STAT 280: OPTIMIZATION
SPRING 2017
PROBLEM SET 2

1. (a) Show that for any $A, B \in \mathbb{R}^{m\times n}$,
$$\operatorname{tr}(A^{\mathsf{T}}B)^2 \le \operatorname{tr}(A^{\mathsf{T}}A)\operatorname{tr}(B^{\mathsf{T}}B).$$
(b) Show that for any $A \in \mathbb{R}^{n\times n}$, the series
$$I + A + \frac{A^2}{2} + \cdots + \frac{A^k}{k!} + \cdots$$
is always convergent.

(c) Show that the set of invertible matrices
$$\mathrm{GL}(n) := \{A \in \mathbb{R}^{n\times n} : \det(A) \neq 0\}$$
is an open set in $\mathbb{R}^{n\times n}$.

(d) Show that the set of nilpotent matrices $\{A \in \mathbb{R}^{n\times n} : A^k = 0 \text{ for some } k \in \mathbb{N}\cup\{0\}\}$ is a closed set in $\mathbb{R}^{n\times n}$.

(e) Show that the set of symmetric positive definite matrices
$$\mathbb{S}^n_{++} := \{A \in \mathbb{S}^n : \mathbf{x}^{\mathsf{T}}A\mathbf{x} > 0 \text{ for all } \mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n\}$$
is an open set in $\mathbb{S}^n := \{A \in \mathbb{R}^{n\times n} : A^{\mathsf{T}} = A\}$ and that its closure is the set of symmetric positive semidefinite matrices
$$\mathbb{S}^n_{+} := \{A \in \mathbb{S}^n : \mathbf{x}^{\mathsf{T}}A\mathbf{x} \ge 0 \text{ for all } \mathbf{x} \in \mathbb{R}^n\}.$$

2. (a) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and $S \subseteq \mathbb{R}^n$ be a compact set. By emulating the proofs for the case $n = 1$ or otherwise, show that (i) there exists $M > 0$ such that
$$|f(\mathbf{x})| \le M \quad \text{for all } \mathbf{x} \in S;$$
and (ii) there exist $\mathbf{x}_{\min}, \mathbf{x}_{\max} \in S$ such that
$$f(\mathbf{x}_{\min}) \le f(\mathbf{x}) \le f(\mathbf{x}_{\max}) \quad \text{for all } \mathbf{x} \in S.$$
(b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be in $C^1(\mathbb{R}^n)$. Suppose f satisfies
$$\left|\frac{\partial f}{\partial x_i}(\mathbf{x})\right| \le K, \quad i = 1, \dots, n,$$
for all $\mathbf{x} \in \mathbb{R}^n$ where $K > 0$ is a constant. Show that
$$|f(\mathbf{x}) - f(\mathbf{y})| \le \sqrt{n}K\|\mathbf{x} - \mathbf{y}\|_2$$
for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ (Hint: Apply the univariate mean-value theorem n times).

(c) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by
$$f(x) = \begin{cases} \left(1 - \cos\dfrac{x^2}{y}\right)\sqrt{x^2 + y^2} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$
Show that f is continuous but not differentiable at $(0, 0)$. For which $\mathbf{v} \in \mathbb{R}^2$ does the directional derivative $d_{\mathbf{v}}f(0, 0)$ exist?

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3. Let $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ be the matrix-valued function

$$A(x) = \begin{bmatrix} a_{11}(x) & \cdots & a_{1n}(x) \\ \vdots & & \vdots \\ a_{n1}(x) & \cdots & a_{nn}(x) \end{bmatrix}$$

where the functions $a_{ij} : \mathbb{R} \to \mathbb{R}$ are all in $C^1(\mathbb{R})$ for all $i, j = 1, \dots, n$.

(a) Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \operatorname{tr}\left(A(x)^3\right)$ is differentiable and that

$$f'(x) = 3 \operatorname{tr}\left(A(x)^2 A'(x)\right)$$

where

$$A'(x) = \begin{bmatrix} a'_{11}(x) & \cdots & a'_{1n}(x) \\ \vdots & & \vdots \\ a'_{n1}(x) & \cdots & a'_{nn}(x) \end{bmatrix}.$$

(b) Let $n = 2$. Suppose $\det A(x) > 0$ for all $x \in \mathbb{R}$ and define $B : \mathbb{R} \to \mathbb{R}^{2\times 2}$ by $B(x) = A(x)^{-1}$. Show that

$$\frac{d}{dx} \log \det A(x) = \sum\nolimits_{i,j=1}^{2} a'_{ij}(x) b_{ji}(x).$$

This is actually true for arbitrary n. For $n = 1$, it reduces to $(\log a(x))' = a'(x)/a(x)$.

4. (a) Find the total derivative of the function $f : \mathbb{R}^4 \to \mathbb{R}^{2\times 2}$ defined by

$$f(x, y, z, w) = \begin{bmatrix} x & y \\ z & w \end{bmatrix}^2 + \begin{bmatrix} x & y \\ z & w \end{bmatrix}^{\mathsf{T}} = X^2 + X^{\mathsf{T}}.$$

Find a point $\mathbf{x} \in \mathbb{R}^4$ where $Df(\mathbf{x})$ is invertible and another where it is not.

(b) Consider the functions $f, g : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ defined by

$$f(X) = X + X^2, \quad g(X) = X^4.$$

Show that the total derivatives $Df(X)$ and $Dg(X)$ are given by

$$[Df(X)](Y) = Y + XY + YX, \quad [Dg(X)](Y) = X^3Y + X^2YX + XYX^2 + YX^3,$$

for all $Y \in \mathbb{R}^{n\times n}$. Are these invertible when $X = 0$, the zero matrix?

5. Find the gradients of the following functions.

(a) $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(\mathbf{x}) = \|\mathbf{x}\|_2^2.$$

(b) $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(\mathbf{x}) = \|\mathbf{x}\|_2.$$

Find also the set of points where f is not differentiable.

(c) $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(\mathbf{x}) = \frac{\mathbf{x}^{\mathsf{T}} A \mathbf{x}}{\mathbf{x}^{\mathsf{T}} B \mathbf{x}}$$

where $A, B \in \mathbb{S}^n_{++}$.

(d) $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ defined by

$$f(X) = \operatorname{tr}(A^{\mathsf{T}} X)$$

where $A \in \mathbb{R}^{m\times n}$.

(e) $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ defined by

$$f(X) = \mathbf{a}^{\mathsf{T}} X \mathbf{b}$$

where $\mathbf{a} \in \mathbb{R}^m$ and $\mathbf{b} \in \mathbb{R}^n$.

(f) $f : \mathbb{S}^n_{++} \to \mathbb{R}$ defined by

$$f(X) = \operatorname{tr}(X^{-1}).$$