STAT 280: OPTIMIZATION
SPRING 2017
PROBLEM SET 3

In the following, we write $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_{++} = (0, \infty)$. So $\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i \ge 0\}$ and $\mathbb{R}^n_{++} = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i > 0\}$.

1. (a) Find all stationary points of the cubic polynomial
$$f(x, y) = x^3 + y^3 - 3x - 12y + 20.$$
Indicate which are the local maximizers and local minimizers.

(b) A rectangular box, open at the top, has a volume of 32 cubic feet. Find the dimensions of the box so that the total surface area is minimized.

(c) Prove that for any $x \ge 0$, $y \ge 0$, we always have
$$\frac{x^2 + y^2}{4} \le e^{x+y-2}.$$

(d) Let $A \in \mathbb{S}^n$, $\mathbf{b} \in \mathbb{R}^n$, and $c \in \mathbb{R}$. Show that $\frac{1}{2}\mathbf{x}^\mathsf{T} A\mathbf{x} + \mathbf{b}^\mathsf{T}\mathbf{x} + c$ has a unique global minimizer iff $A \succ 0$. What is it?

(e) Let $A \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Show that $\mathbf{x}_*$ is a global minimizer of $\|A\mathbf{x} - \mathbf{b}\|^2$ iff $\mathbf{x}_*$ is a solution to $A^\mathsf{T} A\mathbf{x} = A^\mathsf{T}\mathbf{b}$.

2. Compute the Hessians of the following functions and decide if they are convex, concave, or neither on their respective domains.

(a) $e, f, g, h : \mathbb{R}^2_{++} \to \mathbb{R}$ defined by
$$e(x, y) = xy, \quad f(x, y) = \frac{1}{xy}, \quad g(x, y) = \frac{x}{y}, \quad h(x, y) = x^\alpha y^{1-\alpha},$$
where $\alpha \in [0, 1]$.

(b) $\varphi : \mathbb{R} \times \mathbb{R}_{++} \to \mathbb{R}$ defined by $\varphi(x, y) = x^2/y$. (Hint: Write $\nabla^2\varphi(x, y)$ as a rank-1 matrix).

(c) Approximate e, f, g, φ by a 3-term Taylor series about the point $\mathbf{x} = \left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]$. Compare the value of each function at $\mathbf{y} = \left[\begin{smallmatrix}2.1\\3.2\end{smallmatrix}\right]$ and the value of its corresponding 3-term Taylor series approximation.

3. (a) Find the Hessian of the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(\mathbf{x}) = \log(e^{x_1} + \cdots + e^{x_n})$. Show that for any $\mathbf{v} \in \mathbb{R}^n$,
$$\mathbf{v}^\mathsf{T}\nabla^2 f(\mathbf{x})\mathbf{v} = \frac{1}{(e^{x_1} + \cdots + e^{x_n})^2}\left[\left(\sum\nolimits_{i=1}^n e^{x_i}\right)\left(\sum\nolimits_{i=1}^n v_i^2 e^{x_i}\right) - \left(\sum\nolimits_{i=1}^n v_i e^{x_i}\right)^2\right].$$
Hence or otherwise, deduce that f is a convex function.

(b) Find the Hessian of the function $g : \mathbb{R}^n_{++} \to \mathbb{R}$ defined by $g(\mathbf{x}) = (x_1 \cdot \cdots \cdot x_n)^{1/n}$. Show that for any $\mathbf{v} \in \mathbb{R}^n$,
$$\mathbf{v}^\mathsf{T}\nabla^2 g(\mathbf{x})\mathbf{v} = -\frac{g(\mathbf{x})}{n^2}\left[n\sum\nolimits_{i=1}^n \frac{v_i^2}{x_i^2} - \left(\sum\nolimits_{i=1}^n \frac{v_i}{x_i}\right)^2\right]$$
Hence or otherwise, deduce that g is a concave function.

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(c) Find the Hessian of the function $h : \mathbb{R}^n_{++} \to \mathbb{R}$ defined by

$$h(\mathbf{x}) = \frac{1}{1/x_1 + \cdots + 1/x_n}.$$

By emulating what we did in the previous two parts or otherwise, decide if h is convex, concave, or neither on $\mathbb{R}^n_{++}$.

(d) Find the Hessian of the function $\varphi : \mathbb{R}^n_{++} \to \mathbb{R}$ defined by $\varphi(\mathbf{x}) = \log h(\mathbf{x})$. Decide if φ is convex, concave, or neither on $\mathbb{R}^n_{++}$.