Probabilistic Theory of Transport Processes with Polarization

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Abstract. We derive a probabilistic representation for solutions of matrix-valued transport equations that account for polarization effects. Such equations arise in radiative transport for the Stokes parameters that model the propagation of light through turbulent atmospheres. They also arise in radiative transport for seismic wave propagation in the earth's crust. The probabilistic representation involves an augmented scalar transport equation in which the polarization parameters become independent variables. Our main result is that the linear moments of the augmented transport equation with respect to the polarization variables are the solution of the matrix-valued transport equation. The augmented scalar transport equation is well suited to analyzing the hydrodynamic regime of small mean free paths. It is also well suited to getting approximate solutions by Monte Carlo simulation.

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1 Introduction

Radiative transport equations were first used more than a hundred years ago to describe the propagation of light energy through the atmosphere [5]. Transport equations have recently developed into an efficient tool for describing the propagation of energy for other types of waves, like acoustic or elastic waves. They can be derived from first principles, starting with the underlying wave equations in a randomly inhomogeneous medium and in a high frequency regime. We refer to [14] for a detailed analysis and an extended bibliography.

There is no closed equation for the propagation of wave energy, even in the high frequency regime. It is necessary to consider the propagation of energy at time t and position x resolved over angular direction or wave number k. For certain kinds of waves, such as acoustic waves, the phase space energy density, $a(t, x, k)$, is scalar. The radiative transport equation is then a linear integro-differential equation, like the one describing the propagation of neutrons in reactors or electrons in semiconductors.

It is well known that scalar transport equations, as well as some systems of such scalar equations, admit a probabilistic representation [4]. The solution of the transport equation is the probability density of a jump Markov process for the position and velocity of a particle. This representation can be used to solve the transport equation by Monte Carlo simulation [7, 10, 11, 16]. Even though its accuracy is relatively poor, of order $N^{-1/2}$ where N is the number of independent realizations, the Monte Carlo method is a competitive way to obtain three dimensional solutions in complicated geometries.

For polarized waves, such as electromagnetic waves or seismic waves, the corresponding transport equations are matrix valued, or systems of matrix valued transport equations, for a quantity that we denote by $W(t, x, k)$. Systems of matrix valued equations that come from polarized waves cannot be recast as bigger systems of scalar equations with positive components. This is because the components are not only energies but also polarization parameters, such as the ones that determine the polarization ellipse. The way the components of W are transformed by interaction with the underlying scattering medium is different from that of systems of scalar transport equations. The usual probabilistic representation for scalar equations in no longer valid.

The probabilistic representation of such systems of transport equations is what we address in this paper. This is done by introducing a higher dimensional jump Markov process which consists of the position x of a polarized particle, its direction k and its matrix of polarization parameters U. Let the probability density of this process be $\rho(t, x, k, U)$. We first derive a transport equation for the density of this augmented jump Markov process. Our main result is that the solution of the matrix-valued transport equation $W(t, x, k)$ can be expressed in terms of the first order moments of $\rho(t, x, k, U)$ with respect to the matrix of polarization parameters U. This gives the probabilistic representation we want, as well as a theory of existence and uniqueness for a general class of matrix radiative transport equations. It also provides a numerical method to solve them, by first simulating the augmented transport equation by Monte Carlo and then computing its first order moments with respect to the polarization parameters.

The paper is organized as follows. In section 2, we recall briefly the general theory of scalar transport processes given in [4]. This theory was developed with the application of scalar transport equations in mind, with no polarization effects. The main result here is to show that the scalar transport equation is the backward Kolmogorov equation (9) for the expectation of functionals of the random process. The probability density of the process is the solution of a forward Kolmogorov equation (10). This gives a probabilistic representation of the forward and backward transport equations (10) and (9).

We introduce in section 3 the probabilistic theory of matrix transport equations with the simplest physical example that includes polarization: electromagnetic energy transport. In section 4 we generalize the probabilistic theory to any system of transport equations with an arbitrary number of wave types and polarization parameters. We present the electromagnetic case in detail in section 3 because the construction of the augmented process is transparent and explicit. In the general case in section 4 the analysis is more abstract but it can be readily applied to more complex wave propagation phenomena such as elastic waves, which we consider in appendix A.

In section 5 we consider the diffusion approximation of matrix transport equations using the probabilistic representation with the augmented jump Markov process.

2 Scalar Transport Theory

We follow the framework in [4] for scalar equations, and consider a pair of processes $(X(t), Y(t))$, $t \geq 0$ with values in $\mathbb{R}^l = \mathbb{R}^d \times \mathbb{R}^{d'}$. Here d is the space dimension and d' is the "velocity" dimension. The process $Y(t)$ in scalar transport theory corresponds usually to the velocity of the particle undergoing scattering. We will call $X(t)$ the position process and $Y(t)$ the velocity process even though in some applications these quantities have different physical meaning. Given an initial state, $X(t_0) = x$ and $Y(t_0) = y$, the evolution of the jump Markov process $(X(t), Y(t))$ is defined as follows. Let $(\xi(t; x, y), \eta(t; x, y))$ denote the solution of the deterministic system of ordinary differential equations

$$\begin{aligned}\frac{d\xi(t;x,y)}{dt} &= F(\xi(t;x,y),\eta(t;x,y)), \quad \xi(0;x,y)=x \\ \frac{d\eta(t;x,y)}{dt} &= H(\xi(t;x,y),\eta(t;x,y)), \quad \eta(0;x,y)=y.\end{aligned} \tag{1}$$

We assume that F and H are smooth and bounded vector functions with values in $\mathbb{R}^d$ and $\mathbb{R}^{d'}$, respectively so that (1) has a unique solution for all $t \geq 0$. Let $\Sigma(x,y)$ denote the *total scattering cross section*, which we assume is uniformly positive and smooth, and let τ_1 be an exponentially distributed random variable with distribution function

$$P(\tau_1 > t) = \exp\Big(-\int_0^t \Sigma(\xi(s;x,y),\eta(s;x,y))ds\Big). \tag{2}$$

In the time interval $0 \leq t < \tau_1$, we define $(X(t),Y(t))$ by

$$\begin{aligned}X(t) &= \xi(t;x,y) \\ Y(t) &= \eta(t;x,y) \quad 0 \leq t < \tau_1.\end{aligned} \tag{3}$$

At time τ_1, the process Y jumps to a new value. The time τ_1 is the first time of interaction between the particle and the scattering medium. After this interaction, the particle is reemitted with a different velocity vector $Y(\tau_1)$. Let $\Pi(x,y,A)$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^{d'}$, $A \subset \mathbb{R}^{d'}$ be a probability measure for each x and y and a continuous function for each Borel subset A. We define the probability that a scattering takes place before time t and that the velocity $Y(\tau_1)$ is in the subset A after scattering by the expression

$$\begin{aligned}P(\tau_1 \leq t, Y(\tau_1) \in A) = \int_0^t & \Pi(\xi(s;x,y),\eta(s;x,y),A)\Sigma(\xi(s;x,y),\eta(s;x,y))\cdot \\ & \cdot \exp\Big(-\int_0^s \Sigma(\xi(\tau;x,y),\eta(\tau;x,y))d\tau\Big)ds.\end{aligned} \tag{4}$$

We let $x_1 = X(\tau_1) = \xi(\tau_1;x,y)$ and $y_1 = Y(\tau_1)$ be the position and velocity immediately after the jump. Note that only the process Y jumps at time τ_1 while X is continuous.

Let now τ_2 be an independent exponentially distributed random variable with the same distribution function as τ_1, given x_1 and y_1:

$$P(\tau_2 > t) = \exp\Big(-\int_0^t \Sigma(\xi(s;x_1,y_1),\eta(s;x_1,y_1))ds\Big). \tag{5}$$

In the interval $\tau_1 \leq t < \tau_1 + \tau_2$, we define $(X(t),Y(t))$ given x_1 and y_1 by

$$\begin{aligned}X(t) &= \xi(t-\tau_1;x_1,y_1) \\ Y(t) &= \eta(t-\tau_1;x_1,y_1).\end{aligned} \tag{6}$$

This construction is now continued in an obvious manner until a final time T. It is well defined since only finitely many jumps occur in a given interval $[0,T]$ with probability one [4]. We denote by $P_{x,y}$ and $E_{x,y}$ the Markov probability measure and expectation for the position and velocity process starting at (x,y). The measure is defined in the space of trajectories in $\mathbb{R}^l$ with the first d components continuous and remaining d' components right-continuous having left hand limits.

The infinitesimal generator of the process $(X(t),Y(t))$ is given by

$$\mathcal{L}g(x,y) = F(x,y)\cdot\frac{\partial g(x,y)}{\partial x} + H(x,y)\cdot\frac{\partial g(x,y)}{\partial y} + Qg(x,y), \tag{7}$$

where

$$Qg(x,y) = \Sigma(x,y)\int_{\mathbb{R}^{d'}} g(x,z)\Pi(x,y,dz) - \Sigma(x,y)g(x,y). \tag{8}$$

The operator $\mathcal{L}$ is the sum of two terms: the streaming term along the vector fields F and H and the jump operator Q. It characterizes the process $(X(t),Y(t))$ completely. Moreover, given any bounded function $a_0(x,y)$, the function $a(t,x,y) = E_{x,y}\{a_0(X(t),Y(t))\}$ is a solution of the backward Kolmogorov equation

$$\begin{aligned} &\frac{\partial a(t,x,y)}{\partial t} = \mathcal{L}a(t,x,y), \quad t>0,\\ &a(0,x,y) = a_0(x,y). \end{aligned} \tag{9}$$

The probability density $\rho(t,x,y)$ of the jump process satisfies the forward Kolmogorov equation

$$\begin{aligned} &\frac{\partial \rho(t,x,y)}{\partial t} = \mathcal{L}^*\rho(t,x,y), \quad t>0,\\ &\rho(0,x,y) = \rho_0(x,y). \end{aligned} \tag{10}$$

Here $\mathcal{L}^*$ denotes the adjoint operator to $\mathcal{L}$.

Example 1. Let $X(t)\in\mathbb{R}^3$ be the position of a particle and $Y(t)$ its velocity which does not change between the jumps and has unit length so we denote it by $K(t)\in S^2$, the unit sphere. We now have that $F(x,k)=k$, $H(x,k)=0$ and the scattering law is $\Pi(x,k,dk') = \dfrac{\sigma(k,k')dk'}{\Sigma(k)}$ with $\Sigma(k) = \int_{S^2} dk'\sigma(k,k')$. Then the Kolmogorov equation (9) has the form of the usual scalar transport equation

$$\frac{\partial a}{\partial t} = k\cdot\nabla_x a + \int_{S^2} dk'\sigma(k,k')a(x,k') - \Sigma(k)a(x,k). \tag{11}$$

Example 2. A system of coupled transport equations of the form (11) is obtained as follows. The process $X(t)$ is the position of the particle which may be in several states labeled by i: $1\le i\le M$. The velocity process is now $Y(t) = (K(t),i(t))$, and $i\in\{1,2,\ldots,M\}$ with the evolution between jumps given by

$$\frac{dX}{dt} = K(t),\quad \frac{dK}{dt} = 0,\quad \frac{di}{dt} = 0,$$

so that the state of the particle does not change between jumps. The probability distribution of the time between jumps in state i is given by (2) with $\Sigma = \Sigma_i(x)$, that is, the total scattering cross-section may depend on the state. At the time of the jump the probability density of scattering from a state i with velocity k into a state j with velocity k' is $p_{ij} = \dfrac{\sigma_{ij}(x,k,k')}{\Sigma_i(x,k)}$. Then the Kolmogorov equation is a system of coupled scalar transport equations for the functions

$$a_i(t,x,k) = E_{i,x,k}\left\{a_0(X(t),K(t),i(t))\right\},$$

and has the form

$$\frac{\partial a_i}{\partial t} = k \cdot \nabla_x a_i + \int_{S^2} dk' \sum_{j=1}^{M} \sigma_{ij}(x,k,k') a_j(t,x,k') - \Sigma_i(x,k) a_i(t,x,k), \tag{12}$$

$i = 1, 2, ..., M$. This is the system of multi-group transport equations used in reactor theory [10].

3 Electromagnetism and Transport Equations

Electromagnetic waves are polarized and the evolution of their angularly resolved energy density cannot be described by a scalar transport equation of the type considered in the previous section. It is necessary to track the evolution of the 2×2 coherence matrix that satisfies the matrix radiative transport equation (13) given below. The wave energy density is the trace of this coherence matrix [5, 18]. The transport equation does not seem to have a direct probabilistic interpretation in terms of jump processes like the ones in the previous section. However, it is possible to have an indirect probabilistic representation by first introducing a transport process with forward Kolmogorov equation (32) for the energy density resolved not only over directions but also over the polarization states. The augmented transport equation corresponds to a jump Markov process as in section 2. In addition to the position $X(t)$ and wave vector (velocity) $K(t)$ we now have a pair of polarization processes $u_{11}(t)$ and $u_{12}(t)$. The position $X(t)$ is continuous in space, while the generalized velocity $Y(t) = (K(t), u_{11}(t), u_{12}(t))$ jumps at the scattering times. The jump probabilities are defined in such a way that the solution of the matrix radiative transport equation is essentially the matrix of first order moments, with respect to the polarization parameters, of the solution of the augmented transport equation. This is explained in section 3.3.

3.1 Radiative Transport Equations

The evolution of electromagnetic wave energy in phase space is described by a 2×2 coherence matrix $W(t,x,k)$, which is related to the Stokes parameters I_S, Q_S, U_S, V_S (see [5]), by:

$$W(t,x,k) = \frac{1}{2}\begin{pmatrix} I_S + Q_S & U_S + iV_S \\ U_S - iV_S & I_S - Q_S \end{pmatrix}.$$

The trace $\mathrm{Tr}\{W\} = I_S$ is the energy density. The transport equation for the coherence matrix W has the form

$$\begin{aligned}\frac{\partial W}{\partial t} &+ \nabla_k \omega \cdot \nabla_x W - \nabla_x \omega \cdot \nabla_k W + NW - WN \\ &= \int_{|k'|=|k|} dk' \, \sigma(x, k', k) T(k, k') W(x, k') T^*(k, k') - \Sigma W. \end{aligned} \tag{13}$$

Here the dispersion relation is $\omega(x, k) = v(x)|k|$. The initial condition is $W(0, x, k) = W_0(x, k)$.

The scattering cross-section $\sigma(x, k', k)$ is scalar and the 2×2 matrix $T(k, k')$ is defined by

$$T_{ij}(k, k') = (z^{(i)}(k) \cdot z^{(j)}(k')), \quad T^*(k, k') = T(k', k) \tag{14}$$

with the vectors $(\hat{k}, z^{(1)}(k), z^{(2)}(k))$, where $k = \hat{k}|k|$, forming an orthonormal basis in $\mathbb{R}^3$:

$$\hat{k} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \; z^{(1)}(k) = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}, \; z^{(2)}(k) = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}.$$

The coupling matrix N is given by

$$N(x, k) = \sum_{i=1}^{3} \frac{\partial v(x)}{\partial x^i} (z^{(1)}(k) \cdot \frac{\partial z^{(2)}(k)}{\partial k_i}) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = n(x, k) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{15}$$

The differential scattering cross-section $\sigma(x, k', k)$ is a positive function bounded from above and below,

$$0 < C_1 \le \sigma(x, k', k) \le C_2 < \infty,$$

and is symmetric in k, k' and rotation invariant,

$$\sigma(x, k, k') = \sigma(x, k', k) = \sigma(x, k \cdot k').$$

The total scattering cross-section $\Sigma(x, k)$ has no absorption

$$\int_{|k'|=|k|} dk' \sigma(x, k \cdot k') T(k, k') T^*(k, k') = \Sigma(x, k) I_2, \tag{16}$$

where I_2 is the 2×2 identity matrix. Thus the total energy is conserved in time:

$$E(t) = \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dk \mathrm{Tr}\{W(t, x, k)\} = \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dk \mathrm{Tr}\{W_0(x, k)\}.$$

The matrix $W(t, x, k)$ is positive definite and hermitian for all x and k provided that $W_0(x, k)$ is. A consequence of this is that the equation for the Stokes parameter V_S decouples from I_S, Q_S and U_S and we shall assume for simplicity that $V_S = 0$. Then it is convenient to rewrite the matrix W as

$$W(t, x, k) = \frac{\mathrm{Tr} W(t, x, k)}{2} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} u_{11} & u_{12} \\ u_{12} & -u_{11} \end{pmatrix} \right] \tag{17}$$

with

$$u_{11} = \frac{Q_S}{I_S}, \quad u_{12} = \frac{U_S}{I_S}$$

being the two polarization parameters.

3.2 The Augmented Transport Equation

We now construct the jump process

$$(X(t), K(t), u_{11}(t), u_{12}(t)), \tag{18}$$

whose probability density satisfies the forward Kolmogorov equation (32). The process $X(t) \in \mathbb{R}^3$ describes the position of the particle, $K(t) \in \mathbb{R}^3$ the wave vector and $u_{11}(t)$, $u_{12}(t)$ are the polarization parameters as in (17).

Since the matrix $W(t, x, k)$ is positive definite, the matrix

$$\begin{pmatrix} 1 + u_{11} & u_{12} \\ u_{12} & 1 - u_{11} \end{pmatrix} \tag{19}$$

is also positive definite. Thus the process $(u_{11}(t), u_{12}(t))$ takes values in the unit disc in $\mathbb{R}^2$:

$$\mathcal{K} = \left\{(u_{11}, u_{12}) : u_{11}^2 + u_{12}^2 \leq 1\right\}. \tag{20}$$

The processes $X(t)$ and $K(t)$ evolve between jumps by the equations

$$\frac{dX}{dt} = \nabla_k \omega(x, k) = v(x)\hat{k} \tag{21}$$

$$\frac{dK}{dt} = -\nabla_x \omega(x, k) = -|k| \nabla_x v(x). \tag{22}$$

The polarization parameters evolve as follows. Let the matrix U be defined by

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{12} & -u_{11} \end{pmatrix}. \tag{23}$$

Then, because of the form of the left side of the transport equation (13) we have:

$$\frac{dU}{dt} = UN - NU,$$

or in components,

$$\frac{du_{11}}{dt} = 2n(X(t), K(t))u_{12}, \quad \frac{du_{12}}{dt} = -2n(X(t), K(t))u_{11}, \tag{24}$$

with $n(x, k)$ defined in (15). Equations (24) preserve $u_{11}^2 + u_{12}^2$ and describe the rotation of the polarization vector along the ray in a variable medium [8]. In particular, the polarization parameters remain in the unit disc $\mathcal{K}$.

The probability distribution for the time τ_1 between two consecutive jumps is given by

$$P(\tau_1 > t) = \exp\left(-\int_0^t \Sigma(X(s), K(s))ds\right).$$

Let the particle have wave vector k and polarization parameters u_{11} and u_{12} before scattering . Then we define the probability that after scattering the wave vector is k' by

$$\tilde{\pi}(x,k,U,k') = \frac{\sigma(x,k\cdot k')\mathrm{Tr}\left\{T(k',k)[I_2+U]T^*(k',k)\right\}}{2\Sigma(x,k)} \tag{25}$$

with the matrix U defined by (23).

Given the scattered direction, the outgoing polarization parameters u'_{11}, u'_{12} are as follows. Let the traceless symmetric matrix $U_s(k,U,k')$ be defined by

$$\frac{1}{2}(I_2+U_s) = \frac{T(k',k)[I_2+U]T^*(k',k)}{\mathrm{Tr}\{T(k',k)[I_2+U]T^*(k',k)\}}, \tag{26}$$

which is the normalized scattering operator on the right side of Chandrasekhar's equation (13). The average polarization matrix after scattering should be U_s. One way to define the probability density for a transition from wave vector k and polarization matrix U into wave vector k' and polarization matrix U' is

$$\breve{\pi}(x,k,U,k',U') = \frac{\sigma(x,k\cdot k')\mathrm{Tr}\left\{T(k',k)[I_2+U]T^*(k',k)\right\}}{2\Sigma(x,k)}\delta(U'-U_s(k,U,k')). \tag{27}$$

This, however, is not the only way to define jump probabilities consistently with Chandrasekhar's equation (13).

We will show that the linear moments with respect to the polarization parameters of the solution of the Kolmogorov equation for the process defined above solve the matrix valued transport equation (13). However, the choice of the scattering probability (27) that gives this result is not unique. A class of scattering probabilities, generalizing (27), which are consistent with (13) in the above sense, is as follows.

We introduce a function $\phi(k,u_{11},u_{12},k',u'_{11},u'_{12})$ (we shall also write $\phi(k,U,k',U')$) that is positive and bounded from above and below:

$$0 < C_1 \le \phi(k,u_{11},u_{12},k',u'_{11},u'_{12}) \le C_2 < \infty, \tag{28}$$

and has the following moments with respect to U':

$$\int_{\mathcal{K}} du'_{11}du'_{12}\phi(k,U,k',U') = 1 \tag{29}$$

$$\int_{\mathcal{K}} du'_{11}du'_{12}\phi(k,U,k',U')u'_{lr} = u_{s,lr}. \quad (l,r) = (1,1),(1,2), \tag{30}$$

where $\mathcal{K}$ is defined in (20). As in (27), the prototype for the function ϕ is $\delta(U' - U_s(k,U,k'))$ except that it does not satisfy the bounds (28). The ϕ is a smoothed version of the delta function with the correct first two moments. Thus, the probability density to have outgoing polarization matrix U' given the scattered direction k', the incoming wave vector k and polarization matrix U, is $\phi(k, U, k', U')$. The probability density for a jump from wave vector k and polarization matrix U into wave vector k' and polarization matrix U' is

$$\pi(x, k, U, k', U') = \frac{\sigma(x, k \cdot k') \mathrm{Tr}\,\{T(k', k)[I_2 + U]T^*(k', k)\}}{2\Sigma(x, k)} \phi(k, U, k', U'). \tag{31}$$

It can be verified that this is indeed a probability density:

$$\begin{aligned}&\int_{|k'|=|k|} dk' \int_{\mathcal{K}} du'_{11} du'_{12} \pi(x, k, U, k', U') \\ &= \int_{|k'|=|k|} dk' \frac{\sigma(x, k \cdot k') \mathrm{Tr}\,\{T(k', k)[I_2 + U]T^*(k', k)\}}{2\Sigma(x, k)} = 1,\end{aligned}$$

where we used (29) in the first equality, and (14) and (16) in the second one.

There are two aspects of the regularized scattering kernel π that are important. First, it is positive everywhere. This allows a derivation of the diffusion approximation without technical difficulties. We note, however, that the rest of this section is not modified when π is replaced by $\breve{\pi}$ defined in (27). Second, the choice of ϕ allows for more flexibility in the numerical simulation of the augmented transport equation. Indeed, we need to average over the polarization parameters to obtain the solution of (13). This averaging may require a large number of simulated trajectories of the jump process to be accurate. The inherent noise of the simulation can be significantly reduced by a suitable choice of the regularization function ϕ.

The forward Kolmogorov equation for the probability density function $\rho(x, k, U; x_0, k_0, U^0)$, $k \in \mathbb{R}^3$, $(u_{11}, u_{12}) \in \mathcal{K}$ of the jump process $(X(t), K(t), U(t))$ starting at the point (x_0, k_0, U^0) is now given by

$$\begin{aligned}&\frac{\partial \rho}{\partial t} + v(x)\hat{k} \cdot \nabla_x \rho - |k| \nabla_x v \cdot \nabla_k \rho + n(x, k) \left(2u_{12} \frac{\partial \rho}{\partial u_{11}} - 2u_{11} \frac{\partial \rho}{\partial u_{12}} \right) \\ &= \frac{1}{2} \int_{|k'|=|k|} dk' \int_{\mathcal{K}} du'_{11} du'_{12} \sigma(x, k \cdot k') \mathrm{Tr}\,\{T(k, k')[I_2 + U']T^*(k, k')\}\, \phi(k', U', k, U) \\ &\times \rho(t, x, k', u'_{11}, u'_{12}) - \Sigma(x, k) \rho(t, x, k, u_{11}, u_{12})\end{aligned} \tag{32}$$

with the initial data

$$\rho(0, x, k, u_{11}, u_{12}) = \delta(x - x_0)\delta(k - k_0)\delta(u_{11} - u_{11}^0)\delta(u_{12} - u_{12}^0) \tag{33}$$

and the boundary condition

$$\rho(t, x, k, u_{11}, u_{12}; x_0, k_0, U^0) = 0 \quad \text{if} \quad u_{11}^2 + u_{12}^2 = 1. \tag{34}$$

We call this equation, which is the main result of this section, the augmented transport equation, since, as we shall see below, it describes the propagation of the *scalar* energy density of polarized (electromagnetic) waves in an expanded (augmented) phase space that includes polarization.

3.3 Probabilistic Representation for Chandrasekhar's Equations

We now show that the solution of Chandrasekhar's equation (13) may be expressed in terms of the probability density of the jump process introduced in section 3.2. Let $W(t,x,k)$ be the solution of (13) with initial data $W(0,x,k) = W_0(x,k)$. We decompose the initial data as in (17)

$$W_0(x,k) = \frac{w_0(x,k)}{2}\begin{pmatrix} 1+u_{11}^0 & u_{12}^0 \\ u_{12}^0 & 1-u_{11}^0 \end{pmatrix} = \frac{w_0(x,k)}{2}\left[I_2 + U^0(x,k)\right]$$

with $w_0(x,k) = \mathrm{Tr}\{W_0(x,k)\}$. Then we show that $W(t,x,k)$ is given by

$$\begin{aligned} W(t,x,k) &= \int_{\mathbb{R}^3} dx_0 \int_{|k_0|=|k|} dk_0 \int_{\mathcal{K}} du_{11}du_{12}\rho(t,x,k,U;x_0,k_0,U^0(x_0,k_0)) \\ &\times \; w_0(x_0,k_0)\frac{I_2+U}{2} = \int_{\mathcal{K}} du_{11}du_{12}\bar{\rho}(t,x,k,U)\frac{I_2+U}{2} \end{aligned} \tag{35}$$

where the function ρ is the solution of (32) with the initial condition (33) and boundary condition (34), and the matrix U is given by (23). The function $\bar{\rho}$ also solves (32) with the same boundary conditions and the initial condition

$$\bar{\rho}(0,x,k,U) = w_0(x,k)\delta(U - U^0(x,k)). \tag{36}$$

In order to show that (35) satisfies (13), we multiply equation (32) for the function $\bar{\rho}$ by the matrix $\frac{1}{2}[I_2+U]$, with the matrix U given by (19), and integrate with respect to u_{11} and u_{12} over the unit disc $\mathcal{K}$. The first three terms on the left side of (32) are

$$\frac{\partial W}{\partial t} + \nabla_k\omega\cdot\nabla_x W - \nabla_x\omega\cdot\nabla_k W \tag{37}$$

since the dispersion law $\omega(x,k)$ is independent of the polarization parameters. The last term is

$$n(x,k)\int_{\mathcal{K}} du_{11}du_{12}\left(2u_{12}\frac{\partial\bar{\rho}}{\partial u_{11}} - 2u_{11}\frac{\partial\bar{\rho}}{\partial u_{12}}\right)\begin{pmatrix} 1+u_{11} & u_{12} \\ u_{12} & 1-u_{11} \end{pmatrix},$$

which we integrate by parts, using the boundary conditions (34), to get

$$n(x,k)\int_{\mathcal{K}} du_{11}du_{12}\bar{\rho}(t,x,k,U)\begin{pmatrix} -2u_{12} & 2u_{11} \\ 2u_{11} & 2u_{12} \end{pmatrix} = NW - WN. \tag{38}$$

This is the last term on the left side of the Chandrasekhar's equation (13).

The first term on the right side of equation (32) for the function $\bar{\rho}$ is, after multiplication by $(I_2+U)/2$ and integration,

$$\begin{aligned} &\frac{1}{4}\int_{\mathcal{K}} du_{11}du_{12}[I_2+U]\int_{|k'|=|k|} dk' \int_{\mathcal{K}} du'_{11}du'_{12}\sigma(x,k\cdot k') \\ &\quad\times \mathrm{Tr}\left\{T(k,k')[I_2+U']T^*(k,k')\right\}\phi(k',U',k,U)\bar{\rho}(t,x,k',u'_{11},u'_{12}). \end{aligned} \tag{39}$$

The integration in u_{11}, u_{12} is done using (29) for the zeroth and first moments of the function ϕ:

$$\frac{1}{4}\int_{|k'|=|k|} dk' \int_{\mathcal{K}} du'_{11}du'_{12}[I_2+U_s(k',k,U')]\sigma(x,k\cdot k') \\ \times \mathrm{Tr}\left\{T(k,k')[I_2+U']T^*(k,k')\right\}\bar{\rho}(t,x,k',u'_{11},u'_{12}).$$

Finally we use (26), which defines the matrix U_s, to get

$$\frac{1}{2}\int_{|k'|=|k|} dk' \int_{\mathcal{K}} du'_{11}du'_{12}\sigma(x,k\cdot k')T(k,k')[I_2+U']T^*(k,k')\bar{\rho}(t,x,k',u'_{11},u'_{12}) \\ = \int_{|k'|=|k|} dk'\sigma(x,k\cdot k')T(k,k')W(k')T^*(k,k'). \quad (40)$$

After multiplication by $(I_2+U)/2$ and integration over u_{11}, u_{12}, the last term on the right side of (32) gives $\Sigma(x,k)W(t,x,k)$. Combining now this term, (37), (38) and (40) we get Chandrasekhar's equation (13) for the coherence matrix $W(t,x,k)$ defined by (35).

4 General Theory of Transport with Polarization

4.1 Radiative Transport Equations

We now generalize the theory of the previous section to more complex transport phenomena. Consider the transport of M different species of particles which may correspond to different states of particles or waves. The coherence state of each species is represented by a positive hermitian matrix, which we shall take for simplicity to be real symmetric and positive definite, $W^i(x,k)$, for $1\le i\le M$, of size $n^i\times n^i$. The number n^i is the multiplicity of the polarization and $n^i=1$ corresponds to a scalar energy density. For electromagnetic waves, $M=1$ and $n^1=2$. For elastic waves, $M=2$, $n^1=1$ and $n^2=2$ (see appendix A for more details). We define the block-diagonal matrix W by

$$W(t,x,k) = \text{ Diag }(W^i(t,x,k), i=1,\ldots,M). \quad (41)$$

It is positive definite and its size is $\sum_{i=1}^M n^i \times \sum_{i=1}^M n^i$. From the general theory of radiative transport equations [14], it satisfies the following transport equation in a phase space $\mathbb{R}^d_x\times\mathbb{R}^d_k$:

$$\frac{\partial W(t,x,k)}{\partial t} + \nabla_k\Omega\cdot\nabla_x W(t,x,k) - \nabla_x\Omega\cdot\nabla_k W + NW - WN \\ = \int dk'\sigma^*(x,k',k)[W(t,x,k')] - \Sigma(x,k)[W(t,x,k)] \quad (42)$$

with an initial condition $W(0,x,k)=W_0(x,k)$.

The dispersion matrix Ω has the form

$$\Omega = \mathrm{Diag}(\Omega^i, i=1,\ldots,M), \qquad \text{where} \qquad \Omega^i=\omega^i(x,k)I_{n^i}.$$

We have denoted by I_{n^i} the identity matrix in $\mathbb{R}^{n^i}$, and by $\omega^i(x,k)$ the local dispersion relation of each of the species, or wave types. The matrix N is given by

$$N(x,k) = \text{ Diag } (N^i(x,k), i = 1,\dots,M), \tag{43}$$

where all $n^i \times n^i$ sub-matrices N^i are skew-symmetric. They account for the rotation of the polarization parameters along the characteristics, when they are curved. In particular $N^i = 0$ if $\omega^i = \omega^i(k)$ is independent of the position x. The scattering cross section $\sigma^*(k,k')$ is a fourth order tensor. We assume that it preserves the symmetric and positive definite structure of the matrix W, that is, if $W(k')$ is a positive definite symmetric matrix of the form (41) then we have

$$\sigma^*(x,k',k)[W(k')] = \text{ Diag } (W_s^i(x,k',k), i = 1,\dots,M) \quad \text{in } \mathbb{R}^d \times \mathbb{R}^d \tag{44}$$

where the $n^i \times n^i$ matrices W_s^i are symmetric and positive. Moreover, the tensor Σ is diagonal:

$$\Sigma[W] = \text{ Diag } (\Sigma^i(x,k)W^i)$$

and the cross sections $\Sigma^i(x,k)$ are bounded positive scalar functions. We assume that scattering is conservative in the sense that

$$\int dk' \int dk \text{Tr}\{\sigma^*(x,k',k)[W(k')]\} = \int dk \text{Tr}\{\Sigma(x,k)[W(k)]\}. \tag{45}$$

for every matrix W of the form (41).

We rewrite (45) in terms of the blocks σ^{ij} that define the tensor σ when it is acting on W's of the form (41):

$$\sum_{p=1}^{n^l} \sum_{i=1}^{M} \int dk' \sigma_{ppmn}^{li}(x,k,k') = \Sigma^l(k)\delta_{mn}, \tag{46}$$

where $(\sigma^{ij}[W])_{mn} = \sum_{p,q} \sigma_{mnpq}^{ij} W_{pq}$. Here σ is the tensor adjoint to σ^*, defined by

$$\sigma_{mnpq}^{il*}(x,k',k) \quad = \quad \sigma_{mnpq}^{li}(x,k',k), \tag{47}$$

and repeated indices are summed throughout, unless stated otherwise. The conservation relation (46) implies in particular that given any traceless matrix $U(k)$, we have

$$\text{Tr}\{\sum_{i=1}^{M} \int dk' \sigma^{il*}(x,k,k')[U(k)]\} = 0 \tag{48}$$

for all $1 \le l \le M$.

4.2 The Augmented Transport Process

In this section we will extend to general matrix radiative transport equations the construction of the augmented jump Markov process with polarization that we carried out for electromagnetic wave transport in section 3.2.

4.2.1 Jump Process with Polarization

We write the matrix W^i corresponding to the i^{th} species in the form

$$W^i = w^i \tilde{U}^i, \quad \text{where} \quad \left\{ \begin{array}{l} w^i = Tr\{W^i\}, \\ \tilde{U}^i = \dfrac{1}{n^i}(I_{n^i} + U^i). \end{array} \right. \tag{49}$$

Here U^i is a $n^i \times n^i$ symmetric matrix with null trace, $\mathrm{Tr}\{U^i\} = 0$. The energy density of the i-th species is $w^i = \mathrm{Tr}\{W^i\}$. There are $\frac{n^i(n^i+1)}{2} - 1$ polarization parameters since the matrix W^i is symmetric and positive definite.

For $i = 1, \dots, M$ we introduce

$$Y^i(t) = \left(u^i_{mn}(t)\right)_{(m,n)\in I^i}, \tag{50}$$

where

$$I^i = \{(m,n) \in \mathbb{N}^2 \text{ such that } 1 \le m \le n \le n^i \text{ and } m+n \neq 2n^i\}. \tag{51}$$

The matrix U^i has then the form

$$U^i = \begin{pmatrix} u^i_{11} & u^i_{12} & \dots & u^i_{1n_i} \\ u^i_{12} & u^i_{22} & \dots & u^i_{2n_i} \\ \vdots & \vdots & \vdots & \vdots \\ u^i_{1n_i} & u^i_{2n_i} & \dots & -\sum_{j=1}^{n_i-1} u^i_{jj} \end{pmatrix}. \tag{52}$$

The random process Y^i is an element of $\mathbb{R}^{d'_i}$, where

$$d'_i = \# I^i = \frac{n^i(n^i+1)}{2} - 1$$

and n_i, for $1 \le i \le M$, is the size of the matrix of polarization parameters corresponding to species i. The integer M is the total number of species. Since the matrix $\tilde{U}^i$ is positive definite, Y^i takes values in a compact set $\mathcal{K}_i \subset \mathbb{R}^{d_i}$ since

$$||U^i||_2^2 = \sup_{||v||=1} (U^i U^{i*} v, v) \le (n^i - 1)^2,$$

for $Y^i \in \mathcal{K}_i$, because $Tr\{U^i\} = 0$ and $\tilde{U}^i$ in (49) is positive definite.

The full space of velocity and polarization parameters is denoted by

$$Y(t) = \left(K(t), Y^1(t), \dots, Y^N(t), i(t)\right). \tag{53}$$

The index $i(t)$ indicates to which species the particle belongs at time t. The rule is that the state of the particle does not change between jumps. However, it can change by scattering, i.e. at the time of a jump. Moreover, all the vectors Y^j for $j \neq i(t)$ are zero, since only one species is present between jumps. Therefore the state space of $Y(t)$ is the set $\mathcal{S}$ of vectors of the form $(k, 0, \dots, 0, y^i, 0, \dots, 0, i)$ with $y^i \in \mathcal{K}_i$. Here, $y^i \in \mathbb{R}^{d'_i}$ denotes the set of polarization parameters that are currently non-zero. We shall denote such a point in $\mathcal{S}$ by (k, y^i, i).

4.2.2 Construction of the Process

We will apply the general theory of section 2 to processes with polarization effects. The construction of the random process $(X(t), Y(t))$ is threefold: we describe first the trajectory of a particle before the time of the first jump τ_1, then the distribution of the random variable τ_1, and finally the scattering kernel Π, which is a measure on $\mathcal{S}$.

We assume that the particle is in state i_0 at time $t = 0$. The total cross-section for species i_0 is given by $\Sigma^{i_0}(x, y) = \Sigma^{i_0}(x, k)$. The random variable τ_1 has the exponential distribution function

$$P(\tau_1 > t) = \exp\Big(- \int_0^t \Sigma^{i_0}(\xi(s; x, y), \eta(s; x, y)) ds\Big). \tag{54}$$

The type of species does not change between jumps:

$$i(t) = i_0 \quad \text{for} \quad 0 \leq t < \tau_1. \tag{55}$$

The initial conditions for the trajectories are given by

$$\begin{aligned} X(0) &= \xi(0; x, y) = x \\ Y(0) &= \eta(0; x, y) = (k, 0, \ldots, y^{i_0}, \ldots, 0, i_0) \end{aligned} \tag{56}$$

where $Y^{i_0}(0) = y^{i_0} \in \mathcal{K}_{i_0}$ is at position i_0 in the vector $Y(0)$ and corresponds to the initial value of the polarization parameters. The evolution of the variables Y^i for $i \neq i_0$ is given by

$$Y^i(t) = 0 \quad \text{for} \quad i \neq i_0 \text{ and } 0 \leq t < \tau_1.$$

Let $\omega^i(x, k)$ be the dispersion relation for species i. The remaining evolution equations for $(X(t), Y(t))$ are given by

$$\begin{aligned} \frac{dX(t)}{dt} &= F(X(t), K(t)) = \nabla_k \omega^{i_0}(X(t), K(t)) \\ \frac{dK(t)}{dt} &= H(X(t), K(t)) = -\nabla_x \omega^{i_0}(X(t), K(t)) \\ \frac{dU^{i_0}(t)}{dt} &= G^{i_0}(X(t), K(t), U^{i_0}(t)) \\ &= U^{i_0}(t) N^{i_0}(X(t), K(t)) - N^{i_0}(X(t), K(t)) U^{i_0}(t) \end{aligned} \tag{57}$$

with U^{i_0} related to Y^{i_0} by (50) and (52). The skew symmetric matrix N^{i_0} governs the rotation of the polarization parameters along the characteristics. Since N^{i_0} is skew symmetric, the third relation in (57) preserves the symmetry of the matrix U^{i_0}. Moreover, the spectrum of the matrix $U^{i_0}(t)$ is preserved, so it remains traceless, and the matrix $\tilde{U}^{i_0}$ defined in (49) remains positive definite with its trace equal to 1, for all times $t \geq 0$. These equations give us an expression for the stream operators F and H.

It remains to define the scattering probability $\Pi(x, y, A)$, $y = (k, 0, \ldots U^{i_0}, \ldots, 0, i_0)$. It describes the probability that a particle undergoing scattering at time τ_1 at a position x with the incoming wave vector k and polarization matrix U^{i_0} is scattered into a particle of species j with wave vector k' and matrix of polarization parameters U'^j, so that the state vector $(k', 0, \ldots, U'^j, \ldots, 0, j)$ belongs to the set A.

The probability $\Pi(x, y, A)$, $A \subseteq \mathcal{S}$, is constructed using the scattering cross-section tensor σ. Recall that it acts on a matrix $W(x,k) = Diag(W^i(x,k), i = 1, \ldots, M)$ with each W^i being a matrix of size $n^i \times n^i$, as follows,

$$\sigma(x,k,k')[W(x,k)] = Diag(W_s^i(x,k'), i = 1, \ldots, M). \tag{58}$$

We assume that the matrices W_s^j are symmetric positive definite if the matrices W^i are symmetric and positive definite and that the scattering process is conservative, so that (45) holds. The fourth order tensor σ is decomposed into $M \times M$ fourth order tensors σ^{ij} defined by

$$W_s^j(x,k,k',W^i) = \sum_{i=1}^{M} \sigma^{ij}(x,k,k')[W^i(x,k)], \tag{59}$$

where $W^i(x,k')$ and $W_s^j(x,k)$ are the blocks indexed by i and j on the right and left hand side of (58) respectively. We denote by $\pi(x,k,y^i,i,k',z^j,j)$ the probability density of $\Pi(x,y,A)$. Here x is the position of the scattering, $y = (k, y^i, i) \in \mathcal{S}$ records the incident wave vector, the incident polarization parameters $y^i = (u^i_{mn})$, and the type i of species, and $z = (k', z^j, j) \in \mathcal{S}$ is the wave vector and polarization parameters $z^j = (u'^j_{mn})$ of the particle scattered into state j . The density π is defined as follows. First we define the probability of being scattered with velocity k' and in state j:

$$\tilde{\pi}(x,k,y^i,i,k',j) = \frac{\mathrm{Tr}\{\sigma^{ij}(x,k,k')[\tilde{U}^i]\}}{\Sigma^i(x,k)}. \tag{60}$$

Here $\tilde{U}^i$ is a positive definite matrix given in terms of the polarization parameters u^i_{mn} by (49). Since $\sigma^{ij}(x,k,k')[\tilde{U}^i]$ is a non-negative definite matrix, we find that $\tilde{\pi}(x,y^i,i,k',j)$ is non-negative if $\tilde{U}^i$ is non-negative. Moreover, energy conservation (46) implies that

$$\begin{aligned}
\sum_{j=1}^{M} \int dk' \tilde{\pi}(x,k,y^i,i,k',j) &= \sum_{j=1}^{M} \int dk' \frac{\mathrm{Tr}\{\sigma^{ij}(x,k,k')[\tilde{U}^i]\}}{\Sigma^i(x,k)} \\
&= \sum_{j=1}^{M} \int dk' \frac{\sigma^{ij}_{mmpq}(x,k,k')[\tilde{U}^i_{pq}]}{\Sigma^i(x,k)} = 1
\end{aligned} \tag{61}$$

since $\mathrm{Tr}\{\tilde{U}^i\} = 1$.

Let $\phi(k, y^i, i, k', z^j, j)$ be a continuous uniformly positive bounded function such that

$$\int_{\mathcal{K}_j} \phi(k, U^i, i, k', U'^j, j) dU'^j = 1 \tag{62}$$

for all k and U^i. Here

$$dU^j = \prod_{(p,q)\in I^j} du^j_{pq}. \tag{63}$$

Moreover, we impose that the mean value for the emitted polarization parameters be given by

$$\int_{\mathcal{K}_j} \phi(k, U^i, i, k', U'^j, j) u'^j_{lr} dU'^j = u^j_{s,lr}(k, k', U^i), \tag{64}$$

where the matrix U^j_s is determined from

$$\frac{1}{n^j}(I_{n^j} + U^j_s) = \frac{\sigma^{ij}(x, k, k')[\tilde{U}^i]}{\Sigma^i(x, k)\tilde{\pi}(x, y^i, i, k', j)}. \tag{65}$$

The matrix U^j_s is an $n^j \times n^j$ matrix with trace equal to zero. By definition of the scattering tensor σ, the matrix

$$\tilde{U}^j_s = \frac{1}{n^j}(I_{n^j} + U^j_s)$$

is positive definite. We can then define the probability density $\pi(x, k, y^i, k', z^j, j)$ with $y^i = (u^i_{mn})$ and $z^j = (u'^j_{pq})$, as

$$\pi(x, k, y^i, i, k', z^j, j) = \tilde{\pi}(x, k, y^i, i, k', j)\phi(k, U^i, i, k', U'^j, j) \tag{66}$$

if $\tilde{\pi}(x, y^i, i, k', j) > 0$, and $\pi(x, k, y^i, i, k', z^j, j) = 0$ elsewhere. Note that the density $\pi(x, k, y^i, i, k', z^j, j)$ defines a probability measure:

$$\begin{aligned} &\sum_{j=1}^{M} \int dz^j dk' \pi(x, k, y^i, i, k', z^j, j) \\ &= \sum_{j=1}^{M} \int dU'^j dk' \tilde{\pi}(x, k, y^i, i, k', j)\phi(k, U^i, i, k', U'^j, j) = 1 \end{aligned} \tag{67}$$

according to (61) and (62). This completes the construction of the jump process.

4.2.3 Augmented Transport Equation with Polarization

We consider next the backward Kolmogorov equation for the jump process constructed in the previous section. Let $a_0(x, k, y^i, i)$ be a function defined on $\mathcal{S}$ and let

$$a(t, x, k, y^j, j) = E_{x,k,y^j,j}\left\{a_0(X(t), Y(t))\right\}. \tag{68}$$

It is convenient to write the backward Kolmogorov equation as a system of equations for the restrictions of $a(t, x, k, y^i, i)$ to sets of the form $\mathbb{R}^d_x \times \mathbb{R}^d_k \times \{0\} \times \ldots \times \mathcal{K}_i \times \ldots \times \{0\} \times \{i\}$:

$$a_j(t, x, k, U^j) = a(t, x, k, y^j, j)$$

with $y^j = (u^j_{mn})$, $(m, n) \in I^j$. Then these functions satisfy the system of equations:

$$\begin{aligned} \frac{\partial a_i(t, x, k, U^i)}{\partial t} &= \nabla_k \omega^i \cdot \nabla_x a_i - \nabla_x \omega^j \cdot \nabla_k a_i + \sum_{(p,q)\in I^i} G^i_{pq} \partial_{u^i_{pq}} a_i - \Sigma_i(x, k) a_i \\ &+ \sum_{j=1}^{M} \int_{\mathbb{R}^d} dk' \int_{\mathcal{K}_j} dU'^j \mathrm{Tr}\left\{\sigma^{ij}(x, k, k')\left[\frac{I_{n^i} + U^i}{n^i}\right]\right\} \cdot \\ &\cdot \phi(k, U^i, i, k', U'^j, j)\, a_j(t, x, k', U'^j). \end{aligned} \tag{69}$$

The forward Kolmogorov equation can be written as a system of equations for the restriction on the same sets, of the probability density of the jump process:

$$\begin{aligned}
&\frac{\partial \rho_i(t,x,k,U^i)}{\partial t} + \nabla_k \omega^i \cdot \nabla_x \rho_i - \nabla_x \omega^j \cdot \nabla_k \rho_i + \sum_{(p,q)\in I^i} G^i_{pq} \partial_{u^i_{pq}} \rho_i + \Sigma_i(x,k)\rho_i \\
&\quad = \sum_{j=1}^{M} \int_{\mathbb{R}^d} dk' \int_{\mathcal{K}_j} dU'^j \mathrm{Tr}\left\{ \sigma^{ji}(x,k',k) \left[\frac{I_{n^j} + U'^j}{n^j} \right] \right\} \cdot \qquad (70) \\
&\qquad \cdot \phi(k', U'^j, j, k, U^i, i)\, \rho_j(t,x,k',U'^j).
\end{aligned}$$

4.3 Probabilistic Representation of Transport Equations with Polarization

The process constructed in section 4.2 does not directly allow us to solve either the transport equation (42), or its adjoint version. The reason for the introduction of this augmented process is that the solution W of the radiative transport equation is found to be given in terms of linear moments with respect to the polarization parameters, of the solution of the forward Kolmogorov equation $\rho(t,x,y)$. More precisely, we have the following Theorem.

Theorem 4.1 *Let $(X(t), Y(t))$, with $Y(t) = (K(t), 0, \ldots, Y^i(t), \ldots, 0, i(t))$, $Y^i(t) = (u^i_{m,n}(t))$ be the jump Markov process constructed above starting at x_0 in the state i_0 with initial direction k_0 and matrix of polarization parameters $U_0^{i_0}$. Let $\rho(t,x,y;x_0,k_0,U_0^{i_0})$ be its probability density, with the restrictions*

$$\rho_i(t,x,k,U^i) = \rho(t,x,k,0,\ldots,u^i_{mn},\ldots,0,i)$$

satisfying the system of forward transport equations (70) with initial data

$$\rho_i(0,x,k,U^i;x_0,k_0,U_0^{i_0}) = \delta(x-x_0)\delta(k-k_0)\delta(u^i_{mn} - u^i_{mn,0})\delta_{i,i_0}$$

and boundary conditions

$$\rho_i\,|_{\partial \mathcal{K}_i} = 0.$$

Then, for any smooth scalar functions $w_0^i(x,k)$ and matrix-valued functions $U_0^i(x,k)$ taking values in the set of traceless symmetric matrices, the matrix-valued functions

$$\begin{aligned}
W^i(t,x,k) &= \sum_{i_0=1}^{M} \int dx_0\, dk_0 \int_{\mathcal{K}_i} dU^i \rho_i(t,x,k,U^i;x_0,k_0,U_0^{i_0}(x_0,k_0)) \cdot \\
&\qquad \cdot w_0^{i_0}(x_0,k_0) \frac{I_{n^i} + U^i}{n^i} \qquad (71)
\end{aligned}$$

satisfy the system of the transport equations (42). The initial data for (42) are

$$W^i(0,x,k) = W_0^i(x,k) = w_0^i(x,k)\frac{I + U_0^i(x,k)}{n^i}, \quad \text{for } i = 1,\ldots,M.$$

Notice that

$$\mathrm{Tr}\{W^i(t,x,k)\} = \int_{\mathcal{K}_i} dU^i \bar{\rho}_i(t,x,k,U^i),$$

where $\bar{\rho}$ solves (70) with initial conditions $\bar{\rho}(o,x,k,U^i) = w_0^i(x,k)\delta(U^i - U_0^i(x,k))$. Thus, $\bar{\rho}(t,x,y)$ is an energy, which is resolved both in velocity and polarization parameters. Its average over the polarization parameters yields the angularly resolved energy $\mathrm{Tr}\{W^i(t,x,k)\}$.

Our construction provides an existence theory for the matrix-valued transport equation, although classical methods of functional analysis similar to those in [6] may also be used. Since the total energy is conserved (see [14]), the linearity of the transport equation also yields uniqueness.

We prove Theorem 4.1 in the next two sections, first computing the streaming terms of the transport equations, and then the scattering part.

4.3.1 The Liouville Equation

Let us start with the terms corresponding to free transport, i.e. the ones on the left side of the transport equation. The left side of the forward Kolmogorov equation for $\bar{\rho}$ is

$$\frac{\partial \bar{\rho}_i}{\partial t} + \nabla_k \omega^i \cdot \nabla_x \bar{\rho}_i - \nabla_x \omega^i \cdot \nabla_k \bar{\rho}_i + \sum_{(p,q)\in I^i} G^i_{pq} \partial_{u^i_{pq}} \bar{\rho}_i.$$

We multiply this equation by the matrix $\tilde{U}^i$ and integrate over the polarization parameters u^i_{mn}. The two first terms clearly yield

$$\partial_t W^i(t,x,k) + \nabla_k \omega^i \cdot \nabla_x W^i(t,x,k) - \nabla_x \omega^i \cdot \nabla_k W^i(t,x,k)$$

since the dispersion relations ω^i are independent of the polarization parameters. Let us show that the last term yields $W^i N^i - N^i W^i$. Recall that we have for all $(m,n) \in I^i$,

$$G^i_{pq}(x,k,u^i) = \sum_l \Big(u^i_{pl} n^i_{lq}(x,k) - u^i_{lq} n^i_{pl}(x,k) \Big). \tag{72}$$

Since $N(x,k)$ is skew-symmetric, all coefficients $n_{mm} = 0$ and therefore G^i_{mn} does not depend on the polarization parameters u^i_{mn}. Therefore we have

$$\begin{aligned} & \int_{\mathcal{K}_i} dU^i \sum_{(p,q)\in I^i} \partial_{u^i_{pq}} \bar{\rho}_i(t,x,k,U^i) \left(G^i_{pq}(x,k,U^i) \right) \left(I_{n^i} + U^i \right) \\ = \; & - \int_{\mathcal{K}_i} dU^i \bar{\rho}_i(t,x,k,U^i) \sum_{(p,q)\in I^i} \left(G^i_{pq}(x,k,U^i) \right) \Big(\partial_{u^i_{pq}} U^i \Big). \end{aligned}$$

We easily check for $(p,q)\in I^i$ that

$$(\partial_{u^i_{pp}} U^i)_{mn} = \begin{cases} 1 & \text{if } (m,n)=(p,p) \\ -1 & \text{if } (m,n)=(n^i,n^i) \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{for } p\neq q, \qquad (\partial_{u^i_{pq}} U^i)_{mn} = \begin{cases} 1 & \text{if } (m,n)=(p,q) \text{ or } (m,n)=(q,p) \\ 0 & \text{otherwise.} \end{cases}$$

We deduce from (72) that

$$\sum_{(p,q)\in I^i} \left(G^i_{pq}(x,k,U^i)\right)\left(\partial_{u^i_{pq}} U^i\right) = U^i N^i(x,k) - N^i(x,k)U^i.$$

So far we have that the left side is

$$\frac{\partial}{\partial t}W^i(t,x,k) + \Big(\nabla_k\omega^i\cdot\nabla_x W^i - \nabla_x\omega^i\cdot\nabla_k W^i + (N^iW^i - W^iN^i)\Big)(t,x,k).$$

It now remains to add the contribution generated by the scattering operator.

4.3.2 Radiative Transport

In the previous section, we calculated the streaming terms in the transport equation. We now compute the contribution from the scattering operator. First, we have

$$\frac{1}{n^i}\int_{\mathcal{K}_i} dU^i\Sigma^i(x,k)\bar\rho_i(t,x,U^i)[I+U^i] = \Sigma^i(x,k)[W^i(t,x,k)]$$

since Σ^i does not depend on the polarization parameters. Let us compute the second contribution

$$\frac{1}{n_i}\sum_{j=1}^M \int dk' \int_{\mathcal{K}_i} dU^i \int_{\mathcal{K}_j} dU'^j \frac{\mathrm{Tr}\{\sigma^{ji}(x,k',k)[I_{n^j}+U'^j]}{n_j}\}\cdot$$
$$\cdot\,\bar\rho_j(t,x,k',U'^j)\left[U^i+I_{n^i}\right]\phi(k',U'^j,j,k,U^i,i).$$

Using the expressions (62) and (64) for the moments of the function ϕ to evaluate the integral with respect to u^i_{mn} we see that this expression is equal to

$$\sum_{j=1}^M \int dk' \int_{\mathcal{K}_j} dU'^j \frac{\mathrm{Tr}\{\sigma^{ji}(x,k',k)[I_{n^j}+U'^j]\}}{n^j}\frac{(I_{n^i}+U^i_s(k',k,U'^j))\bar\rho_j(k',U'^j)}{n^i}$$
$$=\sum_{j=1}^M \int dk' \int_{\mathcal{K}_j} dU'^j \frac{\mathrm{Tr}\{\sigma^{ji}(x,k',k)[I_{n^j}+U'^j]\}}{n^j}\frac{\sigma^{ji}(x,k',k)[\tilde U'^j]\bar\rho_j(k',U'^j)}{\mathrm{Tr}\{\sigma^{ji}(x,k',k)[\tilde U'^j]\}}$$
$$=\sum_{j=1}^M \int dk'\sigma^{ji}(x,k',k)[W^j(k')] = \sum_{j=1}^M \int dk'\sigma^{ij*}(x,k',k)[W^j(k')].$$

This completes the derivation of the radiative transport equation for the matrix W. The statement regarding the initial value for $W(0,x,k)$ is verified directly, and the proof of Theorem 4.1 is complete.

5 The Diffusive Regime

For long times and large distances, scalar transport processes have asymptotically a simpler form, in which the directional behavior is known and the spatial process is a solution of a diffusion equation. We refer to [4, 9] for more details. Transport processes with polarization effects also converge to diffusion processes, as was shown in [14]. We first recall the diffusion theory for scalar processes, then apply it to the augmented transport process to derive the diffusion approximation for matrix-valued transport equations. We start with electromagnetic wave transport and then consider general matrix-valued transport equations.

5.1 General Theory in the Scalar Case

Let us assume that the transition probability $\Pi(y,dz)$ is independent of the space variable x. For the diffusion limit of the jump Markov process $(X(t),Y(t))$ of section 4.2 we rescale time and space variables

$$x \to \frac{x}{\varepsilon}, t \to \frac{t}{\varepsilon^2}.$$

Here ε is a small dimensionless parameter. We also assume for simplicity that the variable $Y(t)$ does not change between jumps, so that the function $H(x,y)=0$ in (1) and, in addition, that $F(x,y)=F(y)$ is a bounded continuous function independent of the position x. The generator $\mathcal{L}$, given by (7), then becomes

$$\mathcal{L}^\varepsilon = \frac{1}{\varepsilon}F(y)\frac{\partial}{\partial x} + \frac{1}{\varepsilon^2}Q$$

where Q is related to Π by (8). We assume that $Y(t)$ takes values in a compact set $G \subset \mathbb{R}^{d'}$ and that the measure $\Pi(y,dz)$ has a continuous density $\pi(y,z)$ with respect to the Lebesgue measure on G. The density π is bounded from above and below:

$$0 < \pi_l \le \pi(y,z) \le \pi_u < \infty, \quad y,z \in G$$

and the total scattering cross-section $\Sigma(y)$ is continuous and obeys similar bounds

$$0 < \pi_l \le \Sigma(y) \le \pi_u < \infty, \quad y \in G.$$

Under these conditions the Fredholm alternative holds for the operator Q, that is, there exists a unique invariant measure $\bar{P}(dy)$ such that given any bounded continuous function $h(y)$ the equation

$$Qg(y) = h(y)$$

has a bounded solution $g(y)$ if and only if

$$\int h(y)\bar{P}(dy) = 0.$$

We assume that the drift $F(y) = (F_1, \ldots, F_d)$ has mean zero with respect to the invariant measure $\bar{P}(dy)$:

$$\int_G F_j(y)\bar{P}(dy) = 0, \quad j = 1, \ldots, d \tag{73}$$

and that $\bar{P}(y)$ has a density $\bar{p}(y)$. Then there exists a vector valued function $\chi = (\chi_1, \ldots, \chi_d)$ such that

$$Q\chi_j = -F_j(y).$$

Similarly, we denote by χ_j^* one solution of

$$Q^*\chi_j^* = F_j(y)\bar{p}(y).$$

We define an operator $\bar{\mathcal{L}}$ by

$$\bar{\mathcal{L}}f(x) = \sum_{i,j} D_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}, \tag{74}$$

where

$$D_{ij} = \int_G dy\bar{p}(y)F_j(y)\chi_i(y) = -\int_G dy\, F_i(y)\chi_j^*(y)$$

and assume that the coefficient matrix D_{ij} is positive definite. It is always non-negative definite because it is a covariance. The following Theorem was established in [4].

Theorem 5.1 *Under the above hypotheses, the process $X^\varepsilon(t)$, $0 \le t \le T$, converges weakly to the diffusion Markov process generated by $\bar{\mathcal{L}}$.*

Therefore, the solution of the forward transport equation

$$\begin{aligned}\frac{\partial a_\varepsilon}{\partial t} &= \mathcal{L}_\varepsilon^* a_\varepsilon \\ a(0, x, y) &= a_0(x, y)\end{aligned}$$

has a representation as $\varepsilon \to 0$

$$a_\varepsilon(t, x, y) = g(t, x)\bar{p}(y) + O(\varepsilon),$$

with the function $g(t, x)$ satisfying the diffusion equation

$$\begin{aligned}\frac{\partial g(t,x)}{\partial t} &= \bar{\mathcal{L}}^* g(t, x) \\ g(0, x) &= \int_G dy a_0(x, y).\end{aligned} \tag{75}$$

Example. Let us consider the system of appropriately scaled multi-group transport equations as in Example 2 of section 2 with the cross-sections σ^{ij} independent of x and symmetric $\sigma^{ij}(k,k') = \sigma^{ji}(k',k)$:

$$\frac{\partial u_i^\varepsilon}{\partial t} + \frac{1}{\varepsilon} k \cdot \nabla_x u_i^\varepsilon = \frac{1}{\varepsilon^2} \int_{S^2} dk' \sigma^{ij}(k,k') u_j^\varepsilon(x,k') - \frac{1}{\varepsilon^2} \Sigma^j(k) u_j(k) \tag{76}$$
$$u_i^\varepsilon(0,x,k) = u_0^i(x,k).$$

Then the density $\bar{p}$ of the invariant measure is given by $\bar{p} = \frac{1}{4\pi N}(1,\ldots,1)$, and therefore the solution $(u_1^\varepsilon,\ldots,u_N^\varepsilon)$ of (76) converges as $\varepsilon \to 0$ to $(g(x),\ldots,g(x))$, and $g(x)$ satisfies the diffusion equation

$$\frac{\partial g}{\partial t} = \sum_{i,j} D_{ij} \frac{\partial^2 g}{\partial x_i \partial x_j}$$
$$g(0,x) = \frac{1}{4\pi N} \sum_{j=1}^N \int_{S^2} u_0^j(x,k) dk.$$

5.2 Diffusion for Electromagnetic Wave Transport

We consider now the long time, large propagation distance limit for the augmented transport equation (32). Our analysis in this section is by direct asymptotic expansions. In the next section we consider the general case, which can be analyzed by probabilistic methods. In both cases we derive the diffusion limit of the augmented transport equation and from it the corresponding results for the matrix transport equation itself. In the electromagnetic case the results agree with those in [14]. We shall assume for simplicity that the velocity $v = \text{const}$ (and hence $n(x,k) = 0$ in (32)), the differential scattering cross-section $\sigma(k,k') = \sigma(k \cdot k')$ is independent of x and is rotationally invariant, and the total cross-section Σ is independent of x and k. We rescale the time and space variables $x \to x/\varepsilon$, $t \to t/\varepsilon^2$ with $\varepsilon \ll 1$, so that (32) becomes

$$\varepsilon^2 \frac{\partial \bar{\rho}}{\partial t} + \varepsilon v \hat{k} \cdot \nabla_x \bar{\rho} = \int_{S^2 \times \mathcal{K}} dk' du'_{11} du'_{12} \sigma(k \cdot k') \frac{\mathrm{Tr}\,\{T(k,k')[I_2 + U']T^*(k,k')\}}{2}$$
$$\times \phi(k',U',k,U) \bar{\rho}(t,x,k',u'_{11},u'_{12}) - \Sigma \bar{\rho}(t,x,k,u_{11},u_{12}) \tag{77}$$

We expand the solution $\bar{\rho}$ in a series in ε:

$$\bar{\rho} = \bar{\rho}_0 + \varepsilon \bar{\rho}_1 + \varepsilon^2 \bar{\rho}_2 + \ldots \tag{78}$$

which leads to a corresponding expansion for the coherence matrix W (35):

$$W = W_0 + \varepsilon W_1 + \varepsilon^2 W_2 + \ldots \tag{79}$$

We insert the expansion (78) into (77). The leading order term satisfies

$$\int_{S^2 \times \mathcal{K}} dk' du'_{11} du'_{12} \sigma(k \cdot k') \frac{\mathrm{Tr}\,\{T(k,k')[I_2 + U']T^*(k,k')\}}{2}$$
$$\times \phi(k',U',k,U) \bar{\rho}_0(t,x,k',u'_{11},u'_{12}) = \Sigma \bar{\rho}_0(t,x,k,u_{11},u_{12}).$$

This equation has a unique (up to multiplication by a function of t and x only) solution $\bar{p}(k,u_{11},u_{12})$. Indeed, the function $\bar{p}_*=1$ is a positive solution of the adjoint equation, as follows from the normalization (29) and energy conservation (16), and hence it is unique according to the Krein-Rutman theorem [6]. Therefore the leading term $\bar{\rho}_0$ has the form

$$\bar{\rho}_0 = w(t,x)\bar{p}(k,u_{11},u_{12}) \tag{80}$$

with the leading term in the expansion (79) being

$$W_0(t,x,k) = w(t,x)\int_{\mathcal{K}} du_{11}du_{12}\frac{\bar{p}(k,u_{11},u_{12})}{2}\left(I_2 + \begin{pmatrix} u_{11} & u_{12} \\ u_{12} & -u_{11}\end{pmatrix}\right).$$

It is straightforward to verify that the matrix

$$\bar{W} = \int_{\mathcal{K}} du_{11}du_{12}\frac{\bar{p}(k,u_{11},u_{12})}{2}\left(I_2 + \begin{pmatrix} u_{11} & u_{12} \\ u_{12} & -u_{11}\end{pmatrix}\right)$$

is a solution of

$$\int dk'\sigma(k\cdot k')T(k,k')\bar{W}(k')T^*(k,k') = \Sigma\bar{W}(k). \tag{81}$$

By the Krein-Rutman theorem [6], any solution of (81) has the form

$$\bar{W}(k) = CI_2, \quad C\in \mathbb{R}. \tag{82}$$

Therefore, the leading order term in the expansion (79) is proportional to the identity matrix:

$$W_0(t,x,k) = w(t,x)I_2.$$

and waves become depolarized in the limit of long times and large propagation distances, as was observed in [14].

Using the expression (80) for $\bar{\rho}_0$ we find that the next order term $\bar{\rho}_1$ satisfies the equation

$$\begin{aligned} v(\hat{k}\cdot\nabla_x w)\bar{p} &= \int_{S^2\times\mathcal{K}} dk'du'_{11}du'_{12}\sigma(k\cdot k')\frac{\mathrm{Tr}\,\{T(k,k')[I_2+U']T^*(k,k')\}}{2} \\ &\times \phi(k',U',k,U)\bar{\rho}_1(t,x,k',u'_{11},u'_{12}) - \Sigma\bar{\rho}_1(t,x,k,u_{11},u_{12}). \end{aligned}$$

Let the functions $\chi_i^*(k,u_{11},u_{12})$, for $i=1,2,3$, solve the equations

$$\begin{aligned} vk_i\bar{p}(k,u_{11},u_{12}) = \frac{1}{2}\int_{S^2\times\mathcal{K}} dk'du'_{11}du'_{12}\sigma(k\cdot k')\mathrm{Tr}\,[T(k,k')[I_2+U']T^*(k,k')]\,\cdot \\ \cdot\phi(k',U',k,U)\chi_i^*(k',u'_{11},u'_{12}) - \Sigma\chi_i^*(k,u_{11},u_{12}). \end{aligned} \tag{83}$$

Note that (83) has a solution since

$$\int_{S^2\times\mathcal{K}} dkdu_{11}du_{12}\hat{k}_i\bar{p}(k,u_{11},u_{12}) = \int_{S^2} dk\hat{k}_i\mathrm{Tr}\{\bar{W}\} = 0.$$

This follows from the identity

$$\frac{1}{4\pi}\int_{\mathcal{K}} du_{11}du_{12}\bar{p}(k,u_{11},u_{12}) = 1 \qquad (84)$$

implied by (82). Then $\bar{\rho}_1$ is given by

$$\bar{\rho}_1 = \sum_{j=1}^{3}\chi_j^*(k,u_{11},u_{12})\frac{\partial w}{\partial x_j} + \chi_{10}^*(x), \qquad (85)$$

with an unknown function χ_{10}^*.

Using expressions (80) for $\bar{\rho}_0$ and (85) for $\bar{\rho}_1$, the equation for $\bar{\rho}_2$ is

$$\begin{aligned} &\frac{\partial w}{\partial t}\bar{p} + v\hat{k}_i\left[\chi_j^*\frac{\partial^2 w}{\partial x_i \partial x_j} + \frac{\partial \chi_{10}^*}{\partial x_i}\right] \\ = &\int_{S^2\times\mathcal{K}} dk'du'_{11}du'_{12}\sigma(k\cdot k')\frac{\mathrm{Tr}\left\{T(k,k')[I_2+U']T^*(k,k')\right\}}{2} \\ &\times\phi(k',U',k,U)\bar{\rho}_2(t,x,k',u'_{11},u'_{12}) - \Sigma\bar{\rho}_2(t,x,k,u_{11},u_{12}). \end{aligned}$$

The solvability condition for this equation requires that the integral of the left side over $S^2\times\mathcal{K}$ vanishes. Therefore we get the diffusion equation for the function $w(t,x)$:

$$\frac{\partial w}{\partial t} = \sum_{i,j} D_{ij}\frac{\partial^2 w}{\partial x_i \partial x_j}. \qquad (86)$$

The diffusion tensor D_{ij} is given by (89):

$$D_{ij} = -\frac{1}{4\pi^2}\int_{S^2\times\mathcal{K}} dkdu_{11}du_{12}v\hat{k}_i\chi_j^*(k,u_{11},u_{12}).$$

The integral of $\chi_j^*(k,u_{11},u_{12})$ with respect to $u_{11,12}$ may be evaluated explicitly. Let

$$W_\chi^j(k) = \frac{1}{2}\int_{\mathcal{K}} du_{11}du_{12}\chi_j^*(k,u_{11},u_{12})\begin{pmatrix} 1+u_{11} & u_{12} \\ u_{12} & 1-u_{11}\end{pmatrix}.$$

Then, integrating (83) over $\mathcal{K}$ we get, using (84) that the matrix W_χ^j satisfies

$$vk_jI_2 = \int dk'\sigma(k\cdot k')T(k,k')W_\chi^j(k')T^*(k',k) - \Sigma W_\chi^j(k).$$

Straightforward computations give

$$W_\chi^j(k) = \frac{vk_j}{\lambda-\Sigma}I_2$$

where λ is determined by

$$\lambda k_jI_2 = \int dk'k'_j\sigma(k\cdot k')T(k,k')T^*(k',k).$$

Hence we have

$$\int_{\mathcal{K}} du_{11} du_{12} \chi_j^*(k, u_{11}, u_{12}) = \frac{v k_j}{\lambda - \Sigma},$$

and the diffusion tensor D_{ij} has a simple form

$$D_{ij} = \frac{v^2}{3(\Sigma - \lambda)} \delta_{ij}.$$

Note that $\Sigma > \lambda$ by the Krein-Rutman theorem, hence $D > 0$. This expression coincides with the one obtained in [14] by considering asymptotic expansions for $W(t, x, k)$ directly.

5.3 Diffusion Limit for Transport Equation with Polarization

We apply now the general theory of section 5.1 to the jump process for the velocity and polarization parameters constructed in section 4.2.1. We assume for simplicity that the dispersion relations have the form $\omega_i = v_i |k|$ and are independent of the position x. Then the characteristics are straight lines and $N_i = 0$ (see [14]). Moreover, we assume that the length of the vector $K(t)$ does not change during the scattering, so that $K(t) \in S^2$. Let $\bar{p} = (\bar{p}_1(k, U^1), \ldots, \bar{p}_M(k, U^M))$ be the invariant measure of the operator $\mathcal{L}$ acting on vector valued functions as

$$(\mathcal{L}f)_i(k, U^i) = \sum_{j=1}^{M} \mathcal{L}^{ij} f_j(k, U^i) = \sum_{j=1}^{M} \int_{S^2} dk' \int_{\mathcal{K}_j} dU'^j f_j(k', U'^j) \cdot$$

$$\cdot \operatorname{Tr}\{\sigma^{ij}(k, k') \frac{I_{n^i} + U^i}{n^i}\} \phi(k, U^i, i, k', U'^j, j) - \Sigma^i f_i + v_i \hat{k} \cdot \nabla f_j \delta_{ij}, \qquad (87)$$

that is,

$$\sum_{j=1}^{M} \mathcal{L}^{*ij} \bar{p}_j = 0, \qquad \sum_{j=1}^{M} \int_{S^2} dk \int_{\mathcal{K}_j} dU^j \bar{p}_j(k, U^j) = 1.$$

Let us assume that the cross-sections σ^{ij} are bounded and uniformly positive functions, as in the previous section. Moreover, we assume, as in (73), that for $1 \leq l \leq d$,

$$\sum_{j=1}^{M} \int_{S^2} dk \int_{\mathcal{K}_j} dU^j k_l v_j \bar{p}_j(k, U^j) = 0,$$

so that the mean drift is zero. Then the process $X^\varepsilon(t) = \varepsilon X(t/\varepsilon^2)$ converges, according to Theorem 5.1, to a diffusion process with generator $\bar{\mathcal{L}}$ given by (74) and the probability densities $\rho_\varepsilon^i(t, x, k, U^i)$ have the form

$$\rho_\varepsilon^i(t, x, k, U^i) = \bar{\rho}(t, x) \bar{p}_i(k, U^i) + O(\varepsilon).$$

The function $\bar{\rho}(t, x)$ satisfies the diffusion equation

$$\frac{\partial \bar{\rho}}{\partial t} = \sum_{i,j} D_{ij} \frac{\partial^2 \bar{\rho}}{\partial x_i \partial x_j}.$$

The diffusion tensor D_{ij} is defined as follows. Let the functions $\chi^*_{in}(k, U^i)$, for $i = 1, \ldots, M$ and $n = 1, \ldots, d$, be mean zero solutions of

$$v_i k_n \bar{p}_i = \sum_{j=1}^{M} \mathcal{L}^{*ij} \chi^*_{jn}. \tag{88}$$

Then D_{ij} is given by

$$D_{jn} = -\sum_{i=1}^{M} \int_{S^2} dk \int_{\mathcal{K}_i} dU^i v_i k_j \chi^*_{in}(k, U^i). \tag{89}$$

Thus, the solution $W^i_\varepsilon(t, x, k)$ of the system of the radiative transport equations satisfies

$$W^j_\varepsilon(t, x, k) - g(x) \int_{\mathcal{K}_j} dU^j \bar{p}_j(k, U^j) \frac{I_{n^j} + U^j}{n^j} = O(\varepsilon), \tag{90}$$

which may be also established, at least formally, using a multi scale asymptotic expansion for W^i_ε. The scalar function $g(x)$ satisfies a diffusion equation

$$\begin{aligned} \frac{\partial g}{\partial t} &= \sum_{i,j} D_{ij} \frac{\partial^2 g}{\partial x_i \partial x_j} \\ g(0, x) &= g_0(x). \end{aligned} \tag{91}$$

The initial data $g_0(x)$ is given by

$$g_0(x) = \sum_{j=1}^{M} \int_{S^2} dk \mathrm{Tr}\{W^j_0(x, k)\}.$$

The approximation (90) implies that in the diffusion limit, each matrix W^j can be decomposed as the product of two terms. The first term only depends on the velocity k and state j. The second term g is independent of the state j, and describes long time and large distance behavior. This implies that the energy is asymptotically distributed among the states in a prescribed ratio, which is independent of the initial conditions. We will come back to this important observation in section A.1.

6 Summary and Conclusions

We have summarized in Theorem 4.1 the main result of the paper, which is that matrix-valued solutions of radiative transport equations can be represented as the first order moments with respect to polarization parameters of solutions of a scalar, augmented transport equation. Moreover, the solution of the augmented transport equation is the probability law of a jump Markov process. The probabilistic representation can be used to give an elegantly simple theory of existence and uniqueness of solutions of matrix radiative transport equations. In addition, we have shown how to get convergence of the jump Markov processes of radiative transport to a diffusion process, for long times and large distances. This is done by using the general theory of diffusion approximations to the augmented scalar transport equation.

We have restricted ourselves to transport equations in all space, even though boundaries and interfaces are very important in practice. The theory of scalar jump processes can also be carried out when boundaries are present [4]. The generalization of the augmented jump process to account for boundary and interface conditions, like the ones derived in [2], can be done without difficulty.

We have presented a probabilistic theory for general radiative transport equations but the main objective is to formulate effective numerical methods for computing the energy density of electromagnetic and elastic waves. Discretization of the radiative transport equations (13) or (92) by finite differences or finite elements is not useful for genuine three-dimensional geometries. This is because of the large number of degrees of freedom, seven in 3D. Even coarse discretizations of the integro-differential transport equation require a lot of computer memory. However, the propagation of seismic waves generated by an earthquake, which is an application where time dependent 3D radiative transport solutions are required, can be done by Monte Carlo [7, 16] using the probabilistic representation. We can estimate the probability law of the jump Markov process by simulating a large number of its trajectories. Even though the convergence to the solution is slow, like $N^{-1/2}$ where N is the number of realizations, the memory requirements are minimal. The probabilistic representation has been used in [3], where the polarization and depolarization of shear waves is analyzed in detail.

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A Application to Elasticity

We consider in this appendix an example of matrix transport equations with two species. The propagation of the elastic waves, seismic waves, for example, can be modeled in the high frequency regime by matrix transport equations. It is a coupled system of a scalar transport equation for to compressional waves, and a 2×2 matrix transport equation for the coherence matrix of shear waves.

Radiative transport equations have recently been used to interpret the “coda”, or wave trains which appear in the tail portion of seismograms from earthquakes. Pioneered by the work of Aki [1], the subject has developed substantially in the past few years. We refer to [15] for the current state of the art and to other recent contributions [12, 13, 17].

A.1 The Elastic Transport Equations

There are two types of elastic waves: the transverse shear waves, or S waves, and the longitudinal compressional waves, the P waves. Shear waves are polarized, and we describe their energy transport by a coherence matrix W^S, while P waves, which are faster than S waves, are unpolarized. Their energy density is denoted by w^P. The matrix W^S and the scalar w^P satisfy a system of two radiative transport equations which are coupled. This system has the following form, where all integrals run over $k' \in \mathbb{R}^3$,

$$\frac{\partial a^P}{\partial t} + \nabla_k \omega^P \cdot \nabla_x a^P - \nabla_x \omega^P \cdot \nabla_k a^P = \int \sigma^{PP}(k',k) a^P(k') dk' \tag{92}$$
$$+ \int \sigma^{SP}(k',k)[W^S(k')]dk' - [\Sigma^{PP}(k) + \Sigma^{SP}(k)]a^P(k)$$

and

$$\frac{\partial W^S}{\partial t} + \nabla_k \omega^S \cdot \nabla_x W^S - \nabla_x \omega^S \cdot \nabla_k W^S + WN - NW + [\Sigma^{SS} + \Sigma^{PS}]W^S$$
$$= \int \sigma^{SS}(k',k)[W^S(k')]dk' + \int \sigma^{PS}(k',k)[a^P(k')]dk'. \tag{93}$$

The coupling matrix N is the same as for electromagnetic waves, see (15), except that the speed v is now the shear speed v_S. The differential scattering cross-section $\sigma^{PP}(k',k)$ is a scalar function, and the differential scattering tensor $\sigma^{SS}(k',k)$ is similar to the one for electromagnetic waves. They have the form

$$\sigma^{PP}(k',k) = \sigma_{pp}(k',k)\delta(v_P|k| - v_P|k'|) \tag{94}$$

and

$$\begin{aligned} \sigma^{SS}(k',k)[W(k')] = \quad & \{ \quad \sigma_{ss}^{TT} T(k,k')W(k')T(k',k) + \sigma_{ss}^{\Gamma\Gamma}\Gamma(k,k')W(k')\Gamma(k',k) \\ & + \quad \sigma_{ss}^{\Gamma T}[T(k,k')W(k')\Gamma(k',k) + \Gamma(k,k')W(k')T(k',k)] \cdot \} \\ & \quad \cdot \delta(v_S|k| - v_S|k'|). \end{aligned} \tag{95}$$

The 2×2 matrix $\Gamma(k,k')$ is similar to T, and is defined by

$$\Gamma_{ij}(k,k') = (\hat{k} \cdot \hat{k}')(z^{(i)}(k) \cdot z^{(j)}(k')) + (\hat{k} \cdot z^{(j)}(k'))(\hat{k}' \cdot z^{(i)}(k)) \tag{96}$$

while σ_{pp} and σ_{ss} are scalar functions symmetric in k and k'.

The scattering cross-sections for the S to P and P to S coupling terms, σ^{PS} and σ^{SP}, respectively, have the form

$$\sigma^{PS}(k',k)[a^P(k')] = \sigma_{ps}(k',k)G(k',k)a^P(k')\delta(v_S|k| - v_P|k'|) \tag{97}$$
$$\sigma^{SP}(k',k)[W^S(k')] = \mathrm{Tr}\{\sigma_{ps}(k,k')G(k,k')W^S(k')\}\delta(v_P|k| - v_S|k'|) \tag{98}$$

where the 2×2 matrix G is given by

$$G_{ij}(k,k') = (\hat{k} \cdot z^{(i)}(k'))(\hat{k} \cdot z^{(j)}(k')). \tag{99}$$

The function σ_{ps} is scalar. Finally, the total scattering cross-sections Σ are determined from energy conservation, as in (46).

A.2 The Augmented Transport Equation

We have two species, one corresponding to P waves, and one corresponding to S waves, hence $M = 2$. We denote these states by P and S, respectively. There is no polarization effect for particles in state P, while a 2×2 traceless symmetric polarization matrix $U(t)$ describes the polarization of a particle in state S. Its independent elements u_{11} and u_{12} lie inside the unit disc $\mathcal{K}$, as in (20). The evolution between jumps is given by the equations

$$\frac{dX}{dt} = v_P(x)\hat{k}, \quad \frac{dK}{dt} = -|K(t)|\nabla_x v_P$$

for state P, and by

$$\frac{dX}{dt} = v_S(x)\hat{k}, \quad \frac{dK}{dt} = -|K(t)|\nabla_x v_S, \quad \frac{dU}{dt} = UN - NU$$

for state S.

Let us now describe the jump process corresponding to the transport equations (92) and (93). The probability distribution of the time between successive jumps is given by (54) with $\Sigma = \Sigma^{PP} + \Sigma^{PS}$ in state P, and by $\Sigma = \Sigma^{SP} + \Sigma^{SS}$ in state S. We note that P waves with wave vector k_P can be scattered into S waves with wave vector k_S only if the resonance condition $v_S|k_S| = v_P|k_P|$ holds. Therefore the vector $K(t)$ lies on the unit sphere S^2 if our particle is in state P, and $K(t)$ lies on the sphere S_r^2 of radius $r = \dfrac{v_P}{v_S} > 1$ if it is in state S. Let us first give the probability for $K(t)$ to jump from k (with polarization matrix U if the incoming state is S) into direction k' (as in (60))

$$\begin{aligned}
\tilde{\pi}(x,k,P,k',P) &= \frac{\sigma_{pp}(k,k')\delta(v_P|k| - v_P|k'|)}{\Sigma^{PP} + \Sigma^{PS}} \\
\tilde{\pi}(x,k,P,k',S) &= \frac{\sigma_{ps}(k,k')\mathrm{Tr}\{G(k,k')\}\delta(v_S|k'| - v_P|k|)}{\Sigma^{PP} + \Sigma^{PS}} \\
\tilde{\pi}(x,k,U,S,k',P) &= \frac{\sigma_{ps}(k',k)\mathrm{Tr}\{G(k',k)\tilde{U}\}\delta(v_P|k'| - v_S|k|)}{\Sigma^{SS} + \Sigma^{SP}} \\
\tilde{\pi}(x,k,U,S,k',S) &= \frac{\mathrm{Tr}\{\sigma_{SS}(k,k')[\tilde{U}]\}\delta(v_S|k'| - v_S|k|)}{\Sigma^{SS} + \Sigma^{SP}}.
\end{aligned}$$

Here the matrix $\tilde{U}$ is related to U as in (49) by $\tilde{U} = \dfrac{1}{2}[I_2 + U]$.

It remains to prescribe the probability density that a particle scattered into state S, either from state S, or from state P, have the new polarization matrix U'. It is given by two functions: $\phi_P(k,k',U')$, which is the conditional probability to be scattered from state P into state S, and $\phi_S(k,U,k',U')$, which corresponds to scattering from state S into state S. These functions have to satisfy the constraints:

$$\int_{\mathcal{K}} du'_{11}du'_{12}\phi_P(k_P,k',U') = \int_{\mathcal{K}} du'_{11}du'_{12}\phi_S(k_S,U,k',U') = 1$$

for all $k_P \in S^2$, $k_S \in S_r^2$, and

$$\int_{\mathcal{K}} du'_{11} du'_{12} \phi_P(k_P, k', U') u'_{lr} = U^{PS}_{s,lr}(k, k')$$
$$\int_{\mathcal{K}} du'_{11} du'_{12} \phi_S(k_S, k', U') u'_{lr} = U^{SS}_{s,lr}(k, k').$$

Here the matrices U_s^{PS} and U_s^{SS} are determined by

$$\frac{1}{2}(I_2 + U_s^{PS}) = \frac{G(k, k')}{\mathrm{Tr}\{G(k, k')\}}$$

and

$$\frac{1}{2}(I_2 + U_s^{SS}) = \frac{\sigma_{SS}(k, k')[U]}{\mathrm{Tr}\{\sigma_{SS}(k, k')[U]\}}.$$

This completely describes the jump process associated with the transport equations of elastic waves.

A.3 The Diffusion Limit

It is straightforward to consider the long time limit, which leads to the result obtained in [14] by asymptotic expansion. The main result is that

$$W_\varepsilon^S(t, x, k) \to w(t, x) I_2, \quad w_\varepsilon^P \to w(t, x),$$

where $w(t, x)$ is the solution of a diffusion equation

$$\frac{\partial w}{\partial t} = D \Delta w.$$

The diffusion coefficient can be written explicitly in terms of the scattering cross-section as it was done for electromagnetic waves in the previous section. The physical implications is that there exists a universal asymptotic equipartition relation of the energy between P and S waves. This observation is discussed in detail in [13] and [17].

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