

# AN ELEMENTARY AND UNIFIED PROOF OF GROTHENDIECK'S INEQUALITY

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ABSTRACT. We present an elementary, self-contained proof of Grothendieck's inequality that unifies both the real and complex cases and yields both the Krivine and Haagerup bounds, the current best-known explicit bounds for the real and complex Grothendieck constants respectively (although it has recently been shown that Krivine's bound is not sharp).

## 1. INTRODUCTION

We will let  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$  throughout this article. In 1953, Grothendieck proved a powerful result that he called “the fundamental theorem in the metric theory of tensor products” [19]; he showed that there exists a finite constant  $K > 0$  such that for every  $l, m, n \in \mathbb{N}$  and every matrix  $M = (M_{ij}) \in \mathbb{F}^{m \times n}$ ,

$$(1) \quad \max_{\|x_i\|=\|y_j\|=1} \left| \sum_{i=1}^m \sum_{j=1}^n M_{ij} \langle x_i, y_j \rangle \right| \leq K \max_{|\varepsilon_i|=|\delta_j|=1} \left| \sum_{i=1}^m \sum_{j=1}^n M_{ij} \varepsilon_i \delta_j \right|$$

where the maximum on the left is taken over all  $x_i, y_j \in \mathbb{F}^l$  of unit 2-norm, and the maximum on the right is taken over all  $\varepsilon_i, \delta_j \in \mathbb{F}$  of unit absolute value (i.e.,  $\varepsilon_i = \pm 1, \delta_j = \pm 1$  over  $\mathbb{R}$ ;  $\varepsilon_i = e^{i\theta_i}, \delta_j = e^{i\phi_j}$  over  $\mathbb{C}$ ). The inequality (1) has since been christened *Grothendieck's inequality* and the smallest possible constant  $K$  *Grothendieck's constant*. The value of Grothendieck's constant depends on the choice of  $\mathbb{F}$  and we will denote it by  $K_G^{\mathbb{F}}$ .

Developments these past 65 years have led to the revelation that Grothendieck's inequality is not only exceptionally important, but ubiquitous, appearing in crucial ways in diverse topics spanning pure and applied mathematics, computer science, physics, and engineering. Its initial impacts were to Banach space theory and operator theory [33, 39], followed by  $C^*$ -algebra [38, 20, 22], and, more recently, extended within the framework of noncommutative Banach spaces and noncommutative operator spaces [40, 41, 50]. What is perhaps more surprising is the appearance of Grothendieck's inequality in areas far removed from functional analysis and operator theory. By now, it is well-known that the inequality plays a central role in unique games conjecture [27, 28, 29, 42, 43], and in quantum correlations and Bell's inequalities [47, 49, 48, 18, 23]. It even appears in areas as remote as communication complexity [35, 44, 45].

Grothendieck's inequality has also inspired multiple generalizations and variants, all of which are useful and important in their own ways:

- (a) the *Grothendieck constants of finite orders*  $K_G^{\mathbb{F}}(l)$ , obtained by fixing  $l \in \mathbb{N}$  in (1) (the usual Grothendieck constant corresponds to  $K_G^{\mathbb{F}}(\infty)$ ), gives the threshold value for nonlocality of Werner states for projective measurements [1, 17, 24, 30];
- (b) the *Grothendieck constant of a graph*, where one replaces the sums over all  $i, j$  on both sides of (1) by sums over  $\{i, j\} \in E$ , the edge set of a graph (the usual Grothendieck constant corresponds to bipartite graphs), has become a potent technique for graph theory and combinatorial optimization [3, 4, 5, 6];

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2010 *Mathematics Subject Classification*. 47A07, 46B28, 46B85, 81P40, 81P45, 03D15, 97K30, 47N10, 90C27.  
*Key words and phrases*. Grothendieck inequality, Grothendieck constant, Krivine bound, Haagerup bound.

- (c) the *quadratic Grothendieck inequality*, where one requires  $x_i = y_i$  and  $\varepsilon_i = \delta_i$  in (1) (the usual Grothendieck inequality corresponds to the bilinear case), yields a clever way for polynomial-time approximation of graph partitioning via quadratic programming [12];
- (d) the *rank- $r$  Grothendieck inequality*, where the unit scalars  $\varepsilon_i$ 's and  $\delta_j$ 's on the right of (1) are replaced by unit  $r$ -dimensional vectors (the usual Grothendieck inequality corresponds to  $r = 1$ ), arises in the  $n$ -vector model in statistical mechanics and XOR games in quantum information theory [9, 11].
- (e) the *positive semidefinite Grothendieck inequality*, where the matrix  $M$  in (1) is required to be positive semidefinite (the usual Grothendieck constant corresponds to the omission of this requirement), arises in polynomial-time approximability of rank-constrained semidefinite programming problems [10, 37].

In fact, as one can surmise from some of these aforementioned works [3, 4, 5, 6, 10, 11, 9, 12, 27, 28, 29, 42, 43], Grothendieck's inequality provides a key to approximating NP-hard problems as semidefinite programs (which include special cases such as quadratic programs) and Grothendieck's constant provides a key to quantifying the limitations of such approximations.

## 2. BRIEF HISTORY OF THE PROOFS OF GROTHENDIECK'S INEQUALITY

Over the last 65 years, there have been many attempts to improve and simplify the proof of Grothendieck's inequality, and also to obtain better bounds for the Grothendieck constant  $K_G^{\mathbb{R}}$ , whose exact value remains unknown. Major milestones include:

- (i) The central result of Grothendieck's original paper [19] is that his eponymous inequality holds with  $\pi/2 \leq K_G^{\mathbb{R}} \leq \sinh(\pi/2)$  and  $4/\pi \leq K_G^{\mathbb{C}}$ . Grothendieck relied on the sign function for the real case and obtained the complex case from the real case via a complexification argument.
- (ii) The power of Grothendieck's inequality was not generally recognized until the work of Lindenstrauss and Pełczyński [33] 15 years later, which connected the inequality to absolutely  $p$ -summing operators. They elucidated and improved Grothendieck's proof in the real case by computing expectations of sign functions and using Taylor expansions, although they did not get better bounds for  $K_G^{\mathbb{R}}$ .
- (iii) Rietz [46] obtained a slightly smaller bound  $K_G^{\mathbb{R}} \leq 2.261$  in 1974 by averaging over  $\mathbb{R}^n$  with normalized Gaussian measure and using a variational argument to determine an optimal scalar map corresponding to the sign function.
- (iv) Our current best known upper bounds for  $K_G^{\mathbb{R}}$  and  $K_G^{\mathbb{C}}$  are due to Krivine [32], who in 1979 used Banach space theory and ideas in [33] to get

$$K_G^{\mathbb{R}} \leq \frac{\pi}{2 \log(1 + \sqrt{2})} \approx 1.78221;$$

and Haagerup [21], who in 1987 extended Krivine's ideas to  $\mathbb{C}$  to get

$$K_G^{\mathbb{C}} \leq \frac{8}{\pi(x_0 + 1)} \approx 1.40491,$$

where  $x_0 \in [0, 1]$  is the unique solution to:

$$x \int_0^{\pi/2} \frac{\cos^2 t}{\sqrt{1 - x^2 \sin^2 t}} dt = \frac{\pi}{8}(x + 1).$$

- (v) Our current best known lower bounds for  $K_G^{\mathbb{R}}$  and  $K_G^{\mathbb{C}}$  are due to Davie [13, 14], who in 1984 used spherical integrals to get

$$K_G^{\mathbb{R}} \geq \sup_{x \in (0,1)} \frac{1 - \rho(x)}{\max(\rho(x), f(x))} \approx 1.67696,$$

where

$$\rho(x) := \sqrt{\frac{2}{\pi}} x e^{-x^2/2}, \quad f(x) := \frac{2}{\pi} e^{-x^2} + \rho(x) \left[ 1 - \sqrt{\frac{8}{\pi}} \int_x^\infty e^{-t^2/2} dt \right];$$

and

$$K_G^{\mathbb{C}} \geq \sup_{x>0} \frac{1 - \theta(x)}{g(x)} \approx 1.33807,$$

where

$$\begin{aligned} \theta(x) &:= \frac{1}{2} \left[ 1 - e^{-x^2} + x \int_x^\infty e^{-t^2} dt \right], \\ g(x) &:= \left[ \frac{1}{x} (1 - e^{-x^2}) + \int_x^\infty e^{-t^2} dt \right]^2 + \theta(x) \left[ 1 - \frac{2}{x} (1 - e^{-x^2}) \right]. \end{aligned}$$

- (vi) Progress on improving the aforementioned bounds halted for many years. Believing that Krivine's bound is the exact value of  $K_G^{\mathbb{R}}$ , some were spurred to find matrices that yield it as the lower bound of  $K_G^{\mathbb{R}}$  [31]. The belief was dispelled in 2011 in a landmark paper [8], which demonstrated the existence of a positive constant  $\varepsilon$  such that  $K_G^{\mathbb{R}} < \pi / (2 \log(1 + \sqrt{2})) - \varepsilon$  but does not provide an explicit better bound. To date, Krivine's and Haagerup's bounds remain the best known upper bounds for  $K_G^{\mathbb{R}}$  and  $K_G^{\mathbb{C}}$  respectively.
- (vii) There have also been many alternate proofs of Grothendieck's inequality employing a variety of techniques:
- factorization of Hilbert spaces [36, 25, 39];
  - absolutely summing operators [16, 33, 40];
  - geometry of Banach spaces [2, 34];
  - metric theory of tensor product [15];
  - basic probability theory [7];
  - bilinear forms on  $C^*$ -algebra [26].

In this article, we will give yet another proof of Grothendieck's inequality but it differs from previous proofs in that it unifies both the (a) real and (b) complex cases; and that it yields both the (c) Krivine and (d) Haagerup bounds [32, 21]. It is also elementary in that it requires little more than calculus. Our proof will rely on Lemma 4.1, which is a variation of known ideas in [33, 21, 25]. In particular, the idea of using the sign function to establish (1) in the real case was due to Grothendieck himself [19] and later also appeared in [33, 32]; whereas the use of the sign function in the complex case first appeared in [21]. To be clear, all the key ideas in our proof were originally due to Lindenstrauss–Pełczyński, Krivine, and Haagerup [33, 32, 21], our only contribution is pedagogical — combining, simplifying, and streamlining their ideas into what we feel is a more palatable proof.

### 3. A UNIFIED FORM OF GROTHENDIECK'S INEQUALITY

One consequence of our earlier work [51] is that Grothendieck's inequality (1) may be expressed in an equivalent simple form:

$$(2) \quad \max_{X, Y, M \neq 0} \frac{|\operatorname{tr}(XMY)|}{\|X\|_{1,2} \|Y\|_{2,\infty} \|M\|_{\infty,1}} \leq K_G^{\mathbb{F}},$$

for matrices  $M \in \mathbb{F}^{m \times n}$ ,  $X \in \mathbb{F}^{l \times m}$ , and  $Y \in \mathbb{F}^{n \times l}$ . The characterization (2) holds over both  $\mathbb{F} = \mathbb{R}$  and  $\mathbb{C}$ .

The matrix  $(p, q)$ -norms involved are

$$\|X\|_{1,2} := \max_{z \neq 0} \frac{\|Xz\|_2}{\|z\|_1} = \max_{i=1,\dots,m} \|x_i\|_2, \quad \|Y\|_{2,\infty} := \max_{z \neq 0} \frac{\|Yz\|_\infty}{\|z\|_2} = \max_{i=1,\dots,n} \|y_i\|_2,$$

where  $x_1, \dots, x_m$  are the columns of  $X \in \mathbb{F}^{l \times m}$  and  $y_1^\top, \dots, y_n^\top$  are the rows of  $Y \in \mathbb{F}^{n \times l}$ ; also

$$\|M\|_{\infty,1} := \max_{z \neq 0} \frac{\|Mz\|_1}{\|z\|_\infty} = \max_{|\varepsilon_i|=1, |\delta_j|=1} \left| \sum_{i=1}^m \sum_{j=1}^n M_{ij} \varepsilon_i \delta_j \right|.$$

In Theorem 6.1, we will prove Grothendieck's inequality in the form of (2).

#### 4. INTEGRALS OF SIGN FUNCTION

Recall that  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ . For  $z \in \mathbb{F}$ , the sign function is

$$\operatorname{sgn}(z) = \begin{cases} z/|z| & \text{if } z \neq 0, \\ 0 & \text{if } z = 0; \end{cases}$$

and for  $z \in \mathbb{F}^n$ , the Gaussian function is

$$G_n^{\mathbb{F}}(z) = \begin{cases} (2\pi)^{-n/2} \exp(-\|z\|_2^2/2) & \text{if } \mathbb{F} = \mathbb{R}, \\ \pi^{-n} \exp(-\|z\|_2^2) & \text{if } \mathbb{F} = \mathbb{C}. \end{cases}$$

Lemma 4.1 below is based on [25, 21]. It plays an important role in our proof because the right side of (3) depends only on the inner product  $\langle u, v \rangle$ . In addition, the functions on the right are homeomorphisms and admit Taylor expansions, making it possible to expand them in powers  $\langle u, v \rangle^d = \langle u^{\otimes d}, v^{\otimes d} \rangle$ , which will come in useful when we prove Theorem 6.1.

**Lemma 4.1.** *Let  $u, v \in \mathbb{F}^n$  with  $\|u\|_2 = \|v\|_2 = 1$ . Then*

$$(3) \quad \int_{\mathbb{F}^n} \operatorname{sgn}\langle z, \bar{u} \rangle \overline{\operatorname{sgn}\langle z, \bar{v} \rangle} G_n^{\mathbb{F}}(z) dz = \begin{cases} \frac{2}{\pi} \arcsin\langle u, v \rangle & \text{if } \mathbb{F} = \mathbb{R}, \\ \langle u, v \rangle \int_0^{\pi/2} \frac{\cos^2 t}{(1 - |\langle u, v \rangle|^2 \sin^2 t)^{1/2}} dt & \text{if } \mathbb{F} = \mathbb{C}. \end{cases}$$

*Proof.* CASE I:  $\mathbb{F} = \mathbb{R}$ . Let  $\arccos\langle u, v \rangle = \theta$ , so that  $\theta \in [0, \pi]$  and  $\arcsin\langle u, v \rangle = \pi/2 - \theta$ . Choose  $\alpha, \beta$  such that  $0 < \beta - \alpha < \pi$  and define

$$E(\alpha, \beta) = \{(r \cos \theta, r \sin \theta, x_3, \dots, x_n) : r \geq 0, \alpha \leq \theta \leq \beta\}.$$

The Gaussian measure of a measurable set  $A$  is the integral of  $G_n^{\mathbb{R}}(x)$  over  $A$ . Upon integrating with respect to  $x_3, \dots, x_n$ , the following term remains:

$$\frac{1}{2\pi} \int_{E(\alpha, \beta)} e^{-\frac{1}{2}(x_1^2 + x_2^2)} dx_1 dx_2 = \frac{1}{2\pi} \int_\alpha^\beta d\theta \int_0^\infty r e^{-\frac{1}{2}r^2} dr = (\beta - \alpha)/2\pi.$$

Hence the Gaussian measure of  $E(\alpha, \beta)$  is  $(\beta - \alpha)/2\pi$ . Since there is an isometry  $T$  of  $\mathbb{R}^n$  such that  $Tu = e_1$  and  $Tv = (\cos \theta, \sin \theta, 0, \dots, 0)$ , the left side of (3) may be expressed as

$$\int_{\mathbb{R}^n} \operatorname{sgn}\langle Tu, x \rangle \operatorname{sgn}\langle Tv, x \rangle G_n^{\mathbb{R}}(x) dx.$$

The set of  $x$  where  $\langle Tu, x \rangle > 0$  and  $\langle Tv, x \rangle > 0$  is  $E(\theta - \pi/2, \pi/2)$ , which has Gaussian measure  $(\pi - \theta)/2\pi$ ; ditto for  $\langle Tu, x \rangle < 0$  and  $\langle Tv, x \rangle < 0$ . The set of  $x$  where  $\langle Tu, x \rangle < 0$  and  $\langle Tv, x \rangle > 0$  is  $E(\pi/2, \theta + \pi/2)$ , which has Gaussian measure  $\theta/2\pi$ ; ditto for  $\langle Tu, x \rangle > 0$  and  $\langle Tv, x \rangle < 0$ . Hence the value of this integral is  $(\pi - \theta)/2\pi + (\pi - \theta)/2\pi - \theta/2\pi - \theta/2\pi = 2 \arcsin\langle u, v \rangle/\pi$ .

CASE II:  $\mathbb{F} = \mathbb{C}$ . We define vectors  $\alpha, \beta \in \mathbb{R}^{2n}$  with  $\alpha_{2i-1} = \Re(u_i)$ ,  $\alpha_{2i} = -\Im(u_i)$ ,  $\beta_{2i-1} = \Re(v_i)$ ,  $\beta_{2i} = -\Im(v_i)$ ,  $i = 1, \dots, n$ . Then  $\alpha$  and  $\beta$  are unit vectors in  $\mathbb{R}^{2n}$ . For any  $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ , we write

$$x = (\Re(z_1), \Im(z_1), \dots, \Re(z_n), \Im(z_n)) \in \mathbb{R}^{2n}.$$

Then,

$$\Re(\langle z, \bar{u} \rangle) = \sum_{i=1}^n \Re(u_i z_i) = \sum_{i=1}^n (\Re(u_i) \Re(z_i) - \Im(u_i) \Im(z_i)) = \langle x, \alpha \rangle,$$

and likewise  $\Re(\langle z, \bar{v} \rangle) = \langle x, \beta \rangle$ . By a change-of-variables and Case I, we have

$$\begin{aligned} \int_{\mathbb{C}^n} \operatorname{sgn}(\Re\langle z, \bar{u} \rangle) \operatorname{sgn}(\Re\langle z, \bar{v} \rangle) G_n^{\mathbb{C}}(z) dz &= \int_{\mathbb{R}^{2n}} \operatorname{sgn}\langle x, \alpha \rangle \operatorname{sgn}\langle x, \beta \rangle G_{2n}^{\mathbb{R}}(x) dx \\ (4) \qquad \qquad \qquad &= \frac{2}{\pi} \arcsin\langle \alpha, \beta \rangle = \frac{2}{\pi} \arcsin(\Re\langle u, v \rangle). \end{aligned}$$

It is easy to verify that for any  $z \in \mathbb{C}$ ,

$$(5) \qquad \qquad \qquad \operatorname{sgn}(z) = \frac{1}{4} \int_0^{2\pi} \operatorname{sgn}(\Re(e^{-i\theta} z)) e^{i\theta} d\theta.$$

By (4), (5), and Fubini's theorem,

$$\begin{aligned} &\int_{\mathbb{C}^n} \operatorname{sgn}\langle z, \bar{u} \rangle \overline{\operatorname{sgn}\langle z, \bar{v} \rangle} G_n^{\mathbb{C}}(z) dz \\ &= \frac{1}{16} \int_0^{2\pi} \int_0^{2\pi} \int_{\mathbb{C}^n} \operatorname{sgn}(\Re(\langle z, e^{-i\theta} u \rangle)) \operatorname{sgn}(\Re(\langle z, e^{-i\varphi} v \rangle)) e^{i(\theta-\varphi)} G_n^{\mathbb{C}}(z) dz d\theta d\varphi \\ &= \frac{1}{8\pi} \int_0^{2\pi} \int_0^{2\pi} \arcsin(\Re(\langle e^{-i\theta} u, e^{-i\varphi} v \rangle)) e^{i(\theta-\varphi)} d\theta d\varphi \end{aligned}$$

CASE II(a):  $\langle u, v \rangle \in \mathbb{R}$ . The integral above becomes

$$\begin{aligned} &= \frac{1}{8\pi} \int_0^{2\pi} \left[ \int_0^{2\pi} \arcsin(\cos(\theta - \varphi) \langle u, v \rangle) e^{i(\theta-\varphi)} d\theta \right] d\varphi \\ &= \frac{1}{8\pi} \int_0^{2\pi} \left[ \int_{-\varphi}^{2\pi-\varphi} \arcsin(\langle u, v \rangle \cos t) e^{it} dt \right] d\varphi \\ (6) \qquad \qquad \qquad &= \frac{1}{8\pi} \int_0^{2\pi} \left[ \int_0^{2\pi} \arcsin(\langle u, v \rangle \cos t) e^{it} dt \right] d\varphi = \frac{1}{4} \int_0^{2\pi} \arcsin(\langle u, v \rangle \cos t) e^{it} dt. \end{aligned}$$

Since  $\arcsin(\langle u, v \rangle \cos t)$  is an even function with period  $2\pi$ ,

$$\int_0^{2\pi} \arcsin(\langle u, v \rangle \cos t) \sin t dt = 0,$$

the last integral in (6) becomes

$$\frac{1}{4} \int_0^{2\pi} \arcsin(\langle u, v \rangle \cos t) \cos t dt,$$

and as  $\arcsin(\langle u, v \rangle \cos t) \cos t$  is an even function with period  $\pi$ , it becomes

$$\int_0^{\pi/2} \arcsin(\langle u, v \rangle \cos t) \cos t dt = \int_0^{\pi/2} \arcsin(\langle u, v \rangle \sin t) \sin t dt,$$

which, upon integrating by parts, becomes

$$(7) \qquad \qquad \qquad \langle u, v \rangle \int_0^{\pi/2} \frac{\cos^2 t}{(1 - |\langle u, v \rangle|^2 \sin^2 t)^{1/2}} dt.$$

CASE II(b):  $\langle u, v \rangle \notin \mathbb{R}$ . This reduces to Case II(a) by setting  $c \in \mathbb{C}$  of unit modulus so that  $c\langle u, v \rangle = |\langle u, v \rangle|$  and  $\langle cu, v \rangle \in \mathbb{R}$ , then by (7),

$$\begin{aligned} \int_{\mathbb{C}^n} \operatorname{sgn}\langle z, \bar{u} \rangle \overline{\operatorname{sgn}\langle z, \bar{v} \rangle} G_n^{\mathbb{C}}(z) dz &= \bar{c} \int_{\mathbb{C}^n} \operatorname{sgn}\langle z, \bar{cu} \rangle \overline{\operatorname{sgn}\langle z, \bar{v} \rangle} G_n^{\mathbb{C}}(z) dz \\ &= \bar{c} \langle cu, v \rangle \int_0^{\pi/2} \frac{\cos^2 t}{(1 - |\langle cu, v \rangle|^2 \sin^2 t)^{1/2}} dt = \langle u, v \rangle \int_0^{\pi/2} \frac{\cos^2 t}{(1 - |\langle u, v \rangle|^2 \sin^2 t)^{1/2}} dt. \end{aligned}$$

□

**Corollary 4.2.** *Let  $M = (M_{ij}) \in \mathbb{F}^{m \times n}$  be a matrix with  $\|M\|_{\infty,1} \leq 1$ . Let  $x_1, \dots, x_m$  and  $y_1, \dots, y_n$  be unit vectors in a Hilbert space  $\mathcal{H}$  over  $\mathbb{F}$ . Then*

$$\left| \sum_{i=1}^m \sum_{j=1}^n M_{ij} \arcsin \langle x_i, y_j \rangle \right| \leq \frac{\pi}{2} \quad \text{if } \mathbb{F} = \mathbb{R}, \quad \left| \sum_{i=1}^m \sum_{j=1}^n M_{ij} H(\langle x_i, y_j \rangle) \right| \leq 1 \quad \text{if } \mathbb{F} = \mathbb{C},$$

where  $H$  denotes the function on the right side of (3) for  $\mathbb{F} = \mathbb{C}$ .

*Proof.* By restricting to a subspace spanned by the unit vectors, we may assume that  $\mathcal{H} = \mathbb{F}^d$  without loss of generality. The condition  $\|M\|_{\infty,1} \leq 1$  implies that

$$\begin{aligned} \left| \sum_{i=1}^m \sum_{j=1}^n M_{ij} \operatorname{sgn}\langle x_i, x \rangle \operatorname{sgn}\langle y_j, x \rangle G_d^{\mathbb{R}}(z) \right| &\leq G_d^{\mathbb{R}}(z), \\ \left| \sum_{i=1}^m \sum_{j=1}^n M_{ij} \operatorname{sgn}\langle z, \bar{x}_i \rangle \overline{\operatorname{sgn}\langle z, \bar{y}_j \rangle} G_d^{\mathbb{C}}(z) \right| &\leq G_d^{\mathbb{C}}(z), \end{aligned}$$

for any  $x \in \mathbb{R}^d$ ,  $z \in \mathbb{C}^d$  respectively. Integrating over  $\mathbb{R}^d$  or  $\mathbb{C}^d$  respectively and applying Lemma 4.1 give the required results. □

## 5. HAAGERUP FUNCTIONS

We will need to make a few observations regarding the functions on the right side of (3) for the proof of Grothendieck's inequality. Define the *complex Haagerup function* of a complex variable  $z$  by

$$H(z) := z \int_0^{\pi/2} \frac{\cos^2 t}{(1 - |z|^2 \sin^2 t)^{1/2}} dt, \quad |z| \leq 1,$$

and the *real Haagerup function*  $h$  as the restriction of  $H$  to  $[-1, 1] \subseteq \mathbb{R}$ . Observe that  $h : [-1, 1] \rightarrow [-1, 1]$  and is a strictly increasing continuous bijection. Since  $[-1, 1]$  is compact,  $h$  is a homeomorphism of  $[-1, 1]$  onto itself. By the Taylor expansion

$$(1 - x^2 \sin^2 t)^{-1/2} = \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!!} x^{2k} \sin^{2k} t, \quad |x| \leq 1, \quad 0 \leq t < \pi/2,$$

and

$$\int_0^{\pi/2} \cos^2 t \sin^{2k} t dt = \frac{\pi}{4(k+1)} \frac{(2k-1)!!}{(2k)!!},$$

we get

$$h(x) = \sum_{k=0}^{\infty} \frac{\pi}{4(k+1)} \left[ \frac{(2k-1)!!}{(2k)!!} \right]^2 x^{2k+1}, \quad x \in [-1, 1].$$

Since  $h$  is analytic at  $x = 0$  and  $h'(0) \neq 0$ , its inverse function  $h^{-1} : [-1, 1] \rightarrow [-1, 1]$  has an absolutely convergent power series

$$h^{-1}(x) = \sum_{k=0}^{\infty} b_{2k+1} x^{2k+1}, \quad x \in [-1, 1],$$

whose coefficients are given by the Lagrange inversion formula:

$$b_{2k+1} = \frac{1}{(2k+1)!} \lim_{t \rightarrow 0} \left[ \frac{d^{2k}}{dt^{2k}} \left( \frac{t}{h(t)} \right)^{2k+1} \right].$$

It is easy to verify that  $b_1 = 4/\pi$ ,  $b_{2k+1} \leq 0$  for all  $k \geq 1$ , and  $\sum_{k=0}^{\infty} |b_{2k+1}| < \infty$ .

We now turn our attention back to the complex Haagerup function. Observe that  $|H(z)| = h(|z|)$  for all  $z \in D := \{z \in \mathbb{C} : |z| \leq 1\}$  and  $\arg(H(z)) = \arg(z)$  for  $0 \neq z \in D$ . So  $H : D \rightarrow D$  is a homeomorphism of  $D$  onto itself. Let  $H^{-1} : D \rightarrow D$  be its inverse function. Since  $H(z) = \operatorname{sgn}(z)h(|z|)$ , we get

$$(8) \quad H^{-1}(z) = \operatorname{sgn}(z)h^{-1}(|z|) = \operatorname{sgn}(z) \sum_{k=0}^{\infty} b_{2k+1} |z|^{2k+1}.$$

Dini's theorem shows that the function  $\varphi(x) := \sum_{k=0}^{\infty} |b_{2k+1}| x^{2k+1}$  is a strictly increasing and continuous on  $[0, 1]$ , with  $\varphi(0) = 0$  and  $\varphi(1) = \sum_{k=0}^{\infty} |b_{2k+1}| \geq b_1 = 4/\pi > 1$ . Thus there exists a unique  $c_0 \in (0, 1)$  such that  $\varphi(c_0) = 1$ . So

$$1 = \varphi(c_0) = \sum_{k=0}^{\infty} |b_{2k+1}| c_0^{2k+1} = \frac{8}{\pi} c_0 - h^{-1}(c_0),$$

where the last equality follows because  $b_1 = 4/\pi$  and  $b_{2k+1} \leq 0$  for all  $k \geq 1$ . Therefore we obtain  $h^{-1}(c_0) = 8c_0/\pi - 1$ , and if we let  $x_0 := h^{-1}(c_0) \in (0, 1)$ , then  $h(x_0) - \pi(x_0 + 1)/8 = 0$ . From the Taylor expansion of  $h(x)$ , the function  $x \mapsto h(x) - \pi(x + 1)/8$  is increasing and continuous on  $[0, 1]$ . Hence  $x_0$  is the unique solution in  $[0, 1]$  to

$$(9) \quad h(x) - \frac{\pi}{8}(x + 1) = 0$$

and  $c_0 = \pi(x_0 + 1)/8$ .

## 6. A UNIFIED PROOF OF GROTHENDIECK'S INEQUALITY

In this section, we will prove Grothendieck's inequality in a way that simultaneously yields Krivine's and Haagerup's bounds:

$$K_G^{\mathbb{R}} \leq \frac{\pi}{2 \log(1 + \sqrt{2})} =: B^{\mathbb{R}}, \quad K_G^{\mathbb{C}} \leq \frac{8}{\pi(x_0 + 1)} =: B^{\mathbb{C}},$$

where  $x_0$  is as defined in (9).

**Theorem 6.1** (Grothendieck inequality with Krivine and Haagerup bounds). *Let  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ . For any  $l, m, n \in \mathbb{N}$  and nonzero matrices  $X \in \mathbb{F}^{l \times m}$ ,  $Y \in \mathbb{F}^{n \times l}$ ,  $M \in \mathbb{F}^{m \times n}$ , we have*

$$(10) \quad \frac{|\operatorname{tr}(XMY)|}{\|X\|_{1,2} \|Y\|_{2,\infty} \|M\|_{\infty,1}} \leq B^{\mathbb{F}}.$$

*Proof.* Observe that over  $\mathbb{C}$ , since  $\|\bar{Y}\|_{2,\infty} = \|Y\|_{2,\infty}$ ,

$$\begin{aligned} \max_{X, Y, M \neq 0} \frac{|\operatorname{tr}(XMY)|}{\|X\|_{1,2} \|Y\|_{2,\infty} \|M\|_{\infty,1}} &= \max_{X, \bar{Y}, M \neq 0} \frac{|\operatorname{tr}(XMY)|}{\|X\|_{1,2} \|\bar{Y}\|_{2,\infty} \|M\|_{\infty,1}} \\ &= \max_{X, Y, M \neq 0} \frac{|\operatorname{tr}(XMY)|}{\|X\|_{1,2} \|Y\|_{2,\infty} \|M\|_{\infty,1}}. \end{aligned}$$

So it suffices to show that  $|\operatorname{tr}(XM\bar{Y})| \leq B^{\mathbb{R}}$  whenever  $\|X\|_{1,2} \leq 1$ ,  $\|Y\|_{2,\infty} \leq 1$ ,  $\|M\|_{\infty,1} \leq 1$ . Let  $X = [x_1, x_2, \dots, x_m]$  and  $Y^{\top} = [y_1, y_2, \dots, y_n]$ . Then

$$\operatorname{tr}(XM\bar{Y}) = \sum_{i=1}^m \sum_{j=1}^n M_{ij} \langle x_i, y_j \rangle$$

over both  $\mathbb{R}$  and  $\mathbb{C}$ . By the expressions for matrix  $(p, q)$ -norms after (2),  $\|X\|_{1,2} \leq 1$  and  $\|Y\|_{2,\infty} \leq 1$  are equivalent to their column vectors  $x_1, \dots, x_m$  and row vectors  $y_1, \dots, y_n$  lying in the closed unit ball of  $\mathbb{F}^l$ . Since  $\operatorname{tr}(XM\bar{Y})$  is linear/sesquilinear, therefore convex, in each factor when the other two are fixed, it attains the maximum in that factor at an extreme point of the unit ball in the respective norm. As such, we may further assume that all  $x_i$ 's and  $y_j$ 's are unit vectors.

CASE I:  $\mathbb{F} = \mathbb{R}$ . Let  $c := \operatorname{arcsinh}(1) = \log(1 + \sqrt{2})$ . Taylor expansion gives

$$(11) \quad \sin(c \langle x_i, y_j \rangle) = \sum_{k=0}^{\infty} (-1)^k \frac{c^{2k+1}}{(2k+1)!} \langle x_i, y_j \rangle^{2k+1} = \sum_{k=0}^{\infty} (-1)^k \frac{c^{2k+1}}{(2k+1)!} \langle x_i^{\otimes(2k+1)}, y_j^{\otimes(2k+1)} \rangle.$$

Let  $T(x_i)$  and  $S(y_j)$  be vectors in the direct product  $\prod_{k=0}^{\infty} (\mathbb{R}^l)^{\otimes(2k+1)}$  whose  $k$ th coordinates are given respectively by:

$$T(x_i)_k = (-1)^k \sqrt{\frac{c^{2k+1}}{(2k+1)!}} \cdot x_i^{\otimes(2k+1)} \quad \text{and} \quad S(y_j)_k = \sqrt{\frac{c^{2k+1}}{(2k+1)!}} \cdot y_j^{\otimes(2k+1)}.$$

The vector space  $\prod_{k=0}^{\infty} (\mathbb{R}^l)^{\otimes(2k+1)}$  may be given an inner product in the natural way:

$$\langle T, S \rangle := \sum_{k=0}^{\infty} \langle T_k, S_k \rangle_k$$

where  $\langle \cdot, \cdot \rangle_k$  denotes the inner product on tensor product space  $(\mathbb{R}^l)^{\otimes(2k+1)}$  induced by the inner product on  $\mathbb{R}^l$ . This also holds with  $\mathbb{C}$  in place of  $\mathbb{R}$ . With this inner product, (11) becomes:

$$\sin(c \langle x_i, y_j \rangle) = \langle T(x_i), S(y_j) \rangle \quad \text{or} \quad c \langle x_i, y_j \rangle = \arcsin \langle T(x_i), S(y_j) \rangle.$$

Moreover, since  $x_i$  and  $y_j$  are unit vectors in  $\mathbb{R}^l$ , we get

$$\|T(x_i)\|^2 = \sinh(c\|x_i\|^2) = 1 \quad \text{and} \quad \|S(y_j)\|^2 = \sinh(c\|y_j\|^2) = 1.$$

So we may apply Corollary 4.2 to obtain

$$|\operatorname{tr}(XM\bar{Y})| = \left| \sum_{i=1}^m \sum_{j=1}^n M_{ij} \langle x_i, y_j \rangle \right| = \frac{1}{c} \left| \sum_{i=1}^m \sum_{j=1}^n M_{ij} \arcsin \langle T(x_i), S(y_j) \rangle \right| \leq \frac{\pi}{2c},$$

which is Krivine's bound since  $\pi/2c = \pi/(2 \log(1 + \sqrt{2})) = B^{\mathbb{R}}$ .

CASE II:  $\mathbb{F} = \mathbb{C}$ . Let  $c_0 \in (0, 1)$  be the unique constant defined in (9) such that  $\varphi(c_0) = 1$ . By the Taylor expansion in (8),

$$(12) \quad \begin{aligned} H^{-1}(c_0 \langle x_i, y_j \rangle) &= \operatorname{sgn}(c_0 \langle x_i, y_j \rangle) \sum_{k=0}^{\infty} b_{2k+1} |c_0 \langle x_i, y_j \rangle|^{2k+1} \\ &= \operatorname{sgn}(\langle x_i, y_j \rangle) \sum_{k=0}^{\infty} b_{2k+1} |\langle x_i, y_j \rangle|^{2k+1} c_0^{2k+1} \\ &= c_0 b_1 \langle x_i, y_j \rangle + \sum_{k=1}^{\infty} \frac{b_{2k+1}}{(\operatorname{sgn}(x_i, y_j))^{2k}} c_0^{2k+1} \langle x_i^{\otimes(2k+1)}, y_j^{\otimes(2k+1)} \rangle, \end{aligned}$$

where we adopt the convention that any term with  $\langle x_i, y_j \rangle = 0$  is omitted from the sum in (12). Recall from Section 5 that  $b_1 > 0$  and  $b_{2k+1} \leq 0$  for all  $k \geq 1$ .



Analogous to Case I, we define vectors  $T(x_i)$  and  $S(y_j)$  in the direct product  $\prod_{k=0}^{\infty} (\mathbb{C}^l)^{\otimes (2k+1)}$  whose  $k$ th coordinates are given respectively by:

$$T(x_i)_k = \begin{cases} \sqrt{b_1 c_0} \cdot x_i & \text{if } k = 0, \\ \frac{-1}{(\text{sgn}\langle x_i, y_j \rangle)^{2k}} (-b_{2k+1} c_0^{2k+1})^{1/2} \cdot x_i^{\otimes (2k+1)} & \text{if } k \geq 1, \end{cases}$$

$$S(y_j)_k = \begin{cases} \sqrt{b_1 c_0} \cdot y_j & \text{if } k = 0, \\ (-b_{2k+1} c_0^{2k+1})^{1/2} \cdot y_j^{\otimes (2k+1)} & \text{if } k \geq 1, \end{cases}$$

where again any term with  $\langle x_i, y_j \rangle = 0$  is taken to be 0 in the definition of  $T(x_i)_k$ . As in Case I, we may rewrite (12) as

$$H^{-1}(c_0 \langle x_i, y_j \rangle) = \langle T(x_i), S(y_j) \rangle \quad \text{or} \quad c_0 \langle x_i, y_j \rangle = H(\langle T(x_i), S(y_j) \rangle).$$

Moreover, since  $x_i$  and  $y_j$  are unit vectors in  $\mathbb{C}^l$ , we get

$$\|T(x_i)\|^2 = \sum_{k=0}^{\infty} |b_{2k+1}| c_0^{2k+1} = \varphi(c_0) = 1$$

and likewise  $\|S(y_j)\|^2 = 1$ . So we may apply Corollary 4.2 to get

$$|\text{tr}(XMY)| = \left| \sum_{i=1}^m \sum_{j=1}^n M_{ij} \langle x_i, y_j \rangle \right| = \frac{1}{c_0} \left| \sum_{i=1}^m \sum_{j=1}^n M_{i,j} H(\langle T(x_i), S(y_j) \rangle) \right| \leq \frac{1}{c_0},$$

which is Haagerup's bound since  $1/c_0 = 8/\pi(x_0 + 1) = B^{\mathbb{C}}$ .  $\square$

#### ACKNOWLEDGMENT

The work in this article is generously supported by DARPA D15AP00109 and NSF IIS 1546413. LHL is supported by a DARPA Director's Fellowship.

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