STAT 280: OPTIMIZATION
SPRING 2017
PROBLEM SET 1

1. (a) Let $m, n \in \mathbb{N}$. Find all local and global extrema, if any, of the following functions on $\mathbb{R}$,

$$f(x) = \left(1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}\right) e^{-x}, \quad g(x) = x^m(1-x)^n, \quad h(x) = \sin^{2m} x \cdot \cos^{2n} x.$$

(b) Find the local and global extrema, if any, of f on $\mathbb{R}$ and of g on $[-1, 1]$,

$$f(x) = x^{1/3}(1-x)^{2/3}, \quad g(x) = x \sin^{-1} x + \sqrt{1-x^2}.$$

(c) Find the global minimum of the following functions on $\mathbb{R}$,

$$f(x) = -\frac{1}{1+|x|} - \frac{1}{1+|x-1|}, \quad g(x) = \begin{cases} e^{-\frac{1}{|x|}}\left(\sqrt{2} + \frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

2. Let $f \in C^2([a, b])$.

(a) Suppose there exists a constant $\alpha > 0$ such that $f''(x) = \alpha f(x)$. Show that

$$|f(x)| \leq \max\{|f(a)|, |f(b)|\}$$

for all $x \in [a, b]$.

(b) Suppose $f''(x) = e^x f(x)$. Show that f cannot have a positive local maximum or a negative local minimum in (a, b). Deduce that if $f(a) = f(b) = 0$, then f vanishes identically.

(c) Suppose $A := \sup_{x\in\mathbb{R}}|f(x)|$, $B := \sup_{x\in\mathbb{R}}|f'(x)|$, and $C := \sup_{x\in\mathbb{R}}|f''(x)|$ are finite. Prove that

$$B \leq 2\sqrt{AC}.$$

(*Hint:* Consider $f(x + 2h)$ for an appropriate h and apply Taylor's theorem.)

3. The key to the problems below is to figure out the right x to choose so that you get a univariate optimization problem.

(a) Which number is larger, π^e or e^π?

(b) Write down an expression for the area of a rectangle inscribed in the unit circle as a differentiable univariate function $f(x)$. Hence show that the largest inscribed rectangle in a circle must be a square.

(c) Light travels at different speeds in different media (e.g. air and water). Consider the following scenario depicted in Figure 1. Let v_1 be the speed of light in air and v_2 be the speed of light in water. Write down an expression for time required for light to travel from a point in airt to a point in water. Show that the minimum is attained when

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}.$$

4. Let $f \in C^2([a, b])$, $f(a)f(b) < 0$, and f' and f'' do not change signs on $[a, b]$.

(a) Show that $f(x) = 0$ has a unique solution $x_* \in (a, b)$.

(b) Prove if f' and f'' have opposite signs, then Newton method starting at $x_0 = a$ must converge to x_*.

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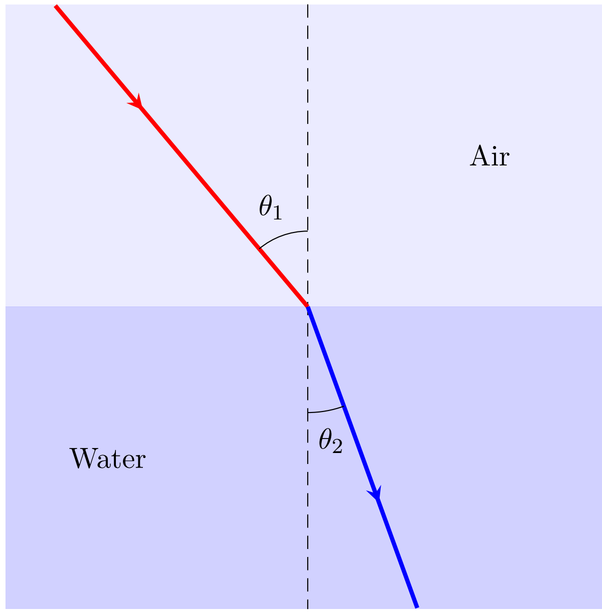

FIGURE 1. Light passing through different media.

(c) What if f' and f'' have the same sign, which starting point should we choose to guarantee convergence? Draw pictures to illustrate four different cases: (i) $f' > 0$, $f'' < 0$, (ii) $f' < 0$, $f'' > 0$, (iii) $f' > 0$, $f'' > 0$, (iv) $f' < 0$, $f'' < 0$.

(d) Let $M = \max\{|f''(x)| : x \in [a, b]\}$ and $m = \min\{|f'(x)| : x \in [a, b]\}$. Show that the errors $e_n = x_n - x_*$, $n = 0, 1, 2, \ldots$, satisfy

$$|e_{n+1}| \leq \frac{M}{2m}|e_n|^2.$$

5. In all of the following you will need to justify your answers.

(a) State whether the following sequences converge linearly, superlinearly, or quadratically (give the fastest rate):

$$\left(\frac{1}{n^2}\right)_{n=1}^{\infty}, \quad \left(\frac{1}{2^{2^n}}\right)_{n=1}^{\infty}, \quad \left(\frac{1}{\sqrt{n}}\right)_{n=1}^{\infty}, \quad \left(e^{-n}\right)_{n=1}^{\infty}, \quad \left(\frac{1}{n^n}\right)_{n=1}^{\infty}.$$

(b) Give an example of a convergent sequence whose convergence rate is slower than linear.

(c) Design an algorithm for computing reciprocal of a positive real number $a > 0$ that requires only addition and multiplication. For what values of x_0 do the algorithm converges? Apply your algorithm to find the decimal expansion of $1/12$ to 10 decimal digits of accuracy starting from $x_0 = 0.1$ and $x_0 = 1$. Discuss your results.