STAT 222 Lecture 3-4
Pairwise Comparisons & Contrasts

Yibi Huang

Outine

Textbook Coverage: Section 4.1-4.3

- Inference for a Single Mean μ_i in a Multi-Sample Problem
- Pairwise Comparisons
- Contrasts

Last Lecture

One-way ANOVA F-test for the Grass/Weed Competition Study:

$$H_0 : \mu_{1\text{N}} = \mu_{1\text{Y}} = \mu_{2\text{N}} = \mu_{3\text{N}} = \mu_{4\text{N}} = \mu_{4\text{Y}}$$

$$H_a : \mu_{1\text{N}}, \mu_{1\text{Y}}, \mu_{2\text{N}}, \mu_{3\text{N}}, \mu_{4\text{N}}, \mu_{4\text{Y}} \text{ not all equal}$$

```
grass = read.table(
  "http://www.stat.uchicago.edu/~yibi/s222/grassweed.txt", h=T)
mod1 = lm(percent ~ trt, data=grass)
anova(mod1)
Analysis of Variance Table

Response: percent
          Df Sum Sq Mean Sq F value  Pr(>F)
trt        5   6398    1280    71.2 3.2e-11
Residuals 18    323      18
```

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- Tiny P-value $\Rightarrow$ significant differences in the means. What should we do next?

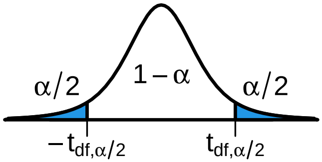
Inference for a Single Group Mean μ_i in a Multi-Sample Problem

Notations for the t-Critical Values

In the remainder of the course, we use $t_{df,\alpha/2}$ to denote the value that

$$P(-t_{df,\alpha/2} < T < t_{df,\alpha/2}) = 1 - \alpha$$

where T has a t-distribution w/ df degrees of freedom



which can be found in R using `qt()`:

```
qt(alpha/2, df, lower.tail=F)
```

		90% CI $t_{df,0.1/2}$ ↓	95% CI $t_{df,0.05/2}$ ↓		99% CI $t_{df,0.01/2}$ ↓
df	α				
	0.1	0.05	0.025	0.01	0.005
1	3.08	6.31	12.71	31.82	63.66
2	1.89	2.92	4.30	6.96	9.92
3	1.64	2.35	3.18	4.54	5.84
4	1.53	2.13	2.78	3.75	4.60
5	1.48	2.02	2.57	3.36	4.03
6	1.44	1.94	2.45	3.14	3.71
7	1.41	1.89	2.36	3.00	3.50
⋮	⋮	⋮	⋮	⋮	⋮

Confidence Interval (CI) for One-Sample Mean (Review)

If $y_1, y_2, \ldots, y_n$ are i.i.d. $\sim (\mu, \sigma^2)$,

$$\text{by CLT} \Rightarrow Z = \frac{\bar{y} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1).$$

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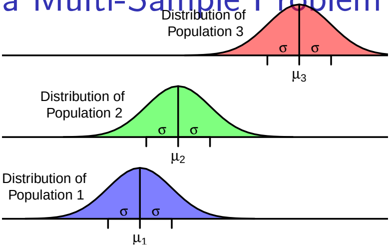
Inverting $P(-t_{n-1,\alpha/2} < t = \frac{\bar{y}-\mu}{s/\sqrt{n}} < t_{n-1,\alpha/2}) = 1 - \alpha$, we get the $(1-\alpha)100\%$ CI for μ:

$$\bar{y} \pm t_{n-1,\alpha/2} \times \frac{s}{\sqrt{n}}$$

A Naive CI for a Group Mean in a Multi-Sample Problem

Model for the multi-sample problem:

$$y_{ij} = \mu_i + \varepsilon_{ij}, \quad \varepsilon_{ij} \sim N(0, \sigma^2)$$



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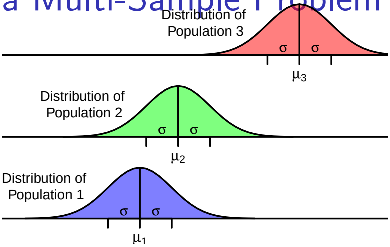

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$$\Rightarrow \quad Z = \frac{\bar{y}_{k\bullet} - \mu_k}{\sigma/\sqrt{n_k}} \sim N(0, 1)$$

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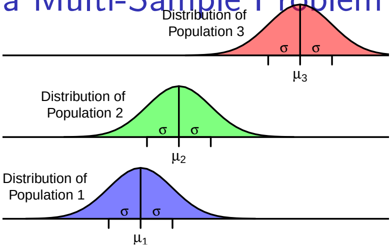

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A naive estimate for the unknown σ is

$$s_k = \text{sample SD of the } k\text{th group} = \sqrt{\frac{\sum_{j=1}^{n_k}(y_{kj} - \bar{y}_{k\bullet})^2}{n_k - 1}}.$$

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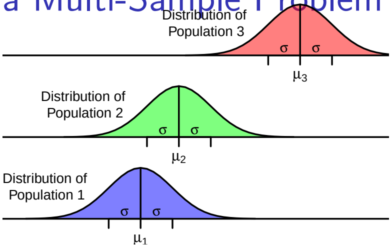

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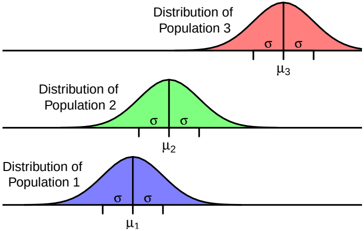
A naive but valid $100(1-\alpha)\%$ CI for μ_k would be

$$\bar{y}_{k\bullet} \pm t_{n_k-1,\alpha/2} \times \frac{s_k}{\sqrt{n_k}} \quad \text{since} \quad t = \frac{\bar{y}_{i\bullet} - \mu_k}{s_k/\sqrt{n_k}} \sim t_{n_k-1}.$$

which uses only data in the kth group, ignoring the rest, *not optimal!*

A Better CI for a Group Mean in a Multi-Sample Problem

As all the groups have a **common SD** σ, data in other groups cannot help estimating μ_k but they can help estimating σ. A better estimate for σ is



$$\widehat{\sigma} = \sqrt{\text{MSE}} = \sqrt{\frac{\sum_{i=1}^{g}\sum_{j=1}^{n_i}(y_{ij} - \bar{y}_{i\bullet})^2}{N-g}}$$

We have

$$t = \frac{\bar{y}_{k\bullet} - \mu_k}{\widehat{\sigma}/\sqrt{n_k}} = \frac{\bar{y}_{i\bullet} - \mu_k}{\sqrt{\text{MSE}}/\sqrt{n_k}} \sim t_{N-g},$$

from which, a better $100(1-\alpha)\%$ CI for μ_k is

$$\bar{y}_{k\bullet} \pm t_{N-g,\alpha/2}\frac{\sqrt{\text{MSE}}}{\sqrt{n_k}}$$

- using observations in all groups to estimate the unknown σ
- higher df $= N - g$, not $n_k - 1$

Example: Grass/Weed Competition

Treatment	$1N$	$1Y$	$2N$	$3N$	$4N$	$4Y$
Mean $\bar{y}_{i\bullet}$	95	82.25	81.5	68.25	50.5	52
SD s_i	2.16	3.775	3.512	5.56	5.00	4.546

, MSE = 17.97

The naive 95% CI for $\mu_{4\text{Y}}$ using only data in Group 4Y:

$$\bar{y}_{4Y\bullet} \pm t_{n_{4\text{Y}}-1,\alpha/2}\frac{s_{4\text{Y}}}{\sqrt{n_{4\text{Y}}}} \approx 52 \pm 3.182 \times \frac{4.546}{\sqrt{4}} \approx 52 \pm 7.23.$$

The better 95% CI for $\mu_{4\text{Y}}$ using the MSE is

$$\bar{y}_{i\bullet} \pm t_{N-g,\alpha/2}\frac{\sqrt{\text{MSE}}}{\sqrt{n_{4\text{Y}}}} = 52 \pm 2.101 \times \frac{\sqrt{17.97}}{\sqrt{4}} \approx 52 \pm 4.45$$

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where $n_{4\text{Y}} = 4$, $N = 24$, $g = 6$, $\alpha = 0.05$. Using R, we can find $t_{n_{4\text{Y}}-1,\alpha/2} = t_{4-1,0.05/2} \approx 3.182$ and $t_{N-g,\alpha/2} = t_{24-6,0.05/2} \approx 2.101$.

```
qt(0.05/2, df = 4-1, lower.tail=F)
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qt(0.05/2, df = 24-6, lower.tail=F)
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```

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```

Observe the naive CI has a bigger margin of error 7.23 than the margin of error 4.45 for the CI using the MSE.

Interpretation of the better 95% CI for $\mu_{4\text{Y}}$: 52 ± 4.45

For plots received 800 mg N/kg soil and 1 cm of irrigation per week, we estimate that 52.0% of living material is bluestem (grass) on average with a margin of error of 4.45% at 95% confidence.

emmeans Library in R

The R library emmeans can produce confidence intervals for each group mean.

Need to install the emmeans library first, by the following command. You only need to install ONCE!

```
install.packages("emmeans")    # JUST RUN THIS ONCE!
```

Once installed, must load emmeans at every R session before it can be used.

```
library(emmeans)
```

The Section 3.9 and 4.7 of the textbook use the library lsmeans, which is now obsolete and replaced by the emmeans library.

```r
grass = read.table(
  "http://www.stat.uchicago.edu/~yibi/s222/grassweed.txt", h=T)
mod1 = lm(percent ~ trt, data=grass)
emmeans(mod1, "trt", level=0.95)
 trt emmean   SE df lower.CL upper.CL
 1N    95.0 2.12 18     90.5     99.5
 1Y    82.2 2.12 18     77.8     86.7
 2N    81.5 2.12 18     77.0     86.0
 3N    68.2 2.12 18     63.8     72.7
 4N    50.5 2.12 18     46.0     55.0
 4Y    52.0 2.12 18     47.5     56.5

Confidence level used: 0.95
```

or

```r
mod2 = aov(percent ~ trt, data=grass)
emmeans(mod2, "trt", level=0.95)
 trt emmean   SE df lower.CL upper.CL
 1N    95.0 2.12 18     90.5     99.5
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 3N    68.2 2.12 18     63.8     72.7
 4N    50.5 2.12 18     46.0     55.0
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Pairwise Comparison

Pairwise Comparison of Group Means

Model for the multi-sample problem:

$$y_{ij} = \mu_i + \varepsilon_{ij}, \quad \varepsilon_{ij} \sim N(0, \sigma^2)$$

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- the estimator is $\bar{y}_{k\bullet} - \bar{y}_{\ell\bullet}$

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- the estimator is $\bar{y}_{k\bullet} - \bar{y}_{\ell\bullet}$
- Since $\bar{y}_{k\bullet}$ and $\bar{y}_{\ell\bullet}$ are independent, we have

$$\mathbb{V}(\bar{y}_{k\bullet} - \bar{y}_{\ell\bullet}) = \mathbb{V}(\bar{y}_{k\bullet}) + \mathbb{V}(\bar{y}_{\ell\bullet}) = \frac{\sigma^2}{n_k} + \frac{\sigma^2}{n_\ell}$$

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- $\mathrm{SD}(\bar{y}_{k\bullet} - \bar{y}_{\ell\bullet}) = \sqrt{\mathbb{V}(\bar{y}_{k\bullet} - \bar{y}_{\ell\bullet})} = \sqrt{\sigma^2 \left(\frac{1}{n_k} + \frac{1}{n_\ell}\right)}$,

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- $\text{SD}(\bar{y}_{k\bullet} - \bar{y}_{\ell\bullet}) = \sqrt{\mathbb{V}(\bar{y}_{k\bullet} - \bar{y}_{\ell\bullet})} = \sqrt{\sigma^2 \left(\frac{1}{n_k} + \frac{1}{n_\ell}\right)}$,
- $\mathrm{SE}(\bar{y}_{k\bullet} - \bar{y}_{\ell\bullet}) = \widehat{\text{SD}}(\bar{y}_{k\bullet} - \bar{y}_{\ell\bullet}) = \sqrt{\text{MSE} \left(\frac{1}{n_k} + \frac{1}{n_\ell}\right)}$.

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- $t = \dfrac{\bar{y}_{k\bullet} - \bar{y}_{\ell\bullet} - (\mu_k - \mu_\ell)}{\mathrm{SE}(\bar{y}_{k\bullet} - \bar{y}_{\ell\bullet})} = \dfrac{\bar{y}_{k\bullet} - \bar{y}_{\ell\bullet} - (\mu_k - \mu_\ell)}{\sqrt{\mathrm{MSE}\left(\frac{1}{n_k} + \frac{1}{n_\ell}\right)}} \sim t_{N-g}$

Confidence Intervals (CIs) for Pairwise Differences

The $100(1-\alpha)\%$ confidence interval (C.I.) for $\mu_k - \mu_\ell$ is

$$\bar{y}_{k\bullet} - \bar{y}_{\ell\bullet} \pm t_{N-g,\alpha/2}\sqrt{\text{MSE}\left(\frac{1}{n_k} + \frac{1}{n_\ell}\right)}.$$

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Note this is neither the two-sample CI assuming equal SDs

$$\bar{y}_{k\bullet} - \bar{y}_{\ell\bullet} \pm t_{n_k+n_\ell-2,\alpha/2}\sqrt{s_p^2\left(\frac{1}{n_k}+\frac{1}{n_\ell}\right)},\text{ where } s_p^2 = \frac{(n_k-1)s_k^2+(n_\ell-1)s_\ell^2}{n_k+n_\ell-2},$$

nor the two-sample CI not assuming equal SDs

$$\bar{y}_{k\bullet} - \bar{y}_{\ell\bullet} \pm t_{df,\alpha/2}\sqrt{\frac{s_k^2}{n_k}+\frac{s_\ell^2}{n_\ell}},\quad \text{where } df = \frac{\left(\frac{s_k^2}{n_k}+\frac{s_\ell^2}{n_\ell}\right)^2}{\frac{1}{n_k-1}\left(\frac{s_k^2}{n_k}\right)^2+\frac{1}{n_\ell-1}\left(\frac{s_\ell^2}{n_\ell}\right)^2}$$

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nor the two-sample CI not assuming equal SDs

$$\bar{y}_{k\bullet} - \bar{y}_{\ell\bullet} \pm t_{df,\alpha/2}\sqrt{\frac{s_k^2}{n_k}+\frac{s_\ell^2}{n_\ell}},\quad \text{where } df = \frac{\left(\frac{s_k^2}{n_k}+\frac{s_\ell^2}{n_\ell}\right)^2}{\frac{1}{n_k-1}\left(\frac{s_k^2}{n_k}\right)^2+\frac{1}{n_\ell-1}\left(\frac{s_\ell^2}{n_\ell}\right)^2}$$

- $\text{MSE} = \frac{\sum_{i=1}^{g}\sum_{j=1}^{n_i}(y_{ij}-\bar{y}_{i\bullet})^2}{N-g}$ calculated using the whole data is a more accurate estimator of σ^2 than s_p^2 or s_k^2, s_ℓ^2 calculated using only data in the two groups compared
- The critical value for the two-sample C.I. is larger

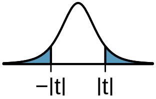
$$t_{n_k+n_\ell-2,\alpha/2} > t_{N-g,\alpha/2}$$

Hypothesis Tests for Pairwise Differences

For testing the hypothesis H_0: $\mu_k - \mu_\ell = 0$, the test statistic is

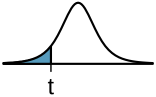
$$t = \frac{\bar{y}_{k\bullet} - \bar{y}_{\ell\bullet}}{\mathrm{SE}(\bar{y}_{k\bullet} - \bar{y}_{\ell\bullet})} = \frac{\bar{y}_{k\bullet} - \bar{y}_{\ell\bullet}}{\sqrt{\mathrm{MSE}\left(\frac{1}{n_k} + \frac{1}{n_\ell}\right)}} \sim t_{N-g}$$

If H_a: $\mu_k \neq \mu_\ell$ (two-sided),

P-value = 

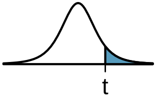
=`2*pt(abs(t), df, lower.tail=F)`

If H_a: $\mu_k < \mu_\ell$ (lower one-sided),

P-value = 

= `pt(t, df)`

If H_a: $\mu_k > \mu_\ell$ (upper one-sided),

P-value = 

= `pt(t, df, lower.tail=F)`

The bell curve above is the t-curve with $\mathrm{df} = N - g$.

Example: CI for Pairwise Diff. (Grass/Weed)

Group	$1N$	$1Y$	$2N$	$3N$	$4N$	$4Y$
Mean $\bar{y}_{i\bullet}$	95	82.25	81.5	68.25	50.5	52
SD s_i	2.16	3.775	3.512	5.56	5.00	4.546

, MSE $= 17.97$

A 95% confidence interval for $\mu_{1\text{N}} - \mu_{1\text{Y}}$ is

$$\bar{y}_{1N\bullet} - \bar{y}_{1Y\bullet} \pm t_{18,0.025} \times \sqrt{\text{MSE}\left(\frac{1}{n_{1\text{N}}} + \frac{1}{n_{1\text{Y}}}\right)}$$

$$= 95 - 82.25 \pm 2.101 \times \sqrt{17.97\left(\frac{1}{4} + \frac{1}{4}\right)} = 12.75 \pm 6.30$$

in which $t_{18,0.025} = 2.101$ is found using the R command

```
qt(0.05/2, df = 18, lower.tail=F)
[1] 2.101
```

Irrigation reduced the percentage of grass (bluestem) by 12.75% on average, with a margin of error of 6.30%, at 95% confidence.

Example: Hyp Tests for Pairwise Diff. (Grass/Weed)

To test whether treatments 1N and 1Y have the same effect

$$H_0 : \mu_{1\text{N}} - \mu_{1\text{Y}} = 0 \quad \text{v.s.} \quad H_a : \mu_{1\text{N}} - \mu_{1\text{Y}} \neq 0$$

the test statistic is

$$t = \frac{\bar{y}_{1N\bullet} - \bar{y}_{1Y\bullet}}{\sqrt{\text{MSE}\left(\frac{1}{n_{1\text{N}}} + \frac{1}{n_{1\text{Y}}}\right)}} = \frac{95 - 82.25}{\sqrt{17.97\left(\frac{1}{4} + \frac{1}{4}\right)}} \approx \frac{12.75}{2.9975} \approx 4.253$$

with $\text{df} = N - g = 24 - 6 = 18$. The two-sided P-value is

```
2*pt(4.235, df = 18, lower.tail=F)
[1] 0.000498
```

As the P-value < 0.05, we again confirm that irrigation made grass (bluestem) less competitive.

Pairwise *t*-Tests using emmeans in R

The R library emmeans can perform pairwise comparisons between all pairs of treatments.

```
library(emmeans)
mod1 = aov(percent ~ trt, data=grass)
mod1emm = emmeans(mod1, "trt")
pairs(mod1emm, infer=c(T,T), level=0.95, adjust="none")
```

See the output on the next page.

The output would include both confidence intervals and hypothesis tests if adding `infer=c(T,T)`.

contrast	estimate	SE	df	lower.CL	upper.CL	t.ratio	p.value
1N - 1Y	12.75	3	18	6.45	19.05	4.253	0.0005
1N - 2N	13.50	3	18	7.20	19.80	4.503	0.0003
1N - 3N	26.75	3	18	20.45	33.05	8.924	<.0001
1N - 4N	44.50	3	18	38.20	50.80	14.845	<.0001
1N - 4Y	43.00	3	18	36.70	49.30	14.344	<.0001
1Y - 2N	0.75	3	18	-5.55	7.05	0.250	0.8053
1Y - 3N	14.00	3	18	7.70	20.30	4.670	0.0002
1Y - 4N	31.75	3	18	25.45	38.05	10.592	<.0001
1Y - 4Y	30.25	3	18	23.95	36.55	10.091	<.0001
2N - 3N	13.25	3	18	6.95	19.55	4.420	0.0003
2N - 4N	31.00	3	18	24.70	37.30	10.341	<.0001
2N - 4Y	29.50	3	18	23.20	35.80	9.841	<.0001
3N - 4N	17.75	3	18	11.45	24.05	5.921	<.0001
3N - 4Y	16.25	3	18	9.95	22.55	5.421	<.0001
4N - 4Y	-1.50	3	18	-7.80	4.80	-0.500	0.6229

Confidence level used: 0.95

The pwpm() function in the emmeans library can display the P-values of pairwise comparison concisely.

```
pwpm(mod1emm, adjust="none")
        1N     1Y     2N     3N     4N     4Y
1N [95.0] 0.0005 0.0003 <.0001 <.0001 <.0001
1Y  12.75 [82.2] 0.8053 0.0002 <.0001 <.0001
2N  13.50   0.75 [81.5] 0.0003 <.0001 <.0001
3N  26.75  14.00  13.25 [68.3] <.0001 <.0001
4N  44.50  31.75  31.00  17.75 [50.5] 0.6229
4Y  43.00  30.25  29.50  16.25  -1.50 [52.0]

Row and column labels: trt
Upper triangle: P values
Diagonal: [Estimates] (emmean)
Lower triangle: Comparisons (estimate)   earlier vs. later
```

Underline Diagrams (p.88, Section 5.4.1 in Oehlert's book)

a concise way to summarize pairwise comparisons

	1N	1Y	2N	3N	4N	4Y
1N	[95.0]	0.0005	0.0003	<.0001	<.0001	<.0001
1Y	12.75	[82.2]	0.8053	0.0002	<.0001	<.0001
2N	13.50	0.75	[81.5]	0.0003	<.0001	<.0001
3N	26.75	14.00	13.25	[68.3]	<.0001	<.0001
4N	44.50	31.75	31.00	17.75	[50.5]	0.6229
4Y	43.00	30.25	29.50	16.25	-1.50	[52.0]

How to make a **underline diagram**?

1. Write out group labels horizontally in **increasing order sorted by group means**
2. (Optional) Write the group mean $\bar{y}_{i\bullet}$ under the corresponding group label
3. Draw a line segment under a set of groups if no two groups in that set of groups are significantly different from each other

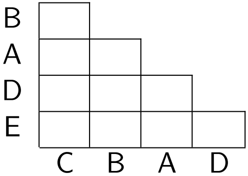
4Y	4N	3N	2N	1Y	1N
50.5	52	68.25	81.5	82.25	95

Underline Diagrams

Answer the following questions based on the underline diagram for pairwise comparisons of the 5 treatments A, B, C, D and E in a randomized experiment.

- Order the means of the 5 groups from low to high.

- Check all the pairs that are significantly different from each other.



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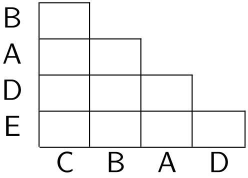
$$C < B < A < D < E$$

- Check all the pairs that are significantly different from each other.



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	C	B	A	D
B	v			
A	v			
D	v			
E	v			

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B	v			
A	v			
D	v			
E	v	v		

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- Order the means of the 5 groups from low to high.

$$C < B < A < D < E$$

- Check all the pairs that are significantly different from each other.

	C	B	A	D
B	v			
A	v			
D	v			
E	v	v	v	

Least Significant Difference (LSD)

- It's an awful lot of work to to compare every pair of groups. One needs to compute the SE, the t-statistic, and P-value for each pair of groups. When there are g groups, there are $\binom{g}{2} = g(g-1)/2$ pairs to compare with.
- When all groups are of the same size n, an easier way to do pairwise comparisons of all treatments is to compute the **least significant difference** (LSD), which is the minimum amount by which two means must differ in order to be considered statistically different.

Least Significant Difference (LSD)

- When all groups are of the same size n, the SEs of pairwise comparisons all equal to

$$\text{SE} = \sqrt{\text{MSE}\left(\frac{1}{n} + \frac{1}{n}\right)}$$

Least Significant Difference (LSD)

- When all groups are of the same size n, the SEs of pairwise comparisons all equal to

$$\text{SE} = \sqrt{\text{MSE}\left(\frac{1}{n} + \frac{1}{n}\right)}$$

- To be significant at level α, the t-statistic for pairwise comparison

$$t = \frac{\bar{y}_{k\bullet} - \bar{y}_{\ell\bullet}}{\text{SE}}$$

must be at least $t_{N-g,\alpha/2}$ in absolute value

Least Significant Difference (LSD)

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- To be significant at level α, the t-statistic for pairwise comparison

$$t = \frac{\bar{y}_{k\bullet} - \bar{y}_{\ell\bullet}}{\mathsf{SE}}$$

must be at least $t_{N-g,\alpha/2}$ in absolute value

- So μ_k and μ_ℓ are significantly different at level α if and only if $\bar{y}_{k\bullet} - \bar{y}_{\ell\bullet}$ is at least

$$t_{N-g,\alpha/2}\sqrt{\mathsf{MSE}\left(\frac{1}{n} + \frac{1}{n}\right)} = \mathsf{LSD}$$

in absolute value, which is called the *least significant difference (LSD)*

Example: Least Significant Difference (Grass/Weed)

For the Grass/Weed experiment, the critical value at $\alpha = 5\%$ significance is $t_{N-g,\alpha/2} = t_{24-6,0.025} \approx 2.101$, the LSD at 5% level is

$$\text{LSD} = t_{N-g,\alpha/2}\sqrt{\text{MSE}\left(\frac{1}{n}+\frac{1}{n}\right)} = 2.101\sqrt{17.97\left(\frac{1}{4}+\frac{1}{4}\right)} \approx 6.30$$

Two treatments are significantly different at 5% level if and only if their means differ by 6.30 or more.

Only the pairs (4Y, 4N) and (2N, 1Y) are not significantly different, as they are the only pairs differ by less than 6.30 in mean.

4Y	4N	3N	2N	1Y	1N
50.5	52	68.25	81.5	82.25	95

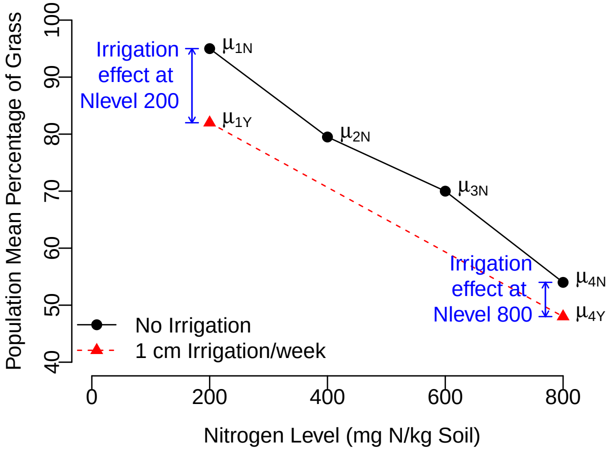
Contrasts

Quantities of Interest Other Than Pairwise Differences (1)

For the Grass/Weed experiment, we are also interested in

Q1 Irrigation effect: $\mu_{1\text{N}} - \mu_{1\text{Y}}$ or $\mu_{4\text{N}} - \mu_{4\text{Y}}$ or the combination

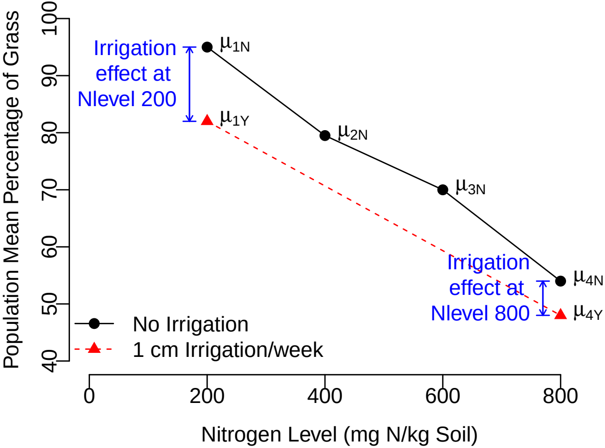
$$\frac{\mu_{1\text{N}} + \mu_{4\text{N}}}{2} - \frac{\mu_{1\text{Y}} + \mu_{4\text{Y}}}{2}$$



Quantities of Interest Other Than Pairwise Differences (2)

Q2 Does the irrigation effect change with nitrogen levels?

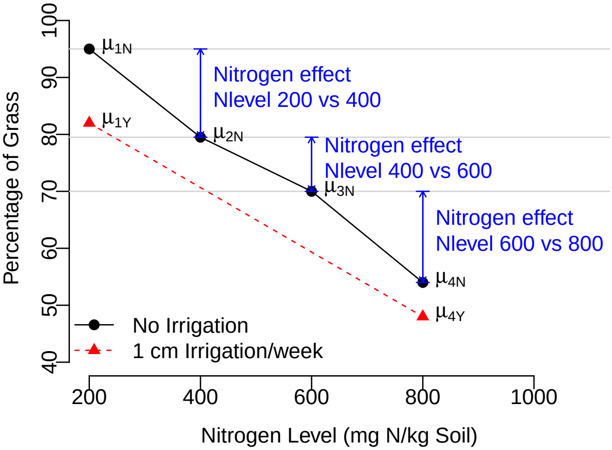
$$\underbrace{(\mu_{1N} - \mu_{1Y})}_{\substack{\text{irrigation effect at} \\ \text{nitrogen level 200}}} - \underbrace{(\mu_{4N} - \mu_{4Y})}_{\substack{\text{irrigation effect at} \\ \text{nitrogen level 800}}}$$



Quantities of Interest Other Than Pairwise Differences (3)

For the Grass/Weed experiment, we are also interested in

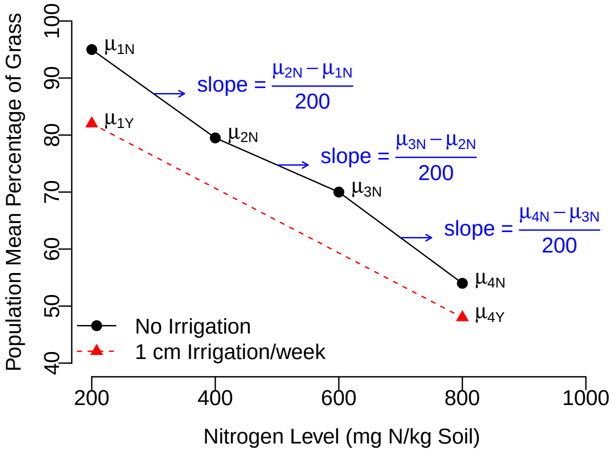
Q3 Nitrogen effect: $\mu_{1N} - \mu_{2N}$, $\mu_{2N} - \mu_{3N}$, etc.



Quantities of Interest Other Than Pairwise Differences (4)

Q4 Is the nitrogen effect **linear**?

$$\frac{\mu_{1N} - \mu_{2N}}{200} - \frac{\mu_{2N} - \mu_{3N}}{200}, \quad \text{or} \quad \frac{\mu_{2N} - \mu_{3N}}{200} - \frac{\mu_{3N} - \mu_{4N}}{200}, \quad \text{etc.}$$



Definition of Contrasts

All the quantities above are **contrasts**.

A **contrast** is a linear combination of group means μ_i's

$$C = \sum_{i=1}^{g} c_i \mu_i$$

where c_i's are known coefficients that add up to 0, $\sum_{i=1}^{g} c_i = 0$.

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Ex. Irrigation Effect Contrast:

$$C = \frac{\mu_{1\text{N}} + \mu_{4\text{N}}}{2} - \frac{\mu_{1\text{Y}} + \mu_{4\text{Y}}}{2}$$

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Ex. Irrigation Effect Contrast:

$$\begin{aligned} C &= \frac{\mu_{1N} + \mu_{4N}}{2} - \frac{\mu_{1Y} + \mu_{4Y}}{2} \\ &= 0.5\,\mu_{1N} + 0.5\,\mu_{4N} + (-0.5)\mu_{1Y} + (-0.5)\mu_{4Y} + \ 0\ \ \mu_{2N} + \ 0\ \ \mu_{3N} \end{aligned}$$

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Ex. Irrigation Effect Contrast:

$$\begin{aligned} C &= \frac{\mu_{1N} + \mu_{4N}}{2} - \frac{\mu_{1Y} + \mu_{4Y}}{2} \\ &= \underset{c_{1N}}{\underset{\downarrow}{0.5}}\,\mu_{1N} + \underset{c_{4N}}{\underset{\downarrow}{0.5}}\,\mu_{4N} + \underset{c_{1Y}}{\underset{\downarrow}{(-0.5)}}\,\mu_{1Y} + \underset{c_{4Y}}{\underset{\downarrow}{(-0.5)}}\,\mu_{4Y} + \underset{c_{2N}}{\underset{\downarrow}{0}}\,\mu_{2N} + \underset{c_{3N}}{\underset{\downarrow}{0}}\,\mu_{3N} \end{aligned}$$

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Ex. Irrigation Effect Contrast:

$$\begin{aligned} C &= \frac{\mu_{1\text{N}} + \mu_{4\text{N}}}{2} - \frac{\mu_{1\text{Y}} + \mu_{4\text{Y}}}{2} \\ &= \underset{\substack{\downarrow \\ c_{1N}}}{0.5}\,\mu_{1\text{N}} + \underset{\substack{\downarrow \\ c_{4N}}}{0.5}\,\mu_{4\text{N}} + \underset{\substack{\downarrow \\ c_{1Y}}}{(-0.5)}\mu_{1\text{Y}} + \underset{\substack{\downarrow \\ c_{4Y}}}{(-0.5)}\mu_{4\text{Y}} + \underset{\substack{\downarrow \\ c_{2N}}}{0}\;\mu_{2\text{N}} + \underset{\substack{\downarrow \\ c_{3N}}}{0}\;\mu_{3\text{N}} \end{aligned}$$

Observe that $c_{1N} + c_{4N} + c_{1Y} + c_{4Y} + c_{2N} + c_{3N}$

$$= 0.5 + 0.5 + (-0.5) + (-0.5) + 0 + 0 = 0$$

Example of Contrasts

Q2 Does the irrigation effect change with nitrogen levels?

$$C = (\mu_{1\mathrm{N}} - \mu_{1\mathrm{Y}}) - (\mu_{4\mathrm{N}} - \mu_{4\mathrm{Y}})$$

Example of Contrasts

Q2 Does the irrigation effect change with nitrogen levels?

$$
\begin{aligned}
C &= (\mu_{1N} - \mu_{1Y}) - (\mu_{4N} - \mu_{4Y}) \\
&= 1\ \mu_{1N} + (-1)\mu_{1Y} + (-1)\mu_{4N} + 1\ \mu_{4Y} + 0\ \mu_{2N} + 0\ \mu_{3N}
\end{aligned}
$$

Example of Contrasts

Q2 Does the irrigation effect change with nitrogen levels?

$$C = (\mu_{1\text{N}} - \mu_{1\text{Y}}) - (\mu_{4\text{N}} - \mu_{4\text{Y}})$$

$$= \underset{\substack{\downarrow \\ c_{1N}}}{1}\ \mu_{1\text{N}} + \underset{\substack{\downarrow \\ c_{1Y}}}{(-1)}\mu_{1\text{Y}} + \underset{\substack{\downarrow \\ c_{4N}}}{(-1)}\mu_{4\text{N}} + \underset{\substack{\downarrow \\ c_{4Y}}}{1}\ \mu_{4\text{Y}} + \underset{\substack{\downarrow \\ c_{2N}}}{0}\ \mu_{2\text{N}} + \underset{\substack{\downarrow \\ c_{3N}}}{0}\ \mu_{3\text{N}}$$

Example of Contrasts

Q2 Does the irrigation effect change with nitrogen levels?

$$C = (\mu_{1\text{N}} - \mu_{1\text{Y}}) - (\mu_{4\text{N}} - \mu_{4\text{Y}})$$

$$= \underset{\substack{\downarrow \\ c_{1N}}}{1}\ \mu_{1\text{N}} + \underset{\substack{\downarrow \\ c_{1Y}}}{(-1)}\mu_{1\text{Y}} + \underset{\substack{\downarrow \\ c_{4N}}}{(-1)}\mu_{4\text{N}} + \underset{\substack{\downarrow \\ c_{4Y}}}{1}\ \mu_{4\text{Y}} + \underset{\substack{\downarrow \\ c_{2N}}}{0}\ \mu_{2\text{N}} + \underset{\substack{\downarrow \\ c_{3N}}}{0}\ \mu_{3\text{N}}$$

Observe that $\sum_i c_i = 1 + (-1) + (-1) + 1 + 0 + 0 = 0$.

Example of Contrasts

Q2 Does the irrigation effect change with nitrogen levels?

$$C = (\mu_{1\text{N}} - \mu_{1\text{Y}}) - (\mu_{4\text{N}} - \mu_{4\text{Y}})$$

$$= \underset{\substack{\downarrow \\ c_{1N}}}{1}\ \mu_{1\text{N}} + \underset{\substack{\downarrow \\ c_{1Y}}}{(-1)}\mu_{1\text{Y}} + \underset{\substack{\downarrow \\ c_{4N}}}{(-1)}\mu_{4\text{N}} + \underset{\substack{\downarrow \\ c_{4Y}}}{1}\ \mu_{4\text{Y}} + \underset{\substack{\downarrow \\ c_{2N}}}{0}\ \mu_{2\text{N}} + \underset{\substack{\downarrow \\ c_{3N}}}{0}\ \mu_{3\text{N}}$$

Observe that $\sum_i c_i = 1 + (-1) + (-1) + 1 + 0 + 0 = 0$.

Q4 Is the nitrogen effect **linear**?

$$C = \frac{\mu_{1\text{N}} - \mu_{2\text{N}}}{200} - \frac{\mu_{2\text{N}} - \mu_{3\text{N}}}{200}$$

Example of Contrasts

Q2 Does the irrigation effect change with nitrogen levels?

$$C = (\mu_{1\text{N}} - \mu_{1\text{Y}}) - (\mu_{4\text{N}} - \mu_{4\text{Y}})$$

$$= \underset{c_{1N}}{\underset{\downarrow}{1}}\ \mu_{1\text{N}} + \underset{c_{1Y}}{\underset{\downarrow}{(-1)}}\mu_{1\text{Y}} + \underset{c_{4N}}{\underset{\downarrow}{(-1)}}\mu_{4\text{N}} + \underset{c_{4Y}}{\underset{\downarrow}{1}}\ \mu_{4\text{Y}} + \underset{c_{2N}}{\underset{\downarrow}{0}}\ \mu_{2\text{N}} + \underset{c_{3N}}{\underset{\downarrow}{0}}\ \mu_{3\text{N}}$$

Observe that $\sum_i c_i = 1 + (-1) + (-1) + 1 + 0 + 0 = 0.$

Q4 Is the nitrogen effect **linear**?

$$C = \frac{\mu_{1\text{N}} - \mu_{2\text{N}}}{200} - \frac{\mu_{2\text{N}} - \mu_{3\text{N}}}{200}$$

$$= \frac{1}{200}\mu_{1\text{N}} + \left(\frac{-2}{200}\right)\mu_{2\text{N}} + \frac{1}{200}\mu_{3\text{N}} + 0\ \mu_{4\text{N}} + 0\ \mu_{1\text{Y}} + 0\ \mu_{4\text{Y}}$$

Example of Contrasts

Q2 Does the irrigation effect change with nitrogen levels?

$$C = (\mu_{1\text{N}} - \mu_{1\text{Y}}) - (\mu_{4\text{N}} - \mu_{4\text{Y}})$$

$$= \underbrace{1}_{c_{1N}} \mu_{1\text{N}} + \underbrace{(-1)}_{c_{1Y}} \mu_{1\text{Y}} + \underbrace{(-1)}_{c_{4N}} \mu_{4\text{N}} + \underbrace{1}_{c_{4Y}} \mu_{4\text{Y}} + \underbrace{0}_{c_{2N}} \mu_{2\text{N}} + \underbrace{0}_{c_{3N}} \mu_{3\text{N}}$$

Observe that $\sum_i c_i = 1 + (-1) + (-1) + 1 + 0 + 0 = 0$.

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$$C = \frac{\mu_{1\text{N}} - \mu_{2\text{N}}}{200} - \frac{\mu_{2\text{N}} - \mu_{3\text{N}}}{200}$$

$$= \underbrace{\tfrac{1}{200}}_{c_{1N}} \mu_{1\text{N}} + \underbrace{(\tfrac{-2}{200})}_{c_{2N}} \mu_{2\text{N}} + \underbrace{\tfrac{1}{200}}_{c_{3N}} \mu_{3\text{N}} + \underbrace{0}_{c_{4N}} \mu_{4\text{N}} + \underbrace{0}_{c_{1Y}} \mu_{1\text{Y}} + \underbrace{0}_{c_{4Y}} \mu_{4\text{Y}}$$

Example of Contrasts

Q2 Does the irrigation effect change with nitrogen levels?

$$C = (\mu_{1\text{N}} - \mu_{1\text{Y}}) - (\mu_{4\text{N}} - \mu_{4\text{Y}})$$

$$= \underset{c_{1N}}{\underset{\downarrow}{1}}\ \mu_{1\text{N}} + \underset{c_{1Y}}{\underset{\downarrow}{(-1)}}\mu_{1\text{Y}} + \underset{c_{4N}}{\underset{\downarrow}{(-1)}}\mu_{4\text{N}} + \underset{c_{4Y}}{\underset{\downarrow}{1}}\ \mu_{4\text{Y}} + \underset{c_{2N}}{\underset{\downarrow}{0}}\ \mu_{2\text{N}} + \underset{c_{3N}}{\underset{\downarrow}{0}}\ \mu_{3\text{N}}$$

Observe that $\sum_i c_i = 1 + (-1) + (-1) + 1 + 0 + 0 = 0.$

Q4 Is the nitrogen effect **linear**?

$$C = \frac{\mu_{1\text{N}} - \mu_{2\text{N}}}{200} - \frac{\mu_{2\text{N}} - \mu_{3\text{N}}}{200}$$

$$= \underset{c_{1N}}{\underset{\downarrow}{\tfrac{1}{200}}}\mu_{1\text{N}} + \underset{c_{2N}}{\underset{\downarrow}{(\tfrac{-2}{200})}}\mu_{2\text{N}} + \underset{c_{3N}}{\underset{\downarrow}{\tfrac{1}{200}}}\mu_{3\text{N}} + \underset{c_{4N}}{\underset{\downarrow}{0}}\ \mu_{4\text{N}} + \underset{c_{1Y}}{\underset{\downarrow}{0}}\ \mu_{1\text{Y}} + \underset{c_{4Y}}{\underset{\downarrow}{0}}\ \mu_{4\text{Y}}$$

Observe that $\sum_i c_i = \frac{1}{200} + (\frac{-2}{200}) + \frac{1}{200} + 0 + 0 + 0 = 0.$

More Examples of Contrasts

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- **A single treatment mean** $C = \mu_k$ **is NOT a contrast**
- Is $C = \dfrac{\mu_1 + \mu_2}{2} - \dfrac{\mu_3 + \mu_4 + \mu_5}{3}$ a contrast?
- Is $C = \dfrac{\mu_1 + \mu_2}{2} - \dfrac{\mu_3 + \mu_4}{2} + \mu_5$ a contrast?

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- Is $C = \dfrac{\mu_1 + \mu_2}{2} - \dfrac{\mu_3 + \mu_4}{2} + \mu_5$ a contrast? No.

$c_1 = c_2 = \frac{1}{2}$, $c_3 = c_4 = -\frac{1}{2}$, $c_5 = 1$, which add up to 1, not 0.

Estimator and Confidence Interval for a Contrast

A natural estimator for a contrast $C = \sum_{i=1}^{g} c_i \mu_i$ is

$$\widehat{C} = \sum_{i=1}^{g} c_i \bar{y}_{i\bullet}$$

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As $\bar{y}_{1\bullet}$, $\bar{y}_{2\bullet}, \ldots$, and $\bar{y}_{g\bullet}$ are indep. of each other, we know

$$\mathbb{V}\left(\sum_{i=1}^{g} c_i \bar{y}_{i\bullet}\right) = \sum_{i=1}^{g} \mathbb{V}(c_i \bar{y}_{i\bullet}) = \sum_{i=1}^{g} c_i^2\, \mathbb{V}(\bar{y}_{i\bullet}) = \sum_{i=1}^{g} c_i^2 \frac{\sigma^2}{n_i}.$$

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The SD and SE of the estimator $\widehat{C}$:

$$\mathrm{SD}(\widehat{C}) = \sqrt{\sigma^2 \sum_{i=1}^{g} \frac{c_i^2}{n_i}}, \qquad \mathrm{SE}(\widehat{C}) = \sqrt{\mathrm{MSE} \times \sum_{i=1}^{g} \frac{c_i^2}{n_i}}$$

A $(1-\alpha)100\%$ confidence interval for the contrast C is

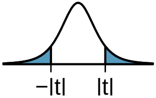
$$\widehat{C} \pm t_{N-g,\alpha/2} \times \mathrm{SE}(\widehat{C})$$

Hypothesis Testing for a Contrast

To test whether a contrast C is 0, $H_0: C = 0$, the test statistic is

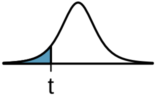
$$t = \frac{\widehat{C}}{\mathrm{SE}(\widehat{C})} = \frac{\sum_{i=1}^{g} c_i \bar{y}_{i\bullet}}{\sqrt{\mathrm{MSE} \times \sum_{i=1}^{g} \frac{c_i^2}{n_i}}} \sim t_{N-g}$$

If H_a: $C \neq 0$ (two-sided),

P-value = 

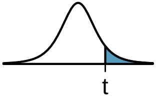
=`2*pt(abs(t), df, lower.tail=F)`

If H_a: $C < 0$ (lower one-sided),

P-value = 

= `pt(t, df)`

If H_a: $C > 0$ (upper one-sided),

P-value = 

= `pt(t, df, lower.tail=F)`

The bell curve above is the t-curve with $\mathrm{df} = N - g$.

Does the Irrigation Effect Change with Nitrogen Levels?

Group	1N	1Y	2N	3N	4N	4Y
$\bar{y}_{i\bullet}$	95	82.25	81.5	68.25	50.5	52

, MSE = 17.97

The contrast we consider is

$$C = \underbrace{(\mu_{1\text{N}} - \mu_{1\text{Y}})}_{\substack{\text{irrigation effect at} \\ \text{nitro level} = 200}} - \underbrace{(\mu_{4\text{N}} - \mu_{4\text{Y}})}_{\substack{\text{irrigation effect at} \\ \text{nitro level} = 800}}$$

in which $(c_{1\text{N}}, c_{1\text{Y}}, c_{2\text{N}}, c_{3\text{N}}, c_{4N}, c_{4\text{Y}}) = (1, -1, 0, 0, -1, 1)$.

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The contrast is estimated by

$$\widehat{C} = \bar{y}_{1N\bullet} - \bar{y}_{1Y\bullet} - (\bar{y}_{4N\bullet} - \bar{y}_{4Y\bullet}) = 95 - 82.25 - (50.5 - 52) = 14.25.$$

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with the standard error

$$\mathrm{SE}(\widehat{C}) = \sqrt{\mathrm{MSE} \sum_{i=1}^{g} \frac{c_i^2}{n_i}} = \sqrt{17.97 \left(\frac{1^2}{4} + \frac{(-1)^2}{4} + \frac{(-1)^2}{4} + \frac{1^2}{4} \right)} \approx 4.24$$

Does the Irrigation Effect Change with Nitrogen Levels?

To test whether the irrigation effect changes with nitrogen level H_0: $C = (\mu_{1N} - \mu_{1Y}) - (\mu_{4N} - \mu_{4Y}) = 0$ v.s. H_a: $C \neq 0$, the t-statistic is

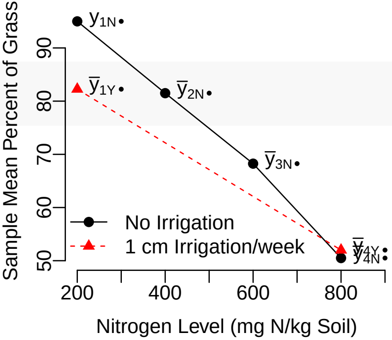
$$t = \frac{\widehat{C}}{\mathrm{SE}(\widehat{C})} = \frac{14.25}{4.24} \approx 3.36$$

with $\text{df} = N - g = 24 - 6 = 18$.

The two-sided p-value is

```
2*pt(3.36,df=18, lower.tail=F)
[1] 0.003487
```

The small P-value indicates the irrigation effects are significantly different at the nitrogen level 200 and 800 mg N/kg soil.



Does the Irrigation Effect Change with Nitrogen Levels?

The 95% confidence interval for $C = (\mu_{1\text{N}} - \mu_{1\text{Y}}) - (\mu_{4\text{N}} - \mu_{4\text{Y}})$ is

$$\widehat{C} \pm t_{N-g,0.025} \times \text{SE}(\widehat{C}) \approx 14.25 \pm 2.101 \times 4.24 \approx (5.34, 23.16)$$

in which $t_{24-6,0.025} \approx 2.101$ is found by the R command

```
qt(0.025,df=18, lower.tail=F)
[1] 2.101
```

This means that the irrigation effect (% of grass w/ irrigation − w/o irrigation) is on average 5.34% to 23.16% higher at nitrogen level 200 than at level 800 mg N/kg soil, with 95% confidence.

Is the Nitrogen Effect Linear?

Treatment	1N	1Y	2N	3N	4N	4Y
Mean $\bar{y}_{i\bullet}$	95	82.25	81.5	68.25	50.5	52

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$$C = \frac{\mu_{1N} - \mu_{2N}}{200} - \frac{\mu_{2N} - \mu_{3N}}{200} = \frac{\mu_{1N} - 2\mu_{2N} + \mu_{3N}}{200}$$

with the coefficients $(c_{1N}, c_{2N}, c_{3N}) = (\frac{1}{200}, \frac{-2}{200}, \frac{1}{200})$.

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The contrast is estimated by

$$\widehat{C} = \frac{\bar{y}_{1N\bullet} - 2\bar{y}_{2N\bullet} + \bar{y}_{3N\bullet}}{200} = \frac{95 - 2 \times 81.5 + 68.25}{200} = \frac{0.25}{200} = 0.00125.$$

with

$$\mathrm{SE}(\widehat{C}) = \sqrt{\mathrm{MSE}\sum_{i=1}^{g}\frac{c_i^2}{n_i}} = \sqrt{17.97\left(\frac{(1/200)^2}{4} + \frac{(\frac{-2}{200})^2}{4} + \frac{(1/200)^2}{4}\right)} \approx 0.026$$

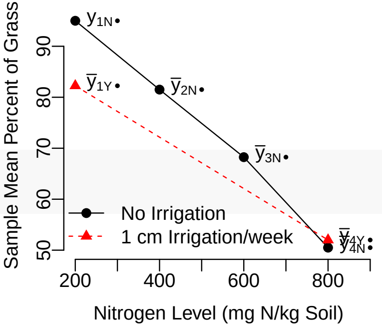
Is the Nitrogen Effect Linear?

To test whether the nitrogen effect is linear, the t-statistic is

$$t = \frac{\widehat{C}}{\mathrm{SE}(\widehat{C})} = \frac{0.00125}{0.026} \approx 0.048$$

with df $= N - g = 24 - 6 = 18$.
The two-sided p-value is

```
2*pt(0.048,df=18, lower.tail=F)
[1] 0.9622
```



Conclusion: The huge P-value means little evidence of nonlinearity (at nitrogen level 1,2, and 3).

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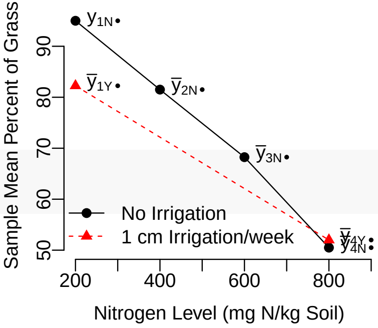

Remark: One can also test the linearity at level 2, 3, and 4

$$C = \frac{\mu_{2N} - \mu_{3N}}{200} - \frac{\mu_{3N} - \mu_{4N}}{200} = \frac{\mu_{2N} - 2\mu_{3N} + \mu_{4N}}{200}.$$

which is left as an exercise.

Inference for Contrasts in R

The R library `emmeans` can produce confidence intervals can conduct tests for contrasts.

e.g., for the interaction contrast $C = (\mu_{1N} - \mu_{1Y}) - (\mu_{4N} - \mu_{4Y})$:

```
grass$trt = as.factor(grass$trt)
levels(grass$trt)
[1] "1N" "1Y" "2N" "3N" "4N" "4Y"
mod1 = aov(percent ~ trt, data=grass)
mod1emm = emmeans(mod1, "trt")
contrast(mod1emm, list(interaction=c(1, -1, 0, 0, -1, 1)),
         infer=c(T,T), level=0.95, side="two-sided")
 contrast    estimate   SE df lower.CL upper.CL t.ratio p.value
 interaction     14.2 4.24 18     5.34     23.2   3.361  0.0035

Confidence level used: 0.95
```

Inference for Contrasts in R

$$\text{Contrast } C = \frac{\mu_{1N} - 2\mu_{2N} + \mu_{3N}}{200}$$

```
contrast(mod1emm, list(lin123=c(1, 0, -2, 1, 0, 0)/200),
         infer=c(T,T), level=0.95, side="two-sided")
 contrast estimate    SE df lower.CL upper.CL t.ratio p.value
 lin123    0.00125 0.026 18  -0.0533   0.0558   0.048  0.9621

Confidence level used: 0.95
```

$$\text{Contrast } C = \frac{\mu_{2N} - 2\mu_{3N} + \mu_{4N}}{200}$$

```
contrast(mod1emm, list(lin234=c(0, 0, 1, -2, 1, 0)/200),
         infer=c(T,T), level=0.95, side="two-sided")
 contrast estimate    SE df lower.CL upper.CL t.ratio p.value
 lin234    -0.0225 0.026 18   -0.077    0.032  -0.867  0.3975

Confidence level used: 0.95
```