Spectrum and Pseudospectrum of a Tensor

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July 11, 2008

Matrix eigenvalues and eigenvectors

- One of the most important ideas ever invented.
  - **R. Coifman et. al.**: "Eigenvector magic: eigenvectors as an extension of Newtonian calculus."
- Normal/Hermitian A
  - Invariant subspace: $A\mathbf{x} = \lambda\mathbf{x}$.
  - Rayleigh quotient: $\mathbf{x}^\top A\mathbf{x}/\mathbf{x}^\top\mathbf{x}$.
  - Lagrange multipliers: $\mathbf{x}^\top A\mathbf{x} - \lambda(\|\mathbf{x}\|^2 - 1)$.
  - Best rank-1 approximation: $\min_{\|\mathbf{x}\|=1}\|A - \lambda\mathbf{x}\mathbf{x}^\top\|$.
- Nonnormal A
  - Pseudospectrum: $\sigma_\varepsilon(A) = \{\lambda \in \mathbb{C} \mid \|(A - \lambda I)^{-1}\| > \varepsilon^{-1}\}$.
  - Numerical range: $W(A) = \{\mathbf{x}^* A\mathbf{x} \in \mathbb{C} \mid \|\mathbf{x}\| = 1\}$.
  - Irreducible representations of $C^*(A)$ with natural Borel structure.
  - Primitive ideals of $C^*(A)$ with hull-kernel topology.
- How can one define these for tensors?

DARPA mathematical challenge eight

One of the twenty three mathematical challenges announced at DARPA Tech 2007.

Problem

Beyond convex optimization: *can linear algebra be replaced by algebraic geometry in a systematic way?*

- **Algebraic geometry in a slogan:** polynomials are to algebraic geometry what matrices are to linear algebra.
- Polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ of degree d can be expressed as

$$f(\mathbf{x}) = a_0 + \mathbf{a}_1^\top \mathbf{x} + \mathbf{x}^\top A_2 \mathbf{x} + \mathcal{A}_3(\mathbf{x}, \mathbf{x}, \mathbf{x}) + \cdots + \mathcal{A}_d(\mathbf{x}, \dots, \mathbf{x}).$$

$a_0 \in \mathbb{R}, \mathbf{a}_1 \in \mathbb{R}^n, A_2 \in \mathbb{R}^{n \times n}, \mathcal{A}_3 \in \mathbb{R}^{n \times n \times n}, \dots, \mathcal{A}_d \in \mathbb{R}^{n \times \cdots \times n}$.

- Numerical linear algebra: $d = 2$.
- Numerical multilinear algebra: $d > 2$.

Hypermatrices

Totally ordered finite sets: $[n] = \{1 < 2 < \cdots < n\}$, $n \in \mathbb{N}$.

- Vector or n-tuple

$$f : [n] \to \mathbb{R}.$$

If $f(i) = a_i$, then f is represented by $\mathbf{a} = [a_1, \dots, a_n]^\top \in \mathbb{R}^n$.

- Matrix

$$f : [m] \times [n] \to \mathbb{R}.$$

If $f(i,j) = a_{ij}$, then f is represented by $A = [a_{ij}]_{i,j=1}^{m,n} \in \mathbb{R}^{m \times n}$.

- Hypermatrix (order 3)

$$f : [l] \times [m] \times [n] \to \mathbb{R}.$$

If $f(i,j,k) = a_{ijk}$, then f is represented by $\mathcal{A} = [\![a_{ijk}]\!]_{i,j,k=1}^{l,m,n} \in \mathbb{R}^{l \times m \times n}$.

Normally $\mathbb{R}^X = \{f : X \to \mathbb{R}\}$. Ought to be $\mathbb{R}^{[n]}, \mathbb{R}^{[m]\times[n]}, \mathbb{R}^{[l]\times[m]\times[n]}$.

Hypermatrices and tensors

Up to choice of bases

- $\mathbf{a} \in \mathbb{R}^n$ can represent a vector in V (contravariant) or a linear functional in V^* (covariant).
- $A \in \mathbb{R}^{m \times n}$ can represent a bilinear form $V \times W \to \mathbb{R}$ (contravariant), a bilinear form $V^* \times W^* \to \mathbb{R}$ (covariant), or a linear operator $V \to W$ (mixed).
- $\mathcal{A} \in \mathbb{R}^{l \times m \times n}$ can represent trilinear form $U \times V \times W \to \mathbb{R}$ (contravariant), bilinear operators $V \times W \to U$ (mixed), etc.

A hypermatrix is the same as a tensor if

1. we give it coordinates (represent with respect to some bases);
2. we ignore covariance and contravariance.

Basic operation on a hypermatrix

- A matrix can be multiplied on the left and right: $A \in \mathbb{R}^{m\times n}$, $X \in \mathbb{R}^{p\times m}$, $Y \in \mathbb{R}^{q\times n}$,

$$(X, Y) \cdot A = XAY^\top = [c_{\alpha\beta}] \in \mathbb{R}^{p\times q}$$

  where

$$c_{\alpha\beta} = \sum\nolimits_{i,j=1}^{m,n} x_{\alpha i} y_{\beta j} a_{ij}.$$

- A hypermatrix can be multiplied on three sides: $\mathcal{A} = [\![a_{ijk}]\!] \in \mathbb{R}^{l\times m\times n}$, $X \in \mathbb{R}^{p\times l}$, $Y \in \mathbb{R}^{q\times m}$, $Z \in \mathbb{R}^{r\times n}$,

$$(X, Y, Z) \cdot \mathcal{A} = [\![c_{\alpha\beta\gamma}]\!] \in \mathbb{R}^{p\times q\times r}$$

  where

$$c_{\alpha\beta\gamma} = \sum\nolimits_{i,j,k=1}^{l,m,n} x_{\alpha i} y_{\beta j} z_{\gamma k} a_{ijk}.$$

Numerical range of a tensor

- Covariant version:

$$\mathcal{A} \cdot (X^\top, Y^\top, Z^\top) := (X, Y, Z) \cdot \mathcal{A}.$$

- Gives convenient notations for multilinear functionals and multilinear operators. For $\mathbf{x} \in \mathbb{R}^l, \mathbf{y} \in \mathbb{R}^m, \mathbf{z} \in \mathbb{R}^n$,

$$\mathcal{A}(\mathbf{x}, \mathbf{y}, \mathbf{z}) := \mathcal{A} \cdot (\mathbf{x}, \mathbf{y}, \mathbf{z}) = \sum_{i,j,k=1}^{l,m,n} a_{ijk} x_i y_j z_k,$$
$$\mathcal{A}(I, \mathbf{y}, \mathbf{z}) := \mathcal{A} \cdot (I, \mathbf{y}, \mathbf{z}) = \sum_{j,k=1}^{m,n} a_{ijk} y_j z_k.$$

- Numerical range of square matrix $A \in \mathbb{C}^{n \times n}$,

$$W(A) = \{\mathbf{x}^* A \mathbf{x} \in \mathbb{C} \mid \|\mathbf{x}\|_2 = 1\} = \{A(\mathbf{x}, \mathbf{x}^*) \in \mathbb{C} \mid \|\mathbf{x}\|_2 = 1\}.$$

- Plausible generalization to cubical hypermatrix $\mathcal{A} \in \mathbb{C}^{n \times \cdots \times n}$,

$$W(\mathcal{A}) = \begin{cases} \{\mathcal{A}(\mathbf{x}, \mathbf{x}^*, \dots, \mathbf{x}) \in \mathbb{C} \mid \|\mathbf{x}\|_k = 1\} & \text{odd order,} \\ \{\mathcal{A}(\mathbf{x}, \mathbf{x}^*, \dots, \mathbf{x}^*) \in \mathbb{C} \mid \|\mathbf{x}\|_k = 1\} & \text{even order.} \end{cases}$$

Symmetric hypermatrices

- Cubical hypermatrix $[\![a_{ijk}]\!] \in \mathbb{R}^{n\times n\times n}$ is **symmetric** if

$$a_{ijk} = a_{ikj} = a_{jik} = a_{jki} = a_{kij} = a_{kji}.$$

- Invariant under all permutations $\sigma \in \mathfrak{S}_k$ on indices.
- $\mathsf{S}^k(\mathbb{R}^n)$ denotes set of all order-k symmetric hypermatrices.

Example

Higher order derivatives of multivariate functions.

Example

Moments of a random vector $\mathbf{x} = (X_1, \dots, X_n)$:

$$m_k(\mathbf{x}) = \left[E(x_{i_1}x_{i_2}\cdots x_{i_k})\right]_{i_1,\dots,i_k=1}^{n} = \left[\int\cdots\int x_{i_1}x_{i_2}\cdots x_{i_k}\; d\mu(x_{i_1})\cdots d\mu(x_{i_k})\right]_{i_1,\dots,i_k=1}^{n}.$$

Symmetric hypermatrices

Example

Cumulants of a random vector $\mathbf{x} = (X_1, \dots, X_n)$:

$$\kappa_k(\mathbf{x}) = \left[\sum_{A_1 \sqcup \dots \sqcup A_p = \{i_1, \dots, i_k\}} (-1)^{p-1}(p-1)! E\left(\prod_{i \in A_1} x_i\right) \cdots E\left(\prod_{i \in A_p} x_i\right) \right]_{i_1, \dots, i_k = 1}^{n}.$$

For $n = 1$, $\kappa_k(x)$ for $k = 1, 2, 3, 4$ are the expectation, variance, skewness, and kurtosis.

- Important in Independent Component Analysis (ICA).

Multilinear spectral theory

Let $\mathcal{A} \in \mathbb{R}^{n \times n \times n}$ (easier if $\mathcal{A}$ symmetric).

1. How should one define its eigenvalues and eigenvectors?
2. What is a decomposition that generalizes the eigenvalue decomposition of a matrix?

Let $\mathcal{A} \in \mathbb{R}^{l \times m \times n}$

1. How should one define its singualr values and singular vectors?
2. What is a decomposition that generalizes the singular value decomposition of a matrix?

Somewhat surprising: (1) and (2) have different answers.

Tensor ranks (Hitchcock, 1927)

- **Matrix rank.** $A \in \mathbb{R}^{m\times n}$.

$$\begin{aligned}\operatorname{rank}(A) &= \dim(\operatorname{span}_{\mathbb{R}}\{A_{\bullet 1},\dots,A_{\bullet n}\}) && \text{(column rank)}\\ &= \dim(\operatorname{span}_{\mathbb{R}}\{A_{1\bullet},\dots,A_{m\bullet}\}) && \text{(row rank)}\\ &= \min\{r \mid A = \textstyle\sum_{i=1}^{r}\mathbf{u}_i\mathbf{v}_i^{\mathsf{T}}\} && \text{(outer product rank).}\end{aligned}$$

- **Multilinear rank.** $\mathcal{A} \in \mathbb{R}^{l\times m\times n}$. $\operatorname{rank}_{\boxplus}(A) = (r_1(A), r_2(A), r_3(A))$,

$$\begin{aligned}r_1(A) &= \dim(\operatorname{span}_{\mathbb{R}}\{A_{1\bullet\bullet},\dots,A_{l\bullet\bullet}\})\\ r_2(A) &= \dim(\operatorname{span}_{\mathbb{R}}\{A_{\bullet 1\bullet},\dots,A_{\bullet m\bullet}\})\\ r_3(A) &= \dim(\operatorname{span}_{\mathbb{R}}\{A_{\bullet\bullet 1},\dots,A_{\bullet\bullet n}\})\end{aligned}$$

- **Outer product rank.** $\mathcal{A} \in \mathbb{R}^{l\times m\times n}$.

$$\operatorname{rank}_{\otimes}(\mathcal{A}) = \min\{r \mid \mathcal{A} = \textstyle\sum_{i=1}^{r}\mathbf{u}_i \otimes \mathbf{v}_i \otimes \mathbf{w}_i\}$$

where $\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} := [\![u_i v_j w_k]\!]_{i,j,k=1}^{l,m,n}$.

Matrix EVD and SVD

- Rank revealing decompositions.
- **Symmetric eigenvalue decomposition** of $A \in \mathsf{S}^2(\mathbb{R}^n)$,

$$A = V\Lambda V^\top = \sum_{i=1}^{r} \lambda_i \mathbf{v}_i \otimes \mathbf{v}_i,$$

  where $\operatorname{rank}(A) = r$, $V \in \mathrm{O}(n)$ eigenvectors, Λ eigenvalues.
- **Singular value decomposition** of $A \in \mathbb{R}^{m\times n}$,

$$A = U\Sigma V^\top = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \otimes \mathbf{v}_i$$

  where $\operatorname{rank}(\mathcal{A}) = r$, $U \in \mathrm{O}(m)$ left singular vectors, $V \in \mathrm{O}(n)$ right singular vectors, Σ singular values.

One plausible EVD and SVD for hypermatrices

- Rank revealing decompositions associated with the outer product rank.
- **Symmetric outer product decomposition** of $\mathcal{A} \in \mathsf{S}^3(\mathbb{R}^n)$,

$$\mathcal{A} = \sum_{i=1}^{r} \lambda_i \mathbf{v}_i \otimes \mathbf{v}_i \otimes \mathbf{v}_i$$

  where $\operatorname{rank}_{\mathsf{S}}(A) = r$, $\mathbf{v}_i$ unit vector, $\lambda_i \in \mathbb{R}$.
- **Outer product decomposition** of $\mathcal{A} \in \mathbb{R}^{l \times m \times n}$,

$$\mathcal{A} = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \otimes \mathbf{v}_i \otimes \mathbf{w}_i$$

  where $\operatorname{rank}_{\otimes}(\mathcal{A}) = r$, $\mathbf{u}_i \in \mathbb{R}^l, \mathbf{v}_i \in \mathbb{R}^m, \mathbf{w}_i \in \mathbb{R}^n$ unit vectors, $\sigma_i \in \mathbb{R}$.

Another plausible EVD and SVD for hypermatrices

- Rank revealing decompositions associated with the multilinear rank.
- **Singular value decomposition** of $\mathcal{A} \in \mathbb{R}^{l \times m \times n}$,

$$\mathcal{A} = (U, V, W) \cdot \mathcal{C}$$

  where $\operatorname{rank}_{\boxplus}(A) = (r_1, r_2, r_3)$, $U \in \mathbb{R}^{l \times r_1}$, $V \in \mathbb{R}^{m \times r_2}$, $W \in \mathbb{R}^{n \times r_3}$ have orthonormal columns and $\mathcal{C} \in \mathbb{R}^{r_1 \times r_2 \times r_3}$.
- **Symmetric eigenvalue decomposition** of $\mathcal{A} \in \mathsf{S}^3(\mathbb{R}^n)$,

$$\mathcal{A} = (U, U, U) \cdot \mathcal{C}$$

  where $\operatorname{rank}_{\boxplus}(A) = (r, r, r)$, $U \in \mathbb{R}^{n \times r}$ has orthonormal columns and $\mathcal{C} \in \mathsf{S}^3(\mathbb{R}^r)$.

Variational approach to eigenvalues/vectors

- $A \in \mathbb{R}^{m \times n}$ symmetric.
- Eigenvalues and eigenvectors are critical values and critical points of
$$\mathbf{x}^\top A\mathbf{x}/\|\mathbf{x}\|_2^2.$$
- Equivalently, critical values/points of $\mathbf{x}^\top A\mathbf{x}$ constrained to unit sphere.
- Lagrangian:
$$L(\mathbf{x}, \lambda) = \mathbf{x}^\top A\mathbf{x} - \lambda(\|\mathbf{x}\|_2^2 - 1).$$
- Vanishing of ∇L at critical $(\mathbf{x}_c, \lambda_c) \in \mathbb{R}^n \times \mathbb{R}$ yields familiar
$$A\mathbf{x}_c = \lambda_c \mathbf{x}_c.$$

Variational approach to singular values/vectors

- $A \in \mathbb{R}^{m \times n}$.
- Singular values and singular vectors are critical values and critical points of
$$\mathbf{x}^\top A\mathbf{y}/\|\mathbf{x}\|_2\|\mathbf{y}\|_2.$$
- Lagrangian:
$$L(\mathbf{x},\mathbf{y},\sigma) = \mathbf{x}^\top A\mathbf{y} - \sigma(\|\mathbf{x}\|_2\|\mathbf{y}\|_2 - 1).$$
- At critical $(\mathbf{x}_c, \mathbf{y}_c, \sigma_c) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}$,
$$A\mathbf{y}_c/\|\mathbf{y}_c\|_2 = \sigma_c\mathbf{x}_c/\|\mathbf{x}_c\|_2, \quad A^\top\mathbf{x}_c/\|\mathbf{x}_c\|_2 = \sigma_c\mathbf{y}_c/\|\mathbf{y}_c\|_2.$$
- Writing $\mathbf{u}_c = \mathbf{x}_c/\|\mathbf{x}_c\|_2$ and $\mathbf{v}_c = \mathbf{y}_c/\|\mathbf{y}_c\|_2$ yields familiar
$$A\mathbf{v}_c = \sigma_c\mathbf{u}_c, \quad A^\top\mathbf{u}_c = \sigma_c\mathbf{v}_c.$$

Eigenvalues/vectors of a tensor

- Extends to hypermatrices.
- For $\mathbf{x} = [x_1, \dots, x_n]^\top \in \mathbb{R}^n$, write $\mathbf{x}^p := [x_1^p, \dots, x_n^p]^\top$.
- Define the 'ℓ^k-norm' $\|\mathbf{x}\|_k = (x_1^k + \cdots + x_n^k)^{1/k}$.
- Define eigenvalues/vectors of $\mathcal{A} \in \mathsf{S}^k(\mathbb{R}^n)$ as critical values/points of the multilinear Rayleigh quotient

$$\mathcal{A}(\mathbf{x}, \dots, \mathbf{x})/\|\mathbf{x}\|_k^k.$$

- Lagrangian

$$L(\mathbf{x}, \lambda) := \mathcal{A}(\mathbf{x}, \dots, \mathbf{x}) - \lambda(\|\mathbf{x}\|_k^k - 1).$$

- At a critical point

$$\mathcal{A}(I_n, \mathbf{x}, \dots, \mathbf{x}) = \lambda \mathbf{x}^{k-1}.$$

Eigenvalues/vectors of a tensor

- If $\mathcal{A}$ is symmetric,

$$\mathcal{A}(I_n, \mathbf{x}, \mathbf{x}, \dots, \mathbf{x}) = \mathcal{A}(\mathbf{x}, I_n, \mathbf{x}, \dots, \mathbf{x}) = \cdots = \mathcal{A}(\mathbf{x}, \mathbf{x}, \dots, \mathbf{x}, I_n).$$

- Also obtained by Liqun Qi independently:
  - L. Qi, "Eigenvalues of a real supersymmetric tensor," *J. Symbolic Comput.*, **40** (2005), no. 6.
  - L, "Singular values and eigenvalues of tensors: a variational approach," *Proc. IEEE Int. Workshop on Computational Advances in Multi-Sensor Adaptive Processing*, **1** (2005).
- For unsymmetric hypermatrices — get different eigenpairs for different modes (unsymmetric matrix have different left/right eigenvectors).
- Falls outside Classical Invariant Theory — not invariant under $Q \in \mathrm{O}(n)$, ie. $\|Q\mathbf{x}\|_2 = \|\mathbf{x}\|_2$.
- Invariant under $Q \in \mathrm{GL}(n)$ with $\|Q\mathbf{x}\|_k = \|\mathbf{x}\|_k$.

Singular values/vectors of a tensor

- Likewise for singular values/vectors of $\mathcal{A} \in \mathbb{R}^{l \times m \times n}$.
- Lagrangian is

$$L(\mathbf{x}, \mathbf{y}, \mathbf{z}, \sigma) = \mathcal{A}(\mathbf{x}, \mathbf{y}, \mathbf{z}) - \sigma(\|\mathbf{x}\|\|\mathbf{y}\|\|\mathbf{z}\| - 1)$$

  where $\sigma \in \mathbb{R}$ is the Lagrange multiplier.
- At a critical point,

$$\begin{aligned} \mathcal{A}\left(I_l, \mathbf{y}/\|\mathbf{y}\|, \mathbf{z}/\|\mathbf{z}\|\right) &= \sigma \mathbf{x}/\|\mathbf{x}\|, \\ \mathcal{A}\left(\mathbf{x}/\|\mathbf{x}\|, I_m, \mathbf{z}/\|\mathbf{z}\|\right) &= \sigma \mathbf{y}/\|\mathbf{y}\|, \\ \mathcal{A}\left(\mathbf{x}/\|\mathbf{x}\|, \mathbf{y}/\|\mathbf{y}\|, I_n\right) &= \sigma \mathbf{z}/\|\mathbf{z}\|. \end{aligned}$$

- Normalize to get

$$\mathcal{A}(I_l, \mathbf{v}, \mathbf{w}) = \sigma \mathbf{u}, \quad \mathcal{A}(\mathbf{u}, I_m, \mathbf{w}) = \sigma \mathbf{v}, \quad \mathcal{A}(\mathbf{u}, \mathbf{v}, I_n) = \sigma \mathbf{w}.$$

Pseudospectrum of a tensor

- Pseudospectrum of square matrix $A \in \mathbb{C}^{n \times n}$,

$$\sigma_\varepsilon(A) = \{\lambda \in \mathbb{C} \mid \|(A - \lambda I)^{-1}\|_2 > \varepsilon^{-1}\}$$
$$= \{\lambda \in \mathbb{C} \mid \sigma_{\min}(A - \lambda I) < \varepsilon\}.$$

- One plausible generalization to cubical hypermatrix $\mathcal{A} \in \mathbb{C}^{n \times \dots \times n}$,

$$\sigma_\varepsilon^\Sigma(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid \sigma_{\min}(\mathcal{A} - \lambda \mathcal{I}) < \varepsilon\}.$$

- Another plausible generalization,

$$\sigma_\varepsilon^\Delta(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid \inf_{\operatorname{Det}_{n,\dots,n}(\mathcal{B})=0} \|\mathcal{A} - \lambda \mathcal{I} - \mathcal{B}\|_F < \varepsilon^{-1}\}.$$

- Fact: hyperdeterminant $\operatorname{Det}_{n,\dots,n}(\mathcal{B}) = 0$ iff 0 is a singular value of $\mathcal{B}$.

Perron-Frobenius theorem for hypermatrices

- An order-k cubical hypermatrix $\mathcal{A} \in \mathsf{T}^k(\mathbb{R}^n)$ is **reducible** if there exist a permutation $\sigma \in \mathfrak{S}_n$ such that the permuted hypermatrix

$$[\![b_{i_1 \cdots i_k}]\!] = [\![a_{\sigma(j_1) \cdots \sigma(j_k)}]\!]$$

  has the property that for some $m \in \{1, \dots, n-1\}$, $b_{i_1 \cdots i_k} = 0$ for all $i_1 \in \{1, \dots, n-m\}$ and all $i_2, \dots, i_k \in \{1, \dots, m\}$.
- We say that $\mathcal{A}$ is **irreducible** if it is not reducible. In particular, if $\mathcal{A} > 0$, then it is irreducible.

Theorem (L)

Let $0 \leq \mathcal{A} = [\![a_{j_1 \cdots j_k}]\!] \in \mathsf{T}^k(\mathbb{R}^n)$ be irreducible. Then $\mathcal{A}$ has

1. *a positive real eigenvalue λ with an eigenvector $\mathbf{x}$;*
2. *$\mathbf{x}$ may be chosen to have all entries non-negative;*
3. *if μ is an eigenvalue of $\mathcal{A}$, then $|\mu| \leq \lambda$.*

Hypergraphs

- $G = (V, E)$ is 3-**hypergraph**.
  - V is the finite set of **vertices**.
  - E is the subset of **hyperedges**, ie. 3-element subsets of V.
- Write elements of E as $[x, y, z]$ $(x, y, z \in V)$.
- G is **undirected**, so $[x, y, z] = [y, z, x] = \cdots = [z, y, x]$.
- Hyperedge is said to **degenerate** if of the form $[x, x, y]$ or $[x, x, x]$ (hyperloop at x). We do not exclude degenerate hyperedges.
- G is m-**regular** if every $v \in V$ is adjacent to exactly m hyperedges.
- G is r-**uniform** if every edge contains exactly r vertices.
- Good reference: D. Knuth, *The art of computer programming*, **4**, pre-fascicle 0a, 2008.

Spectral hypergraph theory

- Define the order-3 **adjacency hypermatrix** $\mathcal{A} = [\![a_{ijk}]\!]$ by

$$a_{xyz} = \begin{cases} 1 & \text{if } [x, y, z] \in E, \\ 0 & \text{otherwise.} \end{cases}$$

- $\mathcal{A} \in \mathbb{R}^{|V| \times |V| \times |V|}$ nonnegative symmetric hypermatrix.
- Consider cubic form

$$\mathcal{A}(f, f, f) = \sum\nolimits_{x,y,z} a_{xyz} f(x) f(y) f(z),$$

  where $f \in \mathbb{R}^V$.
- Eigenvalues (resp. eigenvectors) of A are the critical values (resp. critical points) of $\mathcal{A}(f, f, f)$ constrained to the $f \in \ell^3(V)$, ie.

$$\sum\nolimits_{x \in V} f(x)^3 = 1.$$

Spectral hypergraph theory

We have the following.

Lemma (L)

Let G be an m-regular 3-hypergraph. $\mathcal{A}$ its adjacency hypermatrix. Then

1. *m is an eigenvalue of $\mathcal{A}$;*
2. *if λ is an eigenvalue of $\mathcal{A}$, then $|\lambda| \leq m$;*
3. *λ has multiplicity 1 if and only if G is connected.*

Related work: J. Friedman, A. Wigderson, "On the second eigenvalue of hypergraphs," *Combinatorica*, **15** (1995), no. 1.

Spectral hypergraph theory

- A hypergraph $G = (V, E)$ is said to be k-**partite** or k-**colorable** if there exists a partition of the vertices $V = V_1 \cup \cdots \cup V_k$ such that for any k vertices $u, v, \ldots, z$ with $a_{uv\cdots z} \neq 0$, $u, v, \ldots, z$ must each lie in a distinct V_i ($i = 1, \ldots, k$).

Lemma (L)

Let G be a connected m-regular k-partite k-hypergraph on n vertices. Then

1. *If $k \equiv 1 \bmod 4$, then every eigenvalue of G occurs with multiplicity a multiple of k.*
2. *If $k \equiv 3 \bmod 4$, then the spectrum of G is symmetric, ie. if λ is an eigenvalue, then so is $-\lambda$.*
3. *Furthermore, every eigenvalue of G occurs with multiplicity a multiple of $k/2$, ie. if λ is an eigenvalue of G, then λ and $-\lambda$ occurs with the same multiplicity.*

To do

- Cases $k \equiv 0, 2 \bmod 4$
- Cheeger type isoperimetric inequalities
- Expander hypergraphs
- Algorithms for eigenvalues/vectors of a hypermatrix

Advertisement

Geometry and representation theory of tensors for computer science, statistics, and other areas

1. MSRI Summer Graduate Workshop
   - July 7 to July 18, 2008
   - Organized by J.M. Landsberg, L.-H. Lim, J. Morton
   - Mathematical Sciences Research Institute, Berkeley, CA
   - `http://msri.org/calendar/sgw/WorkshopInfo/451/show_sgw`
2. AIM Workshop
   - July 21 to July 25, 2008
   - Organized by J.M. Landsberg, L.-H. Lim, J. Morton, J. Weyman
   - American Institute of Mathematics, Palo Alto, CA
   - `http://aimath.org/ARCC/workshops/repnsoftensors.html`