Multiple Linear Regression (MLR) Handouts

Yibi Huang

- Data and Models
- Least Squares Estimate, Fitted Values, Residuals
- Sum of Squares
- How to Do Regression in R?
- Interpretation of Regression Coefficients
- t-Tests on Individual Regression Coefficients
- F-Tests for Comparing Nested Models

You may skip this lecture if you have taken STAT 224 or 245.

However, you are encouraged to at least read through the slides if you skip the video lecture.

Data for Multiple Linear Regression Models

Multiple linear regression is a generalized form of simple linear regression, when there are multiple explanatory variables.

	SLR			MLR				
	$\mathbf{x}$	$\mathbf{y}$		$\mathbf{x}_1$	$\mathbf{x}_2$	$\ldots$	$\mathbf{x}_p$	$\mathbf{y}$
case 1:	x_1	y_1		x_{11}	x_{12}	$\ldots$	x_{1p}	y_1
case 2:	x_2	y_2		x_{21}	x_{22}	$\ldots$	x_{2p}	y_2
	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
case n:	x_n	y_n		x_{n1}	x_{n2}	$\ldots$	x_{np}	y_n

- For SLR, we observe pairs of variables.
  For MLR, we observe rows of variables.
  Each row (or pair) is called a *case*, a *record*, or a *data point*
- y_i is the *response* (or *dependent variable*) of the ith case
- There are p *explanatory variables* (or *covariates*, *predictors*, *independent variables*), and x_{ik} is the value of the explanatory variable $\mathbf{x}_k$ of the ith case

Multiple Linear Regression Models

$$y_i = \beta_0 + \beta_1 x_{i1} + \ldots + \beta_p x_{ip} + \varepsilon_i$$

In the model above,

- ε_i's (errors, or noise) are i.i.d. $N(0, \sigma^2)$
- Parameters include:

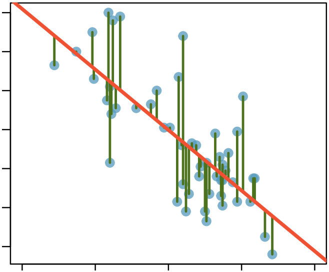
$$\begin{aligned} \beta_0 = &\ \text{intercept;} \\ \beta_k = &\ \text{regression coefficient (slope) for the } k\text{th} \\ &\quad \text{explanatory variable,} \quad k = 1, \ldots, p \\ \sigma^2 = \mathrm{Var}(\varepsilon_i) = &\ \text{the variance of errors} \end{aligned}$$

- Observed (known): $y_i, x_{i1}, x_{i2}, \ldots, x_{ip}$
  Unknown: β_0, $\beta_1, \ldots, \beta_p$, σ^2, ε_i's
- Random variables: ε_i's, y_i's
  Constants (nonrandom): β_k's, σ^2, x_{ik}'s

Fitting the Model — Least Squares Method

Recall for SLR, the least squares estimate $(\widehat{\beta}_0, \widehat{\beta}_1)$ for (β_0, β_1) is the intercept and slope of the straight line with the minimum sum of squared vertical distance to the data points

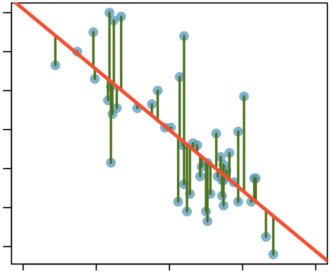
$$\sum_{i=1}^{n}(y_i - \widehat{\beta}_0 - \widehat{\beta}_1 x_i)^2.$$



Fitting the Model — Least Squares Method

Recall for SLR, the least squares estimate $(\widehat{\beta}_0, \widehat{\beta}_1)$ for (β_0, β_1) is the intercept and slope of the straight line with the minimum sum of squared vertical distance to the data points

$$\sum_{i=1}^{n}(y_i - \widehat{\beta}_0 - \widehat{\beta}_1 x_i)^2.$$



MLR is just like SLR. The least squares estimate $(\widehat{\beta}_0, \dots, \widehat{\beta}_p)$ for $(\beta_0, \dots, \beta_p)$ is the intercept and slopes of the (hyper)plane with the minimum sum of squared vertical distance to the data points

$$\sum_{i=1}^{n}(y_i - \widehat{\beta}_0 - \widehat{\beta}_1 x_{i1} - \ldots - \widehat{\beta}_p x_{ip})^2$$

The "Hat" Notation

From now on, we use the "hat" notation to differentiate

- the estimated coefficient $\widehat{\beta}_j$ from
- the actual unknown coefficient β_j

Solving the Least Squares Problem (1)

To find the $(\widehat{\beta}_0, \widehat{\beta}_1, \ldots, \widehat{\beta}_p)$ that minimize

$$L(\widehat{\beta}_0, \widehat{\beta}_1, \ldots, \widehat{\beta}_p) = \sum_{i=1}^{n}(y_i - \widehat{\beta}_0 - \widehat{\beta}_1 x_{i1} - \ldots - \widehat{\beta}_p x_{ip})^2$$

one can set the derivatives of L with respect to $\widehat{\beta}_j$ to 0

$$\frac{\partial L}{\partial \widehat{\beta}_0} = -2\sum_{i=1}^{n}(y_i - \widehat{\beta}_0 - \widehat{\beta}_1 x_{i1} - \ldots - \widehat{\beta}_p x_{ip})$$

$$\frac{\partial L}{\partial \widehat{\beta}_k} = -2\sum_{i=1}^{n} x_{ik}(y_i - \widehat{\beta}_0 - \widehat{\beta}_1 x_{i1} - \ldots - \widehat{\beta}_p x_{ip}),\ k = 1, 2, \ldots, p$$

and then equate them to 0. This results in a system of $(p+1)$ equations in $(p+1)$ unknowns on the next page.

Solving the Least Squares Problem (2)

$$\begin{array}{lllll}
\widehat{\beta}_0 \cdot n & + \widehat{\beta}_1 \sum_{i=1}^n x_{i1} & + \cdots + \widehat{\beta}_p \sum_{i=1}^n x_{ip} & = \sum_{i=1}^n y_i \\
\widehat{\beta}_0 \sum_{i=1}^n x_{i1} & + \widehat{\beta}_1 \sum_{i=1}^n x_{i1}^2 & + \cdots + \widehat{\beta}_p \sum_{i=1}^n x_{i1}x_{ip} & = \sum_{i=1}^n x_{i1}y_i \\
& & \vdots & \\
\widehat{\beta}_0 \sum_{i=1}^n x_{ik} & + \widehat{\beta}_1 \sum_{i=1}^n x_{ik}x_{i1} & + \cdots + \widehat{\beta}_p \sum_{i=1}^n x_{ik}x_{ip} & = \sum_{i=1}^n x_{ik}y_i \\
& & \vdots & \\
\widehat{\beta}_0 \sum_{i=1}^n x_{ip} & + \widehat{\beta}_1 \sum_{i=1}^n x_{ip}x_{i1} & + \cdots + \widehat{\beta}_p \sum_{i=1}^n x_{ip}^2 & = \sum_{i=1}^n x_{ip}y_i
\end{array}$$

- Don't worry about solving the equations.
  R and many other softwares can do the computation for us.
- In general, $\widehat{\beta}_j \neq \beta_j$, but they will be close under some conditions

Solving the Least Squares Problem (2)

$$
\begin{array}{llllll}
\widehat{\beta}_0 \cdot n & + \widehat{\beta}_1 \sum_{i=1}^n x_{i1} & + \cdots + & \widehat{\beta}_p \sum_{i=1}^n x_{ip} & = & \sum_{i=1}^n y_i \\
\widehat{\beta}_0 \sum_{i=1}^n x_{i1} & + \widehat{\beta}_1 \sum_{i=1}^n x_{i1}^2 & + \cdots + & \widehat{\beta}_p \sum_{i=1}^n x_{i1}x_{ip} & = & \sum_{i=1}^n x_{i1}y_i \\
 & & \vdots & & & \\
\widehat{\beta}_0 \sum_{i=1}^n x_{ik} & + \widehat{\beta}_1 \sum_{i=1}^n x_{ik}x_{i1} & + \cdots + & \widehat{\beta}_p \sum_{i=1}^n x_{ik}x_{ip} & = & \sum_{i=1}^n x_{ik}y_i \\
 & & \vdots & & & \\
\widehat{\beta}_0 \underbrace{\sum_{i=1}^n x_{ip}}_{\text{known}} & + \widehat{\beta}_1 \underbrace{\sum_{i=1}^n x_{ip}x_{i1}}_{\text{known}} & + \cdots + & \widehat{\beta}_p \underbrace{\sum_{i=1}^n x_{ip}^2}_{\text{known}} & = & \underbrace{\sum_{i=1}^n x_{ip}y_i}_{\text{known}}
\end{array}
$$

- Don't worry about solving the equations.
  R and many other softwares can do the computation for us.
- In general, $\widehat{\beta}_j \neq \beta_j$, but they will be close under some conditions

Fitted Values

The fitted value or predicted value:

$$\widehat{y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 x_{i1} + \ldots + \widehat{\beta}_p x_{ip}$$

- Again, the "hat" symbol is used to differentiate the fitted value $\widehat{y}_i$ from the actual observed value y_i.

Errors and Residuals

- One cannot directly compute the errors

$$\varepsilon_i = y_i - \beta_0 - \beta_1 x_{i1} - \ldots - \beta_p x_{ip}$$

  since the coefficients $\beta_0, \beta_1, \ldots, \beta_p$ are *unknown*.

- The errors ε_i can be estimated by the **residuals** e_i defined as:

$$\begin{aligned} \text{residual } e_i &= \text{observed } y_i - \text{predicted } y_i \\ &= y_i - \widehat{y}_i \\ &= y_i - \underbrace{(\widehat{\beta}_0 + \widehat{\beta}_1 x_{i1} + \ldots + \widehat{\beta}_p x_{ip})}_{\text{predicted } y_i} \end{aligned}$$

- $e_i \neq \varepsilon_i$ in general since $\widehat{\beta}_j \neq \beta_j$

Properties of Residuals

Recall the least squares estimate $(\widehat{\beta}_0, \widehat{\beta}_1, \ldots, \widehat{\beta}_p)$ satisfies the equations

$$\sum_{i=1}^{n} \underbrace{(y_i - \widehat{\beta}_0 - \widehat{\beta}_1 x_{i1} - \ldots - \widehat{\beta}_p x_{ip})}_{= y_i - \widehat{y}_i = e_i = \text{residual}} = 0 \text{ and}$$

$$\sum_{i=1}^{n} x_{ik} \overbrace{(y_i - \widehat{\beta}_0 - \widehat{\beta}_1 x_{i1} - \ldots - \widehat{\beta}_p x_{ip})} = 0, \ k = 1, 2, \ldots, p.$$

Thus the residuals e_i have the properties

$$\underbrace{\sum\nolimits_{i=1}^{n} e_i = 0}_{\text{Residuals add up to 0.}} \quad , \quad \underbrace{\sum\nolimits_{i=1}^{n} x_{ik} e_i = 0, \ k = 1, 2, \ldots, p.}_{\text{Residuals are orthogonal to covariates.}}$$

Sum of Squares

Observe that

$$y_i - \overline{y} = \underbrace{(\widehat{y}_i - \overline{y})}_{a} + \underbrace{(y_i - \widehat{y}_i)}_{b}$$

Squaring up both sides using the identity $(a+b)^2 = a^2 + b^2 + 2ab$, we get

$$(y_i - \overline{y})^2 = \underbrace{(\widehat{y}_i - \overline{y})^2}_{a^2} + \underbrace{(y_i - \widehat{y}_i)^2}_{b^2} + \underbrace{2(\widehat{y}_i - \overline{y})(y_i - \widehat{y}_i)}_{2ab}$$

Summing up over all the cases $i = 1, 2, \ldots, n$, we get

$$\overbrace{\sum_{i=1}^{n}(y_i - \overline{y})^2}^{\text{SST}} = \overbrace{\sum_{i=1}^{n}(\widehat{y}_i - \overline{y})^2}^{\text{SSR}} + \overbrace{\sum_{i=1}^{n}(\underbrace{y_i - \widehat{y}_i}_{=e_i})^2}^{\text{SSE}}$$

$$+ 2\underbrace{\sum_{i=1}^{n}(\widehat{y}_i - \overline{y})(y_i - \widehat{y}_i)}_{=0\text{, see next page.}}$$

Why $\sum_{i=1}^n (\widehat{y}_i - \overline{y})(y_i - \widehat{y}_i) = 0$?

$$
\begin{aligned}
&\sum_{i=1}^n (\widehat{y}_i - \overline{y})(\underbrace{y_i - \widehat{y}_i}_{=e_i}) \\
&= \sum_{i=1}^n \widehat{y}_i e_i - \sum_{i=1}^n \overline{y} e_i \\
&= \sum_{i=1}^n (\widehat{\beta}_0 + \widehat{\beta}_1 x_{i1} + \ldots + \widehat{\beta}_p x_{ip}) e_i - \sum_{i=1}^n \overline{y} e_i \\
&= \widehat{\beta}_0 \underbrace{\sum_{i=1}^n e_i}_{=0} + \widehat{\beta}_1 \underbrace{\sum_{i=1}^n x_{i1} e_i}_{=0} + \ldots + \widehat{\beta}_p \underbrace{\sum_{i=1}^n x_{ip} e_i}_{=0} - \overline{y} \underbrace{\sum_{i=1}^n e_i}_{=0} \\
&= 0
\end{aligned}
$$

in which we used the properties of residuals that $\sum_{i=1}^n e_i = 0$ and $\sum_{i=1}^n x_{ik} e_i = 0$ for all $k = 1, \ldots, p$.

Interpretation of Sum of Squares

$$\underbrace{\sum_{i=1}^{n}(y_i - \overline{y})^2}_{\text{SST}} = \underbrace{\sum_{i=1}^{n}(\widehat{y}_i - \overline{y})^2}_{\text{SSR}} + \underbrace{\sum_{i=1}^{n}(\overbrace{y_i - \widehat{y}_i}^{=e_i})^2}_{\text{SSE}}$$

- SST = **total sum of squares**
  - total variability of $\mathbf{y}$
  - depends on the response $\mathbf{y}$ only, not on the form of the model

- SSR = **regression sum of squares**
  - variability of $\mathbf{y}$ explained by $\mathbf{x}_1, \ldots, \mathbf{x}_p$

- SSE = **error (residual) sum of squares**
  - $= \min_{\beta_0,\beta_1,\ldots,\beta_p} \sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_{i1} - \cdots - \beta_p x_{ip})^2$
  - variability of $\mathbf{y}$ not explained by $\mathbf{x}$'s

Degrees of Freedom

If the MLR model $y_i = \beta_0 + \beta_1 x_{i1} + \ldots + \beta_p x_{ip} + \varepsilon_i$, ε_i's i.i.d. $\sim N(0, \sigma^2)$ is true, it can be shown that

$$\frac{\text{SSE}}{\sigma^2} \sim \chi^2_{n-p-1},$$

If we further assume that $\beta_1 = \beta_2 = \cdots = \beta_p = 0$, then

$$\frac{\text{SST}}{\sigma^2} \sim \chi^2_{n-1}, \quad \frac{\text{SSR}}{\sigma^2} \sim \chi^2_p$$

and SSR is independent of SSE.
Note the **degrees of freedom** of the 3 chi-square distributions

$$dfT = n - 1, \quad dfR = p, \quad dfE = n - p - 1$$

break down similarly

$$dfT = dfR + dfE$$

just like $SST = SSR + SSE$.

Why SSE Has $n - p - 1$ Degrees of Freedom?

The n residuals $e_1, \dots, e_n$ cannot all vary freely.

There are $p + 1$ constraints:

$$\sum_{i=1}^{n} e_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} x_{ki} e_i = 0 \text{ for } k = 1, \dots, p.$$

So only $n - (p + 1)$ of them can be *freely varying*.

The $p + 1$ constraints comes from the $p + 1$ coefficients $\beta_0, \dots, \beta_p$ in the model, and each contributes one constraint $\frac{\partial}{\partial \beta_k} = 0$.

Mean Square Error (MSE) — Estimate of σ^2

The **mean squares** is the sum of squares divided by its degrees of freedom:

$$\begin{aligned} MST &= \frac{\text{SST}}{dfT} = \frac{\text{SST}}{n-1} = \text{sample variance of } Y, \\ MSR &= \frac{\text{SSR}}{dfR} = \frac{\text{SSR}}{p}, \\ MSE &= \frac{\text{SSE}}{dfE} = \frac{\text{SSE}}{n-p-1} = \widehat{\sigma}^2 \end{aligned}$$

- From the fact $\frac{\text{SSE}}{\sigma^2} \sim \chi^2_{n-p-1}$ and that the mean of a χ^2_k distribution is k, we know that **MSE is an unbiased estimator for σ^2**.

Example: Housing Price

Price	BDR	FLR	FP	RMS	ST	LOT	BTH	CON	GAR	LOC
53	2	967	0	5	0	39	1.5	1	0.0	0
55	2	815	1	5	0	33	1.0	1	2.0	0
56	3	900	0	5	1	35	1.5	1	1.0	0
58	3	1007	0	6	1	24	1.5	0	2.0	0
64	3	1100	1	7	0	50	1.5	1	1.5	0
44	4	897	0	7	0	25	2.0	0	1.0	0
49	5	1400	0	8	0	30	1.0	0	1.0	0
70	3	2261	0	6	0	29	1.0	0	2.0	0
72	4	1290	0	8	1	33	1.5	1	1.5	0
82	4	2104	0	9	0	40	2.5	1	1.0	0
85	8	2240	1	12	1	50	3.0	0	2.0	0
45	2	641	0	5	0	25	1.0	0	0.0	1
47	3	862	0	6	0	25	1.0	1	0.0	1
49	4	1043	0	7	0	30	1.5	0	0.0	1
56	4	1325	0	8	0	50	1.5	0	0.0	1
60	2	782	0	5	1	25	1.0	0	0.0	1
62	3	1126	0	7	1	30	2.0	1	0.0	1
64	4	1226	0	8	0	37	2.0	0	2.0	1
.										
.										
.										
50	2	691	0	6	0	30	1.0	0	2.0	0
65	3	1023	0	7	1	30	2.0	1	1.0	0

Price = Selling price in $1000
BDR = Number of bedrooms
FLR = Floor space in sq. ft.
FP = Number of fireplaces
RMS = Number of rooms
ST = Storm windows (1 if present, 0 if absent)

LOT = Front footage of lot in feet
BTH = Number of bathrooms
CON = Construction (1 if frame, 0 if brick)

GAR = Garage size (0 = no garage, 1 = one-car garage, etc.)

LOC = Location (1 if property is in zone A, 0 otherwise)

How to Do Regression in R?

```
> housing = read.table(
    "http://www.stat.uchicago.edu/~yibi/s222/housing.txt",h=TRUE)
> lm(Price ~ FLR+LOT+BDR+GAR+ST, data=housing)

Call:
lm(formula = Price ~ FLR + LOT + BDR + GAR + ST, data = housing)

Coefficients:
(Intercept)          FLR          LOT          BDR          GAR           ST
   24.63232      0.02009      0.44216     -3.44509      3.35274     11.64033
```

The `lm()` command above asks R to fit the model

$$\text{Price} = \beta_0 + \beta_1\text{FLR} + \beta_2\text{LOT} + \beta_3\text{BDR} + \beta_4\text{GAR} + \beta_5\text{ST} + \varepsilon$$

and R gives us the regression equation

$$\widehat{\text{Price}} = 24.63 + 0.02\text{FLR} + 0.44\text{LOT} - 3.45\text{BDR} + 3.35\text{GAR} + 11.64\text{ST}$$

$$\widehat{\text{Price}} = 24.63 + 0.02\text{FLR} + 0.44\text{LOT} - 3.45\text{BDR} + 3.35\text{GAR} + 11.64\text{ST}$$

- Note here `Price` is in the unit of $1000.

The regression equation tells us:

- an extra square foot in floor area increases the price by ______,
- an extra foot in front footage by . . . . ________,
- an additional bedroom by . . . . __________,
- an additional space in the garage by . . . . _________,
- using storm windows by . . . . _________.

Question:

Why an additional bedroom makes a house less valuable?

$$\widehat{\text{Price}} = 24.63 + 0.02\text{FLR} + 0.44\text{LOT} - 3.45\text{BDR} + 3.35\text{GAR} + 11.64\text{ST}$$

- Note here `Price` is in the unit of $1000.

The regression equation tells us:

- an extra square foot in floor area increases the price by __$20__,
- an extra foot in front footage by . . . . ________,
- an additional bedroom by . . . . ________,
- an additional space in the garage by . . . . ________,
- using storm windows by . . . . ________.

Question:

Why an additional bedroom makes a house less valuable?

$$\widehat{\text{Price}} = 24.63 + 0.02\text{FLR} + 0.44\text{LOT} - 3.45\text{BDR} + 3.35\text{GAR} + 11.64\text{ST}$$

- Note here `Price` is in the unit of $1000.

The regression equation tells us:

- an extra square foot in floor area increases the price by __$20__,
- an extra foot in front footage by . . . . __$440__,
- an additional bedroom by . . . . ________,
- an additional space in the garage by . . . . ________,
- using storm windows by . . . . ________.

Question:

Why an additional bedroom makes a house less valuable?

$$\widehat{\text{Price}} = 24.63 + 0.02\text{FLR} + 0.44\text{LOT} - 3.45\text{BDR} + 3.35\text{GAR} + 11.64\text{ST}$$

- Note here `Price` is in the unit of \$1000.

The regression equation tells us:

- an extra square foot in floor area increases the price by \$20,
- an extra foot in front footage by \$440,
- an additional bedroom by . . . . $-$3450,
- an additional space in the garage by . . . . ________,
- using storm windows by . . . . ________.

Question:

Why an additional bedroom makes a house less valuable?

$$\widehat{\text{Price}} = 24.63 + 0.02\text{FLR} + 0.44\text{LOT} - 3.45\text{BDR} + 3.35\text{GAR} + 11.64\text{ST}$$

- Note here `Price` is in the unit of \$1000.

The regression equation tells us:

- an extra square foot in floor area increases the price by \$20,
- an extra foot in front footage by \$440,
- an additional bedroom by .... −\$3450,
- an additional space in the garage by \$3350,
- using storm windows by .... \$11640.

Question:

Why an additional bedroom makes a house less valuable?

Interpretation of Regression Coefficients

- β_0 = intercept = the mean value of y when all x_j' are 0.
  - may not have practical meaning
    e.g., β_0 is meaningless in the housing price model as no housing unit has 0 floor space.

- β_j: regression coefficient for x_j, is the mean change in the response y when x_j is increased by one unit *holding other x_i's constant*
  - Interpretation of β_j depends on the presence of other covariates in the model
    e.g., the meaning of the 2 β_1's in the following 2 models are different

$$\begin{aligned} \text{Model 1}: &\quad Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \varepsilon \\ \text{Model 2}: &\quad Y = \beta_0 + \beta_1 X_1 + \varepsilon. \end{aligned}$$

What's Wrong?

```
# Model 1
> lm(Price ~ BDR, data=housing)

(Intercept)          BDR
     43.487        3.921
```

The regression coefficient for BDR is 3.921 in the Model 1 above but -3.445 in the Model 2 below.

```
# Model 2
> lm(Price ~ FLR+LOT+BDR+GAR+ST, data=housing)

(Intercept)         FLR         LOT         BDR         GAR          ST
   24.63232     0.02009     0.44216    -3.44509     3.35274    11.64033
```

Considering `BDR` alone, house prices *increase* with `BDR`.

However, an extra bedroom makes a housing unit less valuable when when other covariates (`FLR`, `LOT`, etc) are fixed.

Does this make sense?

More R Commands

```
> lm1 = lm(Price ~  FLR+LOT+BDR+GAR+ST, data=housing)
> lm1$coef
(Intercept)         FLR         LOT         BDR         GAR          ST
24.63231761 0.02009404  0.44216384 -3.44508595  3.35273909 11.64033445

> lm1$fitted        # show the fitted values
> lm1$res           # show the residuals
> cbind(housing$Price, lm1$fit, lm1$res)
  [,1]     [,2]       [,3]
1    53 54.41747 -1.4174732
2    55 55.41567 -0.4156741
3    56 62.85050 -6.8505046
4    58 63.48950 -5.4895039
(...omitted...)

> summary(lm1)      # Regression output with more details
                    # see next page
```

```
> lm1 = lm(Price ~ FLR+LOT+BDR+GAR+ST, data=housing)
> summary(lm1)

Call:
lm(formula = Price ~ FLR + LOT + BDR + GAR + ST, data = housing)

Residuals:
    Min      1Q  Median      3Q     Max
-9.7530 -2.9535  0.1779  3.7183 12.9728

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 24.632318   4.836743   5.093 5.56e-05 ***
FLR          0.020094   0.003668   5.478 2.31e-05 ***
LOT          0.442164   0.150023   2.947 0.007965 **
BDR         -3.445086   1.279347  -2.693 0.013995 *
GAR          3.352739   1.560239   2.149 0.044071 *
ST          11.640334   2.688867   4.329 0.000326 ***
---
Residual standard error: 5.79 on 20 degrees of freedom
Multiple R-squared:  0.8306,Adjusted R-squared:  0.7882
F-statistic: 19.61 on 5 and 20 DF,  p-value: 4.306e-07
```

t-Tests on Individual Regression Coefficients

For a MLR model $Y_i = \beta_0 + \beta_1 X_{i1} + \ldots + \beta_p X_{ip} + \varepsilon_i$, to test the hypotheses,

$$H_0 : \beta_j = c \quad \text{v.s.} \quad H_a : \beta_j \neq c$$

the t-statistic is

$$t = \frac{\widehat{\beta}_j - c}{\text{SE}(\widehat{\beta}_j)}$$

in which $\text{SE}(\widehat{\beta}_j)$ is the standard error for $\widehat{\beta}_j$.

- General formula for $\text{SE}(\widehat{\beta}_j)$ is a bit complicated and hence is omitted
- R can compute $\text{SE}(\widehat{\beta}_j)$ for us
- Formula for $\text{SE}(\widehat{\beta}_j)$ for a few special models will be given later

This t-statistic also has a t-distribution with $n - p - 1$ degrees of freedom

Variable Name ↓ | $\text{SE}(\widehat{\beta}_j)$ ↓ | 2-sided P-value for testing H_0: $\beta_j = 0$ ↓

```
              Estimate Std. Error t value  Pr(>|t|)
(Intercept) 24.632318   4.836743   5.093 5.56e-05 ***
FLR          0.020094   0.003668   5.478 2.31e-05 ***
LOT          0.442164   0.150023   2.947 0.007965 **
BDR         -3.445086   1.279347  -2.693 0.013995 *
GAR          3.352739   1.560239   2.149 0.044071 *
ST          11.640334   2.688867   4.329 0.000326 ***
```

↑ LS estimate of β's | ↓ $t\text{-value} = \frac{\widehat{\beta}_j}{SE(\widehat{\beta}_j)}$

E.g., for `LOT`,

$$\widehat{\beta}_{\text{LOT}} \approx 0.442,\ \text{SE}(\widehat{\beta}_{\text{LOT}}) \approx 0.150,\ t = \frac{\widehat{\beta}_{\text{LOT}}}{\text{SE}(\widehat{\beta}_{\text{LOT}})} \approx \frac{0.442}{0.150} \approx 2.947.$$

The P-value 0.007965 is the 2-sided P-value for testing H_0: $\beta_{\text{LOT}} = 0$

Nested Models

We say Model 1 is **nested in** Model 2 if Model 1 is a special case of Model 2 (and hence Model 2 is an extension of Model 1).

Nested Models

We say Model 1 is **nested in** Model 2 if Model 1 is a special case of Model 2 (and hence Model 2 is an extension of Model 1).
E.g., for the 4 models below,

$$\begin{aligned}
\text{Model A} &: Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \varepsilon \\
\text{Model B} &: Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \varepsilon \\
\text{Model C} &: Y = \beta_0 + \beta_1 X_1 + \beta_3 X_3 + \varepsilon \\
\text{Model D} &: Y = \beta_0 + \beta_1 (X_1 + X_2) + \varepsilon
\end{aligned}$$

- B is nested in A . . . . since A reduces to B when $\beta_3 = 0$

Nested Models

We say Model 1 is **nested in** Model 2 if Model 1 is a special case of Model 2 (and hence Model 2 is an extension of Model 1).
E.g., for the 4 models below,

$$\begin{aligned}
\text{Model A}: Y &= \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \varepsilon \\
\text{Model B}: Y &= \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \varepsilon \\
\text{Model C}: Y &= \beta_0 + \beta_1 X_1 + \beta_3 X_3 + \varepsilon \\
\text{Model D}: Y &= \beta_0 + \beta_1 (X_1 + X_2) + \varepsilon
\end{aligned}$$

- B is nested in A . . . . since A reduces to B when $\beta_3 = 0$
- C is also nested in A . . . . since A reduces to C when $\beta_2 = 0$

Nested Models

We say Model 1 is **nested in** Model 2 if Model 1 is a special case of Model 2 (and hence Model 2 is an extension of Model 1).
E.g., for the 4 models below,

$$\begin{aligned}
\text{Model A}: Y &= \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \varepsilon \\
\text{Model B}: Y &= \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \varepsilon \\
\text{Model C}: Y &= \beta_0 + \beta_1 X_1 + \beta_3 X_3 + \varepsilon \\
\text{Model D}: Y &= \beta_0 + \beta_1 (X_1 + X_2) + \varepsilon
\end{aligned}$$

- B is nested in A . . . . since A reduces to B when $\beta_3 = 0$
- C is also nested in A . . . . since A reduces to C when $\beta_2 = 0$
- D is nested in B . . . . since B reduces to D when $\beta_1 = \beta_2$

Nested Models

We say Model 1 is **nested in** Model 2 if Model 1 is a special case of Model 2 (and hence Model 2 is an extension of Model 1). E.g., for the 4 models below,

$$\begin{aligned}
\text{Model A}: Y &= \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \varepsilon \\
\text{Model B}: Y &= \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \varepsilon \\
\text{Model C}: Y &= \beta_0 + \beta_1 X_1 + \beta_3 X_3 + \varepsilon \\
\text{Model D}: Y &= \beta_0 + \beta_1 (X_1 + X_2) + \varepsilon
\end{aligned}$$

- B is nested in A . . . . since A reduces to B when $\beta_3 = 0$
- C is also nested in A . . . . since A reduces to C when $\beta_2 = 0$
- D is nested in B . . . . since B reduces to D when $\beta_1 = \beta_2$
- B and C are NOT nested in either way

Nested Models

We say Model 1 is **nested in** Model 2 if Model 1 is a special case of Model 2 (and hence Model 2 is an extension of Model 1). E.g., for the 4 models below,

$$\begin{aligned}
\text{Model A}: Y &= \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \varepsilon \\
\text{Model B}: Y &= \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \varepsilon \\
\text{Model C}: Y &= \beta_0 + \beta_1 X_1 + \beta_3 X_3 + \varepsilon \\
\text{Model D}: Y &= \beta_0 + \beta_1 (X_1 + X_2) + \varepsilon
\end{aligned}$$

- B is nested in A .... since A reduces to B when $\beta_3 = 0$
- C is also nested in A .... since A reduces to C when $\beta_2 = 0$
- D is nested in B .... since B reduces to D when $\beta_1 = \beta_2$
- B and C are NOT nested in either way
- D is NOT nested in C

Nested Relationship is Transitive

If Model 1 is nested in Model 2, and Model 2 is nested in Model 3, then Model 1 is also nested in Model 3.

For example, for models in the previous slide,

D is nested in B, and B is nested in A,

implies D is also nested in A, which is clearly true because Model A reduces to Model D when

$$\beta_1 = \beta_2, \text{ and } \beta_3 = 0.$$

Nested Relationship is Transitive

If Model 1 is nested in Model 2, and Model 2 is nested in Model 3, then Model 1 is also nested in Model 3.

For example, for models in the previous slide,

D is nested in B, and B is nested in A,

implies D is also nested in A, which is clearly true because Model A reduces to Model D when

$$\beta_1 = \beta_2, \text{ and } \beta_3 = 0.$$

When two models are nested (Model 1 is nested in Model 2),

- the simpler model (Model 1) is called the **reduced model**,
- the more general model (Model 2) is called the **full model**.

SST of Nested Models

Question: Compare the SST's for Model A, B, C, and D. Which one is the largest? Or are they equal?

SST of Nested Models

Question: Compare the SST's for Model A, B, C, and D. Which one is the largest? Or are they equal?

The 4 models have an identical SST.
SST $= \sum_{i=1}^{n}(y_i - \overline{y})^2$ only depends on the response variable y but not on which explanatory variables are included in the model.

SSE of Nested Models

When a reduced model is nested in a full model, then

$$\text{(i) } \mathsf{SSE}_{reduced} \geq \mathsf{SSE}_{full}, \text{ and (ii) } \mathsf{SSR}_{reduced} \leq \mathsf{SSR}_{full}.$$

SSE of Nested Models

When a reduced model is nested in a full model, then

$$\text{(i) } \text{SSE}_{\textit{reduced}} \geq \text{SSE}_{\textit{full}}, \text{ and (ii) } \text{SSR}_{\textit{reduced}} \leq \text{SSR}_{\textit{full}}.$$

Proof.

- Observe that $\min\{a, b, c, d\} \leq \min\{a, b, c\}$ for any numbers a, b, c, and d

SSE of Nested Models

When a reduced model is nested in a full model, then

$$\text{(i) } \mathrm{SSE}_{reduced} \geq \mathrm{SSE}_{full}, \text{ and (ii) } \mathrm{SSR}_{reduced} \leq \mathrm{SSR}_{full}.$$

Proof.

- Observe that $\min\{a, b, c, d\} \leq \min\{a, b, c\}$ for any numbers a, b, c, and d
- In general, $\min S_1 \leq \min S_2$ if S_2 is a subset of S_1 where S_1 and S_2 are two sets of numbers

SSE of Nested Models

When a reduced model is nested in a full model, then

$$\text{(i) } \text{SSE}_{reduced} \geq \text{SSE}_{full}, \text{ and (ii) } \text{SSR}_{reduced} \leq \text{SSR}_{full}.$$

Proof.

- Observe that $\min\{a, b, c, d\} \leq \min\{a, b, c\}$ for any numbers a, b, c, and d
- In general, $\min S_1 \leq \min S_2$ if S_2 is a subset of S_1 where S_1 and S_2 are two sets of numbers
- We will prove (i) for
  - the full model $y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \varepsilon_i$ and
  - the reduced model $y_i = \beta_0 + \beta_1 x_{i1} + \beta_3 x_{i3} + \varepsilon_i$.

  The proofs for other nested models are similar.

$$\begin{aligned}
\text{SSE}_{full} &= \min_{\beta_0,\beta_1,\beta_2,\beta_3} \sum_{i=1}^{n} (y_1 - \beta_0 - \beta_1 x_{i1} - \beta_2 x_{i2} - \beta_3 x_{i3})^2 \\
&\leq \min_{\beta_0,\beta_1,\beta_3} \sum_{i=1}^{n} (y_1 - \beta_0 - \beta_1 x_{i1} - \beta_3 x_{i3})^2 \\
&= \text{SSE}_{reduced}
\end{aligned}$$

Part (ii) follows directly from (i), the identity SST = SSR + SSE, and the fact that all MLR models of the same data set have a common SST

General Framework for Testing Nested Models

$\boxed{H_0\text{: reduced model is true}}$ v.s. $\boxed{H_a\text{ : full model is true}}$

- As the reduced model is nested in the full model,

$$\text{SSE}_{reduced} \geq \text{SSE}_{full}$$

- Simplicity or Accuracy?
  - The full model fits the data better (with a smaller SSE) but is more complicate

General Framework for Testing Nested Models

$\boxed{H_0\text{: reduced model is true}}$ v.s. $\boxed{H_a\text{ : full model is true}}$

- As the reduced model is nested in the full model,

$$\text{SSE}_{reduced} \geq \text{SSE}_{full}$$

- Simplicity or Accuracy?
  - The full model fits the data better (with a smaller SSE) but is more complicate
  - The reduced model doesn't fit as well but is simpler.

General Framework for Testing Nested Models

$\boxed{\text{H}_0\text{: reduced model is true}}$ v.s. $\boxed{\text{H}_a\text{ : full model is true}}$

- As the reduced model is nested in the full model,

$$\text{SSE}_{reduced} \geq \text{SSE}_{full}$$

- Simplicity or Accuracy?
  - The full model fits the data better (with a smaller SSE) but is more complicate
  - The reduced model doesn't fit as well but is simpler.
  - If $\text{SSE}_{reduced} \approx \text{SSE}_{full}$, one can sacrifice a bit of accuracy in exchange for simplicity

General Framework for Testing Nested Models

$\boxed{\text{H}_0\text{: reduced model is true}}$ v.s. $\boxed{\text{H}_a\text{ : full model is true}}$

- As the reduced model is nested in the full model,

$$\text{SSE}_{reduced} \geq \text{SSE}_{full}$$

- Simplicity or Accuracy?
  - The full model fits the data better (with a smaller SSE) but is more complicate
  - The reduced model doesn't fit as well but is simpler.
  - If $\text{SSE}_{reduced} \approx \text{SSE}_{full}$, one can sacrifice a bit of accuracy in exchange for simplicity
  - If $\text{SSE}_{reduced} \gg \text{SSE}_{full}$, it would cost to much in accuracy in exchange for simplicity. The full model is preferred.

- Hence, a larger difference $\text{SSE}_{reduced} - \text{SSE}_{full}$ is stronger evidence against the reduced model

- Hence, a larger difference $\text{SSE}_{reduced} - \text{SSE}_{full}$ is stronger evidence against the reduced model
- How large $\text{SSE}_{reduced} - \text{SSE}_{full}$ is considered large?

- Hence, a larger difference $\text{SSE}_{reduced} - \text{SSE}_{full}$ is stronger evidence against the reduced model
- How large $\text{SSE}_{reduced} - \text{SSE}_{full}$ is considered large?
  - It depends on the difference in the **complexity** of the two models,

- Hence, a larger difference $\text{SSE}_{reduced} - \text{SSE}_{full}$ is stronger evidence against the reduced model
- How large $\text{SSE}_{reduced} - \text{SSE}_{full}$ is considered large?
  - It depends on the difference in the **complexity** of the two models, which can be reflected by the **difference in the number of parameters** of the two models,

$$dfE_{reduced} - dfE_{full}$$

- Hence, a larger difference $\text{SSE}_{reduced} - \text{SSE}_{full}$ is stronger evidence against the reduced model
- How large $\text{SSE}_{reduced} - \text{SSE}_{full}$ is considered large?
  - It depends on the difference in the **complexity** of the two models, which can be reflected by the **difference in the number of parameters** of the two models,

$$dfE_{reduced} - dfE_{full}$$

  - The larger the magnitude of the noise, σ^2, the larger $\text{SSE}_{reduced} - \text{SSE}_{full}$ is even if H_0 is true

- Hence, a larger difference $\text{SSE}_{reduced} - \text{SSE}_{full}$ is stronger evidence against the reduced model
- How large $\text{SSE}_{reduced} - \text{SSE}_{full}$ is considered large?
  - It depends on the difference in the **complexity** of the two models, which can be reflected by the **difference in the number of parameters** of the two models,

$$dfE_{reduced} - dfE_{full}$$

  - The larger the magnitude of the noise, σ^2, the larger $\text{SSE}_{reduced} - \text{SSE}_{full}$ is even if H_0 is true
- Hence a reasonable test statistic is

$$\frac{(SSE_{reduced} - SSE_{full})/(dfE_{reduced} - dfE_{full})}{\sigma^2}$$

- Hence, a larger difference $\text{SSE}_{reduced} - \text{SSE}_{full}$ is stronger evidence against the reduced model
- How large $\text{SSE}_{reduced} - \text{SSE}_{full}$ is considered large?
  - It depends on the difference in the **complexity** of the two models, which can be reflected by the **difference in the number of parameters** of the two models,

$$dfE_{reduced} - dfE_{full}$$

  - The larger the magnitude of the noise, σ^2, the larger $\text{SSE}_{reduced} - \text{SSE}_{full}$ is even if H_0 is true
- Hence a reasonable test statistic is

$$\frac{(SSE_{reduced} - SSE_{full})/(dfE_{reduced} - dfE_{full})}{\sigma^2}$$

- Need to estimate the unknown σ^2 with the MSE.

- Hence, a larger difference $\text{SSE}_{reduced} - \text{SSE}_{full}$ is stronger evidence against the reduced model
- How large $\text{SSE}_{reduced} - \text{SSE}_{full}$ is considered large?
  - It depends on the difference in the **complexity** of the two models, which can be reflected by the **difference in the number of parameters** of the two models,

$$dfE_{reduced} - dfE_{full}$$

  - The larger the magnitude of the noise, σ^2, the larger $\text{SSE}_{reduced} - \text{SSE}_{full}$ is even if H_0 is true
- Hence a reasonable test statistic is

$$\frac{(SSE_{reduced} - SSE_{full})/(dfE_{reduced} - dfE_{full})}{\sigma^2}$$

- Need to estimate the unknown σ^2 with the MSE.
- Should estimate σ^2 using MSE_{full} rather than $\text{MSE}_{reduced}$ as the full model is always true since the reduced model is special case of the full model

The F-Statistic

$$F = \frac{(SSE_{reduced} - SSE_{full})/(dfE_{reduced} - dfE_{full})}{MSE_{full}}$$

- $dfE_{reduced}$ is the df for SSE for the reduced model.
- dfE_{full} is the df for SSE for the full model.
- $F \geq 0$ since $SSE_{reduced} \geq SSE_{full}$
- The smaller the F-statistic, the more we favor the reduced model
- Under H_0, the F-statistic has an F-distribution with $dfE_{reduced} - dfE_{full}$ and dfE_{full} degrees of freedom.

Testing All Coefficients Equal Zero

Testing the hypotheses

$$\text{H}_0\text{: } \beta_1 = \cdots = \beta_p = 0 \text{ v.s. } \text{H}_a\text{: not all } \beta_1 \ldots, \beta_p = 0$$

is a test to evaluate the **overall significance** of a model.

$$\begin{aligned} \text{Full :} & y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p X_{ip} + \varepsilon_i \\ \text{Reduced :} & y_i = \beta_0 + \varepsilon_i \quad \text{(all covariates are unnecessary)} \end{aligned}$$

- The LS estimate for β_0 in the reduced model is $\widehat{\beta}_0 = \overline{y}$, so

$$SSE_{reduced} = \sum_{i=1}^{n}(y_i - \widehat{\beta}_0)^2 = \sum_i (y_i - \overline{y})^2 = SST_{full}$$

- $dfE_{full} = n - p - 1$.
- $dfE_{reduced} = n - 1$ since the reduced model has 0 explanatory variable.

Testing All Coefficients Equal Zero

Hence

$$F = \frac{(SSE_{reduced} - SSE_{full})/(df_{reduced} - df_{full})}{MSE_{full}}$$
$$= \frac{(SST_{full} - SSE_{full})/[n-1-(n-p-1)]}{SSE_{full}/(n-p-1)}$$
$$= \frac{SSR_{full}/p}{SSE_{full}/(n-p-1)} = \frac{MSR_{full}}{MSE_{full}}.$$

Moreover, $F \sim F_{p,n-p-1}$ under H_0: $\beta_1 = \beta_2 = \cdots = \beta_p = 0$.
In R, the F statistic and p-value are displayed in the last line of the output of the `summary()` command.

```
> lm1 = lm(Price ~ FLR+LOT+BDR+GAR+ST, data=housing)
> summary(lm1)
... (output omitted)

Residual standard error: 5.79 on 20 degrees of freedom
Multiple R-squared:  0.8306,  Adjusted R-squared:  0.7882
F-statistic: 19.61 on 5 and 20 DF,  p-value: 4.306e-07
```

ANOVA and the F-Test

The test of all coefficients equal zero is often summarized in an ANOVA table.

Source	df	Sum of Squares	Mean Squares	F
Regression	$dfR = p$	SSR	$MSR = \frac{SSR}{dfR}$	$F = \frac{MSR}{MSE}$
Error	$dfE = n - p - 1$	SSE	$MSE = \frac{SSE}{dfE}$	
Total	$dfT = n - 1$	SST		

Testing Some Coefficients Equal to Zero

E.g., for the housing price data, we may want to test if we can eliminate BDR and GAR from the model,
i.e., H_0: $\beta_{BDR} = \beta_{GAR} = 0$.

```
> lmfull = lm(Price ~ FLR+LOT+BDR+GAR+ST, data=housing)
> lmreduced = lm(Price ~ FLR+LOT+ST, data=housing)
> anova(lmreduced, lmfull)
Analysis of Variance Table

Model 1: Price ~ FLR + LOT + ST
Model 2: Price ~ FLR + LOT + BDR + GAR + ST
  Res.Df     RSS Df Sum of Sq      F   Pr(>F)
1     22 1105.01
2     20  670.55  2    434.46 6.4792 0.006771 **
```

Note SSE is called RSS (residual sum of square) in R.

Testing Equality of Coefficients

Example. To test H_0: $\beta_1 = \beta_2 = \beta_3$, the reduced model is

$$\begin{aligned} Y &= \beta_0 + \beta_1 X_1 + \beta_1 X_2 + \beta_1 X_3 + \beta_4 X_4 + \varepsilon \\ &= \beta_0 + \beta_1 (X_1 + X_2 + X_3) + \beta_4 X_4 + \varepsilon \end{aligned}$$

1. Make a new variable $W = X_1 + X_2 + X_3$
2. Fit the reduced model by regressing Y on W and X_4
3. Find $\text{SSE}_{reduced}$ and $\text{df}_{reduced} - \text{df}_{full} =$ ____
4. In R

```
> lmfull = lm(Y ~ X1 + X2 + X3 + X4)
> lmreduced = lm(Y ~ I(X1 + X2 + X3) + X4)
> anova(lmreduced, lmfull)
```

The line `lmreduced = lm(Y ~ I(X1 + X2 + X3) + X4)` is equivalent to

```
> W = X1 + X2 + X3
> lmreduced = lm(Y ~ W + X4)
```

Testing Equality of Coefficients

Example. To test H_0: $\beta_1 = \beta_2 = \beta_3$, the reduced model is

$$\begin{aligned} Y &= \beta_0 + \beta_1 X_1 + \beta_1 X_2 + \beta_1 X_3 + \beta_4 X_4 + \varepsilon \\ &= \beta_0 + \beta_1 (X_1 + X_2 + X_3) + \beta_4 X_4 + \varepsilon \end{aligned}$$

1. Make a new variable $W = X_1 + X_2 + X_3$
2. Fit the reduced model by regressing Y on W and X_4
3. Find $\text{SSE}_{reduced}$ and $\text{df}_{reduced} - \text{df}_{full} =$ 2
4. In R

```
> lmfull = lm(Y ~ X1 + X2 + X3 + X4)
> lmreduced = lm(Y ~ I(X1 + X2 + X3) + X4)
> anova(lmreduced, lmfull)
```

The line `lmreduced = lm(Y ~ I(X1 + X2 + X3) + X4)` is equivalent to

```
> W = X1 + X2 + X3
> lmreduced = lm(Y ~ W + X4)
```