Radiative Transfer for Wave Propagation in Random Media. Monte Carlo Simulations of Seismic Waves*

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Abstract

Radiative transfer equations are macroscopic equations capturing the microscopic interactions between propagating waves and random fluctuations of an underlying medium. Recent studies have shown the validity of the radiative transfer theory for describing multiple scattering of seismic waves propagating in the upper lithosphere. We consider here the radiative transfer equations for shear waves taking into account their polarization components. We use a recent probabilistic representation of general radiative transfer equations with polarization parameters to devise a Monte Carlo method for the numerical simulation of shear wave propagation. We show some numerical results emphasizing the prominent role of polarization.

1 Introduction

Since the seminal work by Chandrasekhar [5] on the phenomenological derivation of radiative transfer equations for analyzing the propagation of light through turbulent atmospheres, the theory of radiative transfer has been the interest of many mathematicians and physicists. One crucial breakthrough is the discovery of several methods to obtain radiative transfer equations that model the energy density of quantum and classical waves propagating in rapidly varying inhomogeneous media [6, 8, 10]. Those methods enable us to systematically relate the parameters of the transport equations to some statistical properties of the random medium. One of the most striking examples is the derivation of elastic radiative transfer equations to model the propagation of seismic waves in the Earth's crust. Simplified radiative transfer theories have been applied successfully to the analysis of empirical *codas* generated by seismograms [1, 9]. However, these models are often scalar because of their mathematical relative simplicity, and do not account for polarization effects that are present in rigorously derived radiative transfer equations [8].

In this paper, we analyze the polarization effects of shear waves. We follow the radiative transfer model given in [8]. However, we not consider the coupling between shear and pressure waves for simplicity. Pressure waves generally amount to about 10% of the total seismic energy, and should not qualitatively modify the polarization effects of shear waves. Our numerical simulations are based on the Monte Carlo method. The basic idea is to simulate a large number of paths that the energy can follow and average over these realizations to obtain an estimate of the shear energy density. Classical probabilistic interpretations for particles however do no hold for shear waves. The reason is that the shear

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waves are described by a 2×2 coherence matrix that accounts for its polarization. We use a recent probabilistic representation for general matrix-valued radiative transfer equations [4] that overcomes the difficulty by enlarging the "velocity" space to comprise the polarization parameters. We recall the radiative transfer equations for shear waves in Section 2 and give the probabilistic representation that is used in the Monte Carlo simulations in Section 3. We show some numerical results in Section 4 and state some conclusions. More extended studies of polarization effects for radiative transfer equations can be found in [2, 3].

2 Radiative Transfer Equations

Linearly polarized shear waves cannot be described by a scalar density. They require a 2×2 non-negative hermitian matrix

$$W(t,\mathbf{x},\mathbf{k}) = \frac{1}{2}\begin{pmatrix} I+Q & U \\ U & I-Q \end{pmatrix}, \tag{1}$$

which accounts for their polarized structure. The parameter I describes the total energy of the shear waves, whereas Q and U describe its polarization [5]. We consider a homogeneous constant-velocity background medium with random inhomogeneities on the scale of the wavelength. Inhomogeneities are variations of the elastic coefficients and mass density due to changes in material composition or to the presence of cracks. Because of these fluctuations, a wave with wave vector $\mathbf{k}'$ at a point $\mathbf{x}$ may be scattered into any direction $\hat{\mathbf{k}}$ with wave vector $\mathbf{k}$. To simplify, we do not consider the coupling between shear and pressure modes. With these assumptions, energy transport is modeled by [8]

$$\frac{1}{v}\frac{\partial W(t,\mathbf{x},\mathbf{k})}{\partial t} + \hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}} W(t,\mathbf{x},\mathbf{k}) = \int \sigma(\mathbf{k},\mathbf{k}')[W(\mathbf{k}')]d\mathbf{k}' - \Sigma(\mathbf{k})W(t,\mathbf{x},\mathbf{k}), \tag{2}$$

where $\mathbf{k} = |\mathbf{k}|\hat{\mathbf{k}}$, $v = \sqrt{\mu/\rho}$ is the velocity of the homogeneous background. The elastic constant μ and density ρ are not available everywhere in the lithosphere, and are replaced by the statistical power spectral densities $\hat{R}_{\mu\mu}$ and $\hat{R}_{\rho\rho}$, defined as the Fourier transform or their two point correlation function. The differential scattering tensor has the form

$$\sigma(\mathbf{k},\mathbf{k}')[W(\mathbf{k}')] = [\sigma_{\rho\rho}T(\mathbf{k},\mathbf{k}')W(\mathbf{k}')T(\mathbf{k}',\mathbf{k}) + \sigma_{\mu\mu}\Gamma(\mathbf{k},\mathbf{k}')W(\mathbf{k}')\Gamma(\mathbf{k}',\mathbf{k})], \tag{3}$$

where the scalar functions

$$\sigma_{\rho\rho} = \frac{\pi}{2}v|\mathbf{k}|^2\hat{R}_{\rho\rho}(|\mathbf{k}-\mathbf{k}'|)\delta(v|\mathbf{k}|-v|\mathbf{k}'|),\ \sigma_{\mu\mu} = \frac{\pi}{2}v|\mathbf{k}|^2\hat{R}_{\mu\mu}(|\mathbf{k}-\mathbf{k}'|)\delta(v|\mathbf{k}|-v|\mathbf{k}'|), \tag{4}$$

are the differential scattering cross-sections coming from fluctuations in the density ρ and the Lamé parameter μ, respectively. We assume that the correlation between fluctuations in ρ and μ vanishes for simplicity; i.e., $\hat{R}_{\mu\rho} = 0$. The scalar total cross section $\Sigma(\mathbf{k})$ is such that the scattering operator on the right hand side of (2) is conservative. The 2×2 matrices $T(\mathbf{k},\mathbf{k}')$ and $\Gamma(\mathbf{k},\mathbf{k}')$ are defined by

$$\begin{aligned} T_{ij}(\mathbf{k},\mathbf{k}') &= (\mathbf{z}^{(i)}(\mathbf{k})\cdot\mathbf{z}^{(j)}(\mathbf{k}')) \\ \Gamma_{ij}(\mathbf{k},\mathbf{k}') &= (\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')(\mathbf{z}^{(i)}(\mathbf{k})\cdot\mathbf{z}^{(j)}(\mathbf{k}')) + (\hat{\mathbf{k}}\cdot\mathbf{z}^{(j)}(\mathbf{k}'))(\hat{\mathbf{k}}'\cdot\mathbf{z}^{(i)}(\mathbf{k})), \end{aligned} \tag{5}$$

with the vectors $(\hat{\mathbf{k}},\mathbf{z}^{(1)}(\mathbf{k}),\mathbf{z}^{(2)}(\mathbf{k}))$,

$$\hat{\mathbf{k}} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix},\ \mathbf{z}^{(1)}(\mathbf{k}) = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix},\ \mathbf{z}^{(2)}(\mathbf{k}) = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}, \tag{6}$$

forming a local orthonormal basis in $\mathbb{R}^3$.

3 Probabilistic Representation

We want to solve an adimensionalized version of (2) by the Monte Carlo method. This method enables us to deal with complicated 3D geometries with reasonable storage requirements. It is in addition relatively easy to implement [7, 11].

The simulation of the particle trajectories is threefold: (i) description of the particle trajectories between successive scattering events, (ii) estimation of the time of the next scattering with the underlying medium, and (iii) at the time when the particle undergoes scattering, computation of the outgoing velocity of the particle. The last two steps are random processes. As it is, this method does not hold for coherence matrices, whose entries do not correspond only to energies, but also to polarization parameters. Nonetheless, it is possible to adapt these ideas for general radiative transfer equations (see [4]) by considering the polarization parameters as independent variables, i.e., each particle will be characterized by its position $X(t)$, velocity $K(t)$, and polarization state $\mathcal{P}(t)$.

We rewrite the matrix W as

$$(7)\quad W(t,\mathbf{x},\mathbf{k}) = \frac{I}{2}\begin{pmatrix} 1+Q/I & U/I \\ U/I & 1-Q/I \end{pmatrix} = \frac{I}{2}\left[\mathbb{I}_2 + \mathcal{P}\right], \; \mathcal{P} = \begin{pmatrix} p_1 & p_2 \\ p_2 & -p_1 \end{pmatrix} = \begin{pmatrix} Q/I & U/I \\ U/I & -Q/I \end{pmatrix}.$$

Note that the intensity I is the trace of $W(t,\mathbf{x},\mathbf{k})$. In addition to the position $X(t)$ and wave vector (velocity) $K(t)$, we now have a pair of polarization processes $p_1(t)$ and $p_2(t)$ that describe the polarization parameters Q and U. The position $X(t)$ is continuous in space, while the velocity $K(t)$ and polarization parameters $\mathcal{P}(t)$ jump at the scattering times.

The evolution $(X(t), K(t), \mathcal{P}(t))$ of each particle follows the following rules. It evolves along straight characteristics between successive scatterings:

$$(8)\qquad \frac{dX}{dt} = K(t), \qquad \frac{dK}{dt} = 0, \qquad \frac{dp_i}{dt} = 0, \quad i = 1, 2.$$

At time s, the probability that the next scattering takes place after time t follows the Poisson distribution

$$(9)\qquad P(\text{Time of Next Scattering} > t) = e^{-(t-s)}.$$

When a scattering event takes place, we have to define the outgoing velocity and polarization parameters. The probability density for a jump from wave vector $\mathbf{k}$ and polarization matrix $\mathcal{P}$ into wave vector $\mathbf{k}'$ and polarization matrix $\mathcal{P}'$ is

$$(10)\qquad \pi(\mathbf{k}, \mathcal{P}, \mathbf{k}', \mathcal{P}') = C\,\mathrm{Tr}\left\{S(\mathbf{k}', \mathbf{k})[\mathbb{I}_2 + \mathcal{P}]S(\mathbf{k}, \mathbf{k}')\right\}\;,$$

where the matrix S stands for T or Γ, C is a normalization constant, and Tr stands for trace of the matrix. The matrix of the scattered polarization parameters $\mathcal{P}_s(\mathbf{k}, \mathcal{P}, \mathbf{k}')$, which is symmetric and has null trace, is determined by [4]

$$(11)\qquad \frac{1}{2}(\mathbb{I}_2 + \mathcal{P}_s) = \frac{S(\mathbf{k}', \mathbf{k})[\mathbb{I}_2 + \mathcal{P}]S(\mathbf{k}, \mathbf{k}')}{\mathrm{Tr}\{S(\mathbf{k}', \mathbf{k})[\mathbb{I}_2 + \mathcal{P}]S(\mathbf{k}, \mathbf{k}')\}}.$$

The jump probability does not depend on the outgoing polarization parameters $\mathcal{P}'$ and the outgoing polarization parameters are functions of the incoming and outgoing velocities $\mathbf{k}'$ and $\mathbf{k}$, and of the incoming polarization parameters $\mathcal{P}$.

The history of each particle is constructed as follows. The history $(X(t), K(t), \mathcal{P}(t))$ starts at $(\mathbf{x}_0, \mathbf{k}_0, \mathcal{P}^0)$. The time of first scattering τ_1 is given by the probability density (9).

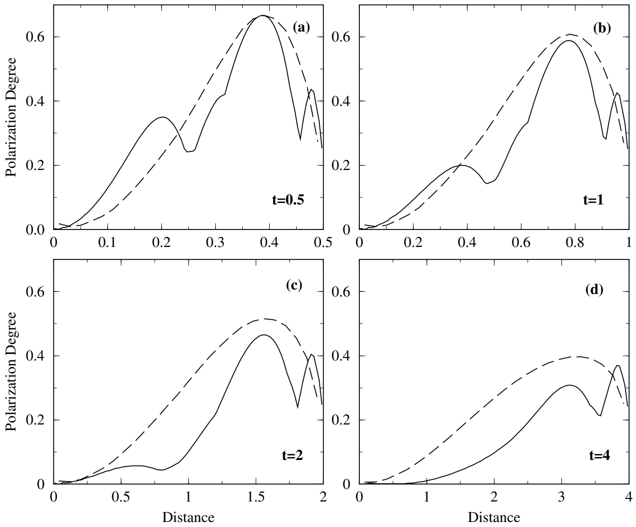

FIG. 1. *Spatial distribution of the rate of polarization for an isotropic pulsed delta source in a spherical medium with* $\hat{R}_{\rho\rho} = 0$ *(solid lines) and a medium with* $\hat{R}_{\mu\mu} = 0$ *(dashed lines) at times* $t = 0.5, 1, 2,$ *and* 4.

The process follows (8) between $0 \le t < \tau_1$. At τ_1 the process jumps to a new velocity $K(\tau_1)$ and a new polarization state $\mathcal{P}(\tau_1)$ according to (10) and (11), respectively. Then, a new scattering time τ_2 is obtained according to (9). The process is continued in an obvious manner until a final time T_f.

The validity of our code was checked by comparing the numerical results with the analytic single scattering solution [2] and with the diffusion solution for large lapse times. The agreement was perfect in both cases.

4 Numerical Results

One of our objectives was to demonstrate that the degree of polarization is sufficiently large to be observed. Moreover, its pattern is quite different for fluctuations in the Lamé parameter μ and the density ρ. We show in Fig. 1 the spatial distribution of the degree of polarization obtained in spherical geometry for different times. The degree of polarization is defined by

$$\alpha(t, \mathbf{x}) = \frac{\displaystyle\int \sqrt{Q^2(t, \mathbf{x}, \mathbf{k}) + U^2(t, \mathbf{x}, \mathbf{k})}\; d\mathbf{k}}{\displaystyle\int I(t, \mathbf{x}, \mathbf{k})\; d\mathbf{k}}. \tag{12}$$

The initial condition is an isotropic unpolarized delta source concentrated at $r = 0$. We see that more than 50% of the energy is polarized at certain locations, even for an unpolarized initial source. Moreover, the degree of polarization behaves very differently for fluctuations in μ and in ρ. We observe that the degree of polarization corresponding to fluctuations of the Lamé parameter μ has three maxima, whereas the degree of polarization for the density has only one maximum. Time traces at a fixed distance to the source display the same behavior. Therefore, qualitative measurements of polarization allow to distinguish between the two kinds of scattering considered here.

We have also simulated three-dimensional configurations and considered the energy

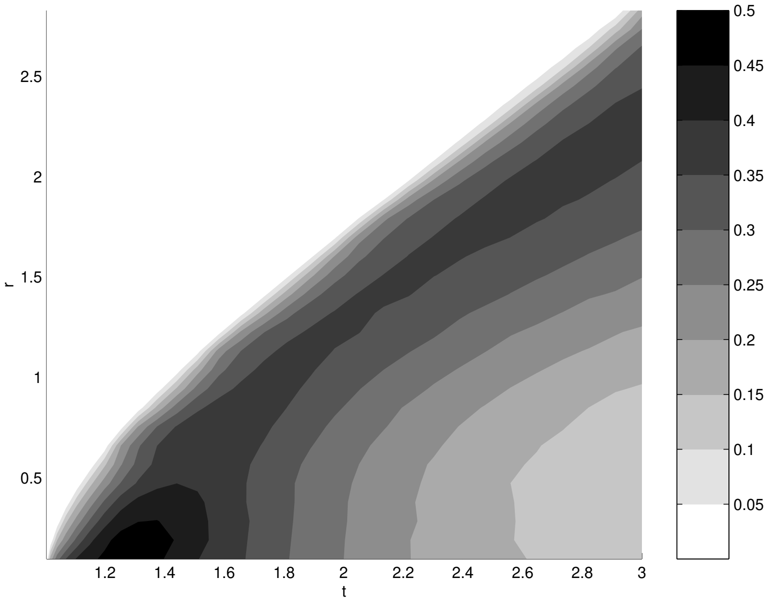

FIG. 2. *Degree of polarization on the surface in the case of density fluctuations. The non polarized isotropic point source is at a distance $z = 1$ from the interface.*

reflection at the interface between the crust and the atmosphere. Time traces of the polarization at the interface –where it can be measured in practice– show that it is still sufficiently large to be detected (see Fig. 2). We believe that polarization is a valuable piece of information to reconstruct unknown statistical properties of the lithosphere.

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