Algebraic Geometry of Matrices IV

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today

- coordinate ring and pullback
- dimension of affine variety
- relate to linear algebra/matrix analysis/operator theory
- Noether's normalization lemma[1]

[1] cf. Ke Ye's tutorial at 3:30pm

Rings $\longleftrightarrow$ Varieties

algebra–geometry revisited

geometry $\longleftrightarrow$ algebra

yesterday:

$$\{\text{affine varieties in } \mathbb{A}^n\} \longleftrightarrow \{\text{radical ideals in } \mathbb{C}[x_1, \ldots, x_n]\}$$

today:

$$\{\text{affine varieties in } \mathbb{A}^n\} \longleftrightarrow \{\text{fin. gen. reduced rings over } \mathbb{C}\}$$

also:

$$\{\text{morphisms } X \to Y\} \longleftrightarrow \{\text{homomorphisms } \mathbb{C}[Y] \to \mathbb{C}[X]\}$$

new terminology later: rings over $\mathbb{C}$ = $\mathbb{C}$-algebra

supplemental glossary

- $\mathfrak{a} \subseteq R$ ideal, then quotient ring is

$$R/\mathfrak{a} := \{[r] = r + \mathfrak{a} : r \in R\}$$

- $[r + s] := [r] + [s]$, $[r][s] := [rs]$
- quotient projection $\pi_\mathfrak{a} : R \to R/\mathfrak{a}$, $r \mapsto [r]$ is homomorphism
- $\pi_\mathfrak{a}$ yields one-to-one correspondence:

$$\{\text{ideals in } R \text{ containing } \mathfrak{a}\} \longleftrightarrow \{\text{ideals in } R/\mathfrak{a}\}$$

- carries maximal/prime/radical ideals to maximal/prime/radical ideals

$\mathbb{C}$-algebra

- $\mathbb{C}$-algebra $\mathcal{A}$ is both
  1. ring (associative, commutative, unital)
  2. vector space over $\mathbb{C}$
- $\mathbb{C}$-subalgebras $J \subseteq \mathcal{A}$ subset that are $\mathbb{C}$-algebras:
  - $\mathbb{C}$-subalgebras generated by set $S \subseteq \mathcal{A}$ is

$$
\begin{aligned}
[S] &= \bigcap\{: S \subseteq J,\ J \subseteq \mathcal{A} \text{ a } \mathbb{C}\text{-subalgebra}\} \\
&= \text{smallest } \mathbb{C}\text{-subalgebra containing } S \\
&= \{f(s_1, \ldots, s_n) \in \mathcal{A} : f \in \mathbb{C}[x_1, \ldots, x_n],\ s_i \in S\}
\end{aligned}
$$

  - J finitely generated if for some $s_1, \ldots, s_m \in \mathcal{A}$,

$$J = [s_1, \ldots, s_m]$$

  - caution: unlike ideals, may not define quotient $\mathcal{A}/J$
- $\mathbb{C}$-algebra homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ is both
  1. ring homomorphism
  2. $\mathbb{C}$-linear transformation

examples

- $\mathbb{C}[x, y]$ is $\mathbb{C}$-algebra finitely generated by x, y
- quotient ring $\mathbb{C}[x, y]/\langle x^2 + y^3\rangle$ also $\mathbb{C}$-algebra
- $\mathbb{C}$-algebra homomorphism $\varphi : \mathbb{C}[x, y]/\langle x^2 + y^3\rangle \to \mathbb{C}[z]$ completely determined by where it sends generators, e.g.

$$\varphi(x) = z^3, \quad \varphi(y) = -z^2,$$
$$\varphi(xy - 3y^2) = \varphi(x)\varphi(y) - 3\varphi(y)^2 = -z^5 - 3z^4$$

- complex conjugation

$$\mathbb{C}[x] \to \mathbb{C}[x]$$
$$a_0 + a_1 x + \cdots + a_n x^n \mapsto \bar{a}_0 + \bar{a}_1 x + \cdots + \bar{a}_n x^n$$

  is ring homomorphism but not $\mathbb{C}$-algebra homomorphism
- most important source of examples for us: *coordinate rings* of affine varieties

$\mathbb{C}$-valued functions

- we studied morphisms $X \to Y$ between affine varieties
- now we consider special case when $Y = \mathbb{C}$
- why: to understand a mathematical object, it helps to understand $\mathbb{C}$-valued functions on that object
- cf. C^*-algebra and von Neumann algebra from yesterday
- want the ‘right level’ of regularity:

  C^*-algebra: continuous functions on locally compact Haudorff space

  von Neumann algebra: L^∞-functions on σ-finite measure space

  finitely generated reduced $\mathbb{C}$-algebra: polynomial functions on affine variety

- *polynomial function* means $f \in \mathbb{C}[x_1, \ldots, x_n]$
  - defines $\mathbb{C}$-valued function $f : \mathbb{A}^n \to \mathbb{C}$
  - restricts to $\mathbb{C}$-valued function $f : V \to \mathbb{C}$ for affine variety $V \subseteq \mathbb{A}^n$

coordinate ring

- coordinate rings of affine variety $V \subseteq \mathbb{A}^n$ is

$$\mathbb{C}[V] := \{f : V \to \mathbb{C} : f \in \mathbb{C}[x_1, \ldots, x_n]\}$$

- clearly a $\mathbb{C}$-algebra
- $\varphi : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[V]$, $f \mapsto f|_V$ is homomorphism and since $\ker(\varphi) = \mathbb{I}(V)$

$$\mathbb{C}[V] \simeq \mathbb{C}[x_1, \ldots, x_n]/\mathbb{I}(V)$$

- for any $V \subseteq \mathbb{A}^n$, $\mathbb{C}[V]$ is always a finitely generated reduced $\mathbb{C}$-algebra by what we saw yesterday:

$$\{\text{affine varieties in } \mathbb{A}^n\} \longleftrightarrow \{\text{radical ideals in } \mathbb{C}[x_1, \ldots, x_n]\}$$

- i.e., $\mathbb{I}(V)$ always a radical ideal

converse also true

- can show: any finitely generated reduced $\mathbb{C}$-algebra is the coordinate ring of some affine variety $V \subseteq \mathbb{A}^n$
- get correspondence

$$\{\text{affine varieties in } \mathbb{A}^n\} \longleftrightarrow \{\text{fin. gen. reduced } \mathbb{C}\text{-algebra}\}$$

- conditions too restrictive: e.g. may have nilpotent elements, infinitely generated, $\mathbb{Z}$-module instead of $\mathbb{C}$-vector space
- e.g. Fermat's last theorem

$$R = \mathbb{Z}[x, y, z]/\langle x^n + y^n - z^n \rangle$$

- Grothendieck's answer: use affine schemes

$$\{\text{affine schemes}\} \longleftrightarrow \{\text{unital commutative rings}\}$$

examples

Hilbert nullstellensatz: $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$

1. $\mathbb{C}[\mathbb{A}^n] = \mathbb{C}[x_1, \ldots, x_n]$
2. $V = \mathbb{V}(x^2 + y^2 + z^2) \subseteq \mathbb{A}^3$, $\mathbb{C}[V] \simeq \mathbb{C}[x, y, z]/\langle x^2 + y^2 + z^2 \rangle$

$$x^2 + y^2 + z^2 = 0 \quad \text{in } \mathbb{C}[V]$$

3. $V = \mathbb{V}(xy - 1) \subseteq \mathbb{A}^2$, $\mathbb{C}[V] \simeq \mathbb{C}[x, y]/\langle xy - 1 \rangle$

$$1/x = y \quad \text{in } \mathbb{C}[V]$$

4. $V = \mathbb{V}(x^2 + y^2 - z^2) \subseteq \mathbb{A}^3$, $\mathbb{C}[V] \simeq \mathbb{C}[x, y, z]/\langle x^2 + y^2 - z^2 \rangle$

$$x^3 + 2xy^2 - 2xz^2 + x = 2x(x^2 + y^2 - z^2) + x - x^3 = x - x^3 \quad \text{in } \mathbb{C}[V]$$

moral: arithmetic on $\mathbb{C}[V]$ is done modulo $\mathbb{I}(V)$

pullback

- morphism $F : V \to W$ of affine varieties induces unique $\mathbb{C}$-algebra homomorphism, called pullback,

$$F^* : \mathbb{C}[W] \to \mathbb{C}[V], \quad g \mapsto g \circ F$$

- converse also true: any $\mathbb{C}$-algebra homomorphism $\varphi : \mathbb{C}[W] \to \mathbb{C}[V]$ induces unique morphism $\varphi^* : V \to W$
- get correspondence

$$\{\text{morphisms } X \to Y\} \longleftrightarrow \{\text{homomorphisms } \mathbb{C}[Y] \to \mathbb{C}[X]\}$$
$$F \longmapsto F^*$$
$$\varphi^* \longleftarrow \varphi$$

- cf. smooth maps $f : M \to N$ of differential manifolds induces pullback f^* from 1-forms on N to 1-forms on M

pullbacks are useful

- pullback $F^* : \mathbb{C}[W] \to \mathbb{C}[V]$ injective iff F is dominant, i.e., image $F(V)$ is dense in W
- pullback $F^* : \mathbb{C}[W] \to \mathbb{C}[V]$ surjective iff F defines isomorphism between V and some affine subvariety of W
- how these may be applied: Ke Ye's talk next week

examples

1. morphism

$$F : \mathbb{A}^3 \to \mathbb{A}^2, \quad (x, y, z) \mapsto (x^2 y, x - z)$$

   induces pullback

$$F^* : \mathbb{C}[u, v] \to \mathbb{C}[x, y, z], \quad u \mapsto x^2 y,\ v \mapsto x - z$$

   completely determined by where it sends generators, e.g.

$$\varphi(u^2 + 5v^3) = (x^2 y)^2 + 5(x - z)^3$$

2. linear morphism

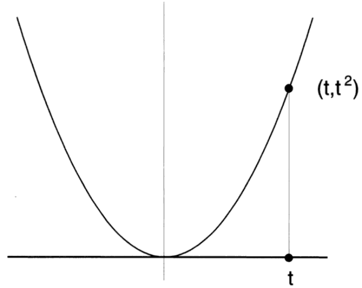
$$F : \mathbb{A}^n \to \mathbb{A}^m, \quad \mathbf{x} \mapsto A\mathbf{x}$$

   for some $A \in \mathbb{C}^{m \times n}$ has pullback

$$F^* : \mathbb{A}^m \to \mathbb{A}^n, \quad \mathbf{y} \mapsto A^{\mathsf{T}}\mathbf{y}$$

an earlier example

- parabola $C = \mathbb{V}(y - x^2) = \{(t, t^2) \in \mathbb{A}^2 : t \in \mathbb{A}\} \simeq \mathbb{A}^1$



- morphism is isomorphism of affine varieties

$$F : \mathbb{A}^1 \to C, \quad t \mapsto (t, t^2)$$

- pullback $F^* : \mathbb{C}[C] \to \mathbb{C}[\mathbb{A}^1]$ surjective with zero kernel

$$\mathbb{C}[x, y]/\langle y - x^2 \rangle \to \mathbb{C}[t], \quad x \mapsto t, \ y \mapsto t^2$$

i.e., isomorphism of $\mathbb{C}$-algebras

exercises for the audience

1. if $F = (F_1, \ldots, F_n) : \mathbb{A}^n \to \mathbb{A}^n$ is an isomorphism of affine varieties, then the Jacobian determinant,

$$\det \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix} \in \mathbb{C}^\times$$

2. show that the converse is also true[2]

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[2] just kidding: this is the Jacobian conjecture

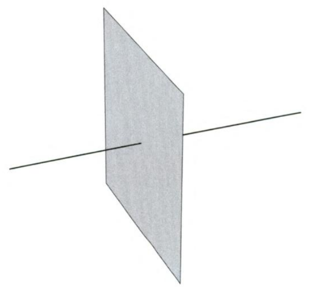
Dimension

dimension

- important notion for graphs, commutative rings, vector spaces, manifolds, metric spaces, topological spaces
- many ways to define dim(V) of affine variety $V \subseteq \mathbb{A}^n$
  1. largest d so that there exists
  $$V_d \supsetneq V_{d-1} \supsetneq \cdots \supsetneq V_1 \supsetneq V_0$$
  where V_i irreducible subvarieties of V for all $i = 1, \dots, d$
  2. largest d so that there exists
  $$\mathfrak{p}_d \supsetneq \mathfrak{p}_{d-1} \supsetneq \cdots \supsetneq \mathfrak{p}_1 \supsetneq \mathfrak{p}_0$$
  where $\mathfrak{p}_i$ prime ideals of $\mathbb{C}[V]$ for all $i = 1, \dots, d$
- second way: Krull dimension of a commutative ring
- several other ways:
  - transcendental degree of $\mathbb{C}[V]$
  - maximal dimensions of tangent space at smooth points
  - number of general hyperplanes needed to intersect V

examples

- $\dim(\mathbb{A}^1) = 1$ since $\{\text{line}\} \supsetneq \{\text{point}\}$
- $\dim(\mathbb{A}^n) = n$
- $\dim(\mathbb{V}(xy, xz)) = 2$



irreducible components $\mathbb{V}(y, z), \mathbb{V}(x)$ different dimensions

- dimension near a point $\dim_p(V)$ is largest d so that

$$V_d \supsetneq V_{d-1} \supsetneq \cdots \supsetneq V_1 \supsetneq V_0 = \{p\}$$

subvariety and dimension

- dimension of irreducible variety is same at all points
- every variety contains dense Zariski-open subset of smooth points
- dimension of variety same as dimension of complex manifold of smooth points
- if $V \subseteq W$, then $\dim(V) \leq \dim(W)$
- if $V \subseteq W$ where W irreducible, then

$$\dim(V) = \dim(W) \quad \Rightarrow \quad V = W$$

more examples

Grassmann variety: dimension same as Grassmann manifold

$$\dim(\mathrm{Gr}(n,k)) = k(n-k)$$

commuting matrix varieties: if $\mathcal{C}(k,n)$ irreducible, then

$$\dim(\mathcal{C}(k,n)) = n^2 + (k-1)n$$

nilpotent matrices: $\mathcal{N}(n) := \{X \in \mathbb{A}^{n\times n} : A^k = 0$ for some $k \in \mathbb{N}\}$ is irreducible and

$$\dim(\mathcal{N}(n)) = n^2 - n$$

morphism and dimension

- V and W vector spaces
  - $F : V \to W$ surjective linear map, then $\dim(V) \geq \dim(W)$
  - $F : V \to W$ surjective linear map, then for all $w \in W$

    $$\dim(F^{-1}(w)) = \dim(V) - \dim(W)$$

    rank-nullity theorem: $\text{nullity}(F) = \dim(V) - \text{rank}(F)$
- $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ affine varieties
  - $F : V \to W$ surjective morphism, then $\dim(V) \geq \dim(W)$
  - $F : V \to W$ surjective morphism, then for all $w \in W$,

    $$\dim(F^{-1}(w)) \geq \dim(V) - \dim(W)$$

    and for generic $w \in W$,

    $$\dim(F^{-1}(w)) = \dim(V) - \dim(W)$$

    'rank-nullity theorem for morphisms'
- why: *linear transformations on vector spaces*
  = *linear morphisms on linear affine varieties*

Acknowledgment

sources

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web:

- Mathematics Stack Exchange
- MathOverflow
- Terry Tao's Blog
- Wikipedia

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