CME 302: NUMERICAL LINEAR ALGEBRA
FALL 2005/06
LECTURE 14

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1. Eigenvalues of Tridiagonal Toeplitz Matrices

We will now show how we can find eigenvalues and eigenvectors of certain tridiagonal toeplitz matrices that frequently arise in difference approximations. Let

$$\hat{T} = \begin{bmatrix} 0 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{bmatrix}, \quad T(a,b) = \begin{bmatrix} a & b & & \\ b & \ddots & \ddots & \\ & \ddots & \ddots & b \\ & & b & a \end{bmatrix} = aI + b\hat{T}.$$

Note that $\lambda_j(T(a,b)) = a + b\lambda_j(\hat{T})$. We first study the case where $a = 0$ and $b = 1$; then we will consider the case $a = 4$, $b = -1$ arising from Poisson's equation.

Consider $\hat{T}\mathbf{v} = \lambda\mathbf{v}$. We can write this as a system of equations

$$\begin{aligned} v_{j-1} + v_{j+1} &= \lambda v_j \\ v_2 &= \lambda v_1 \\ v_{N-1} &= \lambda v_N \end{aligned}$$

Since $\hat{T}$ is symmetric, it has the decomposition $\hat{T} = V\Lambda V^\top$, and therefore we can write $T(a,b) = V\Lambda(a,b)V^\top$ where $\Lambda(a,b) = aI + b\Lambda$.

We guess that

$$v_j = A\sin j\theta + B\cos j\theta.$$

Substituting this representation into $T\mathbf{v} = \lambda\mathbf{v}$ yields

$$\begin{aligned} \lambda v_j &= \lambda(A\sin j\theta + B\cos j\theta) \\ &= A\sin(j-1)\theta + B\cos(j-1)\theta + A\sin(j+1)\theta + B\cos(j+1)\theta \\ &= A[\sin(j-1)\theta + \sin(j+1)\theta] + B[\cos(j-1)\theta + \cos(j+1)\theta] \\ &= A(2\sin j\theta\cos\theta) + B(2\cos\theta\cos j\theta) \\ &= 2\cos\theta v_j \end{aligned}$$

which yields $\lambda = 2\cos\theta$.

We use the boundary conditions to find θ. Our representation of v_j yields

$$\begin{aligned} A\sin 2\theta + B\cos 2\theta &= 2\cos\theta(A\sin\theta + B\cos\theta) \\ A\sin(N-1)\theta + B\cos(N-1)\theta &= 2\cos\theta(A\sin N\theta + B\cos N\theta) \end{aligned}$$

which can be written as a system of two equations for the two unknowns A and B,

$$\begin{aligned} (\sin 2\theta - 2\cos\theta\sin\theta)A + (\cos 2\theta - 2\cos\theta\sin\theta)B &= 0 \\ (\sin(N-1)\theta - 2\cos\theta\sin N\theta)A + (\cos(N-1)\theta - 2\cos\theta\cos\theta) &= 0 \end{aligned}$$

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or, in matrix form,

$$\begin{bmatrix} 0 & -1 \\ \times & \times \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which yields $B = 0$. In order for A to be nonzero, we must have

$$\begin{aligned} 0 &= \sin(N_1)\theta - 2\cos\theta\sin N\theta \\ &= \sin N\theta\cos\theta - \sin\theta\cos N\theta - 2\cos\theta\sin N\theta \\ &= -\sin N\theta\cos\theta - \sin\theta\cos N\theta \\ &= -\sin(N+1)\theta \end{aligned}$$

which yields

$$\theta_k = \frac{j\pi}{N+1}, \quad \lambda_k = 2\cos\left(\frac{k\pi}{N+1}\right).$$

Thus the largest eigenvalue is $\lambda_1 = 2\cos\pi h \approx 2 = \|\hat{T}\|_\infty$. Note that the eigenvalues are not uniformly distributed on the interval $[0, 2]$.

The eigenvectors are given by

$$v_{kj} = A\sin\left(\frac{kj\pi}{N+1}\right).$$

We want normalized eigenvectors, so we take A so that $\|\mathbf{v}_k\|_2^2 = 1$, which yields

$$A = \sqrt{\frac{2}{N+1}}.$$

Recall that $T(a, b) = aI + b\hat{T}$, where $\hat{T} = V\Lambda V^\top$ and $V = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_N \end{bmatrix}$. Thus $\lambda_k(a, b) = a + 2b\cos(k\pi/N + 1)$.

Suppose $T(a, b)\mathbf{u} = \mathbf{e}$. Then the solution $\mathbf{u}$ is given by

$$\mathbf{u} = V\Lambda^{-1}V^\top\mathbf{e} = V\Lambda^{-1}\hat{\mathbf{e}}$$

where

$$\hat{e}_k = \sum_{i=1}^{N} \sqrt{\frac{2}{N+1}} \sin\left(\frac{ik\pi}{N+1}\right) e_i = \mathbf{v}_k^\top\mathbf{e}.$$

This can be computed quickly using the FFT. Similarly, we can use the inverse FFT to compute $V(\Lambda^{-1}\hat{\mathbf{e}})$.

We now wish to find the eigenvalues of

$$A = \begin{bmatrix} T & -I & & & \\ -I & T & -I & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -I \\ & & & -I & T \end{bmatrix}.$$

If we define

$$Q = \begin{bmatrix} V & & \\ & \ddots & \\ & & V \end{bmatrix},$$

then

$$Q^\top AQ = \hat{A} = \begin{bmatrix} \Lambda & -I & & & \\ -I & \Lambda & -I & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -I \\ & & & -I & \Lambda \end{bmatrix}.$$

The system $\hat{A}\mathbf{w} = \mu\mathbf{w}$ has equations of the form

$$-w_{i,j-1} + \lambda w_{ij} - w_{i,j+1} = \mu w_{ij}, \quad i = 1, \ldots, N.$$

If we reorder the unknowns by columns instead of rows, then we obtain a block diagonal matrix where each diagonal block is a tridiagonal block of the form $T_k(\lambda_k, -1)$, where λ_k is an eigenvalue of T. The matrix $T_k(\lambda_k, -1)$ has eigenvalues

$$\lambda_j(T_k(\lambda_k, -1)) = \lambda_k - 2\cos\frac{j\pi}{N+1}, \quad j = 1, \ldots, N+1.$$

Therefore the eigenvalues of A are given by

$$\mu_{rs} = 4 - 2\cos\frac{r\pi}{N+1} - 2\cos\frac{s\pi}{N+1}, \quad r, s = 1, \ldots, N+1.$$

It follows that

$$\mu_{\min} = 4 - 4\cos\frac{\pi}{N+1}, \quad \mu_{\max} = 4 - 4\cos\frac{N\pi}{N+1} = 4 + 4\cos\frac{\pi}{N+1}.$$

Observe that $\mu_{\max} \leq \|A\|_\infty = 8$. However, as $N \to \infty$, $\mu_{\min} \to 0$, so the matrix becomes ill-conditioned quite rapidly as $N \to \infty$.

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