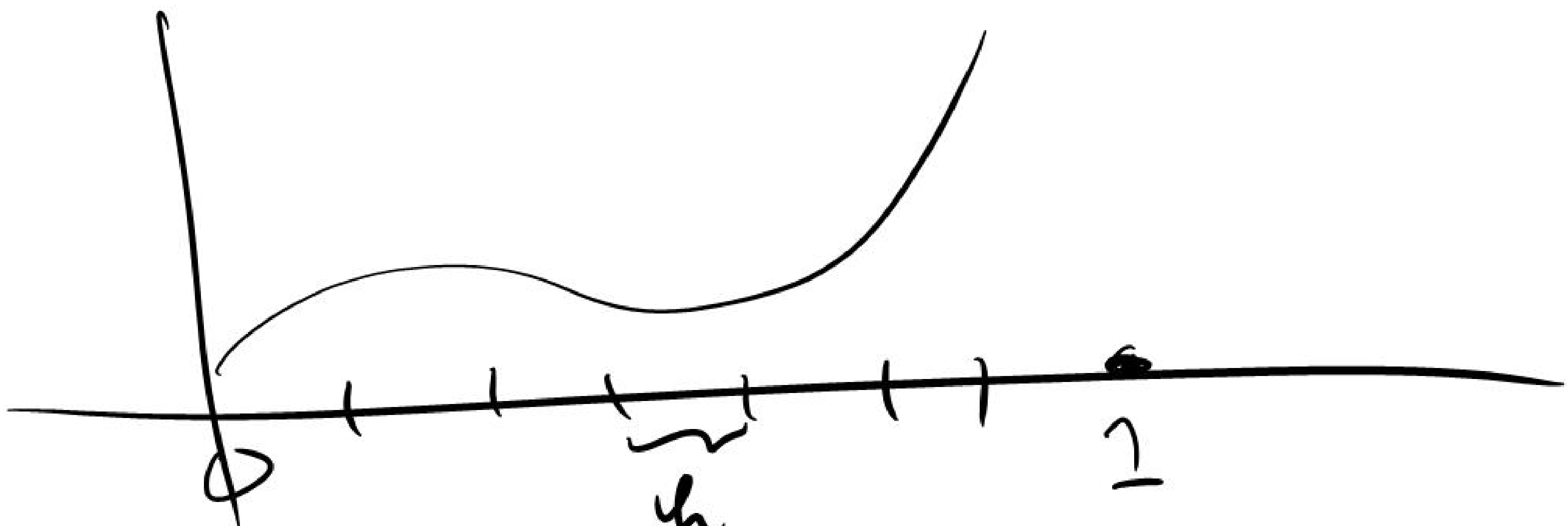
CME 302/
CS 237A

9/28/05

$$-y'' + r y' = f$$

$$0 < x < 1$$

$$y(0) = \alpha, \quad y(1) = \beta$$



$$x_i = ih, \quad h = \frac{1}{N+1}$$

$$-y''(x_i) \approx \frac{-y(x_{i-1}) + 2y(x_i) - y(x_{i+1})}{h^2}$$

$$y'(x_i) \approx \frac{y(x_{i+1}) - y(x_{i-1})}{2h}$$

$$y'(x_i) \approx \frac{y(x_{i+1}) - y(x_i)}{h}$$

$$-y'' + \sigma y' = f \qquad i = 1, \dots, N$$

$$-\frac{y_{i-1} + 2y_i - y_{i+1}}{h^2} + \sigma\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) = f_i$$

$$-\left(1+\frac{\sigma h}{2}\right)y_{i+1} + 2y_i - \left(1-\frac{\sigma h}{2}\right)y_{i-1} = h^2 f_i$$

$$i = 1, 2, \ldots, N$$

$$f_i = f(x_i)$$

$$y(0) = \alpha, \quad y(1) = \beta.$$

$$y_0 = \alpha, \quad y_{N+1} = \beta$$

Toeplitz

$$\begin{pmatrix} a & b & & 0 \\ c & \ddots & \ddots & \\ & \ddots & \ddots & b \\ 0 & & c & a \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = g$$

$$T y = g$$

T: tri-diagonal

$$-y'' + \sigma y' = f$$

$\sigma \equiv 0$, tri-diagonal matrix is symmetric $T^T = T$.

$$|y(x_i) - y_i| = ?$$

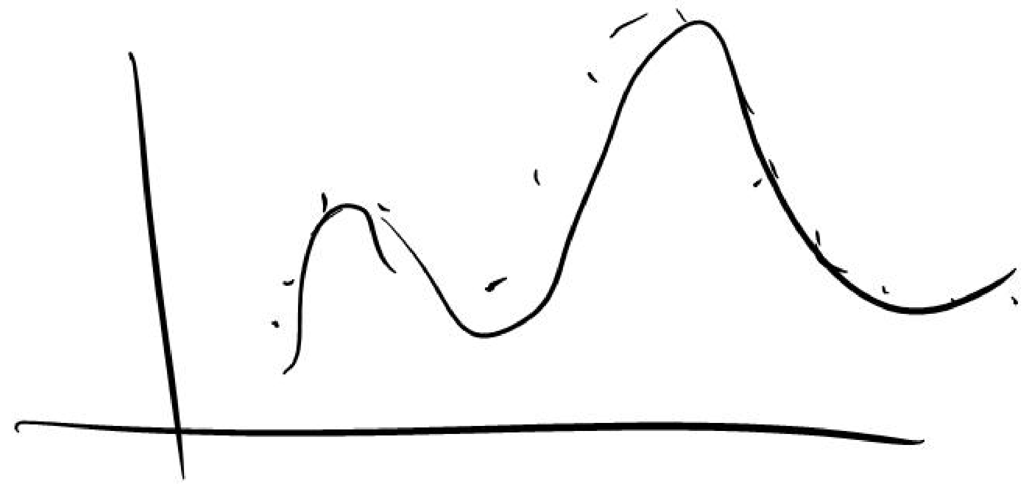
$$\begin{pmatrix} A & B & & & 0 \\ C & A & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & B \\ 0 & & & C & A \end{pmatrix}$$

$A_{p \times p}$, $B_{p \times p}$ $\qquad$ A : $\square$

BLOCK TOEPLITZ MATRIX

BLOCK TRI-DIAGONAL MATRIX

$$\{x_i, y_i\}_{i=1}^{n}$$



$$p_\ell(x) = a_0 + a_1 x + \dots + a_\ell x^\ell$$

$$\sum_{i=1}^{n} (y_i - p_\ell(x_i))^2 = \underline{\min}.$$

$$X = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^\ell \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots & x_2^\ell \\ \vdots & & & & & \\ 1 & x_n & x_n^2 & x_n^3 & \cdots & x_n^\ell \end{pmatrix}_{n \times (\ell+1)}$$

$$\underset{\sim}{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$(\underset{\sim}{y} - X\underset{\sim}{a})^T (\underset{\sim}{y} - X\underset{\sim}{a})$$

$$\boxed{\varphi(\underset{\sim}{a}) = (\underset{\sim}{y} - X\underset{\sim}{a})' (\underset{\sim}{y} - X\underset{\sim}{a})}$$

$$q(\underset{\sim}{a}) = (\underset{\sim}{y} - X\underset{\sim}{a})^T (\underset{\sim}{y} - X\underset{\sim}{a})$$

$$= \underset{\sim}{y}^T \underset{\sim}{y} - \underset{\sim}{y}^T X \underset{\sim}{a} - \underset{\sim}{a}^T X^T \underset{\sim}{y}$$

$$+ \underset{\sim}{a}^T X^T X \underset{\sim}{a}$$

$$(\underset{\sim}{a}^T X^T \underset{\sim}{y})^T = \underset{\sim}{y}^T X \underset{\sim}{a}$$

$$= \underset{\sim}{y}^T \underset{\sim}{y} - 2 a^T X^T \underset{\sim}{y} + \underset{\sim}{a}^T X^T X \underset{\sim}{a}$$

$$\frac{\partial q(a)}{\partial a_i} = -2X^T \underset{\sim}{y} + 2X^T X \underset{\sim}{a},$$

grad $d = 0$

$$\boxed{X^T X \underset{\sim}{a} = X^T y}$$

Normal Equations

$$\underset{\sim}{X}^r = \begin{pmatrix} x_1^r \\ x_2^r \\ \vdots \\ \vdots \\ x_n^r \end{pmatrix}, \quad (X^T X)_{i+1, j+1} = (\underset{\sim}{X}^i)^T \underset{\sim}{x}^j$$

$$= \sum_{k=1}^{n} x_k^{i+j} = \mu_{i+j}$$

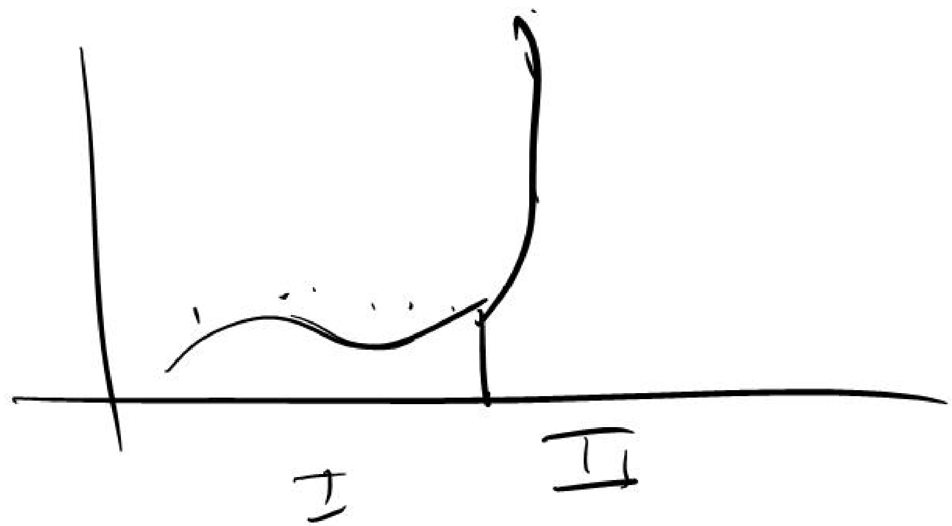
$$
X^T X = \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_\ell \\ \mu_1 & \mu_2 & \cdots & \mu_{\ell+1} \\ \vdots & & & \\ \mu_\ell & \cdots & & \mu_{2\ell} \end{pmatrix}
$$

Hankel Matrix

$$
\boxed{\phi(\hat{\underset{\sim}{a}}) = \min}
$$

Issues

Model: polynomial?



Splines

Numerical procedure: Normal Eq.

x_i : <u>vary greatly</u>

$$f_\ell(x) = a_0 + a_1(x-\mu) + a_2(x-\mu)^2 + \cdots + a_\ell(x-\mu)^\ell$$

$$f_\ell(x) = b_0 q_0(x) + \cdots + b_\ell q_\ell(x)$$

$q_k(x)$: poly of degree k

$$\sum_{i=1}^{k} q_r(x_i)\, q_s(x_i) = 0 \quad r \neq s.$$

ORTHOGONAL.

$$X =$$

$$Q = \begin{pmatrix} q_0(x_1), & q_1(x_1) & \cdot & q_\ell(x_1) \\ \vdots & & & \\ q_0(x_n) & q_1(x_n) & \cdots & q_\ell(x_n) \end{pmatrix}_{n \times (\ell+1)}$$

$$X^T X \underset{\sim}{a} = X^T \underset{\sim}{y}$$

$$Q^T Q \underset{\sim}{b} = Q^T \underset{\sim}{y}$$

$$Q^T Q = \begin{pmatrix} \ddots & & 0 \\ & \ddots & \\ 0 & & \ddots \end{pmatrix}$$

$$X_\ell = \left(\underset{\sim}{x}^0 \; \underset{\sim}{x}^1, \ldots, \underset{\sim}{x}^\ell \right)$$

$$X_{\ell+1} = \left(X_\ell \,\vdots\, \underset{\sim}{x}^{\ell+1} \right)$$

$$X_{\ell+1}^T X_{\ell+1} \underset{\sim}{a} = X_{\ell+1}^T \underset{\sim}{y}$$

$$\tilde{g}_{\ell+1}(x)$$

$$Q_{\ell+1} = \left(\tilde{g}_0(x), \tilde{g}_1(x), \ldots, \tilde{g}_{\ell+1}(x) \right)$$

$$\tilde{g}_j(x) = \begin{pmatrix} \tilde{g}_j(x_1) \\ \vdots \\ \tilde{g}_j(x_\ell) \end{pmatrix}$$

$$Q_{\ell+1}^T Q_{\ell+1} = \begin{pmatrix} \ddots & & 0 \\ & \ddots & \\ 0 & & \ddots \end{pmatrix}$$

$$Q_{l+1}^T Q_{l+1} \underset{\sim}{b}_{l+1} = Q_{l+1}^T \underset{\sim}{y}$$

$$\underset{\sim}{b}_{l+1} = \begin{pmatrix} \underset{\sim}{b}_l \\ x \end{pmatrix}$$

<u>Constraints</u>: $\underset{\sim}{a} \geq \underset{\sim}{0}$

$C^T \underset{\sim}{a} = \underset{\sim}{0}$; $\underset{\sim}{a}^T \underset{\sim}{a} = \alpha^2$

$$\phi(\underset{\sim}{a}) = \sum_{i=1}^{n} (y_i - p_k(x_i))^2$$

$$\phi(\underset{\sim}{a}) = \sum |y_i - p_k(x_i)|$$

or

$$\min_{\sim} \cancel{\max} \; \max |y_i - p_k(x_i)|$$

NORMS

Due to an unfortunate human error, slides 19–24 were lost. Please refer to the lecture notes for this part.

$\|\underset{\sim}{x}\|_2$

1. $\|\underset{\sim}{x}\|_\infty = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}} \le n^{\frac{1}{2}} \max_{i=1} |x_i|$

$= n^{\frac{1}{2}} \|\underset{\sim}{x}\|_\infty$

$c_1 = 1, \quad c_2 = \sqrt{n}$

$\frac{1}{n}\|\underset{\sim}{x}\|_1 \le \|\underset{\sim}{x}\|_\infty \le \|\underset{\sim}{x}\|_1$

$$|\underset{\sim}{x}^T \underset{\sim}{y}| \le \|\underset{\sim}{x}\|_2 \cdot \|\underset{\sim}{y}\|_2$$

Cauchy-Schwartz

Absolute error

$$\|\underset{\sim}{x} - \underset{\sim}{y}\| \quad \checkmark$$

Relative error

$$\frac{\|\underset{\sim}{x} - \underset{\sim}{y}\|}{\|\underset{\sim}{x}\|} \quad \checkmark$$

Point-wise. $\quad \dfrac{x_i - y_i}{x_i} = z_i , \quad \|\underset{\sim}{z}\|_1$