Algebraic Voting Theory

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From Positional Voting to Algebraic Voting Theory

Positional Voting with Three Candidates

Weighting Vector: $w = [1, s, 0]^t \in \mathbb{R}^3$

- 1st: 1 point
- 2nd: s points, $0 \leq s \leq 1$
- 3rd: 0 points

Tally Matrix: $T_{\mathbf{w}} : \mathbb{R}^{3!} \to \mathbb{R}^3$

$$T_{\mathbf{w}}(\mathbf{p}) = \begin{bmatrix} 1 & 1 & s & 0 & s & 0 \\ s & 0 & 1 & 1 & 0 & s \\ 0 & s & 0 & s & 1 & 1 \end{bmatrix} \begin{array}{c} \begin{bmatrix} 2 \\ 3 \\ 0 \\ 4 \\ 0 \\ 2 \end{bmatrix} \end{array} \begin{array}{l} \text{ABC} \\ \text{ACB} \\ \text{BAC} \\ \text{BCA} \\ \text{CAB} \\ \text{CBA} \end{array} = \begin{bmatrix} 5 \\ 4 + 4s \\ 2 + 7s \end{bmatrix} \begin{array}{l} \text{A} \\ \text{B} \\ \text{C} \end{array} = \mathbf{r}$$

Linear Algebra

Tally Matrices

In general, we have a weighting vector $\mathbf{w} = [w_1, \ldots, w_n]^t \in \mathbb{R}^m$ and

$$T_{\mathbf{w}} : \mathbb{R}^{m!} \to \mathbb{R}^m.$$

Profile Space Decomposition

The effective space of $T_{\mathbf{w}}$ is $E(\mathbf{w}) = (\ker(T_{\mathbf{w}}))^{\perp}$. Note that

$$\mathbb{R}^{n!} = E(\mathbf{w}) \oplus \ker(T_{\mathbf{w}}).$$

Questions

What is the dimension of $E(\mathbf{w})$? Given $\mathbf{w}$ and $\mathbf{x}$, what is $E(\mathbf{w}) \cap E(\mathbf{x})$? What about the effective spaces of other voting procedures?

Change of Perspective

Profiles

We can think of our profile

$$\mathbf{p} = \begin{array}{c} \begin{bmatrix} 2 \\ 3 \\ 0 \\ 4 \\ 0 \\ 2 \end{bmatrix} \end{array} \begin{array}{l} \text{ABC} \\ \text{ACB} \\ \text{BAC} \\ \text{BCA} \\ \text{CAB} \\ \text{CBA} \end{array}$$

as an element of the group ring $\mathbb{R}S_3$:

$$\mathbf{p} = 2e + 3(23) + 0(12) + 4(123) + 0(132) + 2(13).$$

Change of Perspective

Tally Matrices

We can think of our tally $T_{\mathbf{w}}(\mathbf{p})$ as the result of $\mathbf{p}$ acting on $\mathbf{w}$:

$$T_{\mathbf{w}}(\mathbf{p}) = \begin{bmatrix} 1 & 1 & s & 0 & s & 0 \\ s & 0 & 1 & 1 & 0 & s \\ 0 & s & 0 & s & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 0 \\ 4 \\ 0 \\ 2 \end{bmatrix}$$

$$= 2\begin{bmatrix} 1 \\ s \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 0 \\ s \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \\ s \end{bmatrix} + 2\begin{bmatrix} 0 \\ s \\ 1 \end{bmatrix}$$

$$= (2e + 3(23) + 4(123) + 2(13)) \cdot \begin{bmatrix} 1 \\ s \\ 0 \end{bmatrix} = \mathbf{p} \cdot \mathbf{w}.$$

Representation Theory

We have just described a situation in which elements of $\mathbb{R}S_m$ act as linear transformations on the vector space $\mathbb{R}^m$. Hiding in the background is a ring homomorphism:

$$\rho : \mathbb{R}S_m \to \mathrm{End}(\mathbb{R}^m) \cong \mathbb{R}^{m\times m}.$$

In other words, each element of $\mathbb{R}S_m$ can be "**represented**" as an $m \times m$ matrix with real entries. This opens the door to a host of useful theorems and machinery from **representation theory**.

Keywords

partial ranking, cosets, orthogonal weighting vectors, Pascal's triangle, harmonic analysis on finite groups, singular value decomposition

Linear Rank Tests of Uniformity

Profiles

Ask n judges to fully rank $A_1, \ldots, A_m$, from most preferred to least preferred, and encode the resulting data as a profile $\mathbf{p} \in \mathbb{R}^{m!}$.

Example

If $m = 3$, and the alternatives are ordered lexicographically, then the profile

$$\mathbf{p} = [10, 15, 2, 7, 9, 21]^t \in \mathbb{R}^6$$

encodes the situation where 10 judges chose the ranking $A_1 A_2 A_3$, 15 chose $A_1 A_3 A_2$, 2 chose $A_2 A_1 A_3$, and so on.

Data from a Distribution

We imagine that the data is being generated using a probability distribution P defined on the permutations of the alternatives.

We want to test the null hypothesis that P is the uniform distribution. A natural starting point is the estimated *probabilities vector*

$$\widehat{P} = (1/n)\mathbf{p}.$$

If $\widehat{P}$ is far from the constant vector $(1/m!)[1, \ldots, 1]^t$, then we would reject the null hypothesis.

In general, given a subspace S that is orthogonal to $[1, \ldots, 1]^t$, we'll compute the projection of $\widehat{P}$ onto S, and we'll use the value

$$nm!\|\widehat{P}^S\|^2$$

as a test statistic.

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Linear Summary Statistics

The *marginals* summary statistic computes, for each alternative, the proportion of times an alternative is ranked first, second, third, and so on.

The *means* summary statistic computes the average rank of obtained by each alternative.

The *pairs* summary statistic computes for each ordered pair (A_i, A_j) of alternatives, the proportion of voters who ranked A_i above A_j.

Key Insight

The linear maps associated with the means, marginals, and pairs summary statistics described above are module homomorphisms. Futhermore, we can use their effective spaces (which are submodules of the data space $\mathbb{R}^{n!}$) to create our subspace S.

Decomposition

If $m \geq 3$, then the effective spaces of the means, marginals, and pairs maps are related by an orthogonal decomposition

$$\mathbb{R}^{n!} = W_1 \oplus W_2 \oplus W_3 \oplus W_4 \oplus W_5$$

into S_m-submodules such that

1. W_1 is the space spanned by the all-ones vector,
2. $W_1 \oplus W_2$ is the effective space for the means,
3. $W_1 \oplus W_2 \oplus W_3$ is the effective space for the marginals, and
4. $W_1 \oplus W_2 \oplus W_4$ is the effective space for the pairs.

Key Insight

The effective spaces for the means, marginals, and pairs summary statistics have some of the W_i in common. Thus the results of one test could have implications for the other tests.

New Directions

Extending Condorcet's Criterion

What if we focused on k candidates at a time? Can we have different "k-winners" for different values of k?

Voting for Committees

When it comes to voting for committees, what is the "best" group of symmetries to use? Does our choice make a difference?

Making Connections

Karl-Dieter Crisman has recently used the symmetry group of the permutahedron (and reversal symmetry) to create a one-parameter family of voting procedures that connects the Borda Count to the Kemeny Rule.

Take Home Message

Looking at voting theory from an algebraic perspective is gratifying and illuminating. In our experience, doing so gives rise to new techniques, deep insights, and interesting questions.

Thanks!

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