CME 302: NUMERICAL LINEAR ALGEBRA
FALL 2005/06
LECTURE 18

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1. Property A

Let A be symmetric positive definite. Then we can use diagonal scaling to obtain a matrix $D^{-1/2}AD^{-1/2}$ with all diagonal elements equal to 1 by setting

$$D = \begin{bmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{bmatrix}.$$

Then we can check whether the new matrix has *Property A*. A matrix A has Property A if there is a permutation matrix Π such that

$$\Pi^\top A \Pi = \begin{bmatrix} I_p & F \\ F^\top & I_q \end{bmatrix}. \tag{1.1}$$

For example, suppose

$$A = \begin{bmatrix} 1 & a_1 & & \\ a_1 & \ddots & \ddots & \\ & \ddots & \ddots & a_{n-1} \\ & & a_{n-1} & 1 \end{bmatrix}.$$

Then by choosing Π so that odd-numbered rows and columns are grouped together, followed by even-numbered rows and columns, we obtain

$$\Pi^\top A \Pi = \begin{bmatrix} 1 & & & & a_1 & & & \\ & \ddots & & & a_2 & \ddots & & \\ & & \ddots & & & \ddots & \ddots & \\ & & & 1 & & & \ddots & \ddots \\ a_1 & a_2 & & & 1 & & & \\ & \ddots & \ddots & & & \ddots & & \\ & & \ddots & \ddots & & & \ddots & \\ & & & \ddots & & & & \ddots \end{bmatrix}.$$

This matrix has all kinds of nice properties. In particular, it allows decoupling of equations.

It should be noted that a matrix arising from the discretization of a PDE in two dimensions using a 5-point stencil has Property A, but a matrix based on a 9-point stencil does not. However,

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the latter matrix does have *block* Property A. For example, if

$$A = \begin{bmatrix} A_1 & B_1 & & \\ B_1^\top & \ddots & \ddots & \\ & \ddots & \ddots & B_{n-1} \\ & & B_{n-1}^\top & A_n \end{bmatrix}$$

then we can choose Π so that

$$\Pi^\top A \Pi = \begin{bmatrix} A_1 & & & & B_1^\top & & & \\ & A_3 & & & B_2^\top & B_3^\top & & \\ & & \ddots & & & \ddots & \ddots & \\ & & & 1 & & & \ddots & \ddots \\ B_1 & B_2 & & & A_2 & & & \\ & B_3 & \ddots & & & A_4 & & \\ & & \ddots & \ddots & & & \ddots & \\ & & & \ddots & & & & \ddots \end{bmatrix}.$$

We now show that for a matrix of the form (1.1), we can choose an optimal parameter ω for the SOR method. Let F be a $p \times q$ matrix with $p \geq q$, and let $F = U\Sigma V^\top$ be the SVD of F. Then

$$\begin{aligned} A &= \begin{bmatrix} UU^\top & U\Sigma V^\top \\ V\Sigma^\top U^\top & VV^\top \end{bmatrix} \\ &= \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} I & \Sigma \\ \Sigma^\top & I \end{bmatrix} \begin{bmatrix} U^\top & 0 \\ 0 & V^\top \end{bmatrix}. \end{aligned}$$

Since the left and right matrices above denote a similarity transformation, it follows that

$$\lambda(A) = \lambda(\tilde{A}), \quad \tilde{A} = \begin{bmatrix} [c]ccI & \Sigma \\ \Sigma^\top & I \end{bmatrix}.$$

Reordering the rows and columns of $\tilde(A)$, we obtain a block diagonal matrix, where each diagonal block is a 2×2 matrix of the form

$$\begin{bmatrix} 1 & \sigma_i \\ \sigma_i & 1 \end{bmatrix}, \quad i = 1, \ldots, q.$$

The eigenvalues of $\tilde{A}$ are the eigenvalues of all of these diagonal blocks, which are $\lambda = 1 \pm \sigma_i$. These eigenvalues must be positive since A is positive definite, so it follows that

$$0 < \sigma_i < 1, \quad i = 1, \ldots, q.$$

2. Optimal parameter for SOR

Now, consider the SOR operator

$$\begin{aligned} \mathcal{L}_\omega &= \left(\frac{1}{\omega} I + L\right)^{-1} \left(\left(\frac{1}{\omega} - 1\right) I - U\right) \\ &= \begin{bmatrix} \frac{1}{\omega} I & 0 \\ F^\top & \frac{1}{\omega} I \end{bmatrix}^{-1} \begin{bmatrix} \left(\frac{1}{\omega} - 1\right) I & -F \\ 0 & \left(\frac{1}{\omega} - 1\right) I \end{bmatrix} \end{aligned}$$

where

$$L = \begin{bmatrix} 0 & 0 \\ F^\top & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}.$$

We can explicitly invert the first matrix to obtain

$$\mathcal{L}_\omega = \begin{bmatrix} \omega I & 0 \\ -\omega^2 F^\top & \omega I \end{bmatrix} \begin{bmatrix} \left(\frac{1}{\omega} - 1\right) I & -F \\ 0 & \left(\frac{1}{\omega} - 1\right) I \end{bmatrix}$$
$$= \begin{bmatrix} (1-\omega) I & -\omega F \\ (\omega^2 - \omega) F^\top & (1-\omega) I + \omega^2 F^\top F \end{bmatrix}.$$

Using the SVD of F again, we obtain

$$\mathcal{L}_\omega = \begin{bmatrix} (1-\omega) UU^\top & -\omega U \Sigma V^\top \\ (\omega^2 - \omega) V \Sigma^\top U^\top & (1-\omega) VV^\top + \omega^2 V \Sigma^\top \Sigma V^\top \end{bmatrix}$$
$$= \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} (1-\omega) I & -\omega \Sigma \\ (\omega^2 - \omega) \Sigma^\top & (1-\omega) I + \omega^2 \Sigma^\top \Sigma \end{bmatrix} \begin{bmatrix} U^\top & 0 \\ 0 & V^\top \end{bmatrix}.$$

Define

$$\Gamma(\omega) = \begin{bmatrix} (1-\omega) I & -\omega \Sigma \\ (\omega^2 - \omega) \Sigma^\top & (1-\omega) I + \omega^2 \Sigma^\top \Sigma \end{bmatrix}.$$

Then $\lambda(\mathcal{L}_\omega) = \lambda(\Gamma(\omega))$ and $\|\mathcal{L}_\omega\|_2 = \|\Gamma(\omega)\|_2$. Recall that

$$\mathbf{e}^k = \mathcal{L}_\omega^k \mathbf{e}^{(0)}.$$

Ideally, we want to choose ω so that $\|\mathcal{L}_\omega^k\|$ is minimized, but this is an open problem. However, Young showed how to compute ω so that $\rho(\mathcal{L}_\omega)$ is minimized. Since each block of $\Gamma(\omega)$ is a diagonal matrix, we can use the same reordering trick as before to obtain a block diagonal matrix, where each diagonal block is a 2×2 matrix of the form

$$\Gamma_i = \begin{bmatrix} (1-\omega) & -\omega \sigma_i \\ (\omega^2 - \omega) \sigma_i & (1-\omega) + \omega^2 \sigma_i^2 \end{bmatrix}, \quad i = 1, \ldots, q.$$

The eigenvalues μ of Γ_i satisfy the characteristic equation

$$(1 - \omega - \mu)^2 - \mu \sigma_i^2 \omega^2 = 0.$$

Note that when $\omega = 0$, then $|\mu| = 1$, indicating divergence. If $\omega = 1$, corresponding to the Gauss-Seidel method, then $\mu = 0$ or $\mu = \sigma_i^2$. If $\omega = 2$, then the eigenvalues are complex conjugates with $|\mu| = 1$. Therefore there exists an ω where μ becomes complex:

$$\hat{\omega} = \frac{2}{1 + \sqrt{1 - \sigma_i^2}}.$$

Thus, $|\mu(\omega_1)| > |\mu(\omega_2)|$ for $\omega_1 > \omega_2 > \hat{\omega}$.

Note that the eigenvalues of the Gauss-Seidel matrix are 0 or σ_i^2, while the eigenvalues of the Jacobi matrix are $\pm\sigma_i$. Therefore we can expect Gauss-Seidel to converge twice as fast as Jacobi for matrices with Property A.

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