Section 4.1-4.2
Pairwise Comparisons & Contrasts

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- Inference for a Single Mean μ_i in a Multi-Sample Problem
- Pairwise Comparisons
- Contrasts

Last Lecture

One-way ANOVA F-test for the Grass/Weed Competition Study:

$$H_0 : \mu_{1N} = \mu_{1Y} = \mu_{2N} = \mu_{3N} = \mu_{4N} = \mu_{4Y}$$
$$H_a : \mu_{1N}, \mu_{1Y}, \mu_{2N}, \mu_{3N}, \mu_{4N}, \mu_{4Y} \text{ are not all equal}$$

```
> lm1 = lm(percent ~ trt, data=grass)
> anova(lm1)
Analysis of Variance Table

Response: percent
          Df Sum Sq Mean Sq F value    Pr(>F)
trt        5 6398.3 1279.67  71.203 3.197e-11 ***
Residuals 18  323.5   17.97
```

- The tiny P-value means there exists differences among the means. What's the next?
- Want to determine which means are different and identify treatments statistically of the same effect

Section 1

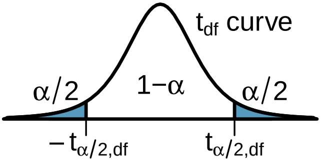
Inference for a Single Group Mean μ_i in a Multi-Sample Problem

Notations for the t-Critical Values

In the remainder of the course, we use $t_{\alpha/2,df}$ to denote the value that

$$P(-t_{\alpha/2,df} < T < t_{\alpha/2,df}) = 1 - \alpha$$

where T has a t-distribution w/ df degrees of freedom



How to find $t_{\alpha/2,df}$ using R and the t-Table?

$t_{0.05/2,3} \approx 3.182$
$t_{0.1/2} \approx 2.101$

```
> qt(0.05/2, df=3, lower.tail=F)
[1] 3.182446
> qt(0.05/2, df=18, lower.tail=F)
[1] 2.100922
```

		$t_{0.1/2,df}$ ↓	$t_{0.05/2,df}$ ↓		$t_{0.01/2,df}$ ↓
$\alpha/2 \rightarrow$ one tail	0.1	0.05	0.025	0.01	0.005
$\alpha \rightarrow$ two tails	0.2	0.10	0.050	0.02	0.010
df 1	3.08	6.31	12.71	31.82	63.66
2	1.89	2.92	4.30	6.96	9.92
3	1.64	2.35	3.18	4.54	5.84
4	1.53	2.13	2.78	3.75	4.60
5	1.48	2.02	2.57	3.36	4.03
6	1.44	1.94	2.45	3.14	3.71
7	1.41	1.89	2.36	3.00	3.50
⋮	⋮	⋮	⋮	⋮	⋮

Confidence Interval (CI) for One-Sample Mean (Review)

If $y_1, y_2, \ldots, y_n$ are i.i.d. $\sim (\mu, \sigma^2)$,

$$\text{by CLT} \Rightarrow Z = \frac{\overline{y} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1).$$

- valid for all n if y_i's are normal
- approx. valid for large n if y_i's are not normal

However, σ is unknown. We estimate it with $s = \sqrt{\frac{\sum_i (y_i - \overline{y})^2}{n-1}}$

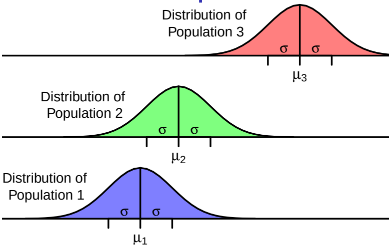
$$t = \frac{\overline{y} - \mu}{s/\sqrt{n}} \sim t_{n-1}$$

- valid for all n if y_i's are normal
- approx. valid for large n if y_i's are not normal

Inverting $P(-t_{\alpha/2,n-1} < t = \frac{\overline{y}-\mu}{s/\sqrt{n}} < t_{\alpha/2,n-1}) = 1 - \alpha$, we get the $(1 - \alpha)100\%$ CI for μ:

$$\overline{y} \pm t_{\alpha/2,n-1} \times \frac{s}{\sqrt{n}}$$

A Naive CI for a Group Mean in a Multi-Sample Problem



Model for the multi-sample problem:

$$y_{ij} = \mu_i + \varepsilon_{ij}, \quad \varepsilon_{ij} \sim N(0, \sigma^2)$$

$$\Rightarrow \quad Z = \frac{\overline{y}_{k\bullet} - \mu_k}{\sigma/\sqrt{n_k}} \sim N(0,1)$$

A naive estimate for the unknown σ is

$$s_k = \text{the sample SD of the } k\text{th group} = \sqrt{\frac{\sum_{j=1}^{n_k}(y_{kj} - \overline{y}_{k\bullet})^2}{n_k - 1}}.$$

From that

$$t = \frac{\overline{y}_{i\bullet} - \mu_k}{s_k/\sqrt{n_k}} \sim t_{n_k-1},$$

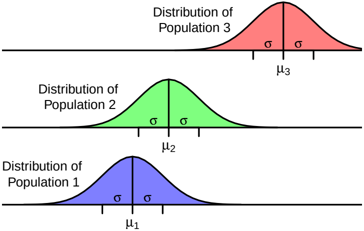
a naive but valid $100(1-\alpha)\%$ CI for μ_k would be

$$\overline{y}_{k\bullet} \pm t_{\alpha/2, n_k-1} \times \frac{s_k}{\sqrt{n_k}}.$$

- using only data in the kth group, ignoring the rest, not optimal

A Better CI for a Group Mean in a Multi-Sample Problem

As all the groups have a **common SD** σ, data in other groups cannot help estimating μ_k but they can help estimating σ. A better estimate for σ is



$$\widehat{\sigma} = \sqrt{\text{MSE}} = \sqrt{\frac{\sum_{i=1}^{g}\sum_{j=1}^{n_i}(y_{ij} - \overline{y}_{i\bullet})^2}{N-g}}$$

We have

$$t = \frac{\overline{y}_{k\bullet} - \mu_k}{\widehat{\sigma}/\sqrt{n_k}} = \frac{\overline{y}_{i\bullet} - \mu_k}{\sqrt{\text{MSE}}/\sqrt{n_k}} \sim t_{N-g},$$

from which, a better $100(1-\alpha)\%$ CI for μ_k is

$$\overline{y}_{k\bullet} \pm t_{\alpha/2,N-g}\frac{\sqrt{\text{MSE}}}{\sqrt{n_k}}$$

- using observations in all groups to estimate the unknown σ
- with a higher df $= N - g$, not $n_k - 1$

Case Study: Grass/Weed Competition

Treatment	1N	1Y	2N	3N	4N	4Y
Mean $\bar{y}_{i\bullet}$	95	82.25	81.5	68.25	50.5	52
SD s_i	2.16	3.775	3.512	5.56	5.00	4.546

, MSE $= 17.97$

The naive 95% CI for μ_{4Y} using only data in Group 4Y:

$$\bar{y}_{4Y\bullet} \pm t_{\alpha/2, n_{4Y}-1} \frac{s_{4Y}}{\sqrt{n_{4Y}}} \approx 52 \pm 3.182 \times \frac{4.546}{\sqrt{4}} \approx 52 \pm 7.23.$$

The better 95% CI for μ_{4Y} using the MSE is

$$\bar{y}_{i\bullet} \pm t_{\alpha/2, N-g} \frac{\sqrt{\text{MSE}}}{\sqrt{n_{4Y}}} = 52 \pm 2.101 \times \frac{\sqrt{17.97}}{\sqrt{4}} \approx 52 \pm 4.45$$

where $n_{4Y} = 4$, $N = 24$, $g = 6$, $\alpha = 0.05$. Using R, we can find $t_{\alpha/2, n_{4Y}-1} = t_{0.05/2, 4-1} \approx 3.182$ and $t_{\alpha/2, N-g} = t_{0.05/2, 24-6} \approx 2.101$.

```
> qt(0.05/2, df = 4-1, lower.tail=F)
[1] 3.182446
> qt(0.05/2, df = 24-6, lower.tail=F)
[1] 2.100922
```

Observe the naive CI has a bigger margin of error 7.23 than the margin of error 4.45 for the CI using the MSE.

Interpretation of the better 95% CI for μ_{4Y}: 52 ± 4.45

For plots received 800 mg N/kg soil and 1 cm of irrigation per week, we estimate that 52.0% of living material is bluestem (grass) on average with a margin of error of 4.45% at 95% confidence.

Section 2

Pairwise Comparison

Pairwise Comparison of Group Means

Model for the multi-sample problem:

$$y_{ij} = \mu_i + \varepsilon_{ij}, \quad \varepsilon_{ij} \sim N(0, \sigma^2)$$

Consider the pairwise comparison of group means $\mu_k - \mu_\ell$:

- the estimator is $\overline{y}_{k\bullet} - \overline{y}_{\ell\bullet}$
- Since $\overline{y}_{k\bullet}$ and $\overline{y}_{\ell\bullet}$ are independent, we have

$$\mathrm{Var}(\overline{y}_{k\bullet} - \overline{y}_{\ell\bullet}) = \mathrm{Var}(\overline{y}_{k\bullet}) + \mathrm{Var}(\overline{y}_{\ell\bullet}) = \frac{\sigma^2}{n_k} + \frac{\sigma^2}{n_\ell}$$

- $\mathrm{SD}(\overline{y}_{k\bullet} - \overline{y}_{\ell\bullet}) = \sqrt{\mathrm{Var}(\overline{y}_{k\bullet} - \overline{y}_{\ell\bullet})} = \sqrt{\sigma^2 \left(\frac{1}{n_k} + \frac{1}{n_\ell}\right)}$,
- $\mathrm{SE}(\overline{y}_{k\bullet} - \overline{y}_{\ell\bullet}) = \widehat{\mathrm{SD}}(\overline{y}_{k\bullet} - \overline{y}_{\ell\bullet}) = \sqrt{\mathrm{MSE} \left(\frac{1}{n_k} + \frac{1}{n_\ell}\right)}$.
- $t = \dfrac{\overline{y}_{k\bullet} - \overline{y}_{\ell\bullet} - (\mu_k - \mu_\ell)}{\mathrm{SE}(\overline{y}_{k\bullet} - \overline{y}_{\ell\bullet})} = \dfrac{\overline{y}_{k\bullet} - \overline{y}_{\ell\bullet} - (\mu_k - \mu_\ell)}{\sqrt{\mathrm{MSE}\left(\frac{1}{n_k} + \frac{1}{n_\ell}\right)}} \sim t_{N-g}$

Confidence Intervals for Pairwise Differences

The $100(1-\alpha)\%$ confidence interval (C.I.) for $\mu_k - \mu_\ell$ is

$$\overline{y}_{k\bullet} - \overline{y}_{\ell\bullet} \pm t_{\alpha/2, N-g} \sqrt{\text{MSE}\left(\frac{1}{n_k} + \frac{1}{n_\ell}\right)}.$$

Note this is neither the two-sample CI assuming equal SDs

$$\overline{y}_{k\bullet} - \overline{y}_{\ell\bullet} \pm t_{\alpha/2, n_k+n_\ell-2} \sqrt{s_p^2\left(\frac{1}{n_k} + \frac{1}{n_\ell}\right)}, \text{ where } s_p^2 = \frac{(n_k-1)s_k^2 + (n_\ell-1)s_\ell^2}{n_k + n_\ell - 2},$$

nor the two-sample CI not assuming equal SDs

$$\overline{y}_{k\bullet} - \overline{y}_{\ell\bullet} \pm t_{\alpha/2, df} \sqrt{\frac{s_k^2}{n_k} + \frac{s_\ell^2}{n_\ell}}, \quad \text{where } df = \frac{\left(\frac{s_k^2}{n_k} + \frac{s_\ell^2}{n_\ell}\right)^2}{\frac{1}{n_k-1}\left(\frac{s_k^2}{n_k}\right)^2 + \frac{1}{n_\ell-1}\left(\frac{s_\ell^2}{n_\ell}\right)^2}$$

- $\text{MSE} = \frac{\sum_{i=1}^{g}\sum_{j=1}^{n_i}(y_{ij} - \overline{y}_{i\bullet})^2}{N-g}$ calculated using the entire dataset is a more accurate estimator of σ^2 than s_p^2 or s_k^2, s_ℓ^2 calculated using only data in the two groups compared
- The critical value for the two-sample C.I. is larger

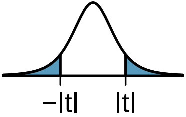
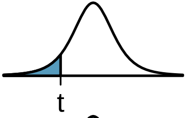
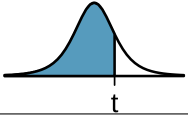
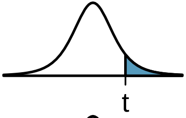
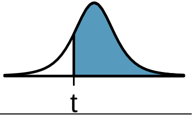
$$t_{\alpha/2, n_k+n_\ell-2} > t_{\alpha/2, N-g}$$

Hypothesis Testing for Difference

For testing the hypothesis H_0: $\mu_k - \mu_\ell = 0$, the test statistic is

$$t = \frac{\overline{y}_{k\bullet} - \overline{y}_{\ell\bullet}}{\mathrm{SE}(\overline{y}_{k\bullet} - \overline{y}_{\ell\bullet})} = \frac{\overline{y}_{k\bullet} - \overline{y}_{\ell\bullet}}{\sqrt{\mathrm{MSE}\left(\frac{1}{n_k} + \frac{1}{n_\ell}\right)}} \sim t_{N-g}$$

The calculation of the p-value depends on H_a as follows

H_a	$\mu_k - \mu_\ell \neq 0$	$\mu_k - \mu_\ell < 0$	$\mu_k - \mu_\ell > 0$
p-value		 	 

The bell curve above is the t-curve with $\mathrm{df} = N - g$.

Case Study: Grass/Weed Competition

Group	1N	1Y	2N	3N	4N	4Y
Mean $\overline{y}_{i\bullet}$	95	82.25	81.5	68.25	50.5	52
SD s_i	2.16	3.775	3.512	5.56	5.00	4.546

, MSE $= 17.97$

A 95% confidence interval for $\mu_{1\text{N}} - \mu_{1\text{Y}}$ is

$$\overline{y}_{1N\bullet} - \overline{y}_{1Y\bullet} \pm t_{0.025,18} \times \sqrt{\text{MSE}\left(\frac{1}{n_{1\text{N}}} + \frac{1}{n_{1\text{Y}}}\right)}$$

$$= 95 - 82.25 \pm 2.101 \times \sqrt{17.97\left(\frac{1}{4} + \frac{1}{4}\right)} = 12.75 \pm 6.65$$

in which $t_{0.025,18} = 2.101$ is found using the R command

```
> qt(0.05/2, df = 18, lower.tail=F)
[1] 2.100922
```

Irrigation reduced the percentage of grass (bluestem) by 12.75% on average, with a margin of error of 6.65%, at 95% confidence.

Case Study: Grass/Weed Competition

To test whether treatment 1N and treatment 1Y have the same effect

$$H_0 : \mu_{1N} - \mu_{1Y} = 0 \quad \text{v.s.} \quad H_a : \mu_{1N} - \mu_{1Y} \neq 0$$

the test statistic is

$$t = \frac{\overline{y}_{1N\bullet} - \overline{y}_{1Y\bullet}}{\sqrt{\text{MSE}\left(\frac{1}{n_{1N}} + \frac{1}{n_{1Y}}\right)}} = \frac{95 - 82.25}{\sqrt{17.97\left(\frac{1}{4} + \frac{1}{4}\right)}} \approx \frac{12.75}{2.9975} \approx 4.253$$

with $\text{df} = N - g = 24 - 6 = 18$. The two-sided P-value is

```
> 2*pt(4.235, df = 18, lower.tail=F)
[1] 0.0004979698
```

As the P-value < 0.05, we again confirm that irrigation made grass (bluestem) less competitive.

Pairwise *t*-Tests in R

The R command `pairwise.t.test` can perform pairwise comparisons between all pairs of treatments, but it shows the *P*-values only.

```
> pairwise.t.test(grass$percent, grass$trt, p.adjust="none")

Pairwise comparisons using t tests with pooled SD
data:  grass$percent and grass$trt
   1N      1Y      2N      3N      4N
1Y 0.00048 -       -       -       -
2N 0.00027 0.80527 -       -       -
3N 5.0e-08 0.00019 0.00033 -       -
4N 1.5e-11 3.7e-09 5.3e-09 1.3e-05 -
4Y 2.7e-11 7.8e-09 1.1e-08 3.8e-05 0.62287

P value adjustment method: none
```

Note that we must include `p.adjust="none"` in the command. Otherwise the *P*-value is not calculated using *t*-tests.

Underline Diagrams (p.88, Section 5.4.1)

a concise way to summarize pairwise comparisons

```
   1N      1Y      2N      3N      4N
1Y 0.00048 -       -       -       -
2N 0.00027 0.80527 -       -       -
3N 5.0e-08 0.00019 0.00033 -       -
4N 1.5e-11 3.7e-09 5.3e-09 1.3e-05 -
4Y 2.7e-11 7.8e-09 1.1e-08 3.8e-05 0.62287
```

How to make a **underline diagram**?

1. Write out group labels horizontally in **increasing order sorted by group means**
2. (Write the group mean $\overline{y}_{i\bullet}$ under each corresponding group) (may skip)
3. Draw a line segment under a set of groups if no two groups in that set of groups are significantly different from each other

4Y	4N	3N	2N	1Y	1N
50.5	52	68.25	81.5	82.25	95

Underline Diagrams

In an experiment with 5 treatments A, B, C, D and E, the underline diagram for all pairwise comparisons of the 5 treatments is as follows.

Answer the following questions:

- Order the means of the 5 groups from low to high.

  $$C < B < A < D < E$$

- Check all the pairs that are significantly different from each other.

	C	B	A	D
B	v			
A	v			
D	v			
E	v	v	v	

Least Significant Difference (LSD)

- It's an awful lot of work to to compare every pair of groups. One needs to compute the SE, the t-statistic, and P-value for each pair of groups. When there are g groups, there are $\binom{g}{2} = g(g-1)/2$ pairs to compare with.
- When all groups are of the same size n, an easier way to do pairwise comparisons of all treatments is to compute the **least significant difference** (LSD), which is the minimum amount by which two means must differ in order to be considered statistically different.

Least Significant Difference (LSD)

- When all groups are of the same size n, the SEs of pairwise comparisons all equal to
$$\text{SE} = \sqrt{\text{MSE}\left(\frac{1}{n} + \frac{1}{n}\right)}$$
- To be significant at level α, the t-statistic for pairwise comparison
$$t = \frac{\overline{y}_{k\bullet} - \overline{y}_{\ell\bullet}}{\text{SE}}$$
must be at least $t_{\alpha/2,N-g}$ in absolute value
- So μ_k and μ_ℓ are significantly different at level α if and only if $\overline{y}_{k\bullet} - \overline{y}_{\ell\bullet}$ is at least
$$t_{\alpha/2,N-g}\sqrt{\text{MSE}\left(\frac{1}{n} + \frac{1}{n}\right)} = \text{LSD}$$
in absolute value, which is called the **least significant difference (LSD)**

Least Significant Difference (LSD)

For the Grass/Weed experiment, the critical value at $\alpha = 5\%$ significance is $t_{\alpha/2,N-g} = t_{0.025,24-6} \approx 2.101$, the LSD at 5% level is

$$\text{LSD} = t_{\alpha/2,N-g}\sqrt{\text{MSE}\left(\frac{1}{n}+\frac{1}{n}\right)} = 2.101\sqrt{17.97\left(\frac{1}{4}+\frac{1}{4}\right)} \approx 6.30$$

Two treatments are significantly different at 5% level if and only if their mean differ by 6.30 or more.

The only two pairs with no significantly difference are (4Y, 4N) and (2N, 1Y), as they are the only pairs differ less than 6.30 in mean.

4Y	4N	3N	2N	1Y	1N
50.5	52	68.25	81.5	82.25	95

Section 3

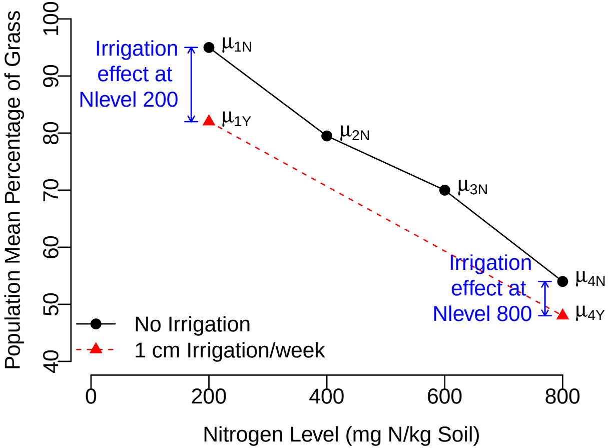
Contrasts

Quantities of Interest Other Than Pairwise Differences (1)

For the Grass/Weed experiment, we are also interested in

Q1 Irrigation effect: $\mu_{1N} - \mu_{1Y}$ or $\mu_{4N} - \mu_{4Y}$ or the combination

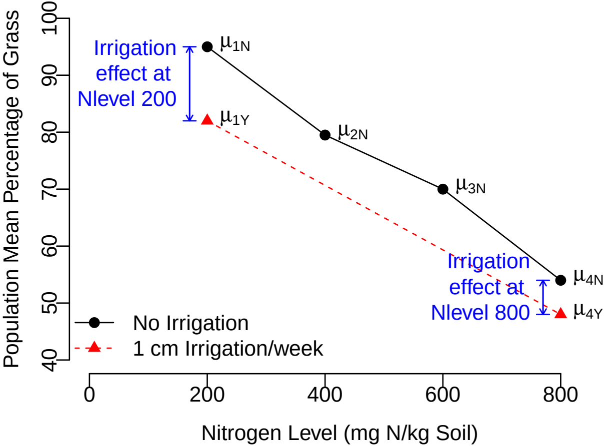
$$\frac{\mu_{1N} + \mu_{4N}}{2} - \frac{\mu_{1Y} + \mu_{4Y}}{2}$$



Quantities of Interest Other Than Pairwise Differences (2)

Q2 Does the irrigation effect change with nitrogen levels?

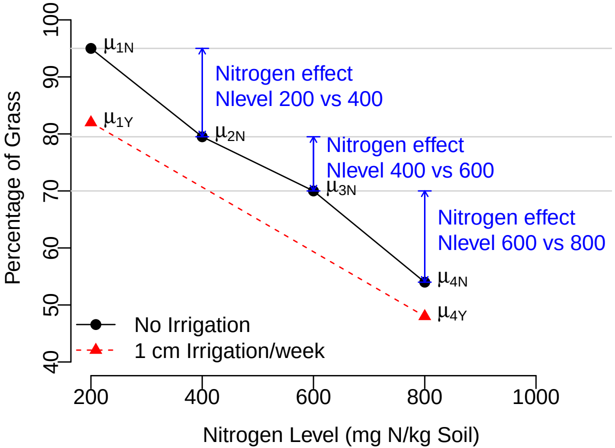
$$\underbrace{(\mu_{1N} - \mu_{1Y})}_{\substack{\text{irrigation effect at} \\ \text{nitrogen level 200}}} - \underbrace{(\mu_{4N} - \mu_{4Y})}_{\substack{\text{irrigation effect at} \\ \text{nitrogen level 800}}}$$



Quantities of Interest Other Than Pairwise Differences (3)

For the Grass/Weed experiment, we are also interested in

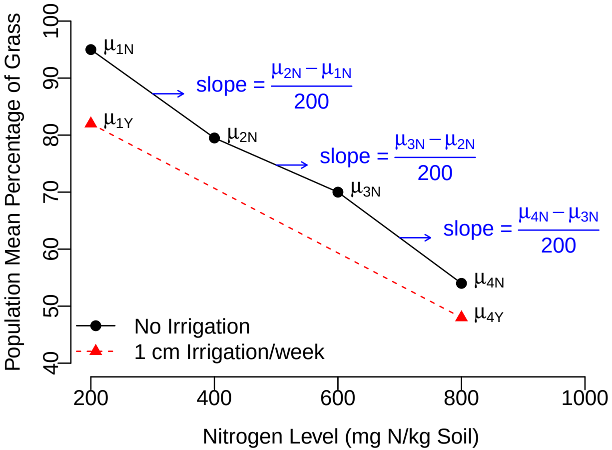
Q3 Nitrogen effect: $\mu_{1N} - \mu_{2N}$, $\mu_{2N} - \mu_{3N}$, etc.



Quantities of Interest Other Than Pairwise Differences (4)

Q4 Is the nitrogen effect **linear**?

$$\frac{\mu_{1N} - \mu_{2N}}{200} - \frac{\mu_{2N} - \mu_{3N}}{200}, \quad \text{or} \quad \frac{\mu_{2N} - \mu_{3N}}{200} - \frac{\mu_{3N} - \mu_{4N}}{200}, \quad \text{etc.}$$



Definition of Contrasts

All the quantities above are **contrasts**.

A **contrast** is a linear combination of group means μ_i's

$$C = \sum_{i=1}^{g} \omega_i \mu_i$$

where the ω_i's are known coefficients that add up to 0, $\sum_{i=1}^{g} \omega_i = 0$.

Ex. Irrigation Effect Contrast:

$$\begin{aligned} C &= \frac{\mu_{1\text{N}} + \mu_{4\text{N}}}{2} - \frac{\mu_{1\text{Y}} + \mu_{4\text{Y}}}{2} \\ &= \underset{\omega_{1N}}{\underbrace{0.5}}\, \mu_{1\text{N}} + \underset{\omega_{4N}}{\underbrace{0.5}}\, \mu_{4\text{N}} + \underset{\omega_{1Y}}{\underbrace{(-0.5)}}\mu_{1\text{Y}} + \underset{\omega_{4Y}}{\underbrace{(-0.5)}}\mu_{4\text{Y}} + \underset{\omega_{2N}}{\underbrace{0}}\ \mu_{2\text{N}} + \underset{\omega_{3N}}{\underbrace{0}}\ \mu_{3\text{N}} \end{aligned}$$

Observe that $\omega_{1N} + \omega_{4N} + \omega_{1Y} + \omega_{4Y} + \omega_{2N} + \omega_{3N}$

$$= 0.5 + 0.5 + (-0.5) + (-0.5) + 0 + 0 = 0$$

Example of Contrasts

Q2 Does the irrigation effect change with nitrogen levels?

$$
\begin{aligned}
C &= (\mu_{1\mathrm{N}} - \mu_{1\mathrm{Y}}) - (\mu_{4\mathrm{N}} - \mu_{4\mathrm{Y}}) \\
&= \underset{\omega_{1N}}{\underset{\downarrow}{1}}\ \mu_{1\mathrm{N}} + \underset{\omega_{1Y}}{\underset{\downarrow}{(-1)}}\mu_{1\mathrm{Y}} + \underset{\omega_{4N}}{\underset{\downarrow}{(-1)}}\mu_{4\mathrm{N}} + \underset{\omega_{4Y}}{\underset{\downarrow}{1}}\ \mu_{4\mathrm{Y}} + \underset{\omega_{2N}}{\underset{\downarrow}{0}}\ \mu_{2\mathrm{N}} + \underset{\omega_{3N}}{\underset{\downarrow}{0}}\ \mu_{3\mathrm{N}}
\end{aligned}
$$

Observe that $\sum_i \omega_i = 1 + (-1) + (-1) + 1 + 0 + 0 = 0.$

Q4 Is the nitrogen effect **linear**?

$$
\begin{aligned}
C &= \frac{\mu_{1\mathrm{N}} - \mu_{2\mathrm{N}}}{200} - \frac{\mu_{2\mathrm{N}} - \mu_{3\mathrm{N}}}{200} \\
&= \underset{\omega_{1N}}{\underset{\downarrow}{\tfrac{1}{200}}}\ \mu_{1\mathrm{N}} + \underset{\omega_{2N}}{\underset{\downarrow}{(\tfrac{-2}{200})}}\mu_{2\mathrm{N}} + \underset{\omega_{3N}}{\underset{\downarrow}{\tfrac{1}{200}}}\ \mu_{3\mathrm{N}} + \underset{\omega_{4N}}{\underset{\downarrow}{0}}\ \mu_{4\mathrm{N}} + \underset{\omega_{1Y}}{\underset{\downarrow}{0}}\ \mu_{1\mathrm{Y}} + \underset{\omega_{4Y}}{\underset{\downarrow}{0}}\ \mu_{4\mathrm{Y}}
\end{aligned}
$$

Observe that $\sum_i \omega_i = \frac{1}{200} + (\frac{-2}{200}) + \frac{1}{200} + 0 + 0 + 0 = 0.$

More Examples of Contrasts

- **Every pairwise comparison is a contrast!** $(C = \mu_k - \mu_\ell)$
  $\omega_k = 1$, $\omega_\ell = -1$, all other ω_i's are 0, and
  $\sum_{i=1}^{g} \omega_i = 1 + (-1) + 0 + \cdots + 0 = 0$

- **A single treatment mean** $C = \mu_k$ **is NOT a contrast**

- Is $C = \dfrac{\mu_1 + \mu_2}{2} - \dfrac{\mu_3 + \mu_4 + \mu_5}{3}$ a contrast? Yes.
  $\omega_1 = \omega_2 = \frac{1}{2}$, $\omega_3 = \omega_4 = \omega_5 = -\frac{1}{3}$, which add up to 0.

- Is $C = \dfrac{\mu_1 + \mu_2}{2} - \dfrac{\mu_3 + \mu_4}{2} + \mu_5$ a contrast? No.

$\omega_1 = \omega_2 = \frac{1}{2}$, $\omega_3 = \omega_4 = -\frac{1}{2}$, $\omega_5 = 1$, which add up to 1, not 0.

Estimator and Confidence Interval for a Contrast

A natural estimator for a contrast $C = \sum_{i=1}^{g} \omega_i \mu_i$ is

$$\widehat{C} = \sum_{i=1}^{g} \omega_i \overline{y}_{i\bullet}$$

As $\overline{y}_{1\bullet}, \overline{y}_{2\bullet}, \ldots,$ and $\overline{y}_{g\bullet}$ are indep. of each other, we know

$$\mathrm{Var}\Big(\sum_{i=1}^{g} \omega_i \overline{y}_{i\bullet}\Big) = \sum_{i=1}^{g} \mathrm{Var}(\omega_i \overline{y}_{i\bullet}) = \sum_{i=1}^{g} \omega_i^2 \mathrm{Var}(\overline{y}_{i\bullet}) = \sum_{i=1}^{g} \omega_i^2 \frac{\sigma^2}{n_i}.$$

The SD and SE of the estimator $\widehat{C}$:

$$\mathrm{SD}(\widehat{C}) = \sqrt{\sigma^2 \sum_{i=1}^{g} \frac{\omega_i^2}{n_i}}, \qquad \mathrm{SE}(\widehat{C}) = \sqrt{\mathrm{MSE} \times \sum_{i=1}^{g} \frac{\omega_i^2}{n_i}}$$

A $(1-\alpha)100\%$ confidence interval for the contrast C is

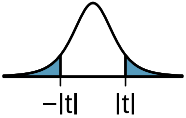
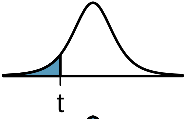
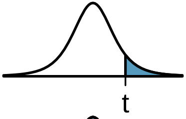
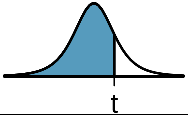
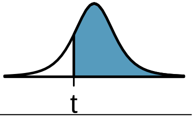
$$\widehat{C} \pm t_{\alpha/2, N-g} \times \mathrm{SE}(\widehat{C})$$

Hypothesis Testing for a Contrast

To test whether a contrast C is 0, $\text{H}_0 : C = 0$, the test statistic is

$$t = \frac{\widehat{C}}{\text{SE}(\widehat{C})} = \frac{\sum_{i=1}^{g} \omega_i \overline{y}_{i\bullet}}{\sqrt{\text{MSE} \times \sum_{i=1}^{g} \frac{\omega_i^2}{n_i}}} \sim t_{N-g}$$

The calculation of the p-value depends on H_a as follows

H_a	$C \neq 0$	$C < 0$	$C > 0$
p-value			
			

The bell curve above is the t-curve with $\text{df} = N - g$.

Does the Irrigation Effect Change with Nitrogen Levels?

Group	1N	1Y	2N	3N	4N	4Y
$\bar{y}_{i\bullet}$	95	82.25	81.5	68.25	50.5	52

, MSE = 17.97

The contrast we consider is

$$C = \underbrace{(\mu_{1\text{N}} - \mu_{1\text{Y}})}_{\substack{\text{irrigation effect at} \\ \text{nitro level} = 200}} - \underbrace{(\mu_{4\text{N}} - \mu_{4\text{Y}})}_{\substack{\text{irrigation effect at} \\ \text{nitro level} = 800}}$$

in which $(\omega_{1\text{N}}, \omega_{1\text{Y}}, \omega_{2\text{N}}, \omega_{3\text{N}}, \omega_{4N}, \omega_{4\text{Y}}) = (1, -1, 0, 0, -1, 1)$.

The contrast is estimated by

$$\widehat{C} = \bar{y}_{1N\bullet} - \bar{y}_{1Y\bullet} - (\bar{y}_{4N\bullet} - \bar{y}_{4Y\bullet}) = 95 - 82.25 - (50.5 - 52) = 14.25.$$

with the standard error

$$\text{SE}(\widehat{C}) = \sqrt{\text{MSE}\sum_{i=1}^{g}\frac{\omega_i^2}{n_i}} = \sqrt{17.97\left(\frac{1^2}{4} + \frac{(-1)^2}{4} + \frac{(-1)^2}{4} + \frac{1^2}{4}\right)} \approx 4.24$$

Does the Irrigation Effect Change with Nitrogen Levels?

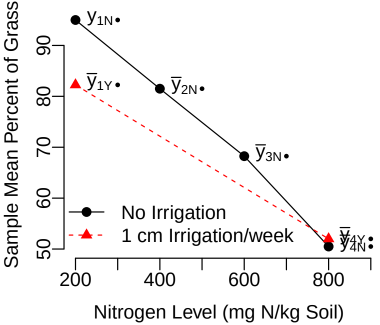
To test whether the irrigation effect changes with nitrogen level H_0: $C = 0$ v.s. H_a: $C \neq 0$, the t-statistic is

$$t = \frac{\widehat{C}}{SE(\widehat{C})} = \frac{14.25}{4.24} \approx 3.36$$

with df $= N - g = 24 - 6 = 18$.
The two-sided p-value is

```
> 2*pt(3.36,df=18, lower.tail=F)
[1] 0.003486951
```

The small P-value indicates the irrigation effects are significantly different at the nitrogen level 200 and 800 mg N/kg soil.



Does the Irrigation Effect Change with Nitrogen Levels?

The 95% confidence interval for $C = (\mu_{1N} - \mu_{1Y}) - (\mu_{4N} - \mu_{4Y})$ is

$$\widehat{C} \pm t_{0.025,N-g} \times \text{SE}(\widehat{C}) \approx 14.25 \pm 2.101 \times 4.24 \approx (5.34, 23.16)$$

in which $t_{0.025,24-6} \approx 2.101$ is found by the R command

```
> qt(0.025,df=18, lower.tail=F)
[1] 2.100922
```

This means that the irrigation effect (% of grass w/ irrigation − w/o irrigation) is on average 5.34% to 23.16% higher at nitrogen level 200 than at level 800 mg N/kg soil, with 95% confidence.

Is the Nitrogen Effect Linear?

Treatment	1N	1Y	2N	3N	4N	4Y
Mean $\overline{y}_{i\bullet}$	95	82.25	81.5	68.25	50.5	52

, MSE = 17.97

The contrast we consider is

$$C = \frac{\mu_{1N} - \mu_{2N}}{200} - \frac{\mu_{2N} - \mu_{3N}}{200} = \frac{\mu_{1N} - 2\mu_{2N} + \mu_{3N}}{200}$$

with the coefficients $(\omega_{1N}, \omega_{2N}, \omega_{3N}) = (\frac{1}{200}, \frac{-2}{200}, \frac{1}{200})$.

The contrast is estimated by

$$\widehat{C} = \frac{\overline{y}_{1N\bullet} - 2\overline{y}_{2N\bullet} + \overline{y}_{3N\bullet}}{200} = \frac{95 - 2 \times 81.5 + 68.25}{200} = \frac{0.25}{200} = 0.00125.$$

with

$$SE(\widehat{C}) = \sqrt{MSE \sum_{i=1}^{g} \frac{\omega_i^2}{n_i}} = \sqrt{17.97\left(\frac{(1/200)^2}{4} + \frac{(\frac{-2}{200})^2}{4} + \frac{(1/200)^2}{4}\right)} \approx 0.026$$

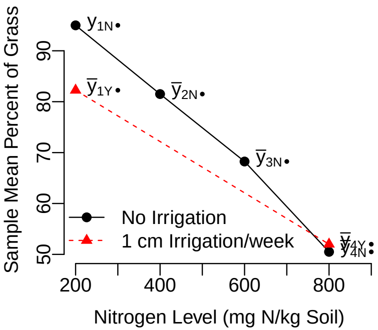
Is the Nitrogen Effect Linear?

To test whether the nitrogen effect is linear, the t-statistic is

$$t = \frac{\widehat{C}}{\mathrm{SE}(\widehat{C})} = \frac{0.00125}{0.026} \approx 0.048$$

with df $= N - g = 24 - 6 = 18$.
The two-sided p-value is

```
> 2*pt(0.048,df=18, lower.tail=F)
[1] 0.9622448
```



Conclusion: The huge P-value gives little evidence of nonlinearity (at nitrogen level 1,2, and 3).

Remark: One can also test the linearity at level 2, 3, and 4

$$C = \frac{\mu_{2N} - \mu_{3N}}{200} - \frac{\mu_{3N} - \mu_{4N}}{200} = \frac{\mu_{2N} - 2\mu_{3N} + \mu_{4N}}{200}.$$

which is left as an exercise.