effect is not of interest, and hence is usually omitted.

ANOVA Tables for CRD and RCBD

If we ignore block effect, and analyze RCBD as a CRD, the ANOVA table becomes

Source	d.f.	SS	MS	F
Treatment	$g-1$	SS_{trt}	MS_{trt}	$F_{trt} = \frac{\text{MS}_{trt}}{\text{MSE}_{CRD}}$
Error	$bg-g$	SSE_{CRD}	MSE_{CRD}	
Total	$bg-1$	SST		

The ANOVA tables for CRD and RCBD have identical SS_{trt}, but the variability due to block is now in the error term

$$\text{SSE}_{CRD} = \text{SSE}_{RCBD} + \text{SS}_{block}.$$

If SS_{block} is large, including block effect can substantially **reduce the size of noise**, easier to detect difference in treatments.

Example: Air Freshener Sales Data

```
freshener$store = factor(freshener$store)
anova(lm(sales ~ store + trtmt, data=freshener))
Analysis of Variance Table

Response: sales
          Df Sum Sq Mean Sq F value     Pr(>F)
store      7 2478.9   354.1  11.536 0.00000595
trtmt      3  329.4   109.8   3.577     0.0312
Residuals 21  644.6    30.7
```

Ignoring block effects, treatment effects become insignificant.

```
anova(lm(sales ~ trtmt, data=freshener))
Analysis of Variance Table

Response: sales
          Df Sum Sq Mean Sq F value Pr(>F)
trtmt      3  329.4   109.8   0.984  0.414
Residuals 28 3123.5   111.5
```

Example: Auditor Training

```
auditor = read.table(
  "http://www.stat.uchicago.edu/~yibi/s222/AuditorTraining", h=T)
auditor$block = as.factor(auditor$block)
auditor$treatment = as.factor(auditor$treatment)
anova(lm(y ~ block + treatment, data=auditor))
Analysis of Variance Table

Response: y
          Df Sum Sq Mean Sq F value         Pr(>F)
block      9  433.4    48.2   7.716       0.000132
treatment  2 1295.0   647.5 103.754 0.000000000132
Residuals 18  112.3     6.2
```

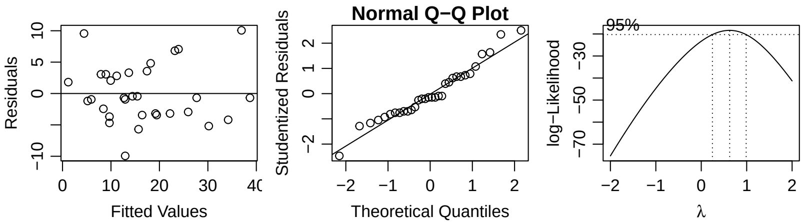
Ignoring block effects, treatment effects become less significant.

```
anova(lm(y ~ treatment, data=auditor))
Analysis of Variance Table

Response: y
          Df Sum Sq Mean Sq F value       Pr(>F)
treatment  2 1295.0   647.5   32.04 0.0000000744
Residuals 27  545.7    20.2
```

Always Check Model Assumptions

```
lm1 = lm(sales ~ store + trtmt, data=freshener)
plot(lm1$fitted, lm1$res, xlab="Fitted Values", ylab="Residuals")
abline(h=0)
qqnorm(rstudent(lm1),ylab="Studentized Residuals")
qqline(rstudent(lm1))
library(MASS); boxcox(lm1)
```



For the RCBD model of the Air Freshener Sales data,

- Residual-vs-fitted value plot and normal QQ plot look fine.
- Box-Cox says it might be better to take square-root of the response ($\lambda = 1/2$), the original response is not too much worse as $\lambda = 1$ is nearly at the upper end of the 95% CI for λ

Latin Square Deaigns

Latin Square Designs — Blocking Two Variations Simultaneously

Sometimes there are two sources of variations in the experimental units we want to eliminate by blocking.

<u>Example</u>: To compare the effects of 5 fertilizers A, B, C, D, E, an experiment is done in a farm that has a north-south variation in sunlight and east-west variation in soil humidity. One can block on the row and column position of plots using the design below.

(dry) $\leftrightarrow$ (humid)
West $\longleftrightarrow$ East
Column

	Row	1	2	3	4	5
(less sunlight) North	*I*	A	B	C	D	E
$\uparrow$	*II*	C	D	E	A	B
$\vert$	*III*	E	A	B	C	D
$\downarrow$	*IV*	B	C	D	E	A
(more sunlight) South	*V*	D	E	A	B	C

Each treatment occurs once in each row and in each column. This is called a **Latin square**.

Example — Automobile Emissions

Variables

- Additives: A, B, C, D (chemicals aimed at reducing pollution)
- Drivers: I, II, III, IV
- Cars: $1, 2, 3, 4$
- Response: Emission reduction index measured for each test drive

The experiment

- Additives as treatments ($i = 1, 2, 3, 4$)
- Drivers as a block variable (row block $j = 1, 2, 3, 4$)
- Cars as another block variable (column block $k = 1, 2, 3, 4$)
- The combination (driver, car) as experimental units
- Latin square of order 4 as the design of the experiment

Data and Design for Automobile Emissions Study

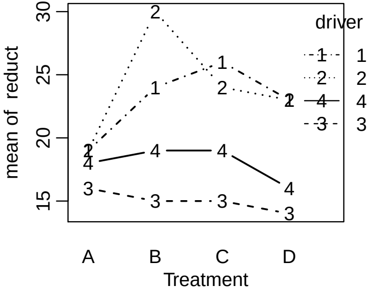
	Cars				
Drivers	1	2	3	4	driver average
I	A	B	D	C	
	19	24	23	26	23
II	D	C	A	B	
	23	24	19	30	24
III	B	D	C	A	
	15	14	15	16	15
IV	C	A	B	D	
	19	18	19	16	18
average					grand mean
per car	19	20	19	22	20

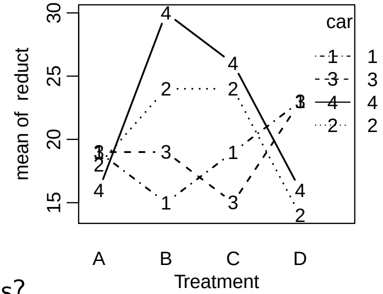
Treatment means

A: 18, B: 22, C: 21, D: 19

Automobile Emissions Data

	driver	car	trt	reduct
1	1	1	A	19
2	1	2	B	24
3	1	3	D	23
4	1	4	C	26
5	2	1	D	23
6	2	2	C	24
7	2	3	A	19
8	2	4	B	30
9	3	1	B	15
10	3	2	D	14
11	3	3	C	15
12	3	4	A	16
13	4	1	C	19
14	4	2	A	18
15	4	3	B	19
16	4	4	D	16





Are there driver effects? Car effects?
Which block effect is stronger?

Model For a Latin Square Design

$$y_{ijk} = \mu + \underset{\text{(trt)}}{\alpha_i} + \underset{\text{(row)}}{\beta_j} + \underset{\text{(column)}}{\gamma_k} + \varepsilon_{ijk}$$

where

$$\sum_i \alpha_i = \sum_j \beta_j = \sum_k \gamma_k = 0, \qquad \varepsilon_{ijk} \sim i.i.d.\ N(0, \sigma^2)$$

We have g^2 experimental units. For given j and k we only have one value $i(j,k)$ corresponding to Treatment i.

The design is balanced, so we have the usual estimates:

$$\widehat{\mu} = \bar{y}_{\bullet\bullet\bullet}, \quad \widehat{\alpha}_i = \bar{y}_{i\bullet\bullet} - \bar{y}_{\bullet\bullet\bullet}$$
$$\widehat{\beta}_j = \bar{y}_{\bullet j\bullet} - \bar{y}_{\bullet\bullet\bullet}$$
$$\widehat{\gamma}_k = \bar{y}_{\bullet\bullet k} - \bar{y}_{\bullet\bullet\bullet}$$

Statistical analysis can be done using familiar R commands.

How Do Latin-Square Designs Work?

Q: What is the mean for $\bar{y}_{A\bullet\bullet}$?

How Do Latin-Square Designs Work?

Q: What is the mean for $\bar{y}_{A\bullet\bullet}$?

For the automobile emission example,

$$\bar{y}_{A\bullet\bullet} = \frac{1}{4}(y_{A11} + y_{A32} + y_{A43} + y_{A24}).$$

driver	car 1	car 2	car 3	car 4
I	A	B	D	C
II	D	C	A	B
III	B	D	C	A
IV	C	A	B	D

Based on the model, we know

$$\mathrm{E}(y_{ijk}) = \mu + \underset{\text{(trt)}}{\alpha_i} + \underset{\text{(car)}}{\beta_j} + \underset{\text{(driver)}}{\gamma_k}$$

Thus

$$\mathrm{E}[\bar{y}_{A\bullet\bullet}] = \frac{1}{4}\begin{pmatrix} \mu + \alpha_A + \beta_1 + \gamma_1 \\ +\mu + \alpha_A + \beta_3 + \gamma_2 \\ +\mu + \alpha_A + \beta_4 + \gamma_3 \\ +\mu + \alpha_A + \beta_2 + \gamma_4 \end{pmatrix}$$

$$= \frac{1}{4}\Big(4\mu + 4\alpha_A + \underbrace{\sum_{j=1}^{4}\beta_j}_{=0} + \underbrace{\sum_{k=1}^{4}\gamma_k}_{=0}\Big) = \mu + \alpha_A$$

ANOVA Table for a Single Latin Square

Source	d.f.	SS	MS	F-value
Row-Block	$g-1$	SS_{row}	$\mathrm{SS}_{row}/(g-1)$	$\mathrm{MS}_{row}/\mathrm{MSE}$
Column-Block	$g-1$	SS_{col}	$\mathrm{SS}_{col}/(g-1)$	$\mathrm{MS}_{col}/\mathrm{MSE}$
Treatment	$g-1$	SS_{trt}	$\mathrm{SS}_{trt}/(g-1)$	$\mathrm{MS}_{trt}/\mathrm{MSE}$
Error	$(g-2)(g-1)$	SSE	$\mathrm{SSE}/[(g-2)(g-1)]$	
Total	g^2-1	SST		

$$
\begin{aligned}
SS_{row} &= \sum\nolimits_{ijk} \widehat{\beta}_j^2 = g\sum\nolimits_j \widehat{\beta}_j^2 = g\sum\nolimits_j (\bar{y}_{\bullet j\bullet} - \bar{y}_{\bullet\bullet\bullet})^2, \\
SS_{col} &= \sum\nolimits_{ijk} \widehat{\gamma}_k^2 = g\sum\nolimits_k \widehat{\gamma}_k^2 = g\sum\nolimits_k (\bar{y}_{\bullet\bullet k} - \bar{y}_{\bullet\bullet\bullet})^2, \\
SS_{trt} &= \sum\nolimits_{ijk} \widehat{\alpha}_i^2 = g\sum\nolimits_i \widehat{\alpha}_i^2 = g\sum\nolimits_i (\bar{y}_{i\bullet\bullet} - \bar{y}_{\bullet\bullet\bullet})^2, \\
SSE &= \sum\nolimits_{ijk} (y_{ijk} - \bar{y}_{i\bullet\bullet} - \bar{y}_{\bullet j\bullet} - \bar{y}_{\bullet\bullet k} + 2\bar{y}_{\bullet\bullet\bullet})^2 \\
&= SST - SS_{row} - SS_{col} - SS_{trt}, \\
SST &= \sum\nolimits_{ijk} (y_{ijk} - \bar{y}_{\bullet\bullet\bullet})^2
\end{aligned}
$$

ANOVA Table for Automobile Emissions Data

```
emis = read.table(
    "http://www.stat.uchicago.edu/~yibi/s222/emission.txt", h=T)
lm1 = lm(reduct ~ as.factor(driver)+as.factor(car)+as.factor(trt),
           data=emis)
anova(lm1)
Analysis of Variance Table

Response: reduct
                  Df Sum Sq Mean Sq F value  Pr(>F)
as.factor(driver)  3    216   72.00    13.5 0.00447
as.factor(car)     3     24    8.00     1.5 0.30717
as.factor(trt)     3     40   13.33     2.5 0.15649
Residuals          6     32    5.33
```

Crossover Design — A Special Latin-Square Design

When a sequence of treatments is given to a subject over several time periods,

- need to block on subjects,
  because each subject tends to respond differently, and
- need to block on time period,
  because there may consistent differences over time due to growth, aging, disease progression, or other factors.

Such a design is called a *crossover design*, which is a common application of the Latin Square designs.

Example: Bioequivalence of Drug Delivery (p.326 in Oehlert's)

- Objectives — Investigate whether different drug delivery systems have similar biological effects.
- Response variable — Average blood concentration of the drug during certain time interval after the drug has been administered.
- Treatments — 3 different drug delivery systems:

  A – solution, B – tablet, C – capsule

- Blocking on both subjects and time period
- Data & design:

	subject		
period	1	2	3
1	1799A	2075C	1396B
2	1846C	1156B	868A
3	2147B	1777A	2291C

Advantages of Latin Squares

- For the same number of experimental units as a randomized complete block design (RCBD) with g treatments and g blocks, we can simultaneously block for a second variable.
- Provide an elegant and efficient use of limited resources for a small experiment.

Disadvantages of Latin Squares

- Cannot identify block-treatment interactions.
- There may be few $d.f.$ left to estimate σ once both block and treatment effects are estimated.
  E.g., in a 3×3 square, block and treatment effects taken up 3×2 $d.f.$, allowing 1 $d.f.$ for the grand mean leaves 2 $d.f.$ for estimating σ^2, and this includes all the treatments.
- One can (and should) **replicate** Latin square.

Replicated Latin Square Designs

In the Bioequivalence of Drug Delivery Example, with a single 3×3 Latin square, there are only 2 degrees of freedom left for estimating the error. One can increase the d.f. for errors by *replicating more Latin squares.*

- Can extend the study from 3 subjects to 12 subjects
- The 12 subjects are divided into 4 groups of 3 each, and a Latin square design is arranged for each group
- Data are shown on the next page and the data file is at

`http://users.stat.umn.edu/~gary/book/fcdae.data/exmpl13.10`

Data and Design for Extended Bioequivalence Study

	square 1			square 2		
	subject					
period	1	2	3	4	5	6
1	1799A	2075C	1396B	3100B	1451C	3174A
2	1846C	1156B	868A	3065A	1217B	1714C
3	2147B	1777A	2291C	4077C	1288A	2919B

	square 3			square 4		
	subject					
period	7	8	9	10	11	12
1	1430C	1186A	1135B	873C	2061A	1053B
2	836A	642B	1305C	1426A	2433B	1534C
3	1063B	1183C	984A	1540B	1337C	1583A

- Each (subject, treatment) combination appears exactly once
- Each (period, treatment) combination appears exactly 4 times
- Note the 4 Latin-squares have a common row block (period). In this case, we say the row block is *reused*.

Model for Replicated Latin Square Design(row block reused)

When Latin squares are replicated (each one separately randomized), the appropriate linear model will depend on which blocks (if any) are **reused**.

In the Bioequivalence study, the row variable, period, is reused. An appropriate model would be

$$y_{ijk} = \mu + \underset{\text{(trt)}}{\alpha_i} + \underset{\text{(period)}}{\beta_j} + \underset{\text{(subject)}}{\gamma_k} + \varepsilon_{ijk}$$

with the constraints

$$\sum_{i=1}^{3} \alpha_i = \sum_{j=1}^{3} \beta_j = \sum_{k=1}^{12} \gamma_k = 0.$$

The parameter estimates are again

$$\widehat{\mu} = \bar{y}_{\bullet\bullet\bullet}, \quad \widehat{\alpha}_i = \bar{y}_{i\bullet\bullet} - \bar{y}_{\bullet\bullet\bullet}$$
$$\widehat{\beta}_j = \bar{y}_{\bullet j\bullet} - \bar{y}_{\bullet\bullet\bullet}$$
$$\widehat{\gamma}_k = \bar{y}_{\bullet\bullet k} - \bar{y}_{\bullet\bullet\bullet}$$

How Do Replicated Latin-Square Designs Row-Block Reused Work? (1)

For the Drug Delivery Study, we can show that $\mathrm{E}[\bar{y}_{B\bullet\bullet}] = \mu + \alpha_B$.

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Subject	1	2	3	4	5	6	7	8	9	10	11	12
1	A	C	B	B	C	A	C	A	B	C	A	B
period 2	C	B	A	A	B	C	A	B	C	A	B	C
3	B	A	C	C	A	B	B	C	A	B	C	A

$$\bar{y}_{B\bullet\bullet} = \frac{1}{12}\big(y_{B31} + y_{B22} + y_{B13} + y_{B14} + y_{B25} + y_{B36} + y_{B37} + y_{B28} + y_{B19} + y_{B3,10} + y_{B2,11} + y_{B1,12}\big).$$

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Subject	1	2	3	4	5	6	7	8	9	10	11	12
period 1	A	C	B	B	C	A	C	A	B	C	A	B
period 2	C	B	A	A	B	C	A	B	C	A	B	C
period 3	B	A	C	C	A	B	B	C	A	B	C	A

$$\bar{y}_{B\bullet\bullet} = \frac{1}{12}\big(y_{B31} + y_{B22} + y_{B13} + y_{B14} + y_{B25} + y_{B36} + y_{B37} + y_{B28} + y_{B19} + y_{B3,10} + y_{B2,11} + y_{B1,12}\big).$$

Since

$$E(y_{ijk}) = \mu + \underset{\text{(trt)}}{\alpha_i} + \underset{\text{(period)}}{\beta_j} + \underset{\text{(subject)}}{\gamma_k}$$

we know

$$E[\bar{y}_{B\bullet\bullet}] = \frac{1}{12}\begin{pmatrix} \mu + \alpha_B + \beta_3 + \gamma_1 \\ + \mu + \alpha_B + \beta_2 + \gamma_2 \\ + \vdots \quad \vdots \quad \vdots \quad \vdots \\ + \mu + \alpha_B + \beta_1 + \gamma_{12} \end{pmatrix} = \frac{1}{12}\Big(12\mu + 12\alpha_B + 4\underbrace{\sum_{j=1}^{3}\beta_j}_{=0} + \underbrace{\sum_{k=1}^{12}\gamma_k}_{=0}\Big)$$

$$= \mu + \alpha_B$$

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Subject	1	2	3	4	5	6	7	8	9	10	11	12
1	A	C	B	B	C	A	C	A	B	C	A	B
period 2	C	B	A	A	B	C	A	B	C	A	B	C
3	B	A	C	C	A	B	B	C	A	B	C	A

$$\bar{y}_{B\bullet\bullet} = \frac{1}{12}(y_{B31} + y_{B22} + y_{B13} + y_{B14} + y_{B25} + y_{B36} + y_{B37} + y_{B28} + y_{B19} + y_{B3,10} + y_{B2,11} + y_{B1,12}).$$

Since

$$E(y_{ijk}) = \mu + \underset{\text{(trt)}}{\alpha_i} + \underset{\text{(period)}}{\beta_j} + \underset{\text{(subject)}}{\gamma_k}$$

we know

$$E[\bar{y}_{B\bullet\bullet}] = \frac{1}{12}\begin{pmatrix} \mu + \alpha_B + \beta_3 + \gamma_1 \\ + \mu + \alpha_B + \beta_2 + \gamma_2 \\ + \vdots \quad \vdots \quad \vdots \quad \vdots \\ + \mu + \alpha_B + \beta_1 + \gamma_{12} \end{pmatrix} = \frac{1}{12}(12\mu + 12\alpha_B + 4\underbrace{\sum_{j=1}^{3}\beta_j}_{=0} + \underbrace{\sum_{k=1}^{12}\gamma_k}_{=0}) = \mu + \alpha_B$$

It works because each treatment shows up in each row 4 times and each column once. Each β_j comes up 4 times and each γ_k once in the sum.

How Do Replicated Latin-Square Designs Row-Block Reused Work? (2)

Similarly, for the period block effect, we can show that $\mathsf{E}[\bar{y}_{\bullet 1 \bullet}] = \mu + \beta_1$.

How Do Replicated Latin-Square Designs Row-Block Reused Work? (2)

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	Subject	1	2	3	4	5	6	7	8	9	10	11	12
	1	A	C	B	B	C	A	C	A	B	C	A	B
period	2	C	B	A	A	B	C	A	B	C	A	B	C
	3	B	A	C	C	A	B	B	C	A	B	C	A

$$\bar{y}_{\bullet 1 \bullet} = \frac{1}{12}\big(y_{A11} + y_{C12} + y_{B13} + y_{B14} + y_{C15} + y_{A16} + y_{C17} + y_{A18} + y_{B19} + y_{C1,10} + y_{A1,11} + y_{B1,12}\big).$$

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Subject	1	2	3	4	5	6	7	8	9	10	11	12
1	A	C	B	B	C	A	C	A	B	C	A	B
period 2	C	B	A	A	B	C	A	B	C	A	B	C
3	B	A	C	C	A	B	B	C	A	B	C	A

$$\bar{y}_{\bullet 1 \bullet} = \frac{1}{12}(y_{A11} + y_{C12} + y_{B13} + y_{B14} + y_{C15} + y_{A16} + y_{C17} + y_{A18} + y_{B19} + y_{C1,10} + y_{A1,11} + y_{B1,12}).$$

Since

$$E(y_{ijk}) = \mu + \underset{\text{(trt)}}{\alpha_i} + \underset{\text{(period)}}{\beta_j} + \underset{\text{(subject)}}{\gamma_k}$$

we know

$$E[\bar{y}_{\bullet 1 \bullet}] = \frac{1}{12}\begin{pmatrix} \mu + \alpha_A + \beta_1 + \gamma_1 \\ + \mu + \alpha_C + \beta_1 + \gamma_2 \\ + \vdots \quad \vdots \quad \vdots \quad \vdots \\ + \mu + \alpha_B + \beta_1 + \gamma_{12} \end{pmatrix} = \frac{1}{12}(12\mu + 4\underbrace{\sum_{j=1}^{3}\alpha_j}_{=0} + 12\beta_1 + \underbrace{\sum_{k=1}^{12}\gamma_k}_{=0})$$

$$= \mu + \beta_1$$

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Similarly, for the period block effect, we can show that $E[\bar{y}_{\bullet 1 \bullet}] = \mu + \beta_1$.

Subject	1	2	3	4	5	6	7	8	9	10	11	12
1	A	C	B	B	C	A	C	A	B	C	A	B
period 2	C	B	A	A	B	C	A	B	C	A	B	C
3	B	A	C	C	A	B	B	C	A	B	C	A

$$\bar{y}_{\bullet 1 \bullet} = \frac{1}{12}(y_{A11} + y_{C12} + y_{B13} + y_{B14} + y_{C15} + y_{A16} + y_{C17} + y_{A18} + y_{B19} + y_{C1,10} + y_{A1,11} + y_{B1,12}).$$

Since

$$E(y_{ijk}) = \mu + \underset{\text{(trt)}}{\alpha_i} + \underset{\text{(period)}}{\beta_j} + \underset{\text{(subject)}}{\gamma_k}$$

we know

$$E[\bar{y}_{\bullet 1 \bullet}] = \frac{1}{12}\begin{pmatrix} \mu + \alpha_A + \beta_1 + \gamma_1 \\ + \mu + \alpha_C + \beta_1 + \gamma_2 \\ + \vdots \quad \vdots \quad \vdots \quad \vdots \\ + \mu + \alpha_B + \beta_1 + \gamma_{12} \end{pmatrix} = \frac{1}{12}(12\mu + 4\underbrace{\sum_{j=1}^{3}\alpha_j}_{=0} + 12\beta_1 + \underbrace{\sum_{k=1}^{12}\gamma_k}_{=0})$$

$$= \mu + \beta_1$$

It works because each treatment shows up in each row 4 times and hence each α_i comes up 4 times in the sum.

How Do Replicated Latin-Square Designs Row-Block Reused Work? (3)

Similarly, for the subject block effect, we can show that $\mathrm{E}[\bar{y}_{\bullet\bullet 3}] = \mu + \gamma_3$.

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Subject	1	2	3	4	5	6	7	8	9	10	11	12
1	A	C	B	B	C	A	C	A	B	C	A	B
period 2	C	B	A	A	B	C	A	B	C	A	B	C
3	B	A	C	C	A	B	B	C	A	B	C	A

$$\bar{y}_{\bullet\bullet 3} = \frac{1}{3}(y_{B13} + y_{A23} + y_{C33})$$

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1	A	C	B	B	C	A	C	A	B	C	A	B
period 2	C	B	A	A	B	C	A	B	C	A	B	C
3	B	A	C	C	A	B	B	C	A	B	C	A

$$\bar{y}_{\bullet\bullet 3} = \frac{1}{3}(y_{B13} + y_{A23} + y_{C33})$$

Since

$$E(y_{ijk}) = \mu + \underset{\text{(trt)}}{\alpha_i} + \underset{\text{(period)}}{\beta_j} + \underset{\text{(subject)}}{\gamma_k}$$

we know

$$E[\bar{y}_{\bullet\bullet 3}] = \frac{1}{3}\begin{pmatrix} \mu + \alpha_B + \beta_1 + \gamma_3 \\ + \mu + \alpha_A + \beta_2 + \gamma_3 \\ + \mu + \alpha_C + \beta_3 + \gamma_3 \end{pmatrix} = \frac{1}{3}(3\mu + \underbrace{\sum_{i=1}^{3} \alpha_i}_{=0} + \underbrace{\sum_{j=1}^{3} \beta_j}_{=0} + 3\gamma_3)$$

$$= \mu + \gamma_3$$

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Similarly, for the subject block effect, we can show that $\text{E}[\bar{y}_{\bullet\bullet 3}] = \mu + \gamma_3$.

Subject	1	2	3	4	5	6	7	8	9	10	11	12
1	A	C	B	B	C	A	C	A	B	C	A	B
period 2	C	B	A	A	B	C	A	B	C	A	B	C
3	B	A	C	C	A	B	B	C	A	B	C	A

$$\bar{y}_{\bullet\bullet 3} = \frac{1}{3}(y_{B13} + y_{A23} + y_{C33})$$

Since

$$\text{E}(y_{ijk}) = \mu + \underset{\text{(trt)}}{\alpha_i} + \underset{\text{(period)}}{\beta_j} + \underset{\text{(subject)}}{\gamma_k}$$

we know

$$\text{E}[\bar{y}_{\bullet\bullet 3}] = \frac{1}{3}\begin{pmatrix} \mu + \alpha_B + \beta_1 + \gamma_3 \\ + \mu + \alpha_A + \beta_2 + \gamma_3 \\ + \mu + \alpha_C + \beta_3 + \gamma_3 \end{pmatrix} = \frac{1}{3}(3\mu + \underbrace{\sum_{i=1}^{3} \alpha_i}_{=0} + \underbrace{\sum_{j=1}^{3} \beta_j}_{=0} + 3\gamma_3)$$

$$= \mu + \gamma_3$$

It works because each treatment shows up in each column exactly once and hence each α_i comes up once in the sum.

ANOVA Table for Latin Square *m* Replicates, Row Block Reused

Number of Latin Squares = m

Source	d.f.	SS	MS	F-value
Row-Block	$g-1$	SS_{row}	$\text{SS}_{row}/(g-1)$	$\text{MS}_{row}/\text{MSE}$
Column-Block	$mg-1$	SS_{col}	$\text{SS}_{col}/(mg-1)$	$\text{MS}_{col}/\text{MSE}$
Treatment	$g-1$	SS_{trt}	$\text{SS}_{trt}/(g-1)$	$\text{MS}_{trt}/\text{MSE}$
Error	$(mg-2)(g-1)$	SSE	$\text{SSE}/[(mg-2)(g-1)]$	
Total	mg^2-1	SST		

where

$$SS_{row} = \sum_{ijk}(\bar{y}_{\bullet j \bullet} - \bar{y}_{\bullet\bullet\bullet})^2 = mg\sum_{j=1}^{g}(\bar{y}_{\bullet j \bullet} - \bar{y}_{\bullet\bullet\bullet})^2,$$

$$SS_{col} = \sum_{ijk}(\bar{y}_{\bullet\bullet k} - \bar{y}_{\bullet\bullet\bullet})^2 = g\sum_{k=1}^{mg}(\bar{y}_{\bullet\bullet k} - \bar{y}_{\bullet\bullet\bullet})^2,$$

$$SS_{trt} = \sum_{ijk}(\bar{y}_{i\bullet\bullet} - \bar{y}_{\bullet\bullet\bullet})^2 = mg\sum_{i=1}^{g}(\bar{y}_{i\bullet\bullet} - \bar{y}_{\bullet\bullet\bullet})^2,$$

$$SST = \sum_{ijk}(y_{ijk} - \bar{y}_{\bullet\bullet\bullet})^2$$

$$SSE = SST - SS_{row} - SS_{col} - SS_{trt}$$

Don't try to memorize the formula for df_{error}. Just keep in mind that

$$df_{error} = df_{total} - df_{row} - df_{col} - df_{trt}$$

```
bioeqv = read.table(
  "http://users.stat.umn.edu/~gary/book/fcdae.data/exmpl13.10", h=T)
anova(lm(area ~ as.factor(subject)+as.factor(period)+
           as.factor(trt), data=bioeqv))
Analysis of Variance Table

Response: area
                   Df   Sum Sq Mean Sq F value    Pr(>F)
as.factor(subject) 11 16385060 1489551   7.255 0.0000747
as.factor(period)   2   737751  368875   1.797     0.192
as.factor(trt)      2    81458   40729   0.198     0.822
Residuals          20  4106500  205325
```

There is no evidence that the 3 drug delivery systems have different biological effects.

Air Freshener Sales Study Uses Replicated Latin Squares

Week	Store							
	1	2	3	4	5	6	7	8
1	B 31	A 23	C 12	D 3	A 10	C 30	B 23	D 14
2	A 19	D 16	B 14	C 4	B 21	D 25	C 17	A 14
3	D 15	C 30	A 12	B 6	C 12	A 47	D 5	B 3
4	C 16	B 27	D 5	A 11	D 12	B 38	A 13	C 6

- The study also blocked on `week` in addition to `store` to account for the possible `week` effect (season, holiday, etc)
- 2 replicates of 4×4 Latin squares with row block (week) reused

```
anova(lm(sales ~ store + as.factor(week) + trtmt, data=freshener))
Analysis of Variance Table

Response: sales
                Df Sum Sq Mean Sq F value    Pr(>F)
store            7 2478.9   354.1  10.310 0.0000348
as.factor(week)  3   26.4     8.8   0.256    0.8561
trtmt            3  329.4   109.8   3.197    0.0484
Residuals       18  618.2    34.3
```

Latin Square w/ Replicates, Neither Row nor Column Reused

Automobile Emissions Example Revisited

Suppose there were 8 cars and 8 drivers available.

We can form 2 Latin Squares.

	Cars			
Drivers	1	2	3	4
I	A	B	D	C
II	D	C	A	B
III	B	D	C	A
IV	C	A	B	D

	Cars			
Drivers	5	6	7	8
V	A	B	C	D
VI	C	D	A	B
VII	D	C	B	A
VIII	B	A	D	C

Note the two squares have different rows (drivers) and different columns (cars). That is, *neither rows nor columns are reused*.

Models for Replicated Latin Squares, not reusing rows or columns

Cannot use the conventional model below if neither row nor column is reused

$$y_{ijk} = \mu + \underset{\text{(trt)}}{\alpha_i} + \underset{\text{(driver)}}{\beta_j} + \underset{\text{(car)}}{\gamma_k} + \varepsilon_{ijk}$$

with constraints $\sum_{i=1}^{4} \alpha_i = \sum_{j=1}^{8} \beta_j = \sum_{k=1}^{8} \gamma_k = 0$ since in this case, even though $\mathrm{E}[\bar{y}_{i\bullet\bullet}] = \mu + \alpha_i$ but

$$\mathrm{E}[\bar{y}_{\bullet j\bullet}] \neq \mu + \beta_j, \quad \mathrm{E}[\bar{y}_{\bullet\bullet k}] \neq \mu + \gamma_k.$$

Observe that $\bar{y}_{\bullet 2\bullet} = (y_{D21} + y_{C22} + y_{A23} + y_{B24})/4$

$$\mathrm{E}[\bar{y}_{\bullet 2\bullet}] = \frac{1}{4}\begin{pmatrix} \mu + \alpha_D + \beta_2 + \gamma_1 \\ + \mu + \alpha_C + \beta_2 + \gamma_2 \\ + \mu + \alpha_A + \beta_2 + \gamma_3 \\ + \mu + \alpha_B + \beta_2 + \gamma_4 \end{pmatrix}$$

$$= \frac{1}{4}\Big(4\mu + \underbrace{\sum_{i=1}^{4} \alpha_i}_{=0} + 4\beta_2 + \underbrace{\sum_{k=1}^{4} \gamma_k}_{\neq 0}\Big) \neq \mu + \beta_2$$

Models for Replicated Latin Squares, Not Reusing Rows or Columns

A better model for m replicated Latin squares not reusing rows or columns is

$$y_{ijk\ell} = \mu + \underset{\text{(trt)}}{\alpha_i} + \underset{\text{(driver)}}{\beta_{j(\ell)}} + \underset{\text{(car)}}{\gamma_{k(\ell)}} + \underset{\text{(square)}}{\delta_\ell} + \varepsilon_{ijk\ell}, \quad \begin{array}{l} i = 1, \ldots, g, \\ \ell = 1, \ldots, m, \\ j = 1, \ldots, g, \\ k = 1, \ldots, g. \end{array}$$

where $y_{ijk\ell}$ = the observation in the jth row and kth column of the ℓth square receiving ith treatment, with the constraints

$$\sum_{i=1}^{g} \alpha_i = \sum_{\ell=1}^{m} \delta_\ell = 0, \text{ and } \sum_{j=1}^{g} \beta_{j(\ell)} = \sum_{k=1}^{g} \gamma_{k(\ell)} = 0, \text{ for } \ell = 1, \ldots, m.$$

The parameter estimates are

$$\widehat{\mu} = \bar{y}_{\bullet\bullet\bullet\bullet}, \qquad \widehat{\alpha}_i = \bar{y}_{i\bullet\bullet\bullet} - \bar{y}_{\bullet\bullet\bullet\bullet}, \qquad \widehat{\beta}_{j(\ell)} = \bar{y}_{\bullet j\bullet\ell} - \bar{y}_{\bullet\bullet\bullet\ell},$$
$$\widehat{\delta}_\ell = \bar{y}_{\bullet\bullet\bullet\ell} - \bar{y}_{\bullet\bullet\bullet\bullet}, \qquad \widehat{\gamma}_{k(\ell)} = \bar{y}_{\bullet\bullet k\ell} - \bar{y}_{\bullet\bullet\bullet\ell}$$

ANOVA Table for Latin Square not reusing rows or columns

Number of Latin Squares = m

Source	d.f.	SS	MS	F-value
Square	$m-1$	SS_{sqr}	$\text{SS}_{sqr}/(m-1)$	$\text{MS}_{sqr}/\text{MSE}$
Row-Block	$m(g-1)$	SS_{row}	$\text{SS}_{row}/[m(g-1)]$	$\text{MS}_{row}/\text{MSE}$
Column-Block	$m(g-1)$	SS_{col}	$\text{SS}_{col}/[m(g-1)]$	$\text{MS}_{col}/\text{MSE}$
Treatment	$g-1$	SS_{trt}	$\text{SS}_{trt}/(g-1)$	$\text{MS}_{trt}/\text{MSE}$
Error	$(mg+m-3)(g-1)$	SSE	SSE/dfE	
Total	mg^2-1	SST		

where
$$\begin{aligned} df_{error} &= df_{total} - df_{sqr} - df_{row} - df_{col} - df_{trt} \\ &= mg^2 - 1 - (m-1) - 2m(g-1) = (mg+m-3)(g-1) \end{aligned}$$

$$SS_{sqr} = \sum_{ijk\ell} (\bar{y}_{\bullet\bullet\bullet\ell} - \bar{y}_{\bullet\bullet\bullet\bullet})^2 = g^2 \sum_{\ell=1}^{m} (\bar{y}_{\bullet\bullet\bullet\ell} - \bar{y}_{\bullet\bullet\bullet\bullet})^2,$$

$$SS_{row} = \sum_{ijk\ell} (\bar{y}_{\bullet j\bullet\ell} - \bar{y}_{\bullet\bullet\bullet\ell})^2 = g \sum_{j\ell} (\bar{y}_{\bullet j\bullet\ell} - \bar{y}_{\bullet\bullet\bullet\ell})^2,$$

$$SS_{col} = \sum_{ijk\ell} (\bar{y}_{\bullet\bullet k\ell} - \bar{y}_{\bullet\bullet\bullet}\ell)^2 = g \sum_{j\ell} (\bar{y}_{\bullet\bullet k\ell} - \bar{y}_{\bullet\bullet\bullet\ell})^2,$$

$$SS_{trt} = \sum_{ijk} (\bar{y}_{i\bullet\bullet\bullet} - \bar{y}_{\bullet\bullet\bullet\bullet})^2 = mg \sum_{i=1}^{g} (\bar{y}_{i\bullet\bullet\bullet} - \bar{y}_{\bullet\bullet\bullet\bullet})^2,$$

$$SST = \sum_{ijk} (y_{ijk\ell} - \bar{y}_{\bullet\bullet\bullet\bullet})^2$$

$$SSE = SST - SS_{sqr} - SS_{row} - SS_{col} - SS_{trt}$$

Skeleton ANOVA for Latin Square Designs with Replicates

	Number of Latin Squares = m			
	row and col. both reused	row not reused	column not reused	neither reused
Source	d.f.	d.f.	d.f.	d.f.
Square				$m-1$
Row-Block	$g-1$	$mg-1$	$g-1$	$m(g-1)$
Column-Block	$g-1$	$g-1$	$mg-1$	$m(g-1)$
Treatment	$g-1$	$g-1$	$g-1$	$g-1$
Error	mg^2-1- (sum of the above)			
total	mg^2-1	mg^2-1	mg^2-1	mg^2-1

How to Create Latin Squares?

See Appendix C.1 (p.607-609) in Oehlert's textbook

http://users.stat.umn.edu/~gary/Book.html

for a list of Latin Squares of various sizes from 2×2 up to 7×7.

Randomization in a Latin Square Design

Randomization the allocation of treatments is necessary in the design of experiment.

How to randomly assign experimental units to treatments while maintaining the Latin-Square structure?

1. First pick a Latin Square of desired size at random from Appendix C.1 in Oehert's textbook
2. Randomly permute the rows of the square
3. Randomly permute the columns of the square
   - When rows or columns are permuted, Latin squares remain to be Latin Squares
4. Randomly assign treatments to the letters.

Contrasts and Multiple Comparisons for Complete Block Designs

Pairwise Comparisons in Complete Block Designs

For complete block designs (RCBD, and Latin-squares replicated or not), one can always do pairwise comparison of treatment effects $\alpha_{i_1} - \alpha_{i_2}$ by comparing the "treatment means"

$$\bar{y}_{i_1\bullet} - \bar{y}_{i_2\bullet} \text{ for RCBD,} \quad \bar{y}_{i_1\bullet\bullet} - \bar{y}_{i_2\bullet\bullet} \text{ for Latin Squares.}$$

The SE's are both in the form

$$\text{SE} = \sqrt{\text{MSE}\left(\frac{1}{r} + \frac{1}{r}\right)}$$

Here r is the total # of replicates for a treatment in the entire data, ignoring blocks, e.g., $r = mg$ in a $g \times g$ Latin-square design with m replicates.

The t-statistic has a t distribution, and d.f. = (d.f. for MSE)

$$t = \frac{\text{diff of 2 trt means}}{\text{SE}} \sim t_{\text{df of MSE}}$$

using which ONE can construct C.I. or do t-test on $\alpha_{i_1} - \alpha_{i_2}$.

Contrasts in Complete Block Designs

A **contrast** $C = \sum_{i=1}^{g} c_i\alpha_i$ of treatment effects in any RCBD or Latin-square (replicated or not) designs can be estimated by

$$\widehat{C} = \sum_{i=1}^{g} c_i(\text{sample mean for trt } i)$$

Here the "sample mean for trt i" means $\bar{y}_{i\bullet}$ or $\bar{y}_{i\bullet\bullet}$. The SE is

$$\mathrm{SE}(\widehat{C}) = \sqrt{\text{MSE} \times \sum_{i=1}^{g} c_i^2 \Big/ r}$$

where r is the total # of replicates for a treatment in all blocks, e.g., $r = mg$ in a $g \times g$ Latin-square design with m replicates.

- The $(1-\alpha)100\%$ C.I. for C: $\widehat{C} \pm t_{\alpha/2,\text{df of MSE}} \times \mathrm{SE}(\widehat{C})$.
- Test statistic for testing $H_0 : C = \sum_{i=1}^{g} c_i\alpha_i = 0$:

$$t_0 = \frac{\widehat{C}}{\mathrm{SE}(\widehat{C})} \sim t_{\text{df of MSE}}$$

Multiple Comparisons in Complete Block Designs

All the multiple comparison procedures apply to all Complete Block Designs. Just change the degree of freedom from $N - g$ to the d.f. of MSE

Method	*Family of Tests*	*Critical Value to Keep FWER* $< \alpha$
Fisher's LSD	a single pairwise comparison	$t_{\alpha/2,\text{df of MSE}}$
Tukey	all pairwise comparisons	$q_\alpha(g, \text{df of MSE})/\sqrt{2}$
Bonferroni	preplanned contrasts	$t_{\text{df of MSE},\alpha/(2m)}$, where $m = \#$ of preplanned contrasts
Scheffe	all contrasts	$\sqrt{(g-1)F_{\alpha,g-1,\text{df of MSE}}}$

Example: Air Freshener Sales — Tukey's HSD

To conduct pairwise comparisons between the 4 treatments, the SE is

$$\mathrm{SE} = \sqrt{\mathrm{MSE}\left(\frac{1}{r}+\frac{1}{r}\right)} = \sqrt{34.3\left(\frac{1}{8}+\frac{1}{8}\right)} \approx 2.9283$$

as there are $r = 8$ because there is 1 obsvertation in each of the 8 blocks and MSE $= 34.3$ is obtained from ANOVA table.

Tukey's critical value at FWER $= 5\%$ is 2.8263 (dfE $= 18$ from ANOVA table)

```
qtukey(0.95, 4, 18)/sqrt(2)
[1] 2.82629
```

Tukey's HSD is hence

$$\text{(critical value)} \times \mathrm{SE} = 2.928 \times 2.8263 \approx 8.276.$$

```r
library(mosaic)
sort(mean(sales ~ trtmt, data = freshener))
     D      C      A      B
11.875 15.875 18.625 20.375
```

Here is the underline diagram based on HSD = 8.276

```
     D      C      A      B
11.875 15.875 18.625 20.375
--------------------
       --------------------
```

Tukey's Method in emmeans Library

```
freshener$week = as.factor(freshener$week)
freshener$store = as.factor(freshener$store)
freshener$trtmt = as.factor(freshener$trtmt)
library(emmeans)
lm1 = lm(sales ~ store + week + trtmt, data=freshener)
lm1em = emmeans(lm1, "trtmt")
pairs(lm1em, infer=c(T,T), level=0.95, adjust="tukey")
 contrast estimate   SE df lower.CL upper.CL t.ratio p.value
 A - B       -1.75 2.93 18  -10.032     6.53  -0.597  0.9316
 A - C        2.75 2.93 18   -5.532    11.03   0.938  0.7849
 A - D        6.75 2.93 18   -1.532    15.03   2.303  0.1343
 B - C        4.50 2.93 18   -3.782    12.78   1.536  0.4381
 B - D        8.50 2.93 18    0.218    16.78   2.901  0.0431
 C - D        4.00 2.93 18   -4.282    12.28   1.365  0.5359

Results are averaged over the levels of: store, week
Confidence level used: 0.95
Conf-level adjustment: tukey method for comparing a family of 4 estimat
P value adjustment: tukey method for comparing a family of 4 estimates
```

Example: Air Freshener Sales — a Contrast

A data analyst looked at the data and noticed Treatment D with no extra display had the lowest mean compared to the other 3 treatments that had an extra display, he thus decided to test the contrast

$$C = \frac{\alpha_A + \alpha_B + \alpha_C}{3} - \alpha_D.$$

- Should use Scheffe as the contrast is suggested by data snooping
- Estimate of C is

$$\widehat{C} = \sum_i c_i \bar{y}_{i\bullet\bullet} = \frac{18.625 + 20.375 + 15.875}{3} - 11.875 \approx 6.417$$

```
mean(sales ~ trtmt, data = freshener)
     A      B      C      D
18.625 20.375 15.875 11.875
```

The SE for the contrast is

$$\mathrm{SE} = \sqrt{\mathrm{MSE}\left(\frac{(1/3)^2}{r} + \frac{(1/3)^2}{r} + \frac{(1/3)^2}{r} + \frac{1}{r}\right)}$$
$$= \sqrt{34.3\left(\frac{(1/3)^2}{8} + \frac{(1/3)^2}{8} + \frac{(1/3)^2}{8} + \frac{1}{8}\right)} \approx \boxed{2.39}.$$

The t-statistic for testing whether $C = 0$ is

$$t = \frac{\widehat{C}}{\mathrm{SE}} \approx \frac{6.417}{2.39} \approx 2.68$$

Scheffe's critical value for FWER at 5% is 3.08.

```
sqrt((4-1)*qf(0.05,4-1,18, lower.tail=F))
[1] 3.07892
```

The contrast is NOT significantly different from 0 as t-stat = 2.68 is below below Scheffe's critical value 3.08 for FWER at 5%.

Caution About Blocking

Blocking Must be Done at the Time of Randomization

- One can't group units into blocks when analyzing data to make treatments more significant after the experiment is done.

- For Air Freshener Sales data, as the `week` block effect is not significant (p-value $= 0.8$), can one ignore `week` and analyze the data as from a RCBD rather than a Latin Square Design?

Cannot Make Causal Conclusions About Block Effects

- We can change the treatment of an experimental unit, but cannot change the block an experimental unit belongs. Blocking variables are a property of the experimental units, not something we can manipulate.
  - e.g., in the Air Freshener Sales example, we can change the `trtmt` but not the `store` of an observation
- Since we cannot experimentally manipulate the blocking variable, block effects are "*observational*", We cannot make causal inference to a blocking variable as to a treatment factor.
  - e.g., cannot conclude that `store` changes `sales`

Power and Sample Size Calculation

Power and Sample Size Calculation

When H_a is true, treatments have different effects, the ANOVA F-statistic also has a non-central F-distribution

$$F = \frac{\mathrm{MS}_{trt}}{\mathrm{MSE}} \sim \begin{cases} F_{g-1,\, dfE} & \text{under } \mathrm{H}_0 \\ F_{g-1,\, dfE, \delta^2} & \text{under } \mathrm{H}_a \end{cases}$$

where the non-centrality parameter δ^2 is

$$\delta^2 = \frac{\sum_{i=1}^{g} r\alpha_i^2}{\sigma^2}.$$

where $r = \#$ of observations per treatment, ignoring blocks.

Note:

- only df changes from $N - g$ to $dfE =$ df of MSE of the model
- ncp δ^2 is calculated using α_i's under the zero-sum constraint $\sum_i \alpha_i = 0$.

Combination of Factorial and Block Designs

Combination of Factorial and Block Designs

The treatments in any block designs can also have factorial structure. The SS and df for treatments can be broken down according like what we have done for factorial data.

Example In a 4×4 Latin square design (no replicates), the $g = 4$ treatments has a 2×2 factorial structure.

Source	d.f.
Row	$g-1=3$
Column	$g-1=3$
Treatment	$g-1=3$
Error	$(g-2)(g-1)=6$
Total	$g^2-1=15$

$\Rightarrow$

Source	d.f.
Row	$g-1=3$
Column	$g-1=3$
A	$a-1=1$
B	$b-1=1$
AB	$(a-1)(b-1)=1$
Error	$(g-2)(g-1)=6$
Total	$g^2-1=15$