CME 302: NUMERICAL LINEAR ALGEBRA
FALL 2005/06
TAKE-HOME FINAL

GENE H. GOLUB

1. Let

$$T = \begin{bmatrix} a_1 & b_1 & & \\ c_2 & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ & & c_n & a_n \end{bmatrix}.$$

(a) Show that if

$$b_i c_{i+1} > 0$$

for all i, there exists a diagonal matrix D such that

$$B = DTD^{-1}$$

is a symmetric matrix. Give an algorithm for computing D.

(b) Consider the difference equation

$$-u_{i-1} + 2u_i - u_{i+1} + \frac{\sigma h}{2}(u_{i+1} - u_{i-1}) = f_i$$

with $u_0 = \alpha$ and $u_{N+1} = \beta$. Develop the corresponding matrix as a tri-diagonal matrix.

(c) Give conditions under which this tri-diagonal matrix may be made symmetric.

(d) From the scaled matrix $B = DTD^{-1}$ compute the eigenvalues of the Jacobi matrix analytically and show that Jacobi converges.

(e) Suppose we wish to solve

$$A\mathbf{x} = \mathbf{b}.$$

Show that if there exists a diagonal matrix D such that

$$B = DAD^{-1}$$

is symmetric and positive definite, then the SOR method converges for the original problem. (Note: This implies that if such a D exists, it is not necessary to construct it).

2. Consider the 2-dimensional Laplace's equation on the domain $0 < x < 1$, $0 < y < 1$, with the following boundary conditions:

$$u(0, y) = 0, \quad u(1, y) = y, \quad u(x, 0) = 0, \quad u(x, 1) = x^2.$$

(a) Apply SOR to solve this problem on an $(N + 1) \times (N + 1)$ grid with $N = 23$. Choose a starting vector $\mathbf{b}$ with random entries $b_i \in (0, 1)$, and iterate until the relative error is less than 10^{-5}. Solve the problem for five values of ω and find an approximate 'optimal' value $\hat{\omega}$. (Hint: You may want to fit a polynomial to the obtained number of iterations as function of ω and find its minimum). Give the optimal value of ω analytically.

(b) Now apply CG method to these equations and plot convergence behavior.

Date: December 9, 2005, version 1.0.

Please submit your solution to Lek-Heng in Gates 286 before 12 Noon, Thursday, December 15, 2005. You may **not** collaborate nor seek help from anyone other than the professor.

3. Consider Laplace's equation as described in Problem **2**. We are interested in applying the block Jacobi to this problem.

(a) By examining the eigenvalues of B_{BJ}, show that the method converges.

(b) Apply block SOR to this problem, using the optimal SOR parameter. The optimal parameter is given by

$$\omega = \frac{2}{1+\sqrt{1-\|B_{BJ}\|^2}}.$$

4. Let A be an $n \times n$ symmetric matrix that is singular. Consider the iteration

$$M\mathbf{x}^{(k+1)} = N\mathbf{x}^{(k)} + \mathbf{b}.$$

(a) Show that at least one eigenvalue of $B = M^{-1}N$ equals 1.

(b) Describe how you could modify a convergent algorithm to obtain a solution to a singular problem.

(c) Using Gauss-Seidel, compute a solution to the problem

$$A\mathbf{x} = \mathbf{b},$$

where A is an $n \times n$ matrix of the form

$$A = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots \\ & & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}$$

and $\mathbf{b}$ is any pre-specified vector. Show how to modify $\mathbf{b}$ so that it is in the range of A. Describe your modification to obtain a solution.

(d) Re-order A using the red-black ordering. Show that $n/2$ eigenvalues are zero for the Gauss-Seidel operator.

5. Suppose we want to solve the linear least squares problem

$$\min \|\mathbf{b} - A\mathbf{x}\|^2 \tag{5.1}$$

where A is $m \times n$ $(m > n)$, $\mathbf{b}$ is $m \times 1$ and $\operatorname{rank}(A) = r \le n$. Consider the iteration scheme

$$(A^\top A + \lambda W)\mathbf{x}_{i+1} = \lambda W \mathbf{x}_i + A^\top \mathbf{b}, \tag{5.2}$$

where $\lambda > 0$ and W is an $n \times n$ symmetric positive definite matrix.

(a) Let $M = A^\top A + \lambda W$ and $N = \lambda W$. Construct the iteration matrix $B = M^{-1}N$. Show that B is diagonalizable, $\rho(B) < 1$ if $A^\top A$ is non-singular.

(b) Assume $\mathbf{x}_0 = \mathbf{0}$. Show that the sequence $\{\mathbf{x}_i\}_{i=1}^{\infty}$ converges to the minimum W-norm solution to the least squares problem (5.1), even when $r < n$. Recall that for a symmetric positive definite W, the W-norm of a vector $\mathbf{x}$ is defined by

$$\|\mathbf{x}\|_W := (\mathbf{x}^\top W \mathbf{x})^{1/2}.$$

(Hint: First consider the case when $W = I$, and show that if $\mathbf{x}_0 = \mathbf{0}$, then $\mathbf{x}_i \to A^+\mathbf{b}$.)

(c) Show that when $A^\top A$ is non-singular,

$$\|\mathbf{e}_{i+1}\|_W \le \|\mathbf{e}_i\|_W,$$

where $\mathbf{e}_i = \mathbf{x} - \mathbf{x}_i$.

(d) Give the rate of convergence for this method.

(e) Consider the iteration

$$\begin{aligned} \mathbf{r}_i &= A^\top \mathbf{b} - A^\top A \mathbf{x}_i, \\ (A^\top A + \lambda W)\mathbf{z}_i &= \mathbf{r}_i, \\ \mathbf{x}_{i+1} &= \mathbf{x}_i + \mathbf{z}_i. \end{aligned}$$

Show that this is equivalent to (5.2).

(f) Going back to (5.2), show how to implement this algorithm using the QR decomposition of

$$\begin{bmatrix} A \\ \sqrt{\lambda} F \end{bmatrix}$$

where $W = F^\top F$ is a factorization of W.