Radiative transport limit for the random Schrödinger equation

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Abstract

We give a detailed mathematical analysis of the radiative transport limit for the average phase space density of solutions of the Schrödinger equation with time dependent random potential. Our derivation is based on the construction of an approximate martingale for the random Wigner distribution.

1 Introduction

The Schrödinger equation with random potential arises in many applications, especially in wave propagation in random media, in the paraxial or parabolic approximation. In this case the time harmonic wave field has the form $u = e^{i\kappa z-\omega t}\phi(z,x)$ where $\kappa = \omega/c$ is the free space wave number, z is the coordinate in the direction of propagation, x are the coordinates in the transverse directions and ϕ satisfies the Schrödinger equation

$$2i\kappa\frac{\partial\phi}{\partial z} + \Delta_x\phi + \kappa^2\mu(z,x)\phi = 0. \tag{1}$$

Here $\mu(z,x) = n^2(z,x) - 1$ denotes the fluctuations of the index of refraction. The original scattering problem for $(\Delta + \kappa^2 n^2)u = 0$ becomes an initial value problem for ψ in z, in this approximation, so ϕ must be given at $z = 0$. The validity of the parabolic approximation in random media under different scaling limits is considered in [1, 18] and in general in [4].

The purpose of this paper is to prove a theorem that establishes the validity of the transport approximation for the average Wigner distribution of ϕ, in a suitable scaling limit and for a class of stochastic models for the index of refraction fluctuations μ that are Markovian in z. For this class of stochastic models it is possible to use martingale methods to simplify the analysis.

Since the coordinate z in the direction of propagation plays the role of time in the parabolic approximation we will denote it by t in the rest of the paper. The problem then is to analyze the Schrödinger equation with time dependent potential and to show that the associated average Wigner distribution converges to the solution of a radiative transport equation.

The propagation of wave energy in a scattering medium is described phenomenologically by radiative transport theory [8] as follows. Multiple scattering creates waves with all wave vectors $k \in \mathbb{R}^d$ at every position $x \in \mathbb{R}^d$. Let us denote by $W(t,x,k)$ the energy density of a wave having wave vector k at position x at time t. The energy balance equation has the form

$$\frac{\partial W(t,x,k)}{\partial t} + k\cdot\nabla_x W(t,x,k) = \int_{\mathbb{R}^d} dp\ \sigma(x,k,p)W(t,x,p) - \Sigma(x,k)W(t,x,k). \tag{2}$$

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Here $\sigma(x,k,p)$ is the probability to scatter from direction p into direction k at position x, and $\Sigma(x,k)$ is the total probability to scatter from direction k into some other direction. When energy is conserved and scattering is symmetric,

$$\Sigma(x,k) = \int_{\mathbb{R}^d} dp\ \sigma(x,k,p), \quad \sigma(x,k,p) = \sigma(x,p,k)$$

equation (2) may be rewritten as

$$\frac{\partial W(t,x,k)}{\partial t} + k \cdot \nabla_x W(t,x,k) = \int_{\mathbb{R}^d} dp\ \sigma(x,k,p)[W(t,x,p) - W(t,x,k)]. \tag{3}$$

While various formal derivations of the radiative transport equation (2), starting from the wave equation in a random medium, have been known since the mid-1960's (see [22] for an extensive bibliography) the mathematical methodology for doing this is not very well developed. A rigorous derivation of a spatially homogeneous transport equation starting from the Schrödinger equation is given by H. Spohn [23] who derived (2) for sufficiently short times $t \in [0, t_0]$ and with a time independent Gaussian potential. This result was extended to higher-order correlation functions by T. Ho, L. Landau and A. Wilkins [15] with the same restrictions. Recently L. Erdös and H-T. Yau [10] removed the small time restriction and considered more general initial data. The idea in these proofs is to consider the Neumann series expansion for the solution of the Schrödinger equation and to infer appropriate estimates from it that allow passage to the limit.

In this paper we deal with time dependent random potentials for which it is possible to analyze the transport approximations in a relatively simple manner, without infinite Neumann expansions. We model the random potential by a Markov process in time so that we can use martingale methods and suitable test function expansions. A limit theorem for one dimensional waves where such methods are used is given in [20] and more general ones in [6]. An analysis of the random Schrödinger equation with rapidly decorrelating in time potential is given in [21]. Limit theorems for linear random operator equations that decorrelate rapidly in time are given in [19]. The random Schrödinger equation with delta function potential is analyzed in [9] using martingale methods and its equilibrium solutions are constructed in [12].

Formal derivations of radiative transport equations for various types of waves in random media are given in [2, 14, 22]. Appendix A contains such a derivation for the time-dependent case. The results presented here extend to linear hyperbolic systems with random coefficients such as those considered in [22]. The transport equation for the Schrödinger equation with a time-dependent potential was used recently in [7] to explain pulse stabilization in time-reversal in a paraxial approximation to the wave equation. We explain in Appendix B how the transport scaling may arise in the parabolic approximation. The same phenomenon for general wave equations in time-independent media was related to transport theory in [3]. Full justification of the results in [7] requires analysis of the higher-order correlation functions and specific scalings, as discussed in [18].

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2 The main result

2.1 The Wigner distribution and the main result

We consider the initial value problem for the Schrödinger equation (1) in dimensionless form

$$i\varepsilon \frac{\partial \phi_\varepsilon}{\partial t} + \frac{\varepsilon^2}{2} \Delta \phi_\varepsilon - \sqrt{\varepsilon} V\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) \phi_\varepsilon = 0. \tag{4}$$

The initial data $\phi_\varepsilon^0(x) = \phi_\varepsilon(0,x)$ are assumed to be uniformly bounded in $L^2(\mathbb{R}^d)$:

$$\|\phi_\varepsilon^0\|_{L^2} \leq C. \tag{5}$$

This implies that

$$\|\phi_\varepsilon(t)\|_{L^2} = \|\phi_\varepsilon^0\|_{L^2} \leq C, \quad t \geq 0 \tag{6}$$

since the L^2-norm of the solution is preserved by the Schrödinger evolution (4). Conservation of the L^2-norm implies that for every realization of the random potential $V_\varepsilon(t,x) = V(t/\varepsilon, x/\varepsilon)$ there is a sequence $\varepsilon_k \to 0$ so that the energy density

$$E_{\varepsilon_k}(t,x) = |\phi_{\varepsilon_k}(t,x)|^2$$

has a weak limit $E(t,x)$ in $L^2([0,T]\times\mathbb{R}^d)$ as $k \to \infty$.

It is well known that the limit $E(t,x)$ does not satisfy a closed equation. A convenient way to study this limit is to consider energy propagation in phase space that includes all positions x and wave vectors k using the Wigner distribution $W_\varepsilon(t,x,k)$, defined by

$$W_\varepsilon(t,x,k) = \int_{\mathbb{R}^d} \frac{dy}{(2\pi)^d} e^{ik\cdot y} \phi_\varepsilon(t, x - \frac{\varepsilon y}{2}) \phi_\varepsilon^*(t, x + \frac{\varepsilon y}{2}). \tag{7}$$

Here $*$ denotes complex conjugation. The basic properties of the Wigner distribution can be found in [13, 16]. In particular,

$$\int dk\, W_\varepsilon(t,x,k) = |\phi_\varepsilon(t,x)|^2 = E_\varepsilon(t,x)$$

but W_ε may not be interpreted as energy density in phase space since it is not necessarily positive. However, the limit of W_ε along a sub-sequence $\varepsilon_k \to 0$ exists in $\mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$, the space of Schwartz distributions, and is positive [13, 16]. We recall the proof of convergence in Section 2.3. We say that the family $\phi_\varepsilon^0(x)$ is pure if the family of its Wigner distributions $W_\varepsilon(x,k)$ converges weakly as $\varepsilon \to 0$ to a distribution $W_0(x,k) \in \mathcal{S}'$ without restriction to a subsequence. We will assume that the initial data form a pure family throughout the paper.

Our main result concerns the convergence of the expectation of the Wigner distribution $W_\varepsilon(t,x,k)$ defined by (7) to the solution of the radiative transport equation (8).

Theorem 2.1 *Let the random potential $V(t,x)$ satisfy the assumptions in Section 2.2 below. Let $W_0(x,k)$ be the limit Wigner measure of the family $\phi_\varepsilon^0(x)$, and let $\overline{W}(t,x,k)$ be the weak probabilistic solution of the transport equation*

$$\frac{\partial \overline{W}}{\partial t} + k \cdot \nabla_x \overline{W} = \mathcal{L}\overline{W} \tag{8}$$

with initial data $W_0(x,k)$ and where the operator $\mathcal{L}$ is defined by

$$\mathcal{L}\lambda = \int \frac{dp}{(2\pi)^d} \hat{R}(\frac{p^2 - k^2}{2}, p - k)(\lambda(p) - \lambda(k)). \tag{9}$$

Here $\hat{R}(\omega, p)$ is the Fourier transform of the correlation function of V, defined by (13). Then the expectation $E\{W_\varepsilon(t,x,k)\}$ of the Wigner distribution $W_\varepsilon(t,x,k)$ of the family $\phi_\varepsilon(t,x)$ of solutions of (4) converges weak-$*$ in $L^\infty([0,T]; \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d))$ to $\overline{W}(t,x,k)$.

By weak probabilistic solution we mean that $\overline{W}(t,x,k)$ satisfies

$$\langle \overline{W}, \lambda\rangle(t) - \langle W_0, \lambda|_{t=0}\rangle = \int_0^t \left\langle \overline{W}, \frac{\partial \lambda}{\partial t} + k \cdot \nabla_x \lambda + \mathcal{L}\lambda \right\rangle(s) ds \tag{10}$$

for all test functions $\lambda \in C^1([0,T]; \mathcal{S})$. We actually show that $E\{W_\varepsilon(t,x,k)\}$ converges to $\overline{W}$ in a smaller space $L^\infty([0,T]; \mathcal{A}'(\mathbb{R}^d \times \mathbb{R}^d))$. The space $\mathcal{A}'$, which is defined below in Section 2.3, is more convenient for the analysis of the Wigner distribution than $\mathcal{S}'$.

2.2 The random potential

The random potential $V(t,x)$ is assumed to be stationary in space and time and to have mean zero. It is constructed in Fourier space as follows. Let $\mathcal{V}$ be the set of measures of bounded total variation with support inside a ball $B_L = \{|p| \le L\}$

$$\mathcal{V} = \left\{ \hat{V} : \int_{\mathbb{R}^d} |d\hat{V}| \le C, \ \mathrm{supp}\ \hat{V} \subset B_L, \ \hat{V}(p) = \hat{V}^*(-p) \right\} \tag{11}$$

and let $\tilde{V}(t,p)$ be a mean-zero Markov process on $\mathcal{V}$ with generator Q. The time-dependent random potential $V(t,x)$ is given by

$$V(t,x) = \int \frac{d\tilde{V}(t,p)}{(2\pi)^d} e^{ip\cdot x}$$

and is real and uniformly bounded:

$$|V(t,x)| \le C.$$

We assume that the process $V(t,x)$ is stationary in t and x with correlation function $R(t,x)$

$$E\left\{V(t,x)V(t+s,x+z)\right\} = R(s,z) \quad \text{for all } x,z \in \mathbb{R}^d, \text{ and } t,s \in \mathbb{R}.$$

In terms of the process $\tilde{V}(t,p)$ this means that given any two bounded continuous functions $\hat{\phi}(p)$ and $\hat{\psi}(p)$ we have

$$E\left\{ \langle \tilde{V}(s), \hat{\phi}\rangle \langle \tilde{V}(t+s), \hat{\psi}\rangle \right\} = (2\pi)^d \int dp \tilde{R}(t,p)\phi(p)\hat{\psi}(-p). \tag{12}$$

Here $\langle\cdot,\cdot\rangle$ is the usual duality product on $\mathbb{R}^d \times \mathbb{R}^d$, and the power spectrum $\tilde{R}$ is the Fourier transform of $R(t,x)$ in x:

$$\tilde{R}(t,p) = \int dx e^{-ip\cdot x} R(t,x).$$

We assume that $\tilde{R}(t,p) \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^d)$ for simplicity and define $\hat{R}(\omega,p)$ as

$$\hat{R}(\omega,p) = \int dt e^{-i\omega t}\tilde{R}(t,p), \tag{13}$$

which is the space-time Fourier transform of R.

We assume that the generator Q is a bounded operator on $L^\infty(\mathcal{V})$ with a unique invariant measure $\pi(\hat{V})$

$$Q^*\pi = 0.$$

and that there exists $\alpha > 0$ such that if $\langle g, \pi\rangle = 0$ then

$$\|e^{rQ} g\|_{L^\infty_{\mathcal{V}}} \le C\|g\|_{L^\infty_{\mathcal{V}}} e^{-\alpha t}. \tag{14}$$

The simplest example of a generator with gap in the spectrum and invariant measure π is a jump process on $\mathcal{V}$ where

$$Qg(\hat{V}) = \int_{\mathcal{V}} g(\hat{V}_1)d\pi(\hat{V}_1) - g(\hat{V}), \qquad \int_{\mathcal{V}} d\pi(\hat{V}) = 1.$$

Given (14), the Fredholm alternative holds for the Poisson equation

$$Qf = g,$$

provided that g satisfies $\langle \pi, g\rangle = 0$. It has a unique solution f with $\langle \pi, f\rangle = 0$ and $\|f\|_{L^\infty_{\mathcal{V}}} \le C\|g\|_{L^\infty_{\mathcal{V}}}$. The solution f is given explicitly by

$$f(\hat{V}) = -\int_0^\infty dr e^{rQ} g(\hat{V}),$$

and the integral converges absolutely because of (14).

2.3 General convergence of the Wigner distribution

Existence of the limit of the Wigner family $W_\varepsilon(t,x,k)$ defined by (7) is shown as follows. We introduce the space $\mathcal{A}$, as in [16], of functions $\lambda(x,k)$ of x and k such that

$$\|\lambda\|_{\mathcal{A}} = \int_{\mathbb{R}^{2d}} dy \sup_x |\tilde{\lambda}(x,y)| < \infty,$$

where

$$\tilde{\lambda}(x,y) = \int_{\mathbb{R}^d} dk e^{-ik\cdot y}\lambda(x,k) \tag{15}$$

is the Fourier transform of λ in k. Convergence in the space $\mathcal{A}$ is easier to establish than in $\mathcal{S}$ because its definition does not involve derivatives. Moreover, as the following lemma shows, the distributions W_ε are uniformly bounded in $\mathcal{A}'$, the dual space to $\mathcal{A}$.

Lemma 2.2 *[16] The family $W_\varepsilon(t,x,k)$ is uniformly bounded in $\mathcal{A}'$, that is, there exists a constant $C>0$ independent of t so that:*

$$\|W_\varepsilon(t)\|_{\mathcal{A}'} \le C \tag{16}$$

for all $\varepsilon > 0$ and $t \ge 0$.

Proof. Let $\lambda(x,k) \in \mathcal{A}$. Then,

$$\begin{aligned}\langle W_\varepsilon(t),\lambda\rangle &= \int dxdk W_\varepsilon(t,x,k)\lambda(x,k) = \int \frac{dxdkdy}{(2\pi)^d} e^{ik\cdot y}\phi_\varepsilon(t,x-\frac{\varepsilon y}{2})\phi_\varepsilon^*(t,x+\frac{\varepsilon y}{2})\lambda(x,k)\\ &= \int \frac{dxdy}{(2\pi)^d}\phi_\varepsilon(t,x+\frac{\varepsilon y}{2})\phi_\varepsilon^*(t,x-\frac{\varepsilon y}{2})\tilde{\lambda}(x,y).\end{aligned}$$

Therefore, using the Cauchy-Schwartz inequality in x

$$|\langle W_\varepsilon(t),\lambda\rangle| \le \int \frac{dydx}{(2\pi)^{2d}}\left|\tilde{\lambda}(x,y)\right|\left|\phi_\varepsilon(t,x-\frac{\varepsilon y}{2})\right|\left|\phi_\varepsilon(t,x+\frac{\varepsilon y}{2})\right| \le \|\phi_\varepsilon(t)\|_{L_2}^2\|\lambda\|_{\mathcal{A}} \le C\|\lambda\|_{\mathcal{A}},$$

where we use the conservation of the L^2-norm (6) in the last step. This gives (16).

Lemma 2.2 implies that at every time $t \ge 0$ we can choose a sequence $\varepsilon_j \to 0$ so that W_{ε_j} converges weakly in $\mathcal{A}' \subset \mathcal{S}'$ to a limit distribution $W(t)$. One can show [13] that the limit measure $W(t)$ is non-negative and may thus be interpreted as the limit energy density in phase space. Moreover, if there are no oscillations in the initial data on scales smaller than ε then the limit captures correctly the behavior of the energy $E_\varepsilon(t,x)$. More precisely, if

$$\varepsilon\|\nabla\phi_\varepsilon^0\|_{L_x^2} \le C \tag{17}$$

then for any test function $\theta(x) \in \mathcal{S}(\mathbb{R}^d)$ we have

$$\int dxdk\theta(x)W(t,x,k) = \lim_{\varepsilon\to 0}\int dx\theta(x)|\phi_\varepsilon(t,x)|^2.$$

Condition (17) is sufficient but not necessary for this.

3 Convergence of the expectation

We prove Theorem 2.1 in this section. The proof is based on the method of [20] and proceeds as follows. The distribution W_ε defined by (7) satisfies

$$\begin{aligned}&\frac{\partial W_\varepsilon}{\partial t} + k\cdot\nabla_x W_\varepsilon = \frac{1}{i\sqrt{\varepsilon}}\int\frac{d\tilde{V}(t/\varepsilon,p)}{(2\pi)^d}e^{ip\cdot x/\varepsilon}\left[W_\varepsilon(t,x,k-\frac{p}{2}) - W_\varepsilon(t,x,k+\frac{p}{2})\right] \quad (18)\\ &W_\varepsilon(0,x,k) = W_\varepsilon^0(x,k).\end{aligned}$$

Here W_ε^0 is the Wigner distribution of the family $\phi_\varepsilon^0(x)$, the initial data for (4). The Cauchy problem (18) generates a measure P_ε on the space $C([0,T];\mathcal{A}')$ of continuous functions in time with values in $\mathcal{A}'$. It is supported on paths inside a ball $X = \{W \in \mathcal{A}' : \|W\|_{\mathcal{A}'} \le C\}$ with the constant C as in (16). The set X is the state space for the random process $W_\varepsilon(t)$. The joint process $(\tilde{V}(t/\varepsilon), W_\varepsilon(t))$ takes values in the space $\mathcal{V}\times X$. We will denote by $\tilde{P}_\varepsilon$ the corresponding measure on the space $\mathcal{V}\times X$ generated by (18) and the process $\tilde{V}(t/\varepsilon)$. Let us fix a deterministic function $\lambda \in C^1([0,T];\mathcal{S})$. We will show that the functional $G_\lambda : C([0,T];X) \to C[0,T]$ defined by

$$G_\lambda[W](t) = \langle W, \lambda\rangle(t) - \int_0^t \langle W, \frac{\partial \lambda}{\partial t} + k\cdot\nabla_x\lambda + \mathcal{L}\lambda\rangle(s)ds$$

is an approximate P_ε-martingale. More precisely, we will show that

$$\left|E^{P_\varepsilon}\left\{G_\lambda[W](t)|\mathcal{F}_s\right\} - G_\lambda[W](s)\right| \le C_{\lambda,T}\sqrt{\varepsilon} \tag{19}$$

uniformly for all $W \in C([0,T];X)$ and $0 \le s < t \le T$.

The next step is to show that the measures P_ε form a tight family, so that there exists a subsequence $\varepsilon_j \to 0$ so that P_ε converges weakly to a measure P supported on $C([0,T];X)$. Weak convergence of P_ε and the strong convergence (19) together imply that $G_\lambda[W](t)$ is a P-martingale so that

$$E^P\left\{G_\lambda[W](t)|\mathcal{F}_s\right\} - G_\lambda[W](s) = 0. \tag{20}$$

Taking $s = 0$ in this we obtain the transport equation (8) for $\overline{W} = E^P\{W(t)\}$, in its weak formulation (10).

The limit measure P may depend on the choice of the subsequence $\varepsilon_j \to 0$ but the expectation $\overline{W}$ being the unique solution of (8) does not depend on it. Therefore the whole family $E\{W_\varepsilon\}$ converges to $\overline{W}$ as $\varepsilon \to 0$ in $L^\infty([0,T];\mathcal{S}')$. Furthermore, the a priori bound (16) implies that $W_\varepsilon(t,x,k)$ converges weak-$*$ in $L^\infty([0,T];\mathcal{A}')$ for every realization of the random potential. Therefore the result above implies that actually $E\{W_\varepsilon\}$ converges to $\overline{W}$ in $L^\infty([0,T];\mathcal{A}')$.

The proof is organized as follows. The approximate martingale property (19) is proved in Sections 3.1 and 3.2. The weak compactness of the family P_ε is proved in Section 3.3.

3.1 Construction of the test functions

In order to obtain the approximate martingale property (19) one has to consider conditional expectation of functions $F(W,\hat{V})$. The only functions we will need to consider are those of the form $F(W,\hat{V}) = \langle W, \lambda(\hat{V})\rangle$ with $\lambda \in L^\infty(\mathcal{V}; C^1([0,T];\mathcal{S}(\mathbb{R}^d\times\mathbb{R}^d)))$. Given a function $F(W,\hat{V})$ let us define the conditional expectation

$$E^{\tilde{P}_\varepsilon}_{W,\hat{V},t}\left\{F(W,\hat{V})\right\}(\tau) = E^{\tilde{P}_\varepsilon}\left\{F(W(\tau),\tilde{V}(\tau))|\; W(t) = W, \tilde{V}(t) = \hat{V}\right\},\quad \tau \ge t.$$

The weak form of the infinitesimal generator of the Markov process generated by $\tilde{P}_\varepsilon$ is given by

$$\left.\frac{d}{dh}E^{\tilde{P}_\varepsilon}_{W,\hat{V},t}\left\{\langle W,\lambda(V)\rangle\right\}(t+h)\right|_{h=0} = \frac{1}{\varepsilon}\langle W, Q\lambda\rangle + \left\langle W, \left(\frac{\partial}{\partial t} + k\cdot\nabla_x + \frac{1}{\sqrt{\varepsilon}}\mathcal{K}[\hat{V},\frac{x}{\varepsilon}]\right)\lambda\right\rangle \tag{21}$$

and hence

$$G_\lambda^\varepsilon = \langle W,\lambda(V)\rangle(t) - \int_0^t \left\langle W, \left(\frac{1}{\varepsilon}Q + \frac{\partial}{\partial t} + k\cdot\nabla_x + \frac{1}{\sqrt{\varepsilon}}\mathcal{K}[\hat{V},\frac{x}{\varepsilon}]\right)\lambda\right\rangle(s)ds$$

is a $\tilde{P}_\varepsilon$-martingale. The operator $\mathcal{K}$ is defined by

$$\mathcal{K}[\hat{V},z]\psi(x,z,k,\hat{V}) = \frac{1}{i}\int\frac{d\hat{V}(p)}{(2\pi)^d}e^{ip\cdot z}\left[\psi(x,z,k-\frac{p}{2}) - \psi(x,z,k+\frac{p}{2})\right]. \tag{22}$$

The generator (21) comes from equation (18) written in the form

$$\partial_t W_\varepsilon + k\cdot\nabla_x W_\varepsilon = \frac{1}{\sqrt{\varepsilon}}\mathcal{K}[\tilde{V}(t/\varepsilon), x/\varepsilon]W_\varepsilon. \tag{23}$$

Given a test function $\lambda(t,x,k)\in C^1([0,T];\mathcal{S})$ we construct a function

$$\lambda_\varepsilon(t,x,k,\hat{V}) = \lambda(t,x,k) + \sqrt{\varepsilon}\lambda_1^\varepsilon(t,x,k,\hat{V}) + \varepsilon\lambda_2^\varepsilon(t,x,k,\hat{V}) \tag{24}$$

with $\lambda_{1,2}^\varepsilon(t)$ bounded in $L^\infty(\mathcal{V};\mathcal{A}(\mathbb{R}^d\times\mathbb{R}^d))$ uniformly in $t\in[0,T]$. It is sufficient for us to prove the simpler bound for the correctors in $\mathcal{A}$ instead of $\mathcal{S}$ because of the a priori bound (16) for W_ε in $\mathcal{A}'$. The functions $\lambda_{1,2}^\varepsilon$ will be chosen so that

$$\|G^\varepsilon_{\lambda_\varepsilon}(t) - G_\lambda(t)\|_{L^\infty(\mathcal{V})} \le C_\lambda\sqrt{\varepsilon}$$

for all $t\in[0,T]$. The approximate martingale property (19) follows from this. The approximate test function $\lambda_\varepsilon(t,x,k)$ in (24) is constructed in a manner similar to the formal asymptotic expansion (43) considered in Appendix A.

The functions λ_1^ε and λ_2^ε are as follows. Let $\lambda_1(t,x,z,k,\hat{V})$ be the mean-zero solution of the Poisson equation

$$k\cdot\nabla_z\lambda_1 + Q\lambda_1 = -\mathcal{K}\lambda. \tag{25}$$

It is given explicitly by

$$\lambda_1(t,x,z,k,\hat{V}) = \frac{1}{i}\int_0^\infty dr e^{rQ}\int\frac{d\hat{V}(p)}{(2\pi)^d}e^{ir(k\cdot p)+i(z\cdot p)}\left[\lambda(t,x,k-\frac{p}{2}) - \lambda(t,x,k+\frac{p}{2})\right].$$

Then we let $\lambda_1^\varepsilon(t,x,k,\hat{V}) = \lambda_1(t,x,x/\varepsilon,k,\hat{V})$. The second order corrector is $\lambda_2^\varepsilon(t,x,k,\hat{V}) = \lambda_2(t,x,x/\varepsilon,k,\hat{V})$ where $\lambda_2(t,x,z,k,\hat{V})$ is the mean-zero solution of

$$k\cdot\nabla_z\lambda_2 + Q\lambda_2 = \mathcal{L}\lambda - \mathcal{K}\lambda_1, \tag{26}$$

which exists because $E\{\mathcal{K}\lambda_1\} = \mathcal{L}\lambda$, and is given by

$$\lambda_2(t,x,z,k,\hat{V}) = -\int_0^\infty dr e^{rQ}\left[\mathcal{L}\lambda(t,x,k) - [\mathcal{K}\lambda_1](t,x,z+rk,k,\hat{V})\right].$$

Using (25) and (26) we have

$$\begin{aligned}
\frac{d}{dh}E^{\tilde{P}_\varepsilon}_{W,\hat{V},t}\left\{\langle W,\lambda_\varepsilon\rangle\right\}(t+h)\Big|_{h=0} &= \left\langle W, \left(\frac{\partial}{\partial t} + k\cdot\nabla_x + \frac{1}{\sqrt{\varepsilon}}\mathcal{K}[\hat{V},\frac{x}{\varepsilon}] + \frac{1}{\varepsilon}Q\right)\left(\lambda + \sqrt{\varepsilon}\lambda_1^\varepsilon + \varepsilon\lambda_2^\varepsilon\right)\right\rangle \\
&= \left\langle W, \left(\frac{\partial}{\partial t} + k\cdot\nabla_x\right)\lambda + \mathcal{L}\lambda\right\rangle + \left\langle W, \left(\frac{\partial}{\partial t} + k\cdot\nabla_x\right)\left(\sqrt{\varepsilon}\lambda_1^\varepsilon + \varepsilon\lambda_2^\varepsilon\right) + \sqrt{\varepsilon}\mathcal{K}[\hat{V},\frac{x}{\varepsilon}]\lambda_2^\varepsilon\right\rangle \\
&= \left\langle W, \left(\frac{\partial}{\partial t} + k\cdot\nabla_x\right)\lambda + \mathcal{L}\lambda\right\rangle + \sqrt{\varepsilon}\langle W,\zeta_\varepsilon^\lambda\rangle
\end{aligned}$$

with

$$\zeta_\varepsilon^\lambda = \left(\frac{\partial}{\partial t} + k\cdot\nabla_x\right)\lambda_1^\varepsilon + \sqrt{\varepsilon}\left(\frac{\partial}{\partial t} + k\cdot\nabla_x\right)\lambda_2^\varepsilon + \mathcal{K}[\hat{V},\frac{x}{\varepsilon}]\lambda_2^\varepsilon.$$

The terms $k\cdot\nabla_x\lambda_{1,2}^\varepsilon$ above are understood as differentiation with respect to the slow variable x only, and not with respect to x/ε. It follows that $G^\varepsilon_{\lambda_\varepsilon}$ is given by

$$G^\varepsilon_{\lambda_\varepsilon}(t) = \langle W(t),\lambda_\varepsilon\rangle - \int_0^t ds\left\langle W, \left(\frac{\partial}{\partial t} + k\cdot\nabla_x + \mathcal{L}\right)\lambda\right\rangle(s) - \sqrt{\varepsilon}\int_0^t ds\langle W,\zeta_\varepsilon^\lambda\rangle(s)$$

and is a martingale with respect to the measure $\tilde{P}_\varepsilon$ defined on $D([0,T];X\times\mathcal{V})$, the space of right-continuous paths with left-side limits [5]. The estimate (19) follows from the following two lemmas.

Lemma 3.1 *Let $\lambda \in C^1([0,T];\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d))$. Then there exists a constant $C_\lambda > 0$ independent of time $t \in [0,T]$ so that the correctors $\lambda_1^\varepsilon(t)$ and $\lambda_2^\varepsilon(t)$ satisfy the uniform bounds*

$$\|\lambda_1^\varepsilon(t)\|_{L^\infty(\mathcal{V};\mathcal{A})} + \|\lambda_2^\varepsilon(t)\|_{L^\infty(\mathcal{V};\mathcal{A})} \leq C_\lambda \tag{27}$$

and

$$\left\|\frac{\partial\lambda_1^\varepsilon(t)}{\partial t} + k\cdot\nabla_x\lambda_1^\varepsilon(t)\right\|_{L^\infty(\mathcal{V};\mathcal{A})} + \left\|\frac{\partial\lambda_2^\varepsilon(t)}{\partial t} + k\cdot\nabla_x\lambda_2^\varepsilon(t)\right\|_{L^\infty(\mathcal{V};\mathcal{A})} \leq C_\lambda. \tag{28}$$

Lemma 3.2 *There exists a constant C_λ such that*

$$\|\mathcal{K}[\hat{V}, x/\varepsilon]\|_{\mathcal{A}\to\mathcal{A}} \leq C$$

for any $\hat{V} \in \mathcal{V}$ and all $\varepsilon \in (0,1]$.

Indeed, (27) implies that $|\langle W,\lambda\rangle - \langle W,\lambda_\varepsilon\rangle| \leq C\sqrt{\varepsilon}$ for all $W \in X$ and $V \in \mathcal{V}$, while (28) and Lemma 3.2 imply that for all $t \in [0,T]$

$$\|\zeta_\varepsilon^\lambda(t)\|_{\mathcal{A}} \leq C \tag{29}$$

for all $V \in \mathcal{V}$ so that (19) follows.

Proof of Lemma 3.2. Let $g_\varepsilon(x,k) = \mathcal{K}[V,x/\varepsilon]\eta(x,k)$ with $\eta(x,k) \in \mathcal{A}$. Then

$$\tilde{g}_\varepsilon(x,y) = \int \frac{d\hat{V}(p)}{(2\pi)^d} e^{ip\cdot x/\varepsilon}\tilde{\eta}(x,y)\left[e^{-ip\cdot y/2} - e^{ip\cdot y/2}\right]$$

and thus

$$|\tilde{g}_\varepsilon(x,y)| \leq C|\tilde{\eta}(x,y)|$$

and the conclusion of Lemma 3.2 follows.

3.2 Bounds on the correctors

We now prove Lemma 3.1. We will omit the time dependence of the test function λ to simplify the notation.

Proof of Lemma 3.1. We first prove (27). The Fourier transform of λ_1^ε in k is given by

$$\tilde{\lambda}_1^\varepsilon(x,y,\hat{V}) = \frac{1}{i}\int_0^\infty dr e^{rQ}\int \frac{d\hat{V}(p)}{(2\pi)^d}\tilde{\lambda}(x,y-rp)e^{ix/\varepsilon\cdot p}\left[e^{-ip\cdot(y-rp)/2} - e^{ip\cdot(y-rp)/2}\right]. \tag{30}$$

Therefore using (14) we obtain

$$\|\tilde{\lambda}_1^\varepsilon(x,y,\hat{V})\|_{L^\infty_{x,y}} \leq \frac{C}{\alpha}\|\tilde{\lambda}\|_{L^\infty_{x,y}} \tag{31}$$

uniformly for all $\hat{V} \in \mathcal{V}$. It is therefore sufficient to consider $|y| > 2$. Let $S(y) = (|y|-1)/4L$ with L as in the definition (11) of the set $\mathcal{V}$. We write (30) as

$$\tilde{\lambda}_1^\varepsilon(x,y,\hat{V}) = J_{r<S(y)} + J_{r>S(y)}$$

with

$$J_{r<S(y)} = \frac{1}{i}\int_0^{S(y)} dr e^{rQ}\int \frac{d\hat{V}(p)}{(2\pi)^d}\tilde{\lambda}(x,y-rp)e^{ix/\varepsilon\cdot p}\left[e^{-ip\cdot(y-rp)/2} - e^{ip\cdot(y-rp)/2}\right]$$

and

$$J_{r>S(y)} = \frac{1}{i}\int_{S(y)}^\infty dr e^{rQ}\int \frac{d\hat{V}(p)}{(2\pi)^d}\tilde{\lambda}(x,y-rp)e^{ix/\varepsilon\cdot p}\left[e^{-ip\cdot(y-rp)/2} - e^{ip\cdot(y-rp)/2}\right].$$

We estimate each of these two terms separately.

To bound $J^1_{r<S(y)}$ we note that since λ is of the Schwartz class we have for $|p| \leq L$ and $r < S(y)$

$$\left|\tilde{\lambda}(x, y-rp)\right| \leq \sup_{|z-y|\leq rL} \left|\tilde{\lambda}(x,z)\right| \leq \sup_{|z|\geq|y|/2} \left|\tilde{\lambda}(x,z)\right| \leq \frac{C_\lambda}{|y|^{5d}}.$$

Then we obtain

$$\int_{|y|\geq 2} dy \sup_{x,V} \left|J_{r<S(y)}(x,y,\hat{V})\right| \leq C_\lambda \int_{|y|\geq 2} \frac{dy}{|y|^{5d}} \int_0^{S(y)} dr e^{-\alpha r} \leq C_{\lambda,\alpha}. \tag{32}$$

Next we note that

$$\int_{|y|\geq 2} dy \sup_{x,\hat{V}} |J_{r>S(y)}(x,y,\hat{V})| \leq \|\tilde{\lambda}\|_{L^\infty_{x,y}} \int_{|y|\geq 2} dy \int_{S(y)}^\infty dr e^{-\alpha r} \leq C_\alpha \|\tilde{\lambda}\|_{L^\infty_{x,y}}. \tag{33}$$

Therefore (31), (32) and (33) imply that

$$\|\lambda_1^\varepsilon\|_{\mathcal{A}} \leq C_{\lambda,\alpha} \tag{34}$$

for all $\hat{V} \in \mathcal{V}$.

We show next that λ_2^ε is uniformly bounded. This is done in several steps that we formulate as separate lemmas. Define

$$I(x,y) = \int_0^\infty dr e^{-\alpha r} \sup_{\hat{V}} \int \frac{|d\hat{V}(p)|}{(2\pi)^d} \int_0^\infty ds e^{-\alpha s} \sup_{\hat{V}_1} \int \frac{|d\hat{V}_1(q)|}{(2\pi)^d} |\tilde{\lambda}(x, y-rp-(r+s)q)|.$$

Lemma 3.3 *We have the estimate*

$$|\tilde{\lambda}_2^\varepsilon(x,y,\hat{V})| \leq C_\alpha \left[I(x,y) + |\widetilde{\mathcal{L}\lambda}(x,y)|\right] \tag{35}$$

Lemma 3.4 *For the limit operator $\mathcal{L}$ we have the bound*

$$\|\mathcal{L}\lambda\|_{\mathcal{A}} \leq \int d\xi dy |\widehat{\mathcal{L}\lambda}(\xi,y)| \leq C \int d\xi dy |\hat{\lambda}(\xi,y)|. \tag{36}$$

Here $\hat{f}$ denotes the Fourier transform of f both in x and k.

We split $I(x,y)$ as

$$I(x,y) = I_{r<S(y)} + I_{r>S(y)} = I^{s<S(y)}_{r<S(y)} + I^{s>S(y)}_{r<S(y)} + I_{r>S(y)}. \tag{37}$$

Lemma 3.5 *We have the following bounds:*

$$\int dy \sup_x \left[I_{r>S(y)}(x,y) + I^{s>S(y)}_{r<S(y)}\right] \leq C_\alpha \|\tilde{\lambda}\|_{L^\infty_{x,y}} \tag{38}$$

and

$$\int dy \sup_x I^{s<S(y)}_{r<S(y)} \leq C_\alpha \sup_{x\in\mathbb{R}^d, |y|\geq 1} \left\{|y|^{5d}|\tilde{\lambda}(x,y)|\right\}. \tag{39}$$

Lemmas 3.3, 3.4 and 3.5 imply clearly that $\|\lambda_\varepsilon^2\|_{\mathcal{A}} \le C$ for all $\hat{V} \in \mathcal{V}$. This finishes the proof of (27). The proof of (28) is quite similar and is therefore omitted.

We now prove Lemmas 3.3-3.5 to conclude the proof of Lemma 3.1.

Proof of Lemma 3.3. The Fourier transform of λ_2^ε in k is given by

$$\tilde{\lambda}_2^\varepsilon(x,y,\hat{V}) = -\int_0^\infty dr e^{rQ} \int dk e^{-ik\cdot y}\left[\mathcal{L}\lambda(x,k) - \frac{1}{i}\int \frac{d\hat{V}(p)}{(2\pi)^d} e^{ip\cdot(x/\varepsilon+rk)}\right.$$
$$\left.\times\left[\lambda_1(x,\frac{x}{\varepsilon}+rk,k-\frac{p}{2},\hat{V}) - \lambda_1(x,\frac{x}{\varepsilon}+rk,k+\frac{p}{2},\hat{V})\right]\right].$$

The second term above may be written as

$$\frac{1}{i}\int dk e^{-ik\cdot y}\int \frac{d\hat{V}(p)}{(2\pi)^d} e^{ip\cdot(x/\varepsilon+rk)}\left[\lambda_1(x,\frac{x}{\varepsilon}+rk,k-\frac{p}{2},\hat{V}) - \lambda_1(x,\frac{x}{\varepsilon}+rk,k+\frac{p}{2},\hat{V})\right]$$
$$= -\int dk e^{-ik\cdot y}\int d\hat{V}(p) e^{ip\cdot(x/\varepsilon+rk)}\int_0^\infty ds e^{sQ}\int \frac{d\hat{V}(q)}{(2\pi)^d} e^{is(k-p/2)\cdot q+i(x/\varepsilon+rk)\cdot q}$$
$$\times\left[\lambda(k-\frac{p}{2}-\frac{q}{2}) - \lambda(k-\frac{p}{2}+\frac{q}{2})\right]$$
$$+\int dk e^{-ik\cdot y}\int \frac{d\hat{V}(p)}{(2\pi)^d} e^{ip\cdot(x/\varepsilon+rk)}\int_0^\infty ds e^{sQ}\int \frac{d\hat{V}(q)}{(2\pi)^d} e^{is(k+p/2)\cdot q+i(x/\varepsilon+rk)\cdot q}$$
$$\times\left[\lambda(k+\frac{p}{2}-\frac{q}{2}) - \lambda(k+\frac{p}{2}+\frac{q}{2})\right].$$

This is further transformed to

$$\int dk e^{-ik\cdot y}\int \frac{d\hat{V}(p)}{(2\pi)^d} e^{ip\cdot(x/\varepsilon+rk)}\int_0^\infty ds e^{sQ}\int \frac{d\hat{V}(q)}{(2\pi)^d} e^{i(x/\varepsilon+rk)\cdot q}\int dy' \tilde{\lambda}(x,y')$$
$$\times\left[-e^{is(k-p/2)\cdot q}\left\{e^{iy'\cdot(k-p/2-q/2)} - e^{iy'\cdot(k-p/2+q/2)}\right\}\right.$$
$$\left.+ e^{is(k+p/2)\cdot q}\left\{e^{iy'\cdot(k+p/2-q/2)} - e^{iy'\cdot(k+p/2+q/2)}\right\}\right]$$
$$= \int \frac{d\hat{V}(p)}{(2\pi)^d} e^{ip\cdot(x/\varepsilon)}\int_0^\infty ds e^{sQ}\int \frac{d\hat{V}(q)}{(2\pi)^d} e^{ix/\varepsilon\cdot q}\tilde{\lambda}(x,y-rp-(r+s)q)$$
$$\times\left\{-e^{-isp\cdot q/2-i(p+q)\cdot(y-rp-(r+s)q)/2} + e^{-isp\cdot q/2-i(p-q)\cdot(y-rp-(r+s)q)/2}\right.$$
$$\left.+ e^{isp\cdot q/2+i(p-q)\cdot(y-rp-(r+s)q)/2} - e^{isp\cdot q/2+i(p+q)\cdot(y-rp-(r+s)q)/2}\right\}.$$

Therefore we obtain

$$|\tilde{\lambda}_2^\varepsilon(x,y,\hat{V})| \le C\int_0^\infty dr e^{-\alpha r}\sup_{\hat{V}}\int \frac{|d\hat{V}(p)|}{(2\pi)^d}\int_0^\infty ds e^{-\alpha s}\sup_{\hat{V}_1}\int \frac{|d\hat{V}_1(q)|}{(2\pi)^d}$$
$$\times|\tilde{\lambda}(x,y-rp-(r+s)q)| + C\int_0^\infty dr e^{-\alpha r}|\widetilde{\mathcal{L}\lambda}(x,y)|,$$

which is (35).

Proof of Lemma 3.4. The first inequality in (36) follows form the definition of $\|\cdot\|_{\mathcal{A}}$, and the second is shown as follows. Let us define

$$g(x,k) = \mathcal{L}\lambda(x,k) = \int \frac{dp}{(2\pi)^d}\hat{R}(\frac{k^2-p^2}{2},k-p)\left[\lambda(x,p)-\lambda(x,k)\right].$$

Taking the Fourier transform in x and k we obtain

$$\hat{g}(\xi,y) = \int \frac{dxdkdpdy'}{(2\pi)^{2d}} e^{-i\xi\cdot x-ik\cdot y}\hat{R}(\frac{k^2-p^2}{2},k-p)\tilde{\lambda}(x,y')\left[e^{ip\cdot y'}-e^{ik\cdot y'}\right].$$

Integrating x out we obtain

$$\hat{g}(\xi,y)=\int\frac{dkdpdy'}{(2\pi)^{2d}}e^{-ik\cdot y}\hat{R}(\frac{k^2-p^2}{2},k-p)\hat{\lambda}(\xi,y')\left[e^{ip\cdot y'}-e^{ik\cdot y'}\right].$$

We make a change of variables $k'=k-p$, $p'=(k+p)/2$ and drop the primes to get

$$\hat{g}(\xi,y)=\int\frac{dkdpdy'}{(2\pi)^{2d}}e^{-ip\cdot y-ik\cdot y/2+ip\cdot y'}\hat{R}(k\cdot p,k)\hat{\lambda}(\xi,y')$$
$$\times\left[e^{-ik\cdot y'/2}-e^{ik\cdot y'2}\right]=\int\frac{dkdpdy'ds}{(2\pi)^{2d}}e^{-isk\cdot p-ip\cdot y-ik\cdot y/2+ip\cdot y'}$$
$$\times\tilde{R}(s,k)\hat{\lambda}(\xi,y)\left[e^{-ik\cdot y'/2}-e^{ik\cdot y'/2}\right].$$

We may now integrate p and y' out to obtain

$$|\hat{g}(\xi,y)|\leq 2\int\frac{dkds}{(2\pi)^d}|\hat{R}(s,k)||\hat{\lambda}(\xi,y+sk)|.$$

Therefore we have

$$\int d\xi dy|\hat{g}(\xi,y)|\leq 2\|\tilde{R}(s,p)\|_{L^1_{p,s}}\int d\xi dy|\hat{\lambda}(\xi,y)|$$

and thus (36) holds.

Proof of Lemma 3.5. Clearly we have

$$|I(x,y)|\leq\frac{C}{\alpha^2}\|\tilde{\lambda}\|_{L^\infty_{x,y}}\tag{40}$$

and thus it suffices to look at $|y|>2$. We observe that

$$I_{r>S}(x,y)=\int_{S(y)}^\infty dre^{-\alpha r}\sup_{\hat{V}}\int\frac{|d\hat{V}(p)|}{(2\pi)^d}\int_0^\infty dse^{-\alpha s}\sup_{\hat{V}_1}\int\frac{|d\hat{V}_1(q)|}{(2\pi)^d}|\tilde{\lambda}(x,y-rp-(r+s)q)|$$
$$\leq\frac{C}{\alpha}e^{-\alpha S(y)}\|\tilde{\lambda}\|_{L^\infty_{x,y}}.$$

Therefore we have

$$\int dy\sup_x I_{r>S}(x,y)\leq C_\alpha\|\tilde{\lambda}\|_{L^\infty_{x,y}}.$$

Now we look at $I_{r<S}$ and split it as:

$$I_{r<S}(x,y)=\int_0^{S(y)}dre^{-\alpha r}\sup_{\hat{V}}\int\frac{|d\hat{V}(p)|}{(2\pi)^d}\int_0^\infty dse^{-\alpha s}\sup_{\hat{V}_1}\int\frac{|d\hat{V}_1(q)|}{(2\pi)^d}|\tilde{\lambda}(x,y-rp-(r+s)q)|$$
$$\leq I_{r<S}^{s<S}(x,y)+I_{r<S}^{s>S}(x,y).$$

Observe that

$$I_{r<S}^{s>S}(x,y)\leq C_\alpha e^{-\alpha S(y)}\|\tilde{\lambda}\|_{L^\infty_{x,y}}$$

so that

$$\int dy\sup_x I_{r<S}^{s>S}(x,y)\leq C_\alpha\|\tilde{\lambda}\|_{L^\infty_{x,y}}.$$

It remains to bound $I_{r<S}^{s<S}$. Note that for $r,s\leq S(y)$, $|p|,|q|\leq L$ and λ in the Schwartz class we have

$$|\tilde{\lambda}(x,y-rp-(r+s)q)|\leq\sup_{|z-y|\leq 2(r+s)L}|\tilde{\lambda}(x,z)|\leq\sup_{|z|\geq|y|/2}|\tilde{\lambda}(x,z)|\leq\frac{C_\lambda}{|y|^{5d}}.$$

Therefore we have

$$\int_{|y|\geq 2} dy I^{s<S}_{r<S}(x,y) = \int_{|y|\geq 2} dy \int_0^{S(y)} dre^{-\alpha r} \sup_{\hat{V}} \int \frac{|d\hat{V}(p)|}{(2\pi)^d} \int_0^{S(y)} dse^{-\alpha s} \sup_{\hat{V}_1} \int \frac{|d\hat{V}_1(q)|}{(2\pi)^d}$$
$$\times |\tilde{\lambda}(x, y - rp - (r+s)q)| \leq \int_{|y|\geq 2} dy \frac{C_\lambda}{|y|^{5d}} \leq C_\lambda.$$

This finishes the proof of Lemma 3.5.

3.3 The tightness of the measures $\mathcal{P}_\varepsilon$.

The process $W_\varepsilon(t)$ generates a probability measure P_ε on the space $C([0,T];X)$ with the space X defined as before $X = \{W \in \mathcal{S}' : \|W\|_{\mathcal{A}'} \leq C\}$. This family is tight.

Lemma 3.6 *The family of measures P_ε is weakly compact.*

Proof. We follow the corresponding proof of Blankenship and Papanicolaou [6] for oscillatory ordinary differential equations with random coefficients. A theorem of Mitoma and Fouque [17, 11] implies that in order to verify tightness of the family P_ε it is enough to check that for each $\lambda \in C^1([0,T], \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d))$ the family of measures $\mathcal{P}_\varepsilon$ on $C([0,T];\mathbb{R})$ generated by the random processes $W^\varepsilon_\lambda(t) = \langle W_\varepsilon(t), \lambda\rangle$ is tight. Tightness of $\mathcal{P}_\varepsilon$ would follow from the following two conditions. First, a Kolmogorov moment condition [5] in the form

$$E^{P_\varepsilon}\left\{|\langle W,\lambda\rangle(t) - \langle W,\lambda\rangle(t_1)|^\gamma\, |\langle W,\lambda\rangle(t_1) - \langle W,\lambda\rangle(s)|^\gamma\right\} \leq C_\lambda (t-s)^{1+\beta}, \quad 0 \leq s \leq t \leq T \quad (41)$$

should hold with $\gamma > 0$, $\beta > 0$ and C_λ independent of ε. Second, we should have

$$\lim_{R\to\infty} \limsup_{\varepsilon\to 0} \mathrm{Prob}^{\mathcal{P}_\varepsilon}\left\{\sup_{0\leq t\leq T} |\langle W,\lambda\rangle(t)| > R\right\} = 0.$$

The second condition holds automatically in our case since the process $W^\varepsilon_\lambda(t)$ is uniformly bounded for all $t > 0$ and $\varepsilon > 0$. In order to verify (41), note that we have

$$\langle W(t),\lambda\rangle = G^\varepsilon_{\lambda_\varepsilon}(t) - \sqrt{\varepsilon}\langle W, \lambda_1^\varepsilon\rangle - \varepsilon\langle W,\lambda_2^\varepsilon\rangle + \int_0^t ds \langle W, \frac{\partial\lambda}{\partial t} + k\cdot\nabla_x\lambda + \mathcal{L}\lambda\rangle(s) + \sqrt{\varepsilon}\int_0^t ds\langle W, \zeta^\lambda_\varepsilon\rangle(s).$$

The uniform bound (29) on $\zeta^\lambda_\varepsilon$ and the bounds on $\|\lambda^\varepsilon_{1,2}(t)\|_{\mathcal{A}}$ in Lemma 3.1 imply that it suffices to check (41) for

$$x_\varepsilon(t) = G^\varepsilon_{\lambda_\varepsilon}(t) + \int_0^t ds\langle W, \frac{\partial\lambda}{\partial t} + k\cdot\nabla_x\lambda + \mathcal{L}\lambda\rangle(s).$$

We have

$$E\left\{|x_\varepsilon(t) - x_\varepsilon(s)|^2 \Big| \mathcal{F}_s\right\} \leq 2E\left\{\left|\int_s^t d\tau\langle W, \frac{\partial\lambda}{\partial t} + k\cdot\nabla_x\lambda + \mathcal{L}\lambda\rangle(\tau)\right|^2 \Bigg| \mathcal{F}_s\right\}$$
$$+2E\left\{\left|G^\varepsilon_{\lambda_\varepsilon}(t) - G^\varepsilon_{\lambda_\varepsilon}(s)\right|^2 \Big| \mathcal{F}_s\right\} \leq C(t-s)^2 + 2E\left\{\langle G^\varepsilon_{\lambda_\varepsilon}\rangle(t) - \langle G^\varepsilon_{\lambda_\varepsilon}\rangle(s) \big| \mathcal{F}_s\right\}.$$

Here $\langle G^\varepsilon_{\lambda_\varepsilon}\rangle$ is the increasing process associated with $G^\varepsilon_{\lambda_\varepsilon}$. We will now compute it explicitly. First we obtain that

$$\frac{d}{dh} E^{P_\varepsilon}_{W,\hat{V},t}\left\{\langle W,\lambda_\varepsilon\rangle^2(t+h)\right\}\Big|_{h=0} = 2\langle W,\lambda_\varepsilon\rangle\langle W, \frac{\partial\lambda}{\partial t} + k\cdot\nabla_x\lambda_\varepsilon + \frac{1}{\sqrt{\varepsilon}}\mathcal{K}[\hat{V}, x/\varepsilon]\lambda_\varepsilon\rangle + \frac{1}{\varepsilon}Q\left[\langle W,\lambda_\varepsilon\rangle^2\right]$$

so that

$$\langle W,\lambda_\varepsilon\rangle^2(t)-\int_0^t\left(2\langle W,\lambda_\varepsilon\rangle(s)\langle W,\frac{\partial\lambda}{\partial t}+k\cdot\nabla_x\lambda_\varepsilon+\frac{1}{\sqrt{\varepsilon}}\mathcal{K}[\hat V,x/\varepsilon]\lambda_\varepsilon\rangle(s)+\frac{1}{\varepsilon}Q\left[\langle W,\lambda_\varepsilon\rangle^2\right](s)\right)ds$$

is a martingale. Therefore we have

$$\begin{aligned}\langle G^\varepsilon_{\lambda_\varepsilon}(t)\rangle&=\int_0^t ds\left[\frac{1}{\varepsilon}Q[\langle W,\lambda_\varepsilon\rangle^2]-\frac{2}{\varepsilon}\langle W,\lambda_\varepsilon\rangle\langle W,Q\lambda_\varepsilon\rangle\right](s)\\&=\int_0^t ds\left(Q\left[\langle W,\lambda_1^\varepsilon\rangle^2\right]-\langle W,\lambda_1^\varepsilon\rangle\langle W,Q\lambda_1^\varepsilon\rangle(s)\right)+\sqrt{\varepsilon}\int_0^t dsH_\varepsilon(s)\end{aligned}$$

with

$$\begin{aligned}H_\varepsilon=&\,2\sqrt{\varepsilon}\left(Q[\langle W,\lambda_1^\varepsilon\rangle\langle W,\lambda_2^\varepsilon\rangle]-\langle W,\lambda_1^\varepsilon\rangle\langle W,Q\lambda_2^\varepsilon\rangle-\langle W,\lambda_2^\varepsilon\rangle\langle W,Q\lambda_1^\varepsilon\rangle\right)\\&+\varepsilon\left(Q[\langle W,\lambda_2^\varepsilon\rangle^2]-2\langle W,\lambda_2^\varepsilon\rangle\langle W,Q\lambda_2^\varepsilon\rangle\right).\end{aligned}$$

Lemma 3.1 and the boundedness of Q on $L^\infty(\mathcal{V})$ imply that $|H_\varepsilon(s)|\le C$ for all $V\in\mathcal{V}$. This yields

$$E\left\{\langle G^\varepsilon_{\lambda_\varepsilon}\rangle(t)-\langle G^\varepsilon_{\lambda_\varepsilon}\rangle(s)\middle|\mathcal{F}_s\right\}\le C(t-s)$$

and hence

$$E\left\{|x_\varepsilon(t)-x_\varepsilon(s)|^2\middle|\mathcal{F}_s\right\}\le C(t-s).$$

In order to obtain (41) we note that

$$\begin{aligned}&E^{P_\varepsilon}\left\{|x_\varepsilon(t)-x_\varepsilon(t_1)|^\gamma|x_\varepsilon(t_1)-x_\varepsilon(s)|^\gamma\right\}\\&=E^{P_\varepsilon}\left\{E^{P_\varepsilon}\left\{|x_\varepsilon(t)-x_\varepsilon(t_1)|^\gamma\middle|\mathcal{F}_{t_1}\right\}|x_\varepsilon(t_1)-x_\varepsilon(s)|^\gamma\right\}\\&\le E^{P_\varepsilon}\left\{\left[E^{P_\varepsilon}\left\{|x_\varepsilon(t)-x_\varepsilon(t_1)|^2\middle|\mathcal{F}_{t_1}\right\}\right]^{\gamma/2}|x_\varepsilon(t_1)-x_\varepsilon(s)|^\gamma\right\}\\&\le C(t-t_1)^{\gamma/2}E^{P_\varepsilon}\left\{|x_\varepsilon(t_1)-x_\varepsilon(s)|^\gamma\right\}\le C(t-t_1)^{\gamma/2}E^{P_\varepsilon}\left\{E^{P_\varepsilon}\left\{|x_\varepsilon(t_1)-x_\varepsilon(s)|^\gamma|\mathcal{F}_s\right\}\right\}\\&\le C(t-t_1)^{\gamma/2}E^{P_\varepsilon}\left\{\left[E^{P_\varepsilon}\left\{|x_\varepsilon(t_1)-x_\varepsilon(s)|^2\middle|\mathcal{F}_s\right\}\right]^{\gamma/2}\right\}\le C(t-t_1)^{\gamma/2}(t_1-s)^{\gamma/2}\\&\le C(t-s)^\gamma.\end{aligned}$$

Choosing now $\gamma>1$ we get (41) which finishes the proof of Lemma 3.6. This also finishes the proof of Theorem 2.1.

4 Conclusions

We have presented a proof of the transport limit (2) for the Schrödinger equation with a time-dependent random potential. Our proof is relatively simple and does not involve infinite Neumann expansions because it relies on the Markovian property of the potential, which allows us to construct approximate martingales and to show weak compactness of the family of probability measures P_ε generated by the dynamics (18) of the Wigner transform on $C([0,T];\mathcal{A}')$. However, we only show convergence for the average Wigner distribution, which is the the first moment of P_ε. We do not have a rigorous convergence result for the higher moments of the Wigner distribution and thus are not able to fully characterize the set of accumulation points of the family P_ε, although we believe that the limit measure P is unique, based on the formal analysis in [18] of a similar problem in the white noise limit.

A The formal perturbation expansion

We present the formal derivation of the transport equation for the limit Wigner distribution $W(t,x,k)$ similar to the one in [22] for a time-independent potential. Recall that the Cauchy problem for the Wigner distribution is

$$\begin{aligned}&\frac{\partial W_\varepsilon}{\partial t}+k\cdot\nabla_x W_\varepsilon=\frac{1}{i\sqrt{\varepsilon}}\int\frac{d\tilde{V}(t/\varepsilon,p)}{(2\pi)^d}e^{ip\cdot x/\varepsilon}\left[W_\varepsilon(t,x,k-\frac{p}{2})-W_\varepsilon(t,x,k+\frac{p}{2})\right]\\&W_\varepsilon(0,x,k)=W_\varepsilon^0(x,k).\end{aligned}\tag{42}$$

Here W_ε^0 is the Wigner distribution of the family $\phi_\varepsilon^0(x)$, the initial data for (4). We will construct a formal perturbation expansion for W_ε and derive the transport equation for the average $\overline{W}(t,x,k)$. We seek an expansion of W_ε with multiple scales

$$W_\varepsilon(t,x,k)=W^{(0)}(t,x,k)+\sqrt{\varepsilon}W^{(1)}(t,\frac{t}{\varepsilon},x,\frac{x}{\varepsilon},k)+\varepsilon W^{(2)}(t,\frac{t}{\varepsilon},x,\frac{x}{\varepsilon},k)+\dots\tag{43}$$

and introduce the fast time and spatial variables $\tau=t/\varepsilon$, $z=x/\varepsilon$. We assume that the leading term $W^{(0)}$ is deterministic and independent of the fast scale variables. We insert (43) into (42) and obtain at the order $O(1/\sqrt{\varepsilon})$:

$$\partial_\tau W^{(1)}+k\cdot\nabla_z W^{(1)}=\mathcal{K}[\hat{V},z]W^{(0)}.$$

Then $W^{(1)}$ has the form

$$W^{(1)}(t,\tau,x,z,k)=\frac{1}{i}\int\frac{dpd\omega e^{ip\cdot z+i\omega s}}{(2\pi)^{d+1}}\frac{\hat{V}(\omega,p)}{i\omega+ip\cdot k+\delta}\left[W^{(0)}(x,k-\frac{p}{2})-W^{(0)}(x,k+\frac{p}{2})\right].\tag{44}$$

Here $\delta\ll 1$ is a regularization parameter that we will send to zero at the end of the calculation and $\hat{V}$ denotes the Fourier transform in time:

$$\hat{V}(\omega,p)=\int_{\mathbb{R}}dse^{-i\omega s}\tilde{V}(s,p).$$

The term of order $O(1)$ in (18) gives

$$\frac{\partial W^{(0)}}{\partial t}+\frac{\partial W^{(2)}}{\partial\tau}+k\cdot\nabla_x W^{(0)}+k\cdot\nabla_z W^{(2)}=\frac{1}{i}\int\frac{dp\tilde{V}(\tau,p)e^{ip\cdot z}}{(2\pi)^d}\left[W^{(0)}(k-\frac{p}{2})-W^{(0)}(k+\frac{p}{2})\right].$$

We average the above equation assuming formally that $E\left\{\frac{\partial W^{(2)}}{\partial\tau}+k\cdot\nabla_z W^{(2)}\right\}=0$. This gives an equation for the leading order term $W^{(0)}$:

$$\frac{\partial W^{(0)}}{\partial t}+k\cdot\nabla_x W^{(0)}=E\left\{\frac{1}{i}\int\frac{dp\tilde{V}(\tau,p)e^{ip\cdot z}}{(2\pi)^d}\left[W^{(1)}(x,z,k-\frac{p}{2})-W^{(1)}(x,z,k+\frac{p}{2})\right]\right\}.\tag{45}$$

The average on the right side of (45) may be computed explicitly using (44) and spatial homogeneity (12):

$$\begin{aligned}&E\left\{\frac{1}{i}\int\frac{dp\tilde{V}(\tau,p)e^{ip\cdot z}}{(2\pi)^d}\left[W^{(1)}(x,k-\frac{p}{2})-W^{(1)}(x,z,k+\frac{p}{2})\right]\right\}\\&\to\int\frac{dp}{(2\pi)^d}\hat{R}(\frac{p^2-k^2}{2},p-k)\left[W^{(0)}(p)-W^{(0)}(k)\right]\end{aligned}$$

as $\delta \to 0$. Here $\hat{R}$ is the Fourier transform of $\tilde{R}$ in time:

$$\hat{R}(\omega, p) = \int_{\mathbb{R}} ds e^{-i\omega s} \tilde{R}(s, p).$$

Therefore we obtain the transport equation for the leading order term $W^{(0)}$:

$$\frac{\partial W^{(0)}}{\partial t} + k \cdot \nabla_x W^{(0)} = \int \frac{dp}{(2\pi)^d} \hat{R}(\frac{p^2 - k^2}{2}, p - k) \left[W^{(0)}(p) - W^{(0)}(k) \right]. \tag{46}$$

The formal asymptotic expansion (43) may not be justified but the final equation (8) for the expectation of the limit Wigner distribution $E\{W(t, x, k)\}$ is correct. Moreover, the test functions that we used in our proof of Theorem 2.1 are based on the formal expressions for $W^{(1)}$ and $W^{(2)}$. The role of the regularization parameter δ is played by the spectral gap of the generator Q because of the bound (14).

B The parabolic approximation

We recall briefly the formal passage from the reduced wave equation to the parabolic equation (1). In particular we would like to explain how the particular scaling in (4) arises in this context. We start with the reduced wave equation

$$\Delta\psi + k^2 n^2(x, z)\psi = 0, \tag{47}$$

and look for solutions of (47) in the form

$$\psi(x, z) = e^{ikz}\phi(x, z).$$

Then we obtain

$$\frac{\partial^2 \phi}{\partial z^2} + 2ik\frac{\partial \phi}{\partial z} + \Delta_x \phi + k^2(n^2 - 1)\phi = 0. \tag{48}$$

We assume that the refraction index $n(x, z)$ is fluctuating so that

$$n^2(x, z) = 1 + 2\sigma\mu\left(\frac{x}{l_x}, \frac{z}{l_z}\right).$$

Here l_x and l_z are the correlation lengths of μ, and the small parameter σ measures the strength of the fluctuation. We assume that wave is propagating over distance L_x in the x-direction and L_z in the z-direction, and rescale x and z accordingly. We also introduce a carrier wave number k_0 and replace $k = k_0 k'$, k' being the non-dimensional wave number (we drop the prime below). The physical parameters are l_x, l_z and σ, we explain now how the spatial scales L_x and L_z, and the carrier frequency k_0 should be chosen so that the parabolic approximation in the scaling of (4) is valid. Equation (48) becomes

$$\frac{1}{L_z^2}\frac{\partial^2 \phi}{\partial z^2} + \frac{2ikk_0}{L_z}\frac{\partial \phi}{\partial z} + \frac{1}{L_x^2}\Delta_x \phi + 2k^2 k_0^2 \sigma\mu\left(\frac{xL_x}{l_x}, \frac{zL_z}{l_z}\right)\phi = 0. \tag{49}$$

We introduce the following small parameters:

$$\delta_x = \frac{l_x}{L_x}, \quad \delta_z = \frac{l_z}{L_z}, \quad \gamma_x = \frac{1}{k_0 l_x}, \quad \gamma_z = \frac{1}{k_0 l_z}.$$

We rewrite (49) as

$$\gamma_z\delta_z\frac{\partial^2 \phi}{\partial z^2} + 2ik\frac{\partial \phi}{\partial z} + \frac{\delta_x^2\gamma_x^2}{\delta_z\gamma_z}\Delta_x \phi + \frac{2k^2\sigma}{\gamma_z\delta_z}\mu\left(\frac{x}{\delta_x}, \frac{z}{\delta_z}\right)\phi = 0. \tag{50}$$

Let us assume now that

$$\delta_x = \delta_z \ll 1, \quad \gamma_z = \gamma_x^2 \ll 1, \quad \sigma = \gamma_z \sqrt{\delta_x} \tag{51}$$

and denote $\varepsilon = \delta_x$. Then (50) becomes after multiplication by $\varepsilon/2$

$$\frac{\gamma_z \varepsilon^2}{2} \frac{\partial^2 \phi}{\partial z^2} + ik\varepsilon \frac{\partial \phi}{\partial z} + \frac{\varepsilon^2}{2} \Delta_x \phi + k^2 \sqrt{\varepsilon} \mu \left(\frac{x}{\varepsilon}, \frac{z}{\varepsilon} \right) \phi = 0. \tag{52}$$

Observe that when $k = O(1)$ and $\gamma_z \ll 1$, the first term in (52) is small since $|\varepsilon^2 \psi_{zz}| = O(1)$, and may be neglected in the leading order. Then (52) becomes

$$ik\varepsilon \frac{\partial \phi}{\partial z} + \frac{\varepsilon^2}{2} \Delta_x \phi + k^2 \sqrt{\varepsilon} \mu \left(\frac{x}{\varepsilon}, \frac{z}{\varepsilon} \right) \phi = 0$$

which is (4) when $k = 1$.

The second condition in (51) implies that the carrier wave number should be chosen as $k_0 = l_z / l_x^2$. Then $\gamma_z = l_x^2 / l_z^2 \ll 1$ implies that $l_x \ll l_z$. Therefore the correlation length in the longitudinal direction z should be much longer than in the transverse direction x. Furthermore, we should have $\varepsilon = (\sigma/\gamma_z)^2 = \sigma^2 l_z^4 / l_x^4 \ll 1$ which in turn implies that the fluctuation strength $\sigma \ll (l_x/l_z)^4$. Given that these constraints are satisfied the last condition in (51) implies that the spatial scales L_x and L_z should be chosen according to

$$L_x = l_x \frac{l_x^4}{\sigma^2 l_z^4}, \quad L_z = l_z \frac{l_x^4}{\sigma^2 l_z^4}.$$

Note that the constraint $l_x \ll l_z$ implies that $L_x \ll L_z$ which is the usual constraint on the general validity of the parabolic approximation. The scaling described in this appendix leads to the particular scaling in the Schrödinger equation (4) where transport regime is valid. General connection between the relations of the physical parameters and the resulting scaling of the parabolic approximation is discussed in [18]. Rigorous passage to the parabolic approximation in one dimension in a similar scaling is discussed in [1].

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