STAT 220 Lecture Slides
Inference for Linear Regression

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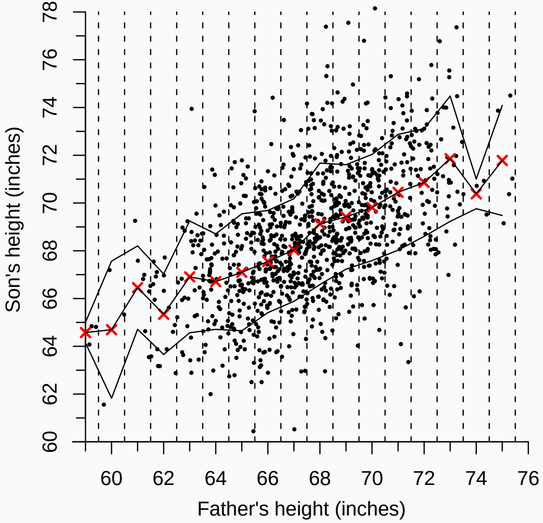
Simple Linear Regression Models

Example: Pearson's Father-and-Son Data

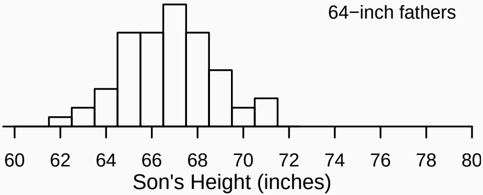
Father-son pairs are grouped by father's height, to the nearest inch.

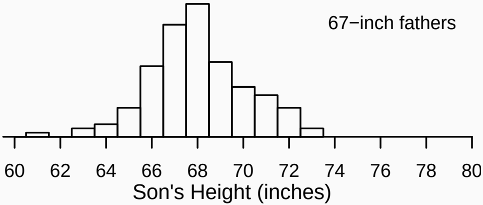


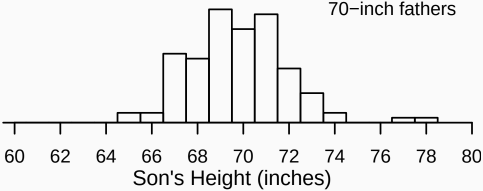
How do the

- mean of son's height (SH),
- SD of SH, and
- distribution of SH (histogram of SH)?

within each group change with father's height (FH)?


64−inch fathers
60
62
64
66
68
70
72
74
76
78
80
Son's Height (inches)


67−inch fathers
60
62
64
66
68
70
72
74
76
78
80
Son's Height (inches)


70−inch fathers
60
62
64
66
68
70
72
74
76
78
80
Son's Height (inches)

Simple Linear Regression Model

Pearson's father-and-son data inspire the following assumptions for the simple linear regression (SLR) model:

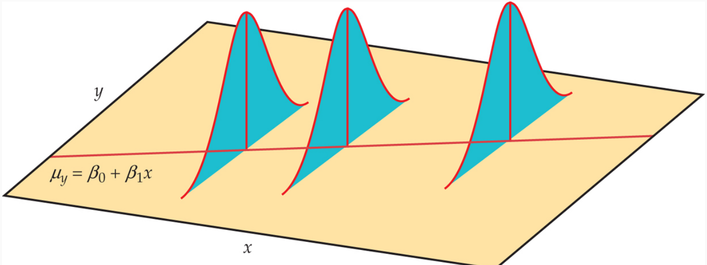
1. The means of Y is a linear function of X, i.e.,

$$E(Y\,|\,X=x)=\beta_0+\beta_1 x$$

2. The SD of Y does not change with x, i.e.,

$$SD(Y\,|\,X=x)=\sigma \quad \text{for every } x$$

3. (Optional) Within each subpopulation, the distribution of Y is normal.



Simple Linear Regression Model

Equivalently, the SLR model asserts the values of X and Y for individuals in a population are related as follows

$$Y = \beta_0 + \beta_1 X + \varepsilon,$$

- the value of ε, called the **error** or the **noise**, varies from observation to observation, follows a normal distribution

$$\varepsilon \sim N(0, \sigma)$$

In the model, the line $y = \beta_0 + \beta_1 x$ is called the **population regression line**.

Inference for Simple Linear Regression Models

Data for a Simple Linear Regression Model

Suppose we have a SRS of n individuals from a population.

From individual i we observe the response y_i and the explanatory variable x_i:

$$(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)$$

The SLR model states that

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

Recall in the previous lecture, the least square line of the data above is

$$y = b_0 + b_1 x$$

in which

$$b_1 = r\frac{s_y}{s_x} = \frac{\sum_i (x_i - \overline{x})(y_i - \overline{y})}{\sum_i (x_i - \overline{x})^2}, \quad b_0 = \overline{y} - b_1 \overline{x}$$

We can use b_1 to estimate β_1 and b_0 to estimate β_0.

Caution: Sample v.s. Population

Note the population regression line

$$y = \beta_0 + \beta_1 x$$

is <u>different</u> from the least square regression line

$$y = b_0 + b_1 x$$

we learned in the previous lecture.

- The latter is merely the least square line for a sample, while the former is the least square line for the entire population.
- The values of b_0 and b_1 will change from sample to sample.

$$b_1 = r\frac{s_y}{s_x} = \frac{\sum_i (x_i - \overline{x})(y_i - \overline{y})}{\sum_i (x_i - \overline{x})^2}, \quad b_0 = \overline{y} - b_1 \overline{x}$$

- We are interested in the population intercept β_0 and slope β_1, NOT the sample counterparts b_0 and b_1.

How Close Is b_1 to β_1?

Recall the slope of the least square line is

$$b_1 = \frac{\sum_i (x_i - \overline{x})(y_i - \overline{y})}{\sum_i (x_i - \overline{x})^2}$$

Under the SLR model: $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$, replacing y_i in the formula above by $\beta_0 + \beta_1 x_i + \varepsilon_i$, we can show after some algebra that

$$b_1 = \beta_1 + \frac{\sum_i (x_i - \overline{x})\varepsilon_i}{\sum_i (x_i - \overline{x})^2}$$

From the above, one can get the mean, the SD, and the **sampling distribution** of b_1.

- $E(b_1) = \beta_1$ . . . . (b_1 is an **unbiased** estimate of β_1)
- $\text{SD}(b_1) = ?$ . . . . (See the next slide)

Variability of b_1

One can show that

$$SD(b_1) = \frac{\sigma}{\sqrt{\sum(x_i - \overline{x})^2}} = \frac{\sigma}{s_x\sqrt{n-1}},$$

where $s_x = \sqrt{\frac{\sum(x_i-\overline{x})^2}{n-1}}$ is the sample SD of x_i's.

How to reduce the SD of b_1 (and make b_1 closer to β_1):

- increase the sample size n
- increase the range of x_i's (and hence s_x is increased)

But σ is unknown, we need to estimate it.

Estimate of σ

We want to estimate σ, SD of the error ε_i.

- An intuitive estimate of σ is the sample SD of the *errors ε_i*

$$\widehat{\sigma} = \sqrt{\frac{\sum(\varepsilon_i - \overline{\varepsilon})^2}{n-1}} \quad \text{where} \quad \varepsilon_i = y_i - \beta_0 - \beta_1 x_i$$

However, this is not possible β_0 and β_1 are unknown.

- We can estimate β_0 and β_1 with b_0 and b_1 and approximate the errors ε_i with the **residuals**

$$e_i = y_i - (b_0 + b_1 x_i) = y_i - \widehat{y_i}$$

We use the "sample SD" of the residuals e_i to estimate σ:

$$s_e = \sqrt{\frac{\sum e_i^2}{n - 2}}$$

Estimate of σ

We use the *“sample SD” of the residuals e_i* to estimate σ:

$$s_e = \sqrt{\frac{\sum(e_i - \overline{e})^2}{n - 2}} = \sqrt{\frac{\sum e_i^2}{n - 2}}$$

- Recall that the mean of residuals is 0, $\overline{e} = \sum_i e_i / n = 0$
- Note here we divide by $n - 2$, not $n - 1$. Why?
  - We lose two degrees of freedom because we estimate two parameters, β_0 and β_1.

Standard Error of b_1

Recall that

$$SD(b_1) = \frac{\sigma}{\sqrt{\sum(x_i - \overline{x})^2}}.$$

But σ is unknown, we estimate it with s_e. The estimated SD of b_1 is called the **standard error (SE)** of b_1

$$SE(b_1) = \frac{s_e}{\sqrt{\sum(x_i - \overline{x})^2}}$$

Sampling distribution of β_1

The **sampling distribution** of b_1 is normal

$$b_1 \sim N\left(\beta_1, \frac{\sigma}{\sqrt{\sum(x_i - \overline{x})^2}}\right) \quad \Rightarrow \quad z = \frac{b_1 - \beta_1}{\sigma / \sqrt{\sum(x_i - \overline{x})^2}} \sim N(0, 1)$$

This is (approx.) valid

- either if the errors ε_i are i.i.d. $N(0, \sigma)$
- or if the errors ε_i are independent and the sample size n is large

As σ is unknown, if replaced with s_e, the t-statistic below has a t-distribution with $n - 2$ degrees of freedom

$$T = \frac{b_1 - \beta_1}{s_e / \sqrt{\sum(x_i - \overline{x})^2}} = \frac{b_1 - \beta_1}{SE(b_1)} \sim t_{n-2},$$

Confidence Intervals for β_1

The $(1 - \alpha)$ **confidence interval for** β_1 is given as

$$b_1 \pm t^* SE(b_1)$$

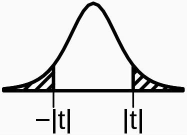
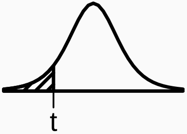
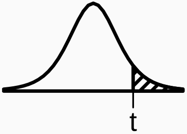
where t^* is the critical value for the $t_{(n-2)}$ distribution at confidence level $1 - \alpha$.

Tests for β_1

To **test the hypothesis** $H_0 : \beta_1 = a$, we use the t-statistic

$$t = \frac{b_1 - a}{SE(b_1)} \sim t_{n-2}$$

The p-value can be computed using the t-table based on the H_a:

H_a	$\beta_1 \neq a$	$\beta_1 < a$	$\beta_1 > a$
P-value			

Observe that testing $H_0 : \beta_1 = 0$ is equivalent to testing whether x is useful in predicting y linearly.

- It is possible that r is small but β_1 is significantly different from 0.

Inference for the Intercept β_0

Though **the population intercept β_0 is rarely of interest**, all the results for the population slope β_1 have their counterparts for β_0.

- $b_0 = \beta_0 + \overline{\varepsilon} - \frac{\sum_i \overline{x}(x_i - \overline{x})\varepsilon_i}{\sum_i (x_i - \overline{x})^2}$
- $E(b_0) = \beta_0$ .... (b_0 is an **unbiased** estimate of β_0)
- $\text{SD}(b_0) = \sigma \sqrt{\frac{1}{n} + \frac{\overline{x}^2}{\sum (x_i - \overline{x})^2}}$
- $\text{SE}(b_0) = s_e \sqrt{\frac{1}{n} + \frac{\overline{x}^2}{\sum (x_i - \overline{x})^2}}$
- The sampling distribution of b_0 (when n is large) is

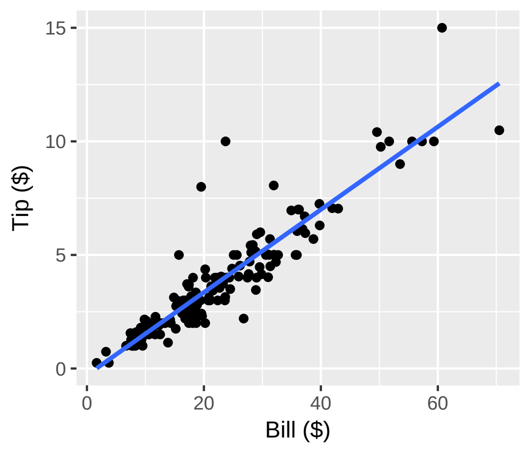
$$b_0 \sim N\left(\beta_0, \sigma \sqrt{\frac{1}{n} + \frac{\overline{x}^2}{\sum (x_i - \overline{x})^2}}\right)$$

- $(1 - \alpha)$ C.I. for β_0: $b_0 \pm t^* SE(b_0)$
- The test statistic for $H_0 : \beta_0 = a$ is $t = \dfrac{b_0 - a}{SE(b_0)} \sim t_{n-2}$ and the P-value can be computed similarly as for β_1

Example: Restaurant Tips

The owner of a bistro called *First Crush* in Potsdam, NY, collected 157 restaurant bills over a 2-week period that he believes provide a good sample of his customers.

He wanted to study the payment and tipping patterns of its patrons.



Regression in R

Regression in R is as simple as `lm(y ~ x)`, in which “lm” stands for “linear model”

```
> tips = read.table("RestaurantTips.txt",h=T)
> lm(Tip ~ Bill, data=tips)

Call:
lm(formula = Tip ~ Bill, data = tips)

Coefficients:
(Intercept)         Bill
    -0.2923       0.1822
```

It is better to save the model as an object,

```
lmtips = lm(Tip ~ Bill, data=tips)
```

and then we can get a more detailed output by viewing the `summary()` of the model object. The output is shown in the next slide

Regression in R

```
> summary(lmtips)
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.292267   0.166160  -1.759   0.0806 .
Bill         0.182215   0.006451  28.247   <2e-16 ***
---
Residual standard error: 0.9795 on 155 degrees of freedom
Multiple R-squared:  0.8373,Adjusted R-squared:  0.8363
F-statistic: 797.9 on 1 and 155 DF,  p-value: < 2.2e-16
```

- The column “`Estimate`” gives the LS estimate for the intercept $b_0 = -0.292267$ and the slope $b_1 = 0.182215$
- The column “`Std. Error`” gives $\mathrm{SE}(b_0)$ and $\mathrm{SE}(b_1)$:

$$SE(b_0) = 0.166160, \quad SE(b_1) = 0.006451$$

Example: Confidence Interval for β_1

```
             Estimate  Std. Error  t value  Pr(>|t|)
(Intercept) -0.292267   0.166160   -1.759  0.0806
Bill         0.182215   0.006451   28.247  <2e-16
```

As df $= n - 2 = 157 - 2 = 155$, t^* for a 95% CI is 1.975 (between 1.97 and 1.98).

one tail	0.1	0.05	0.025	0.01	0.005
two tails	0.2	0.10	0.050	0.02	0.010
df 150	1.29	1.66	1.98	2.35	2.61
200	1.29	1.65	1.97	2.35	2.60

Hence the 95% CI for β_1 is

$$\begin{aligned} b_1 \pm t^* SE(b_1) &= 0.182215 \pm 1.975 \times 0.006451 \\ &= 0.182215 \pm 0.01274 \approx (0.169, 0.195). \end{aligned}$$

Interpretation: With 95% confidence, for each additional dollar in the bill, the customers gave 16.9 cents to 19.5 cents more tips on average.

```
Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
(Intercept) -0.292267   0.166160  -1.759   0.0806 .  
Bill         0.182215   0.006451  28.247   <2e-16 ***
```

- Note t-values $b_i/SE(b_i)$ are simply the ratio of the numbers in the “`Estimate`” column and the numbers in the “`Std. Error`” column, e.g.,

$$-1.759 = \frac{-0.292267}{0.166160}, \quad 28.247 = \frac{0.182215}{0.006451}$$

- Testing $\mathrm{H}_0 : \beta_1 = 0$ is equivalent to testing whether the amount of tips is linearly related to the amount of the bill. The small P-value $< 2 \times 10^{-16}$ asserts that the relation is significant

Example: Test for the Slope β_1

A general rule for waiters is to tip 15 to 20% of the pre-tax bill. That is, $\beta_0 = 0$ and β_1 is between 0.15 to 0.20.

```
              Estimate Std. Error t value  Pr(>|t|)
(Intercept) -0.292267   0.166160  -1.759 0.0806
Bill         0.182215   0.006451  28.247 <2e-16
```

- R tests $\beta_0 = 0$ for us: t-statistic $= -1.759$, 2-sided p-value $= 0.0806$
- To test $H_0 : \beta_1 = 0.2$ v.s. $H_A : \beta_1 < 0.2$. The t-statistic is

$$t = \frac{b_1 - 0.2}{SE(b_1)} = \frac{0.182215 - 0.2}{0.006451} = -2.757$$

with df $= 155$, the one-sided p-value is < 0.005.

one tail	0.1	0.05	0.025	0.01	0.005
two tails	0.2	0.10	0.050	0.02	0.010
df 150	1.29	1.66	1.98	2.35	2.61
200	1.29	1.65	1.97	2.35	2.60

Conclusion: Customers of this restaurant gave less than 20% the bill as tips on average.

How to Read R Outputs for Regression?

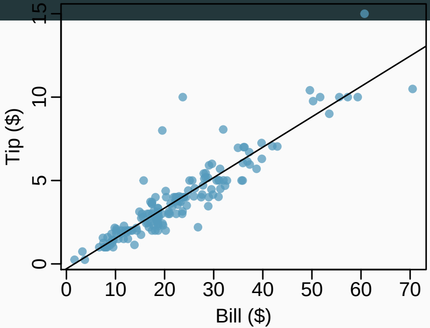
```
Residual standard error: 0.9795 on 155 degrees of freedom
Multiple R-squared:  0.8373,Adjusted R-squared:  0.8363
F-statistic: 797.9 on 1 and 155 DF,  p-value: < 2.2e-16
```

- `Residual standard error: 0.9795 on 155 degrees of freedom`
  This gives the estimate s_e of σ, which is 0.9795.
  df $= n - 2 = 157 - 2 = 155$
- `Multiple R-squared: 0.8373` gives $r^2 = 0.8373$, Bill size explained 83.73% of the variation in tipping amount.
  The correlation between bill size and tips is
  $r = \sqrt{r^2} = \sqrt{0.8373} = 0.915$.
- `Adjusted R-squared`: Ignore this.
- `F-statistic: 797.9 on 1 and 155 DF,  p-value: < 2.2e-16` Skip.

Checking Conditions for Simple Linear Regression Model

Conditions for Simple Linear Regression Model

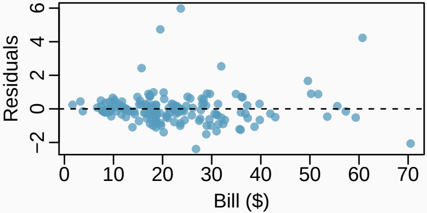
1. Linearity
2. Constant variability
3. (Optional) Nearly normal residuals



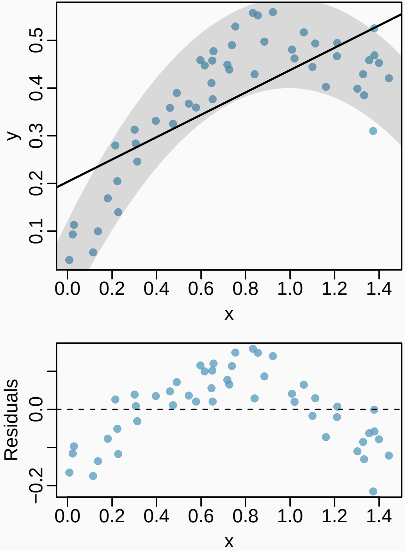
Tools for checking conditions:

- Residual plot

If conditions are satisfied, points should scatter evenly around the zero line in the residual plot.



Checking Conditions – Linearity



What condition is this linear model obviously violating?

(a) Constant variability

(b) Linear relationship

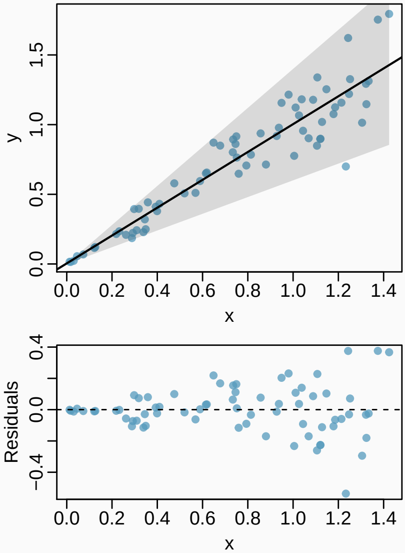
(c) *Linear relationship*

(d) Normal residuals

(e) No extreme outliers

Note the correlation between the residuals and *x* remains zero, but zero correlation ≠ no association. It can be a non-linear association

Checking Conditions – Constant Variability

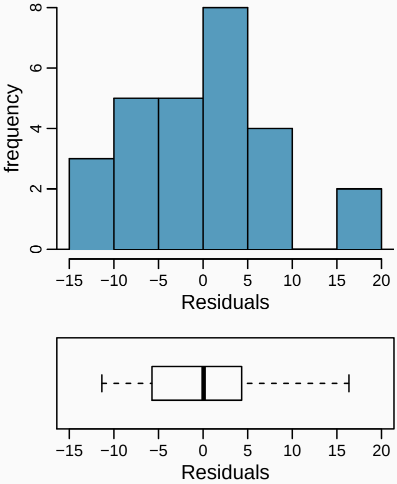


The variability of points around the least-squares line should be roughly constant, implying the variability of residuals around the 0 line should be roughly constant as well, called *homoscedasticity.*

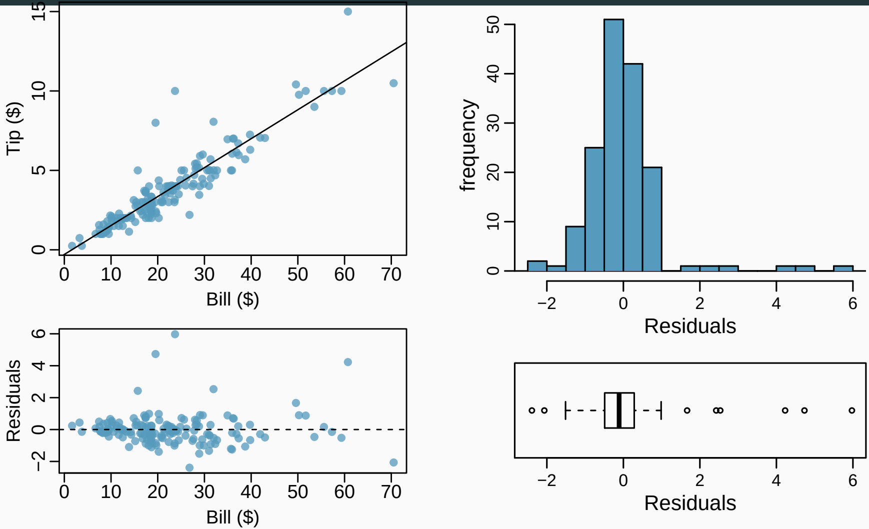
If not, called *heterocedasticity*, predictions made in areas of larger variability will be worse. May try weighted least-square method or transforming the response.

Conditions: Nearly Normal Residuals

- Less relevant than the first two conditions
- Diagnosis: Check the histogram or boxplot of residuals
- If the linearity or constant variability condition is clearly violated, there is no need to check the normality of residuals.



Checking Conditions for the Restaurant Tip Data

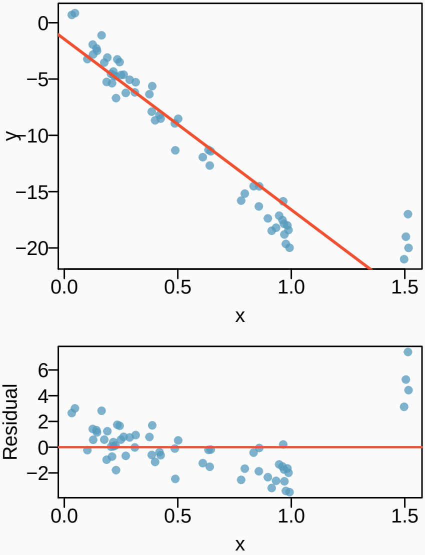


The constant variability condition seems to be violated.

The size of residual seems to increase with Bill.

Types of Outliers

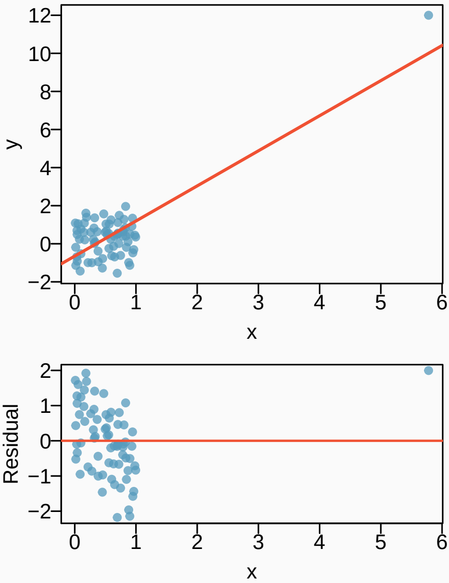
Types of Outliers



How do outliers influence the least squares line in this plot?

To answer this question think of where the regression line would be with and without the outlier(s). Without the outliers the regression line would be steeper, and lie closer to the larger group of observations. With the outliers the line is pulled up and away from some of the observations in the larger group.

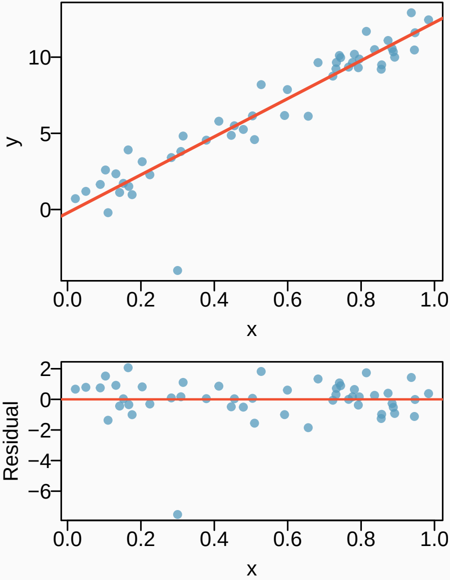
Types of Outliers — Influential Points



How do outliers influence the least squares line in this plot?

Without the outlier there is no evident relationship between x and y.

Types of Outliers — Outlier but Not Influential



Does this outlier influence the slope of the regression line?

Not much...

Some Terminology

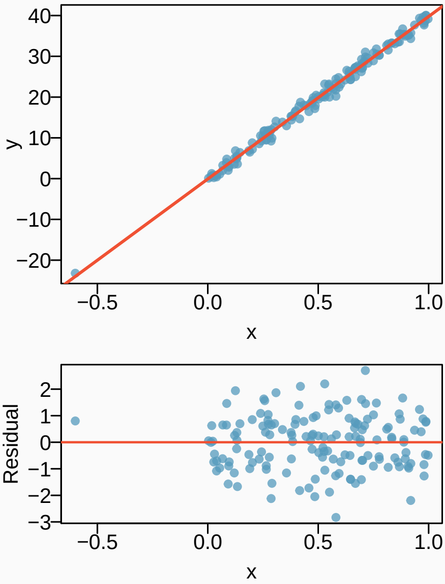
- *Outliers* are points that lie away from the cloud of points.

- Outliers that lie away from the center of the cloud in the x-direction are called *high leverage* points.

- A point is *influential* if including or excluding the point would considerably change the slope of the regression line.

- Influential points must be outliers with high leverages.

Types of Outliers



Which of the below best describes the outlier?

(a) influential

(b) high leverage

(c) *high leverage*

(d) none of the above

(e) there are no outliers

Recap

Which of following is true?

(a) Influential points always change the intercept of the regression line. . . . . change the *slope*

(b) Influential points always reduce R^2. . . False. See the next slide

(c) It is much more likely for a low leverage point to be influential, than a high leverage point.

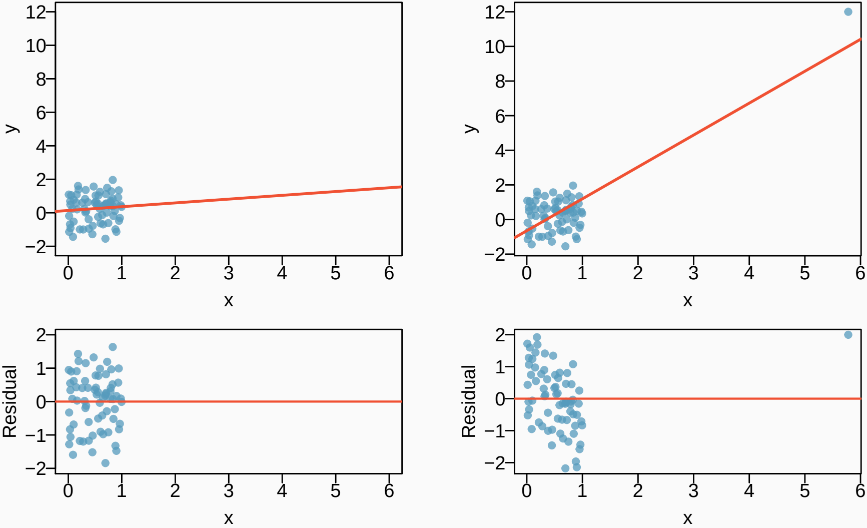
(d) When the data set includes an influential point, the relationship between the explanatory variable and the response variable is always nonlinear.

(e) None of the above.

(f) *None of the above.*

Recap (cont.)

Influential point may also increase R^2



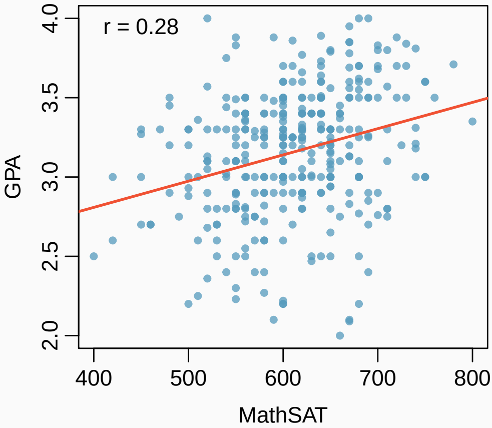
$r = 0.08,\ R^2 = r^2 = 0.0064$

$r = 0.79,\ R^2 = r^2 \approx 0.627$

More Examples

Example: GPA and MathSAT

The scatter plot below shows the GPA and MathSAT of a random sample of 345 students in a college.



The correlation $r = 0.28$ is weak.

Can the slope β_1 of the line be significantly different from 0?

```
> summary(lm(GPA ~ VerbalSAT, data=stu))
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 2.1466877  0.1867166  11.497  < 2e-16 ***
MathSAT     0.0016544  0.0003036   5.449 9.68e-08 ***
```

To test $H_0 : \beta_1 = 0$ v.s. $H_1 : \beta_1 \neq 0$, the t-statistic is

$$t = \frac{b_1}{SE(b_1)} = \frac{0.0016544}{0.0003036} = 5.449$$

with $\text{df} = 345 - 2 = 343$. Two-sided P-value $= 9.68 \times 10^{-8}$.

- There is strong evidence that students' GPA is linearly related with their MathSAT, despite of their small correlation $r = 0.28$.
- It is possible that r is small but β_1 is significantly different from 0, especially when the sample size n is large.
- Students with higher MathSAT indeed have significantly higher GPA on average, despite of the huge variability in GPA.
- As $R^2 = r^2 = (0.28)^2 = 0.0784$, MathSAT merely explains 7.84% of the variation in GPA.

Example: GPA and MathSAT – 95% CI for β_1

```
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 2.1466877  0.1867166  11.497  < 2e-16 ***
MathSAT     0.0016544  0.0003036   5.449 9.68e-08 ***
```

df $= 345 - 2 = 343$. The t^* for a 95% CI is 1.97.

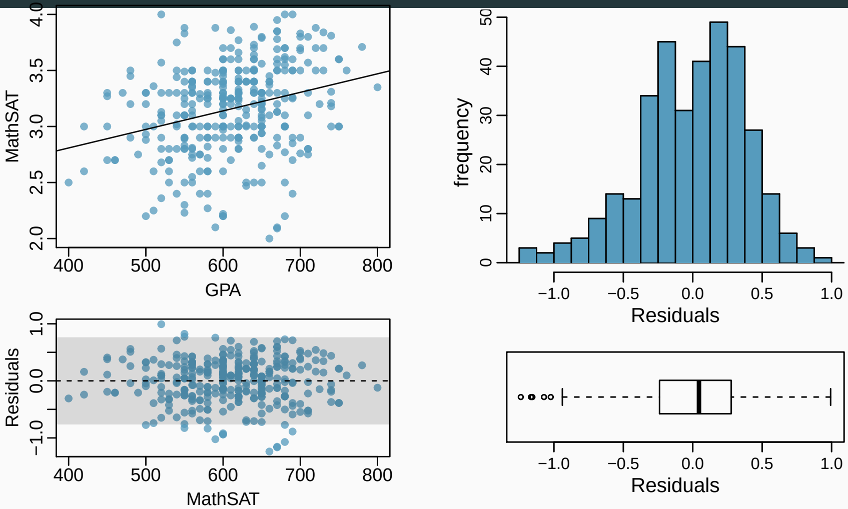
one tail	0.1	0.05	0.025	0.01	0.005
two tails	0.2	0.10	0.050	0.02	0.010
df 300	1.28	1.65	1.97	2.34	2.59
400	1.28	1.65	1.97	2.34	2.59

So a 95% confidence interval for β_1 is

$$b_1 \pm t^* SE(b_1) = 0.0016544 \pm 1.97 \times 0.0003036 \approx (0.00106, 0.00225)$$

Interpretation: We have 95% confidence that for students with 100 more points in their MathSAT scores, their GPA are 0.106 to 0.225 higher on average.

Example: GPA and MathSAT – Checking Conditions



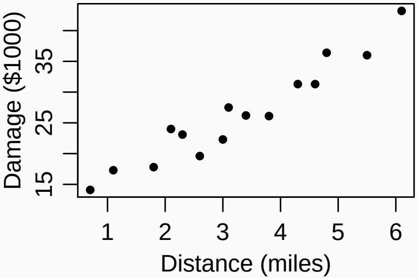
The linearity and constant variability conditions are fine.

The slight left-skewness of residuals is fine because of the large sample size

Example: Fire Damage and Distance to Fire Station

A fire insurance company wanted to relate the amount of fire damage in major residential fires to the distance between the burning house and the nearest fire station. The study was conducted in a large suburb of a major city; a sample of 15 recent fires in this suburb was selected. The amount of damage and the distance between the fire and the nearest fire station were recorded in each fire.

Distance (mile)	Damage ($1000)
0.7	14.1
1.1	17.3
1.8	17.8
2.1	24.0
2.3	23.1
2.6	19.6
3.0	22.3
3.1	27.5
3.4	26.2
3.8	26.1
4.3	31.3
4.6	31.3
4.8	36.4
5.5	36.0
6.1	43.2



```
> fire = read.table("fire.txt",h=T)
> summary(lm(damage ˜ dist, data=fire))
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  10.2779     1.4203   7.237 6.59e-06 ***
dist          4.9193     0.3927  12.525 1.25e-08 ***
```

- estimate for the intercept $b_0 = 10.2779$ and the slope $b_1 = 4.9193$
- $SE(b_0) = 1.4203$, $SE(b_1) = 0.3927$

one tail two tails	0.1 0.2	0.05 0.10	0.025 0.050	0.01 0.02	0.005 0.010
df 13	1.35	1.77	2.16	2.65	3.011

So a 95% confidence interval for β_1 is

$$b_1 \pm t^* SE(b_1) = 4.9193 \pm 2.16 \times 0.3927 \approx 4.919 \pm 0.848 \approx (4.071, 5.767)$$

Interpretation: We have 95% confidence that every extra mile from the nearest fire station increases the amount of damage by $4071 to $5767.

Example: Test for the Slope β_1

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  10.2779     1.4203   7.237 6.59e-06 ***
dist          4.9193     0.3927  12.525 1.25e-08 ***
```

To test $H_0 : \beta_1 = 4$ v.s. $H_1 : \beta_1 > 4$, the t-statistic is

$$t = \frac{b_1 - 4}{SE(b_1)} = \frac{4.9193 - 4}{0.3927} = 2.3409$$

Looking at the t-table for the row with df $= 13$, the one-sided P-value is between 0.01 and 0.025.

one tail	0.1	0.05	0.025	0.01	0.005
two tails	0.2	0.10	0.050	0.02	0.010
df 13	1.35	1.77	2.16	2.65	3.011

Conclusion: At 5% level, the extra amount of damage for every extra mile from the nearest fire station is significantly higher than $4000.