STAT22200 Chapter 13 Complete Block Designs

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Block Designs

- A **block** is a set of experimental units that are homogeneous in some sense. Hopefully, units in the same block will have similar responses (if applied with the same treatment.)

- **Block designs**: randomize the units within each block to the treatments.

Randomized Complete Block Designs (RCBD)

We want to test g treatments. There are b blocks of units available, each block contains $k = rg$ units.

- ▶ Within each block, the $k = rg$ units are randomized to the g treatments, r units each.
- ▶ "Complete" means each of the g treatments appears the same number of times (r) in every block.
- ▶ Mostly, block size $k = \#$ of treatments g, i.e., $r = 1$.
- ▶ Matched-Pair design is a special case of RCBD in which the block size $k = 2$.

	Block 1	Block 2	$\cdots$	Block b
Treatment 1	y_{11}	y_{12}	$\cdots$	y_{1b}
Treatment 2	y_{21}	y_{22}	$\cdots$	y_{2b}
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
Treatment g	y_{g1}	y_{g2}	$\cdots$	y_{gb}

Normally, data are shown arranged by block and treatment.
Cannot tell from the data what was/was not randomized.

Things One Can Block On

- Block when you can identify a source of variation (e.g., age, gender, medical history, etc)
- Block on machine/operator/batch (e.g., milk produced in a day)
- Block spatially
- Block on time
- Block on . . .

Advantages of Blocking

- Blocking is the second basic principle of experimental design after randomization.

  > "Block what you can, randomize everything else."

- If units are highly variable, grouping them into more similar blocks can lead to a large increase in efficiency (more power to detect difference in treatment effects).

- The choice of blocks is crucial

Example 13.1 (Mealybugs on Cycads)

A Cycad

Mealybugs
on a Cycad Branch

Example 13.1 (Mealybugs on Cycads)

- ▶ Treatment: water (control), fungal spores, and horticultural oil
- ▶ 5 infested cycads, 3 branches are randomly chosen on each cycad, and 2 patches (3 cm $\times$ 3 cm) are marked on each branch
- ▶ 3 branches on each cycad are randomly assigned to the 3 treatments
- ▶ Response: difference of the # of mealybugs in the patches before and 3 days after treatments are applied

	Plant				
	1	2	3	4	5
Water	-9	18	10	9	-6
	-6	5	9	0	13
Spores	-4	29	4	-2	11
	7	10	-1	6	-1
Oil	4	29	14	14	7
	11	36	16	18	15

Example 13.1 (Mealybugs on Cycads)

What are the experimental units, branches or patches? Why?

```
> cycad = read.table(
        "http://www.stat.uchicago.edu/~yibi/s222/cycad.txt",h=T)
> cycad
   trt plant change
1    1    1     -9
2    1    1     -6
3    2    1     -4
4    2    1      7
5    3    1      4
6    3    1     11
... (omitted) ...
25   1    5     -6
26   1    5     13
27   2    5     11
28   2    5     -1
29   3    5      7
30   3    5     15
```

Example 13.1 (Mealybugs on Cycads)

As the patches are measurement units, we take the average of the two patches on each branch as the response

```
> avechange = (cycad$change[2*(1:15)-1]+cycad$change[2*(1:15)])/2
> cycad = data.frame(trt = cycad$trt[2*(1:15)],
                     plant = cycad$plant[2*(1:15)],
                     avechange)
> cycad
   trt plant avechange
1    1     1      -7.5
2    2     1       1.5
3    3     1       7.5
4    1     2      11.5
5    2     2      19.5
6    3     2      32.5
7    1     3       9.5
(... omitted ...)
14   2     5       5.0
15   3     5      11.0
```

Example 13.1 (Mealybugs on Cycads)

```
cycad$trt = factor(cycad$trt,labels=c("Water","Spore", "Oil"))
interaction.plot(cycad$trt, cycad$plant, cycad$avechange,
  type="b", xlab="Treatments",
  ylab="Avg. changes in mealybug counts\n before and after treatment",
  legend=FALSE,ylim=c(-10,32))
legend("top","Labels = Plant ID",bty="n",cex=.8)
```
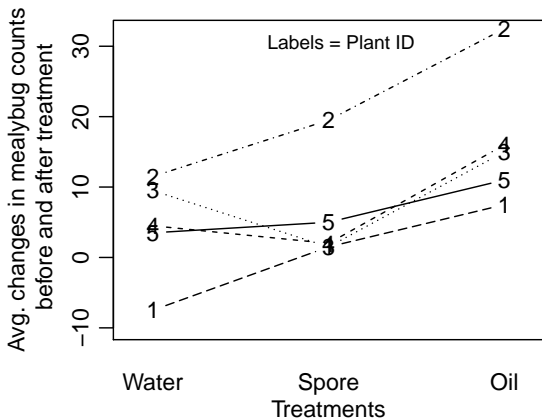
Models for RCBDs

$$y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}$$
$$\text{(trt)} \quad \text{(block)}$$

in which

- y_{ij} = response of the unit receiving treatment i in block j
- μ = the grand mean
- α_i = the treatment effects
- β_j = the block effects
- ε_{ij} = measurement errors, i.i.d. $\sim N(0, \sigma^2)$

Q: In an RCBD model, are we more interested in α_i's or β_j's?

Just like the models for factorial design, the model above is over-parameterized. Need to impose constraints on parameters. The commonly used constraints are

$$\sum_{i=1}^{g} \alpha_i = 0 \quad \text{and} \quad \sum_{j=1}^{b} \beta_j = 0.$$

Parameter Estimates for RCBD Models

The model for a RCBD

$$y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij} \quad \varepsilon_{ij}\text{'s are i.i.d. } N(0, \sigma^2).$$

has the same format as the additive model for a balanced 2-way factorial design. So the formulae for the parameter estimates are identical

$$\begin{aligned}
\widehat{\mu} &= \overline{y}_{\bullet\bullet} \\
\widehat{\alpha}_i &= \overline{y}_{i\bullet} - \overline{y}_{\bullet\bullet} & \text{for } i = 1, \ldots, g \\
\widehat{\beta}_j &= \overline{y}_{\bullet j} - \overline{y}_{\bullet\bullet} & \text{for } j = 1, \ldots, b
\end{aligned}$$

Questions: Why not including treatment-block interactions?

Fitted Values for RCBD

The fitted values for RCBD is then

$$
\begin{aligned}
\widehat{y}_{ij} &= \widehat{\mu} + \widehat{\alpha}_i + \widehat{\beta}_j \\
&= \overline{y}_{\bullet\bullet} + (\overline{y}_{i\bullet} - \overline{y}_{\bullet\bullet}) + (\overline{y}_{\bullet j} - \overline{y}_{\bullet\bullet}) \\
&= \overline{y}_{i\bullet} + \overline{y}_{\bullet j} - \overline{y}_{\bullet\bullet} \\
&= \text{row mean} + \text{column mean} - \text{grand mean}
\end{aligned}
$$

just like in an additive model for a balanced two-way design.

	Block 1	Block 2	$\cdots$	Block b	row mean
Treatment 1	y_{11}	y_{12}	$\cdots$	y_{1b}	$\overline{y}_{1\bullet}$
Treatment 2	y_{21}	y_{22}	$\cdots$	y_{2b}	$\overline{y}_{2\bullet}$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
Treatment g	y_{g1}	y_{g2}	$\cdots$	y_{gb}	$\overline{y}_{g\bullet}$
column mean	$\overline{y}_{\bullet 1}$	$\overline{y}_{\bullet 2}$	$\cdots$	$\overline{y}_{\bullet b}$	grand mean $\overline{y}_{\bullet\bullet}$

Sum of Squares and Degrees of Freedom

The sum of squares and degrees of freedom for RCBD are just like those for the additive models:

$$SST = SS_{trt} + SS_{block} + SSE$$

where

$$SST = \sum_{i=1}^{g} \sum_{j=1}^{b} (y_{ij} - \overline{y}_{\bullet\bullet})^2$$

$$SS_{trt} = \sum_{i=1}^{g} \sum_{j=1}^{b} (\overline{y}_{i\bullet} - \overline{y}_{\bullet\bullet})^2 = b \sum_{i=1}^{g} (\overline{y}_{i\bullet} - \overline{y}_{\bullet\bullet})^2$$

$$SS_{block} = \sum_{i=1}^{g} \sum_{j=1}^{b} (\overline{y}_{\bullet j} - \overline{y}_{\bullet\bullet})^2 = g \sum_{j=1}^{b} (\overline{y}_{\bullet j} - \overline{y}_{\bullet\bullet})^2$$

$$SSE = \sum_{i=1}^{g} \sum_{j=1}^{b} (y_{ij} - \overline{y}_{i\bullet} - \overline{y}_{\bullet j} + \overline{y}_{\bullet\bullet})^2.$$

Total	Treatment	Block	Error
$dfT = bg - 1$	$df_{trt} = g - 1$	$df_{block} = b - 1$	$dfE = (g-1)(b-1)$

Expected Values for the Mean Squares

Just like CRD, the mean squares for RCBD is the sum of squares divided by the corresponding d.f.

$$\text{MS}_{trt} = \frac{\text{SS}_{trt}}{g-1}, \quad \text{MS}_{block} = \frac{\text{SS}_{block}}{b-1}, \quad \text{MSE} = \frac{\text{SSE}}{(g-1)(b-1)}.$$

Under the model for RCBD,

$$y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij} \quad \varepsilon_{ij}\text{'s are i.i.d. } N(0, \sigma^2),$$

with the zero-sum constraints $\sum_{i=1}^{g} \alpha_i = \sum_{j=1}^{b} \beta_j = 0$, one can show that

$$\mathbb{E}(\text{MS}_{trt}) = \sigma^2 + \frac{b}{g-1} \sum_{i=1}^{g} \alpha_i^2$$

$$\mathbb{E}(\text{MS}_{block}) = \sigma^2 + \frac{g}{b-1} \sum_{j=1}^{b} \beta_j^2$$

$$\mathbb{E}(\text{MSE}) = \sigma^2$$

Thus the MSE is again an unbiased estimator of σ^2.

ANOVA *F*-Test for Treatment Effect

To test whether there is an treatment effect

$$H_0 : \alpha_1 = \alpha_2 = \cdots = \alpha_g \quad \text{v.s.} \quad H_a : \text{not all } \alpha_i\text{'s are equal}$$

is equivalent to testing whether all α_i's are zero

$$H_0 : \alpha_1 = \alpha_2 = \cdots = \alpha_g = 0 \quad \text{v.s.} \quad H_a : \text{not all } \alpha_i\text{'s are zero}$$

because of the constraint $\sum_{i=1}^{g} \alpha_i = 0$. This test is also equivalent to a comparison between 2 models:

full model: $y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}$ reduced model: $y_{ij} = \mu + \beta_j + \varepsilon_{ij}$

The test statistic is

$$F_{trt} = \frac{\text{MS}_{trt}}{\text{MSE}} \sim F_{g-1,\,(g-1)(b-1)}\text{under } H_0.$$

The ANOVA table is given in the next slide

ANOVA Table for RCBD

Source	d.f.	Sum of Squares	Mean Squares	F
Treatment	$g-1$	SS_{trt}	MS_{trt}	$F_{trt} = \frac{MS_{trt}}{MSE}$
Block	$b-1$	SS_{block}	MS_{block}	$(F_{block} = \frac{MS_{block}}{MSE})$
Error	$(b-1)(g-1)$	SSE	MSE	
Total	$bg-1$	SST		

▶ The F statistic F_{block} for testing the block effect, which is not interesting, and is often omitted.

ANOVA Table for CRD and RCBD

If we ignore the block effect, and analyzed the experiment as a CRD, the ANOVA table becomes

Source	d.f.	Sum of Squares	Mean Squares	F
Treatment	$g - 1$	SS_{trt}	MS_{trt}	$F_{trt} = \frac{MS_{trt}}{MSE_{CRD}}$
Error	$bg - g$	SSE_{CRD}	MSE_{CRD}	
Total	$bg - 1$	SST		

In the table SS_{trt} and MS_{trt} is the same in CRD and RCBD, but the variability due to block is now in the error term

$$SSE_{CRD} = SSE_{RCBD} + SS_{block}.$$

If SS_{block} is large, by considering the block effect, one can substantially reduce the size of noise. With a smaller MSE, it is easier to detect difference in treatment effects.

RCBD can be a very effective noise-reducing technique if SS_{block} is large.

Example 13.1 (Mealybugs on Cycads)

```
> cycad$trt = factor(cycad$trt,labels=c("Water","Spore", "Oil"))
> anova(lm(avechange ~ trt + as.factor(plant), data=cycad))
Analysis of Variance Table

Response: avechange
                 Df Sum Sq Mean Sq F value   Pr(>F)
trt               2 432.03 216.017 12.1871 0.003729 **
as.factor(plant)  4 686.40 171.600  9.6812 0.003708 **
Residuals         8 141.80  17.725
```

If we ignore the block effect . . .

```
> anova(lm(avechange ~ trt, data=cycad))
Analysis of Variance Table

Response: avechange
          Df Sum Sq Mean Sq F value  Pr(>F)
trt        2 432.03 216.017  3.1299 0.08056 .
Residuals 12 828.20  69.017
```

The treatment effect become less significant.

Caution

Blocking must be done at the time of randomization; you can't group units into blocks when analyzing data to make treatments more significant after the experiment is done.

Model Diagnosis

Just like CRD, one should also do model checking after fitting a RCBD model, including all kinds of residual plots, normal QQ plot, time plot, Box-Cox, etc.
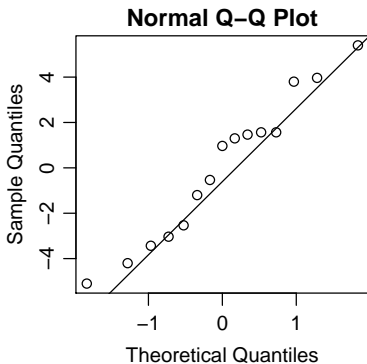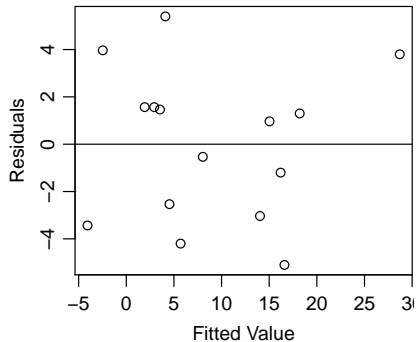
For the Mealybug example, both the residual plot and the normal QQ plot look fine.

Why not checking Box-Cox?

A Blocking Factor is Not A Treatment Factor

It is sometimes easy to confuse blocking variables with treatment factors. Remember that

- ▶ We can assign an experimental unit to any treatment, but cannot assign an unit to any block.
  Blocking variables are a property of the experimental units, not something we can manipulate.
  e.g., in the mealybug example, we can change treatment, but cannot change the plant a branch comes from.

- ▶ Since we cannot experimentally manipulate the blocking variable, block effects are "observational". We cannot make causal inference to a blocking variable as to a treatment factor.

Contrasts in RCBD (1)

A natural estimator of a **contrast** $C = \sum_{i=1}^{g} w_i \alpha_i$ is

$$\widehat{C} = \sum_{i=1}^{g} w_i \widehat{\alpha}_i = \sum_{i=1}^{g} w_i (\overline{y}_{i\bullet} - \overline{y}_{\bullet\bullet}) = \sum_{i=1}^{g} w_i \overline{y}_{i\bullet},$$

which is identical to the estimator in CRD. (Note a contrast must have $\sum_{i=1}^{g} w_i = 0$). So is the SE of the estimator (group size n_i becomes b)

$$\text{SE}(\widehat{C}) = \sqrt{\text{MSE} \times \sum_{i=1}^{g} \frac{w_i^2}{b}}$$

The $(1 - \alpha)100\%$ C.I. and test for C remain the same form, only the degrees of freedom changes from $N - g$ to $(g - 1)(b - 1)$.

$$\widehat{C} \pm t_{\alpha/2, (g-1)(b-1)} \times \text{SE}(\widehat{C})$$

Contrasts in RCBD (2)

As for testing

$$\begin{cases} H_0: & C = \sum_{i=1}^{g} w_i \alpha_i = 0 \\ H_a: & C = \sum_{i=1}^{g} w_i \alpha_i \neq 0 \end{cases}$$

the test statistic remain the same form,

$$t_0 = \frac{\widehat{C}}{SE(\widehat{C})} = \frac{\sum_{i=1}^{g} w_i \overline{y}_{i\bullet}}{\sqrt{MSE \times \sum_{i=1}^{g} \frac{w_i^2}{b}}} \sim t_{(g-1)(b-1)}$$

only the $t-$distribution changes its d.f. form $N - g$ to $(g - 1)(b - 1)$. At test level of α, reject H_0 when

$$|t_0| > t_{\alpha/2, \, (g-1)(b-1)}$$

For one-sided alternative, we reject H_0 at level α when

$$\begin{aligned} t_0 > & \quad t_{\alpha, \, (g-1)(b-1)}, & \text{if } H_a: C > 0, \\ t_0 < & \, -t_{\alpha, \, (g-1)(b-1)}, & \text{if } H_a: C < 0. \end{aligned}$$

Example 13.1 (Mealybugs on Cycads)

```
> library(mosaic)
> mean(avechange ~ trt, data = cycad)
Water Spore   Oil
  4.3   5.9  16.4
```

Recall the MSE we obtained from the ANOVA table is 17.725.
To test whether two treatments, Water and Spore, have the same
mean, H_0: $\mu_{water} = \mu_{spore}$, the t-statistic is

$$t = \frac{\overline{y}_{water} - \overline{y}_{spore}}{\sqrt{\text{MSE}\left(\frac{1}{b} + \frac{1}{b}\right)}} = \frac{5.9 - 4.3}{\sqrt{17.725 \times \left(\frac{1}{5} + \frac{1}{5}\right)}} \approx 0.60 \sim t_8$$

which is not significant.
We can perform pairwise comparisons between the 3 treatments,
and found that the mean change after applying Oil is significantly
higher than the other two treatments, but Water and Spore are
not significantly different effect. Here is the underline diagram:

<u>Water   Spore</u>   Oil

Coming Up Next...

There are several extensions of RCBD

- ▶ Chapter 14 Incomplete Block Design
  (what if block size $\neq$ number of treatments?)

13.3 Latin Square Designs — Blocking Two Variations Simultaneously

Sometimes there are more than one source of variations or disturbance that can be eliminated by blocking.

Example Suppose in a farm there is a north-south variation in sunlight and east-west variation in soil humidity. It is natural to block on row and column position of plots. Suppose the treatments are 5 fertilizers A, B, C, D, E. One can consider the design

			(dry)	$\leftrightarrow$	(humid)		
			West	$\longleftrightarrow$	East		

	Row	Column 1	2	3	4	5
(less sunlight) North	I	A	B	C	D	E
	II	C	D	E	A	B
↑	III	E	A	B	C	D
↓	IV	B	C	D	E	A
(more sunlight) South	V	D	E	A	B	C

Each treatment occurs once in each row and in each column. This is called a **Latin square**. When we estimate treatment effects, the row and column effects will cancel out.

Example — Automobile Emissions

Variables

> Additives: A, B, C, D (chemicals aimed at reducing pollution)
>
> Drivers: I, II, III, IV
>
> Cars: $1, 2, 3, 4$
>
> Response: Emission reduction index measured for each test drive

The experiment

> Additives as treatments ($i = 1, 2, 3, 4$)
>
> Drivers as a block variable (row block $j = 1, 2, 3, 4$)
>
> Cars as another block variable (column block $k = 1, 2, 3, 4$)
>
> The combination (driver, car) as experimental units
>
> Latin square of order 4 as the design of the experiment

Data and Design for the Automobile Emissions Example

| | Cars | | | | |
Drivers	1	2	3	4	driver average
I	A	B	D	C	
	19	24	23	26	23
II	D	C	A	B	
	23	24	19	30	24
III	B	D	C	A	
	15	14	15	16	15
IV	C	A	B	D	
	19	18	19	16	18
average					grand mean
per car	19	20	19	22	20

<u>Treatment means</u>

$$A: 18, \quad B: 22, \quad C: 21, \quad D: 19$$

Data Set of the Automobile Emissions Example

	driver	car	trt	reduct
1	1	1	A	19
2	1	2	B	24
3	1	3	D	23
4	1	4	C	26
5	2	1	D	23
6	2	2	C	24
7	2	3	A	19
8	2	4	B	30
9	3	1	B	15
10	3	2	D	14
11	3	3	C	15
12	3	4	A	16
13	4	1	C	19
14	4	2	A	18
15	4	3	B	19
16	4	4	D	16
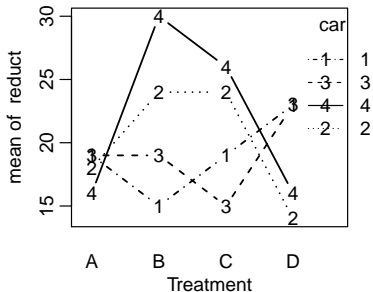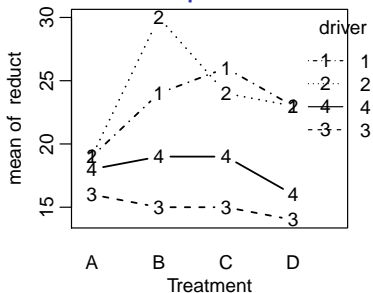
Are there driver effects? Car effects?
Which block effect is stronger?

Model For a Latin Square Design

If we label the two blocking variables j and k and the treatment i with $1 \leq i, j, k \leq g$, we write

$$y_{ijk} = \mu + \underset{\text{(trt)}}{\alpha_i} + \underset{\text{(row)}}{\beta_j} + \underset{\text{(column)}}{\gamma_k} + \varepsilon_{ijk}$$

where

$$\sum_i \alpha_i = \sum_j \beta_j = \sum_k \gamma_k = 0, \qquad \varepsilon_{ijk} \sim i.i.d.\ N(0, \sigma^2)$$

We have g^2 experimental units. For given j and k we only have one value $i(j, k)$ corresponding to *Treatment i*.

The design is balanced, so we have the usual estimates:

$$\widehat{\mu} = \bar{y}_{\bullet\bullet\bullet}, \quad \widehat{\alpha}_i = \bar{y}_{i\bullet\bullet} - \bar{y}_{\bullet\bullet\bullet}$$
$$\widehat{\beta}_j = \bar{y}_{\bullet j \bullet} - \bar{y}_{\bullet\bullet\bullet}$$
$$\widehat{\gamma}_k = \bar{y}_{\bullet\bullet k} - \bar{y}_{\bullet\bullet\bullet}$$

Statistical analysis can be done using familiar R commands.

How Do Latin-Square Designs Work?

Q: What is the mean for $\bar{y}_{A\bullet\bullet}$?

For the automobile emission example,

	car			
driver	1	2	3	4
I	A	B	D	C
II	D	C	A	B
III	B	D	C	A
IV	C	A	B	D

$$\bar{y}_{A\bullet\bullet} = \frac{1}{4}(y_{A11} + y_{A32} + y_{A43} + y_{A24}).$$

Based on the model, we know

$$\mathbb{E}(y_{ijk}) = \mu + \underset{\text{(trt)}}{\alpha_i} + \underset{\text{(car)}}{\beta_j} + \underset{\text{(driver)}}{\gamma_k}$$

Thus

$$\mathbb{E}[\bar{y}_{A\bullet\bullet}] = \frac{1}{4}\begin{pmatrix} \mu + \alpha_A + \beta_1 + \gamma_1 \\ + \mu + \alpha_A + \beta_3 + \gamma_2 \\ + \mu + \alpha_A + \beta_4 + \gamma_3 \\ + \mu + \alpha_A + \beta_2 + \gamma_4 \end{pmatrix}$$

$$= \frac{1}{4}\Big(4\mu + 4\alpha_A + \underbrace{\sum_{j=1}^{4}\beta_j}_{=0} + \underbrace{\sum_{k=1}^{4}\gamma_k}_{=0}\Big) = \mu + \alpha_A$$

ANOVA Table for a Latin Square — (w/o Replicates)

Source	d.f.	SS	MS	F-value
Row-Block	$g-1$	SS_{row}	$SS_{row}/(g-1)$	MS_{row}/MSE
Column-Block	$g-1$	SS_{col}	$SS_{col}/(g-1)$	MS_{col}/MSE
Treatment	$g-1$	SS_{trt}	$SS_{trt}/(g-1)$	MS_{trt}/MSE
Error	$(g-2)(g-1)$	SSE	$SSE/[(g-2)(g-1)]$	
Total	g^2-1	SST		

where $(g-2)(g-1) = g^2 - 1 - 3(g-1)$.

$$SS_{row} = \sum_{ijk} \widehat{\beta}_j^2 = \sum_{ijk} (\bar{y}_{\bullet j \bullet} - \bar{y}_{\bullet\bullet\bullet})^2 = g \sum_j (\bar{y}_{\bullet j \bullet} - \bar{y}_{\bullet\bullet\bullet})^2,$$

$$SS_{col} = \sum_{ijk} \widehat{\gamma}_k^2 = \sum_{ijk} (\bar{y}_{\bullet\bullet k} - \bar{y}_{\bullet\bullet\bullet})^2 = g \sum_k (\bar{y}_{\bullet\bullet k} - \bar{y}_{\bullet\bullet\bullet})^2,$$

$$SS_{trt} = \sum_{ijk} \widehat{\alpha}_i^2 = \sum_{ijk} (\bar{y}_{i\bullet\bullet} - \bar{y}_{\bullet\bullet\bullet})^2 = g \sum_i (\bar{y}_{i\bullet\bullet} - \bar{y}_{\bullet\bullet\bullet})^2,$$

$$SSE = \sum_{ijk} (y_{ijk} - \bar{y}_{i\bullet\bullet} - \bar{y}_{\bullet j \bullet} - \bar{y}_{\bullet\bullet k} + 2\bar{y}_{\bullet\bullet\bullet})^2$$

$$= SST - SS_{row} - SS_{col} - SS_{trt},$$

$$SST = \sum_{ijk} (y_{ijk} - \bar{y}_{\bullet\bullet\bullet})^2$$

Decomposition of the Latin Square (Automobile Emissions)

	1	2	3	4
I	A	B	D	C
	19	24	23	26
II	D	C	A	B
	23	24	19	30
III	B	D	C	A
	15	14	15	16
IV	C	A	B	D
	19	18	19	16

$=$

$\bar{y}_{\bullet\bullet\bullet}$

20	20	20	20
20	20	20	20
20	20	20	20
20	20	20	20

$+$

$y_{ijk} - \bar{y}_{\bullet\bullet\bullet}$

-1	4	3	6
3	4	-1	10
-5	-6	-5	-4
-1	-2	-1	-4

$y_{ijk} - \bar{y}_{\bullet\bullet\bullet}$

-1	4	3	6
3	4	-1	10
-5	-6	-5	-4
-1	-2	-1	-4

$=$

$y_{\bullet j \bullet} - \bar{y}_{\bullet\bullet\bullet}$

3	3	3	3
4	4	4	4
-5	-5	-5	-5
-2	-2	-2	-2

$+$

$y_{\bullet\bullet k} - \bar{y}_{\bullet\bullet\bullet}$

-1	0	-1	2
-1	0	-1	2
-1	0	-1	2
-1	0	-1	2

$+$

$y_{i\bullet\bullet} - \bar{y}_{\bullet\bullet\bullet}$

-2	2	-1	1
-1	1	-2	2
2	-1	1	-2
1	-2	2	-1

$+$

residuals

-1	-1	2	0
1	-1	-2	2
-1	0	0	1
1	2	0	-3

$\mathbf{Y} - \bar{\mathbf{Y}}$	$\mathbf{D}$	$\mathbf{C}$	$\mathbf{T}$	$\mathbf{R}$
$SST = 312$	$SS_{Driver} = 216$	$SS_{Car} = 24$	$SS_{trt} = 40$	$SSE = 32$
$d.f. = 15$	$d.f. = 3$	$d.f. = 3$	$d.f. = 3$	$d.f. = 6$

Example: ANOVA Table for Automobile Emissions Data

```
> emis = read.table(
        "http://www.stat.uchicago.edu/~yibi/s222/emission.txt", h=T)
> lm1 = lm(reduct ~ as.factor(driver)+as.factor(car)+as.factor(trt),
          data=emis)
> anova(lm1)
Analysis of Variance Table

Response: reduct
                 Df Sum Sq Mean Sq F value   Pr(>F)
as.factor(driver) 3    216  72.000    13.5 0.004466 **
as.factor(car)    3     24   8.000     1.5 0.307174
as.factor(trt)    3     40  13.333     2.5 0.156490
Residuals         6     32   5.333
```

13.3.1 Crossover Design (A Special Latin-Square Design)

When a sequence of treatments is given to a subject over several time periods,

- ▶ need to block on subjects,
  because each subject tends to respond differently, and

- ▶ need to block on time period,
  because there may consistent differences over time due to growth, aging, disease progression, or other factors.

Such a design is called a **crossover design**, which is a common application of the Latin Square designs.

Example 13.5 — Bioequivalence of Drug Delivery (p.326)

- ▶ Objectives — Investigate whether different drug delivery systems have similar biological effects.

- ▶ Response variable — Average blood concentration of the drug during certain time interval after the drug has been administered.

- ▶ Treatments — 3 different drug delivery systems:

  A – solution, B – tablet, C – capsule

- ▶ Blocking on both subjects and time period

- ▶ Data & design:

	subject		
period	1	2	3
1	1799A	2075C	1396B
2	1846C	1156B	868A
3	2147B	1777A	2291C

Advantages of Latin Squares

▶ For the same number of experimental units as a randomized complete block design (RCBD) with g treatments and g blocks, we can simultaneously block for a second variable.

▶ Provide an elegant and efficient use of limited resources for a small experiment.

Disadvantages of Latin Squares

▶ Cannot identify block-treatment interactions.

▶ There may be few $d.f.$ left to estimate σ once both block and treatment effects are estimated.
E.g., in a 3×3 square, block and treatment effects taken up 3×2 $d.f.$, allowing 1 $d.f.$ for the grand mean leaves 2 $d.f.$ for estimating σ^2, and this includes all the treatments.

▶ One can (and should) **replicate** Latin square.

13.3.4 Replicated Latin Square Designs

In Example 13.5, with a single 3×3 Latin square, there are only 2 degrees of freedom left for estimating the error. One can increase the d.f. for errors by *replicating more Latin squares.*

Example 13.6 is a continuation of Example 13.5.

- ▶ There are 12 subjects, instead of 3
- ▶ The 12 subjects are divided into 4 groups of 3 each, and a Latin square design is arranged for each group
- ▶ Data are shown on the next page and the data file is available at

  http://users.stat.umn.edu/~gary/book/fcdae.data/exmpl13.10

Data and Design for Example 13.6

	square 1			square 2		
	subject					
period	1	2	3	4	5	6
1	1799A	2075C	1396B	3100B	1451C	3174A
2	1846C	1156B	868A	3065A	1217B	1714C
3	2147B	1777A	2291C	4077C	1288A	2919B

	square 3			square 4		
	subject					
period	7	8	9	10	11	12
1	1430C	1186A	1135B	873C	2061A	1053B
2	836A	642B	1305C	1426A	2433B	1534C
3	1063B	1183C	984A	1540B	1337C	1583A

▶ Each (subject, treatment) combination appears exactly once
▶ Each (period, treatment) combination appears exactly 4 times
▶ Note the 4 Latin-squares have a common row block (period). In this case, we say the row block is **reused**.

A Model for the Replicated Latin Square Design

When Latin squares are replicated (each one separately randomized), the appropriate linear model will depend on which blocks (if any) are **reused**.

In Example 13.6, the row variable, period, is reused. An appropriate model would be

$$y_{ijk} = \mu + \underset{\text{(trt)}}{\alpha_i} + \underset{\text{(period)}}{\beta_j} + \underset{\text{(subject)}}{\gamma_k} + \varepsilon_{ijk}$$

with the constraints

$$\sum_{i=1}^{3} \alpha_i = \sum_{j=1}^{3} \beta_j = \sum_{k=1}^{12} \gamma_k = 0.$$

The parameter estimates are again

$$\widehat{\mu} = \bar{y}_{\bullet\bullet\bullet}, \quad \widehat{\alpha}_i = \bar{y}_{i\bullet\bullet} - \bar{y}_{\bullet\bullet\bullet}$$
$$\widehat{\beta}_j = \bar{y}_{\bullet j\bullet} - \bar{y}_{\bullet\bullet\bullet}$$
$$\widehat{\gamma}_k = \bar{y}_{\bullet\bullet k} - \bar{y}_{\bullet\bullet\bullet}$$

For the Drug Delivery Study, we can show that $\mathbb{E}[\overline{y}_{B\bullet\bullet}] = \mu + \alpha_B$.

Subject	1	2	3	4	5	6	7	8	9	10	11	12
1	A	C	B	B	C	A	C	A	B	C	A	B
period 2	C	B	A	A	B	C	A	B	C	A	B	C
3	B	A	C	C	A	B	B	C	A	B	C	A

$$\overline{y}_{B\bullet\bullet} = \frac{1}{12}(y_{B31} + y_{B22} + y_{B13} + y_{B14} + y_{B25} + y_{B36}$$
$$+ y_{B37} + y_{B28} + y_{B19} + y_{B3,10} + y_{B2,11} + y_{B1,12}).$$

Since
$$\mathbb{E}(y_{ijk}) = \mu + \underset{\text{(trt)}}{\alpha_i} + \underset{\text{(period)}}{\beta_j} + \underset{\text{(subject)}}{\gamma_k}$$

we know
$$\mathbb{E}[\overline{y}_{B\bullet\bullet}] = \frac{1}{12} \begin{pmatrix} \mu + \alpha_B + \beta_3 + \gamma_1 \\ + \mu + \alpha_B + \beta_2 + \gamma_2 \\ + \vdots \quad \vdots \quad \vdots \quad \vdots \\ + \mu + \alpha_B + \beta_1 + \gamma_{12} \end{pmatrix} = \frac{1}{12}(12\mu + 12\alpha_B + 4\underbrace{\sum_{j=1}^{3}\beta_j}_{=0} + \underbrace{\sum_{k=1}^{12}\gamma_k}_{=0})$$
$$= \mu + \alpha_B$$

This is because each treatment shows up in each row 4 times and each column once. Hence each β_j comes up 4 times and each γ_k once in the sum.

Similar, for the period block effect, we can show that $\mathbb{E}[\overline{y}_{\bullet 1 \bullet}] = \mu + \beta_1$.

Subject	1	2	3	4	5	6	7	8	9	10	11	12
1	A	C	B	B	C	A	C	A	B	C	A	B
period 2	C	B	A	A	B	C	A	B	C	A	B	C
3	B	A	C	C	A	B	B	C	A	B	C	A

$$\overline{y}_{\bullet 1 \bullet} = \frac{1}{12}(y_{A11} + y_{C12} + y_{B13} + y_{B14} + y_{C15} + y_{A16}$$
$$+ y_{C17} + y_{A18} + y_{B19} + y_{C1,10} + y_{A1,11} + y_{B1,12}).$$

Since
$$\mathbb{E}(y_{ijk}) = \mu + \underset{\text{(trt)}}{\alpha_i} + \underset{\text{(period)}}{\beta_j} + \underset{\text{(subject)}}{\gamma_k}$$

we know
$$\mathbb{E}[\overline{y}_{\bullet 1 \bullet}] = \frac{1}{12} \begin{pmatrix} \mu + \alpha_A + \beta_1 + \gamma_1 \\ + \mu + \alpha_C + \beta_1 + \gamma_2 \\ + \vdots \quad \vdots \quad \vdots \quad \vdots \\ + \mu + \alpha_B + \beta_1 + \gamma_{12} \end{pmatrix} = \frac{1}{12}(12\mu + 4\underbrace{\sum_{j=1}^{3} \alpha_j}_{=0} + 12\beta_1 + \underbrace{\sum_{k=1}^{12} \gamma_k}_{=0})$$
$$= \mu + \beta_1$$

This is because each treatment shows up in each row 4 times and hence each α_i comes up 4 times in the sum.

Similar, for the subject block effect, we can show that $\mathbb{E}[\bar{y}_{\bullet\bullet 3}] = \mu + \gamma_3$.

Subject	1	2	3	4	5	6	7	8	9	10	11	12
1	A	C	B	B	C	A	C	A	B	C	A	B
period 2	C	B	A	A	B	C	A	B	C	A	B	C
3	B	A	C	C	A	B	B	C	A	B	C	A

$$\bar{y}_{\bullet\bullet 3} = \frac{1}{3}(y_{B13} + y_{A23} + y_{C33})$$

Since

$$\mathbb{E}(y_{ijk}) = \mu + \underset{\text{(trt)}}{\alpha_i} + \underset{\text{(period)}}{\beta_j} + \underset{\text{(subject)}}{\gamma_k}$$

we know

$$\mathbb{E}[\bar{y}_{\bullet\bullet 3}] = \frac{1}{3}\begin{pmatrix} \mu + \alpha_B + \beta_1 + \gamma_3 \\ + \mu + \alpha_A + \beta_2 + \gamma_3 \\ + \mu + \alpha_C + \beta_3 + \gamma_3 \end{pmatrix} = \frac{1}{3}(3\mu + \underbrace{\sum_{i=1}^{3}\alpha_i}_{=0} + \underbrace{\sum_{j=1}^{3}\beta_j}_{=0} + 3\gamma_3)$$

$$= \mu + \gamma_3$$

This is because each treatment shows up in each column exactly once and hence each α_i comes up once in the sum.

ANOVA Table for Latin Square w/ *m* Replicates, Row Block Reused

Number of Latin Squares = m

Source	d.f.	SS	MS	F-value
Row-Block	$g-1$	SS_{row}	$SS_{row}/(g-1)$	MS_{row}/MSE
Column-Block	$mg-1$	SS_{col}	$SS_{col}/(mg-1)$	MS_{col}/MSE
Treatment	$g-1$	SS_{trt}	$SS_{trt}/(g-1)$	MS_{trt}/MSE
Error	$(mg-2)(g-1)$	SSE	$SSE/[(mg-2)(g-1)]$	
Total	mg^2-1	SST		

where

$$SS_{row} = \sum_{ijk}(\bar{y}_{\bullet j \bullet} - \bar{y}_{\bullet\bullet\bullet})^2 = mg\sum_{j=1}^{g}(\bar{y}_{\bullet j \bullet} - \bar{y}_{\bullet\bullet\bullet})^2,$$

$$SS_{col} = \sum_{ijk}(\bar{y}_{\bullet\bullet k} - \bar{y}_{\bullet\bullet\bullet})^2 = g\sum_{k=1}^{mg}(\bar{y}_{\bullet\bullet k} - \bar{y}_{\bullet\bullet\bullet})^2,$$

$$SS_{trt} = \sum_{ijk}(\bar{y}_{i\bullet\bullet} - \bar{y}_{\bullet\bullet\bullet})^2 = mg\sum_{i=1}^{g}(\bar{y}_{i\bullet\bullet} - \bar{y}_{\bullet\bullet\bullet})^2,$$

$$SST = \sum_{ijk}(y_{ijk} - \bar{y}_{\bullet\bullet\bullet})^2$$

$$SSE = SST - SS_{row} - SS_{col} - SS_{trt}$$

Don't try to memorize the formula for df_{error}. Just keep in mind that

$$df_{error} = df_{total} - df_{row} - df_{col} - df_{trt}$$

```
Response: area
                   Df    Sum Sq Mean Sq F value     Pr(>F)
as.factor(period)   2    737751  368875  1.7965    0.1916
as.factor(subject) 11 16385060 1489551  7.2546 7.475e-05 ***
as.factor(trt)      2     81458   40729  0.1984    0.8217
Residuals          20   4106500  205325
```

There is no evidence that the 3 drug delivery systems have
different biological effects.

Latin Square w/ Replicates, Neither Row nor Column Reused

Automobile Emissions Example Revisited
Suppose there were 8 cars and 8 drivers available.
We can from 2 Latin Squares.

	Cars			
Drivers	1	2	3	4
I	A	B	D	C
II	D	C	A	B
III	B	D	C	A
IV	C	A	B	D

	Cars			
Drivers	5	6	7	8
V	A	B	C	D
VI	C	D	A	B
VII	D	C	B	A
VIII	B	A	D	C

Note the two squares have neither rows (drivers) in common nor columns (cars) in common. That is, **neither rows nor columns are reused**.

Models for Replicated Latin Squares, w/o reusing rows or columns

In this case, it will be problematic to use the conventional model below

$$y_{ijk} = \mu + \underset{\text{(trt)}}{\alpha_i} + \underset{\text{(driver)}}{\beta_j} + \underset{\text{(car)}}{\gamma_k} + \varepsilon_{ijk}$$

with constraints $\sum_{i=1}^4 \alpha_i = \sum_{j=1}^8 \beta_j = \sum_{k=1}^8 \gamma_k = 0$ since in this case, even though $\mathbb{E}[\bar{y}_{i\bullet\bullet}] = \mu + \alpha_i$ but

$$\mathbb{E}[\bar{y}_{\bullet j\bullet}] \neq \mu + \beta_j, \quad \mathbb{E}[\bar{y}_{\bullet\bullet k}] \neq \mu + \gamma_k.$$

Observe that $\bar{y}_{\bullet 2\bullet} = (y_{D21} + y_{C22} + y_{A23} + y_{B24})/4$

$$
\mathbb{E}[\bar{y}_{\bullet 2\bullet}] = \frac{1}{4} \begin{pmatrix} \mu + \alpha_D + \beta_2 + \gamma_1 \\ + \mu + \alpha_C + \beta_2 + \gamma_2 \\ + \mu + \alpha_A + \beta_2 + \gamma_3 \\ + \mu + \alpha_B + \beta_2 + \gamma_4 \end{pmatrix}
$$

$$
= \frac{1}{4}\Big(4\mu + \underbrace{\sum_{i=1}^4 \alpha_i}_{=0} + 4\beta_2 + \underbrace{\sum_{k=1}^4 \gamma_k}_{\neq 0}\Big) \neq \mu + \beta_2
$$

Models for Replicated Latin Squares, w/o reusing rows or columns

A better model for replicated Latin squares w/o reusing rows or columns is

$$y_{ijk\ell} = \mu + \underset{\text{(trt)}}{\alpha_i} + \underset{\text{(driver)}}{\beta_{j(\ell)}} + \underset{\text{(car)}}{\gamma_{k(\ell)}} + \underset{\text{(square)}}{\delta_\ell} + \varepsilon_{ijk\ell}$$

with constraints $\sum_{i=1}^{4} \alpha_i = \sum_{\ell=1}^{2} \delta_\ell = 0$ and

$$\sum_{j=1}^{4} \beta_{j(1)} = \sum_{j=5}^{8} \beta_{j(2)} = \sum_{k=1}^{4} \gamma_{k(1)} = \sum_{k=5}^{8} \gamma_{k(2)} = 0.$$

The parameter estimates are again

$$\widehat{\mu} = \bar{y}_{\bullet\bullet\bullet\bullet}, \qquad \widehat{\alpha}_i = \bar{y}_{i\bullet\bullet\bullet} - \bar{y}_{\bullet\bullet\bullet\bullet}$$

$$\widehat{\delta}_\ell = \bar{y}_{\bullet\bullet\bullet\ell} - \bar{y}_{\bullet\bullet\bullet\bullet}$$

$$\widehat{\beta}_{j(\ell)} = \bar{y}_{\bullet j\bullet\ell} - \bar{y}_{\bullet\bullet\bullet\ell}$$

$$\widehat{\gamma}_{k(\ell)} = \bar{y}_{\bullet\bullet k\ell} - \bar{y}_{\bullet\bullet\bullet\ell}$$

ANOVA Table for Latin Square w/o reusing rows or columns

Number of Latin Squares $= m$

Source	d.f.	SS	MS	F-value
Square	$m-1$	SS_{sqr}	$SS_{sqr}/(m-1)$	MS_{sqr}/MSE
Row-Block	$m(g-1)$	SS_{row}	$SS_{row}/[m(g-1)]$	MS_{row}/MSE
Column-Block	$m(g-1)$	SS_{col}	$SS_{col}/[m(g-1)]$	MS_{col}/MSE
Treatment	$g-1$	SS_{trt}	$SS_{trt}/(g-1)$	MS_{trt}/MSE
Error	$(mg+m-3)(g-1)$	SSE	SSE/dfE	
Total	mg^2-1	SST		

where $df_{error} = df_{total} - df_{sqr} - df_{row} - df_{col} - df_{trt}$

$$= mg^2 - 1 - (m-1) - 2m(g-1) = (mg+m-3)(g-1)$$

$$SS_{sqr} = \sum_{ijk\ell} (\bar{y}_{\bullet\bullet\bullet\ell} - \bar{y}_{\bullet\bullet\bullet\bullet})^2 = g^2 \sum_{\ell=1}^{m} (\bar{y}_{\bullet\bullet\bullet\ell} - \bar{y}_{\bullet\bullet\bullet\bullet})^2,$$

$$SS_{row} = \sum_{ijk\ell} (\bar{y}_{\bullet j\bullet\ell} - \bar{y}_{\bullet\bullet\bullet\ell})^2 = g \sum_{j\ell} (\bar{y}_{\bullet j\bullet\ell} - \bar{y}_{\bullet\bullet\bullet\ell})^2,$$

$$SS_{col} = \sum_{ijk\ell} (\bar{y}_{\bullet\bullet k\ell} - \bar{y}_{\bullet\bullet\bullet\ell})^2 = g \sum_{j\ell} (\bar{y}_{\bullet\bullet k\ell} - \bar{y}_{\bullet\bullet\bullet\ell})^2,$$

$$SS_{trt} = \sum_{ijk} (\bar{y}_{i\bullet\bullet\bullet} - \bar{y}_{\bullet\bullet\bullet\bullet})^2 = mg \sum_{i=1}^{g} (\bar{y}_{i\bullet\bullet\bullet} - \bar{y}_{\bullet\bullet\bullet\bullet})^2,$$

$$SST = \sum_{ijk} (y_{ijk\ell} - \bar{y}_{\bullet\bullet\bullet\bullet})^2$$

$$SSE = SST - SS_{sqr} - SS_{row} - SS_{col} - SS_{trt}$$

Skeleton ANOVA for Latin Square Designs with Replicates

	Number of Latin Squares $= m$			
	row and col. both reused	row not reused	column not reused	neither reused
Source	d.f.	d.f.	d.f.	d.f.
Square				$m-1$
Row-Block	$g-1$	$mg-1$	$g-1$	$m(g-1)$
Column-Block	$g-1$	$g-1$	$mg-1$	$m(g-1)$
Treatment	$g-1$	$g-1$	$g-1$	$g-1$
Error	mg^2-1- (sum of the above)			
total	mg^2-1	mg^2-1	mg^2-1	mg^2-1

13.4 Graeco-Latin Squares — Blocking Three Variations Simultaneously

A **Graeco-Latin square** is a $g \times g$ table where each entry has a Greek and a Latin letter, so that

▶ the Greek letters form a Latin square

▶ the Latin letters form a Latin square

▶ each Greek letter appears exactly once with each Latin letter.

Example:

$$
\begin{array}{ccc}
A\alpha & B\beta & C\gamma \\
B\gamma & C\alpha & A\beta \\
C\beta & A\gamma & B\alpha
\end{array}
\qquad
\begin{array}{cccc}
A\alpha & B\beta & C\gamma & D\delta \\
D\gamma & C\delta & B\alpha & A\beta \\
B\delta & A\gamma & D\beta & C\alpha \\
C\beta & D\alpha & A\delta & B\gamma
\end{array}
$$

▶ Graeco-Latin square designs can block 3 variables simultaneously — corresponding to rows, columns and Greek letters.

▶ The design is balanced, in that row, column, Latin and Greek effects are all orthogonal. A Graeco-Latin square can be produced from any orthogonal pair of Latin squares.

Exercise 13.5 — Disk Drive

- ▶ Objectives: Investigate whether different materials of disk drive substrates may affect the amplitude of the signal obtained during readback.

- ▶ Response variable: Amplitudes of the signals during readback measured in microvolts $\times 0.01$.

- ▶ Treatments — 4 types of substrate materials:
  (A) aluminum   (C) glass type I
  (B) nickle-plated aluminum   (D) glass type II

- ▶ Blocking variables
  - ▶ Machines: Four machines
  - ▶ Operators: Four operators
  - ▶ Days: Four different days

- ▶ Design: Graeco-Latin Square

Data and the Design Structure

Machine	Operator 1	2	3	4		Row Mean
I	A	C	D	B	(treatments)	
	α	γ	δ	β	(days)	
	8	11	2	8	(responses)	7.25
II	C	A	B	D		
	δ	β	α	γ		
	7	5	2	4		4.5
III	D	B	A	C		
	β	δ	γ	α		
	3	9	7	9		7
IV	B	D	C	A		
	γ	α	β	δ		
	4	5	9	3		5.25
Column Mean	5.5	7.5	5	6		6

treatment	A	B	C	D
mean	5.75	5.75	9	3.5

day	α	β	γ	δ
mean	6	6.25	6.5	5.25

Data file: http://users.stat.umn.edu/~gary/book/fcdae.data/ex13.5

Models for Graeco-Latin Square Designs

The model for the example of Graeco-Latin design is

$$y_{ijk\ell} = \mu + \underset{\text{(trt)}}{\theta_i} + \underset{\text{(machine)}}{\tau_j} + \underset{\text{(operator)}}{\omega_k} + \underset{\text{(day)}}{\phi_\ell} + \varepsilon_{ijk\ell}$$

with

$$\sum_{i=1}^{4} \theta_i = \sum_{j=1}^{4} \tau_j = \sum_{k=1}^{4} \omega_k = \sum_{\ell=1}^{4} \phi_\ell = 0.$$

The design is balanced, so we have the usual estimates:

$$\widehat{\mu} = \overline{y}_{\bullet\bullet\bullet\bullet}, \quad \widehat{\theta}_i = \overline{y}_{i\bullet\bullet\bullet} - \overline{y}_{\bullet\bullet\bullet\bullet}$$
$$\widehat{\tau}_j = \overline{y}_{\bullet j\bullet\bullet} - \overline{y}_{\bullet\bullet\bullet\bullet}$$
$$\widehat{\omega}_k = \overline{y}_{\bullet\bullet k\bullet} - \overline{y}_{\bullet\bullet\bullet\bullet}$$
$$\widehat{\phi}_\ell = \overline{y}_{\bullet\bullet\bullet\ell} - \overline{y}_{\bullet\bullet\bullet\bullet}$$

Statistical analysis can be done using familiar R commands.

Why Do Graeco-Latin Square Designs Work?

What is the mean for $\overline{y}_{C\bullet\bullet\bullet}$?

For the design of the disk drive example,

$$\overline{y}_{C\bullet\bullet\bullet} = \frac{1}{4}(y_{C123} + y_{C214} + y_{C341} + y_{C432}).$$

Based on the model, we know $\mathbb{E}[y_{ijk\ell}] = \mu + \theta_i + \tau_j + \omega_k + \phi_\ell$.

Thus

$$
\begin{aligned}
\mathbb{E}[\overline{y}_{C\bullet\bullet\bullet}] &= \frac{1}{4}
\begin{pmatrix}
\mu + \theta_C + \tau_1 + \omega_2 + \phi_3 \\
+ \mu + \theta_C + \tau_2 + \omega_1 + \phi_4 \\
+ \mu + \theta_C + \tau_3 + \omega_4 + \phi_1 \\
+ \mu + \theta_C + \tau_4 + \omega_3 + \phi_2
\end{pmatrix} \\
&= \frac{1}{4}\Big(4\mu + 4\theta_C + \underbrace{\sum_{j=1}^{4}\tau_j}_{=0} + \underbrace{\sum_{k=1}^{4}\omega_k}_{=0} + \underbrace{\sum_{\ell=1}^{4}\phi_\ell}_{=0}\Big) \\
&= \mu + \theta_C
\end{aligned}
$$

ANOVA Table for Graeco-Latin Square Designs

Source	d.f.	SS	MS	F-value
Row-Block	$g-1$	SS_{row}	$SS_{row}/(g-1)$	MS_{row}/MSE
Column-Block	$g-1$	SS_{col}	$SS_{col}/(g-1)$	MS_{col}/MSE
Greek-Block	$g-1$	SS_{grk}	$SS_{grk}/(g-1)$	MS_{grk}/MSE
Treatment	$g-1$	SS_{trt}	$SS_{trt}/(g-1)$	MS_{trt}/MSE
Error	$(g-3)(g-1)$	SSE	$SSE/[(g-3)(g-1)]$	
Total	g^2-1	SST		

where $(g-3)(g-1) = g^2 - 1 - 4(g-1)$.

$$SS_{row} = \sum_{ijk\ell}(\overline{y}_{\bullet j \bullet \bullet} - \overline{y}_{\bullet\bullet\bullet\bullet})^2 = g\sum_j(\overline{y}_{\bullet j \bullet \bullet} - \overline{y}_{\bullet\bullet\bullet\bullet})^2,$$

$$SS_{col} = \sum_{ijk\ell}(\overline{y}_{\bullet\bullet k \bullet} - \overline{y}_{\bullet\bullet\bullet\bullet})^2 = g\sum_k(\overline{y}_{\bullet\bullet k \bullet} - \overline{y}_{\bullet\bullet\bullet\bullet})^2,$$

$$SS_{grk} = \sum_{ijk\ell}(\overline{y}_{\bullet\bullet\bullet \ell} - \overline{y}_{\bullet\bullet\bullet\bullet})^2 = g\sum_\ell(\overline{y}_{\bullet\bullet\bullet \ell} - \overline{y}_{\bullet\bullet\bullet\bullet})^2,$$

$$SS_{trt} = \sum_{ijk\ell}(\overline{y}_{i\bullet\bullet\bullet} - \overline{y}_{\bullet\bullet\bullet\bullet})^2 = g\sum_i(\overline{y}_{i\bullet\bullet\bullet} - \overline{y}_{\bullet\bullet\bullet\bullet})^2,$$

$$SST = \sum_{ijk\ell}(y_{ijk} - \overline{y}_{\bullet\bullet\bullet\bullet})^2,$$

$$SSE = SST - SS_{row} - SS_{col} - SS_{grk} - SS_{trt}$$

ANOVA Table For the Disk Drive Example

```
> diskd =
read.table("http://users.stat.umn.edu/~gary/book/fcdae.data/ex13.5",
 header=T)
> diskd$machine = as.factor(diskd$machine)
> diskd$operator = as.factor(diskd$operator)
> diskd$day = as.factor(diskd$day)
> diskd$trt = as.factor(diskd$trt)
> mylm1 = lm(y ~ machine + operator + day + trt, data = diskd)
> anova(mylm1)
          Df Sum Sq Mean Sq F value Pr(>F)
machine    3   21.5  7.1667  1.0000 0.5000
operator   3   14.0  4.6667  0.6512 0.6335
day        3    3.5  1.1667  0.1628 0.9149
trt        3   61.5 20.5000  2.8605 0.2055
Residuals  3   21.5  7.1667
```

Just like RCBD and Latin-square designs, Graeco-Latin square
designs are "balanced", that switching the order of terms will not
change the SS.

When one blocking variable is found unimportant, it can be removed and a Graeco-Latin square design becomes a Latin-square design. The d.f.s and SS for the blocking variable being removed are simply "pooled" into the errors.

After removing the day block variable:

```
> mylm2 = lm(y ~ machine + operator + trt, data = diskd)
> anova(mylm2)
          Df Sum Sq Mean Sq F value  Pr(>F)
machine    3   21.5  7.1667    1.72 0.26166
operator   3   14.0  4.6667    1.12 0.41241
trt        3   61.5 20.5000    4.92 0.04671 *
Residuals  6   25.0  4.1667
```

Similarly, when two blocking variable are found unimportant and are removed, a Graeco-Latin square design becomes a RCBD. Their d.f.s and SS are again "pooled" into the errors.

Pairwise Comparisons in Balanced Block Designs

No matter it's RCBD, Latin-squares (w/ replicates or not), or Graeco-Latin squares, one can always do pairwise comparison of treatment effects $\alpha_{i_1} - \alpha_{i_2}$ by comparing the "treatment means"

$$\overline{y}_{i_1\bullet} - \overline{y}_{i_2\bullet}, \quad \overline{y}_{i_1\bullet\bullet} - \overline{y}_{i_2\bullet\bullet}, \quad \text{or} \quad \overline{y}_{i_1\bullet\bullet\bullet} - \overline{y}_{i_2\bullet\bullet\bullet}$$

and the SE is

$$SE = \sqrt{\text{MSE}\left(\frac{1}{r} + \frac{1}{r}\right)}$$

here r is the total # of replicates for a treatment in all blocks, e.g., $r = mg$ in a $g \times g$ Latin-square design with m replicates. The t-statistic has a t distribution, and d.f. = (d.f. for MSE)

$$t = \frac{\text{diff of 2 trt means}}{SE} \sim t_{\text{df of MSE}}$$

using which can construct C.I. or do t-test on $\alpha_{i_1} - \alpha_{i_2}$.

Contrasts in Balanced Block Designs

Like RCBD, a **contrast** $C = \sum_{i=1}^{g} w_i \alpha_i$ of treatment effects in any Latin-square (w/ replicates or not), or Graeco-Latin square designs can be estimated by

$$\widehat{C} = \sum_{i=1}^{g} w_i(\text{sample mean for trt } i)$$

Here the "sample mean for trt i" means $\overline{y}_{i\bullet}$, $\overline{y}_{i\bullet\bullet}$ or $\overline{y}_{i\bullet\bullet\bullet}$. The SE is

$$\text{SE}(\widehat{C}) = \sqrt{\text{MSE} \times \sum_{i=1}^{g} w_i^2/r}$$

where r is the total # of replicates for a treatment in all blocks, e.g., $r = mg$ in a $g \times g$ Latin-square design with m replicates.

▶ The $(1 - \alpha)100\%$ C.I. for C: $\widehat{C} \pm t_{\alpha/2, \text{df of MSE}} \times \text{SE}(\widehat{C})$.
▶ Test statistic for testing $H_0 : C = \sum_{i=1}^{g} w_i \alpha_i = 0$:

$$t_0 = \frac{\widehat{C}}{\text{SE}(\widehat{C})} \sim t_{\text{df of MSE}}$$

Combination of Factorial and Block Designs

The treatments in any block designs can also have factorial structure. The SS and df for treatments can be broken down according like what we have done in Chapter 8-10.

<u>Example</u> In a 4×4 Latin square design (no replicates), the $g = 4$ treatments has a 2×2 factorial structure.

Source	d.f.
Row	$g - 1 = 3$
Column	$g - 1 = 3$
Treatment	$g - 1 = 3$
Error	$(g - 2)(g - 1) = 6$
Total	$g^2 - 1 = 15$

$\Rightarrow$

Source	d.f.
Row	$g - 1 = 3$
Column	$g - 1 = 3$
A	$a - 1 = 1$
B	$b - 1 = 1$
AB	$(a - 1)(b - 1) = 1$
Error	$(g - 2)(g - 1) = 6$
Total	$g^2 - 1 = 15$

Exercise 13.2

Grains or crystals adversely affect the sensory qualities of foods using dried fruit pulp.

- ▶ Treatment factors:
    - ▶ A, drying temperature (3 levels),
    - ▶ B, acidity (pH) of pulp (2 levels), and
    - ▶ C, sugar content (2 levels).
- ▶ Block: 2 batches of pulp, 12 observations per batach
- ▶ Response: a measure of graininess

This is a $3 \times 2 \times 2$ design with 2 replicates in 2 different blocks, RCBD with 12 treatments and 2 blocks.

Source	d.f.
Block	$2 - 1 = 1$
Treatment	$12 - 1 = 11$
Error	$= 11$
Total	23

$\Rightarrow$

Source	d.f.
Block	$2 - 1 = 1$
A	$3 - 1 = 2$
B	$2 - 1 = 1$
C	$2 - 1 = 1$
AB	$(3 - 1)(2 - 1) = 2$
BC	$(2 - 1)(2 - 1) = 1$
AC	$(3 - 1)(2 - 1) = 2$
ABC	$(3 - 1)(2 - 1)(2 - 1) = 2$
Error	11
Total	23

The SS for each effect (e.g., A main effects or AC interactions) can be computed ignoring the blocking structure.
SSA = sum of (y_{i..}-y_{...})^2