Inverse problems for homogeneous transport equations. Part II. Multidimensional case.

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Abstract. A companion paper [1] and this paper are part I and II of a series dealing with the reconstruction from boundary measurements of the scattering operator of homogeneous linear transport equations. This part II deals with the case of convex bounded domains in higher dimensions than one. We distinguish the analysis of smooth boundaries from that of boundaries with discontinuities like corners. We propose a reconstruction in the case of degenerate symmetric scattering operators and show the well-posedness of the inverse problem. The proof of well-posedness is based on a decomposition of angular moments of the transport solution into unbounded and bounded components. This decomposition allows us to show the linear independence of a sufficiently large number of angular moments of the transport solution that are used to construct an invertible system for the scattering coefficients to be reconstructed.

Submitted to: *Inverse Problems*

1. Introduction

This paper considers the reconstruction from boundary measurements of the scattering operator of multi-dimensional homogeneous linear transport equations in a convex bounded domain. We treat the one-dimensional (1D) in the companion paper [1]. We also refer to that paper for some references on the forward and inverse problems in linear transport theory.

In one-dimensional geometries, all measurements are made at a single point in space for half space problems, and at two points for slab problems. In higher dimensions, the geometry of the boundary allows for more measurements. The inverse problem is thus expected to be easier to solve. However, in one-dimensional geometries we have access to the invariant imbedding technique. This does not carry to higher dimensions and no closed equation can be obtained for the response operator. Hence, information on the scattering operator must be directly obtained from the transport equation, which involves non measurable quantities inside the domain.

Inverse problems for three dimensional transport have been addressed in the literature. Recently, it was shown theoretically [2] that fairly general scattering operators can be reconstructed from boundary measurements. A more constructive approach using similar ideas was proposed in [3]. In both works, the reconstruction is obtained by splitting the solution into components of different regularity. The ballistic part, which does not scatter, is used to reconstruct the total cross-section, whereas single scattering, the part that scatters only once, is used to reconstruct the scattering cross-section. The method requires that one be able to measure delta functions in space and velocity and be able to separate these components from the rest of the solution. We refer to the review papers for recent information on transport inverse problems [4, 5].

Our reconstruction is based on the local behavior of the solution to the transport equation at the boundary of the domain in the direction that is tangent to the boundary. We obtain equations that only involve the measurable solution at the boundary of the domain. Using the singular part of single scattering, we obtain an invertible system for the coefficients we want to reconstruct. Even if regularity is used in the proof of inversibility, the method does not necessitate to differentiate singular and regular parts of the measured data, which is difficult to implement in practice.

To simplify the presentation we consider here only two-dimensional (2D) geometries. The 2D transport is of lesser practical interest than the three-dimensional (3D) transport, although some physical applications have been proposed recently [6, 7]. Nonetheless all our results and techniques extend to the 3D problem. The notation and proofs become more involved and we focus only on the 2D case here. Notice that whereas general scattering kernels can be reconstructed from boundary measurements in 3D, the problem remains open in 2D (see [2]). The reason is that the inverse problem is formally overdetermined in 3D, but not in 2D. Here, since the scattering kernel is homogeneous (spatially invariant) and degenerate, boundary measurements provide an overdetermined inverse problem in 2D as well. The crux of the inversion procedure is Proposition 3.3 below, where it is shown that single scattering of delta incoming distributions is logarithmically singular. Single scattering remains singular in higher dimensions (although with a different singularity) and the arguments given in this paper for the two-dimensional case carry with no major difficulty to higher dimensions.

The paper is organized as follows. Section 2 presents the 2D transport equation. The inverse problem for smooth boundaries is analyzed in section 3. The case of boundaries with corners is dealt with in section 4. Section 5 gives some numerical evidence that the solutions of the transport equation fulfill the conditions imposed to obtain the well-posedness of the inverse problem.

2. Steady Transport Problem

The two-dimensional monoenergetic steady-state transport equation is given by

$$\begin{aligned} &v \cdot \nabla u(x,v) + \Sigma u(x,v) = \Sigma \int_V k(v,v')u(x,v')d\mu(v') \qquad \text{in} \quad \Omega \times V, \\ &u(x,v) = g(x,v) \qquad \text{on} \quad \Gamma_-. \end{aligned} \tag{1}$$

Here Ω is the physical domain; the velocity space $V = S^1$ is the unit circle; Γ_- is the set of incoming boundary conditions

$$\Gamma_- = \{(x,v) \in \partial\Omega \times V \quad \text{such that} \quad v \cdot n(x) < 0\}, \tag{2}$$

where $n(x)$ is the outward unit normal to Ω at $x \in \partial\Omega$. The incoming distribution $g(x,v) \in L^\infty(\Gamma_-)$. The domain Ω is an open convex bounded subset of $\mathbb{R}^2$. The total cross section Σ is a positive constant. We assume that the medium is dissipative or conservative; i.e.,

$$\int_V k(v',v)d\mu(v') \le 1 \qquad \text{for all} \quad v \in V, \tag{3}$$

where $\mu(v)$ is the Lebesgue measure on the unit circle. In addition we assume that $k(v',v)$ is bounded from above and below by positive constants. With these assumptions, the problem is well-posed in $L^\infty(\Omega \times V)$ (see the integral formulation of (1) in [8]). Denoting by $\Gamma_+ = \{(x,v) \in \partial\Omega \times V$ such that $v \cdot n(x) > 0\}$, we deduce from the same integral formulation that the trace $u_{|\Gamma_+} \in L^\infty(\Gamma_+)$.

We want to know whether the total cross-section Σ and the scattering kernel $k(v,v')$ can be reconstructed from boundary measurements. The answer to this question is positive for fairly general scattering operators [2, 3]. The result in [2] is actually more general and also applies when Σ and k are spatially non homogeneous (at least in 3D). However, the proof of well-posedness of the inverse problem relies on a decomposition of the response operator into components of different regularity that is difficult to implement in practice.

This paper presents a simple method to reconstruct the scattering operator and its kernel $k(v,v')$ assuming that it takes the form

$$k(v,v') = \sum_{m=1}^{M} \sigma_m \alpha_m(v)\alpha_m(v'), \tag{4}$$

where the functions $\alpha_m(v)$ are smooth, linearly independent, and vanish at most at a finite number of points. Many scattering operators of practical interest satisfy this decomposition [9, 10, 11]. We also aim at recovering the total cross-section Σ. The reconstruction does not require to know how to measure delta functions. In other words, if the outgoing distribution is the sum of a delta function in space and/or velocity and of a smooth function, we do not need to know how to separate them.

3. Case of Smooth Boundary

We assume in this section that $\partial\Omega$ is a smooth curve and that the curvature is strictly positive everywhere. We first want to reconstruct the M coefficients $\Sigma\sigma_m$. So we need an invertible system of M equations. Since we have access only to boundary measurements, these M equations can only involve measured data at the boundary and the unknowns $\Sigma\sigma_m$. We make use of the convexity of the domain to obtain these equations.

Let us consider a point $x_0 \in \partial\Omega$ and assume that the boundary condition $g(x, v)$ vanishes for x in the vicinity of x_0. We denote by $v(x)$ the unit vector in S^1 that is tangent to $\partial\Omega$ at $x \in \partial\Omega$ defined by $v(x) = n^\perp(x)$, where $n^\perp$ is the counter-clockwise rotation of angle $\pi/2$ of n. Let $v_0 = v(x_0)$. We have that $u(x_0, v_0) = g(x_0, v_0) = 0$. However, for those points $x \in \partial\Omega$ close to x_0 and such that $x - x_0$ is close to v_0, the solution $u(x, v_0)$ is small but does not vanish, and is asymptotically (as $x \to x_0$) related to the boundary measurements and the coefficients $\Sigma\sigma_m$.

More specifically, let $\delta > 0$ small enough. We denote by $\tilde{x}_0(\delta) \in \partial\Omega$ the closest point to x_0 such that $d(\tilde{x}_0(\delta), v_0) = \delta$, where $d(x, v)$ is the distance from a point $x \in \bar{\Omega}$ to the boundary $\partial\Omega$ in direction $v \in S^1$, defined by

$$d(x, v) : \quad \text{unique positive real number such that} \quad x - d(x, v)v \in \partial\Omega. \tag{5}$$

We also define

$$\bar{x}(x, v) = x - d(x, v)v \in \partial\Omega. \tag{6}$$

So $\tilde{x}_0(\delta)$ is the closest point to x_0 (there are two such points) such that the half line $\{\tilde{x}_0(\delta) - tv_0,\ t \geq 0\}$ intersects Ω on an interval of size δ (see Fig. 1). We will show that

$$\sum_{m=1}^{M} \Sigma\sigma_m \alpha_m(v_0)\phi_m(x_0) = \lim_{\delta\to 0} \frac{u(\tilde{x}_0(\delta), v_0)}{\delta}, \tag{7}$$

where the moments ϕ_m are given by

$$\phi_m(x) = \int_V \alpha_m(v)u(x, v)d\mu(v). \tag{8}$$

Notice that since $x_0 \in \partial\Omega$, $\phi_m(x_0)$ is known. We obtain as many equations as we consider points x_0. So a system of M equations can be built up for the coefficients $\Sigma\sigma_m$. We will give examples of experiments allowing us to obtain an invertible system for the coefficients $\Sigma\sigma_m$. By ‘experiment’ is meant a choice of incoming boundary condition, together with determination of the corresponding outgoing solution at some boundary points.

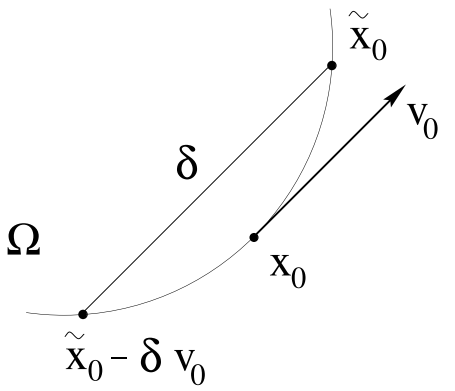

Figure 1. Local Geometry in the vicinity of $x_0 \in \partial\Omega$. We recall that $\|v_0\| = 1$.

It now remains to find an equation for Σ. This is done as follows. Assume that the boundary condition has the form $g(x,v) = \delta_{v_0}(x)g(x)$ on Γ_- and that $g(x) = 1$ for $x \in \partial\Omega$ in the vicinity of x_0. Then we will see that

$$\Sigma = \lim_{\delta\to 0} \frac{1 - u(\tilde{x}_0(\delta), v_0)}{\delta} + \sum_{m=1}^{M} \Sigma\sigma_m \alpha_m(v_0)\phi_m(x_0). \tag{9}$$

Since the $\Sigma\sigma_m$ are already known, this equation gives Σ explicitly.

We state our main result:

Theorem 3.1 *The inverse problem consisting in reconstructing the coefficients σ_m and Σ from the boundary measurement of two well-chosen experiments is well-posed. We can construct a first experiment for which the family $\phi_k(x)\alpha_k(v(x))$ is linearly independent; i.e.,*

$$\sum_{k=1}^{M} \lambda_k \phi_k(x)\alpha_k(v(x)) = 0 \ \ \textit{for almost all} \ \ x \in \partial\Omega \quad \Rightarrow \quad \lambda_k = 0, \ 1 \le k \le M.$$

We can choose M points x_i such that $T_{ik} = \phi_k(x_i)\alpha_k(v(x_i))$ forms an invertible matrix T whose inverse is given by T^{-1}_{ij} and such that (7) holds for every x_i. Denoting by

$$b_i = \lim_{\delta\to 0} \frac{u(\tilde{x}_i(\delta), v_i)}{\delta},$$

the scattering coefficients are given by

$$\Sigma\sigma_m = \sum_{i=1}^{M} T^{-1}_{mi} b_i. \tag{10}$$

We can then construct a second experiment for which (9) holds, yielding Σ and the coefficients σ_m.

Remark 3.2 With some slight modifications in the proof of the theorem, we can replace the measurement of a single experiment at M different points by the measurement of M different experiments at the same point. The corresponding matrix T of coefficients $T_{ik} = \phi^{(i)}_k(x_0)\alpha_k(v(x_0))$, where $\phi^{(i)}_k(x_0)$ is the kth moment at x_0 corresponding to the ith experiment, is also invertible so that (10) holds.

Proof. We first consider the integral formulation of the transport equation (1) [8]

$$\begin{aligned} u(x,v) &= e^{-\Sigma d(x,v)} g(\bar{x}(x,v)) + \int_0^{d(x,v)} e^{-\Sigma t} S(x-vt,v)\,dt \\ S(x,v) &= \sum_{m=1}^{M} \Sigma \sigma_m \alpha_m(v) \phi_m(x), \end{aligned} \tag{11}$$

where $\phi_m(x)$, $d(x,v)$, and $\bar{x}(x,v)$ are defined in (8), (5), and (6), respectively.

Let us consider a point $x_0 \in \partial\Omega$. We assume that $g = 0$ in the vicinity of the point x_0 and that all moments ϕ_m are continuous at x_0. Then for δ small enough, we have from (11)

$$\frac{u(\tilde{x}_0(\delta), v_0)}{\delta} = \frac{1}{\delta} \int_0^{\delta} e^{-\Sigma t} S(\tilde{x}_0(\delta) - v_0 t, v_0)\,dt,$$

since $d(\tilde{x}_0(\delta), v_0) = \delta$. In addition we have $e^{-\Sigma t} = 1 + O(\delta)$ and since S is continuous, $S(\tilde{x}_0(\delta) - v_0 t, v_0) \to S(x_0, v_0)$ as $\delta \to 0$. So the right-hand side in the above equation converges to $S(x_0, v_0)$ as $\delta \to 0$. This implies (7).

Let us now assume that $g(x,v) = \delta_{v_0}(v)$ in the vicinity of x_0. Then (11) yields

$$u(\tilde{x}_0(\delta), v_0) = e^{-\Sigma\delta} + \int_0^{\delta} e^{-\Sigma t} S(\tilde{x}_0(\delta) - v_0 t, v_0)\,dt.$$

Sending δ to 0 and remarking that $e^{-\Sigma\delta} = 1 - \Sigma\delta + o(\delta)$, we obtain (9) as previously. So (7) and (9) require that the moments ϕ_m be continuous in the vicinity of x_0. In the first case, the boundary condition has to vanish in the vicinity of the point x_0, whereas in the second case it has to be constantly equal to 1.

We consider these two problems in the following proposition, which is interesting in itself, and whose proof is postponed to the end of this section.

Proposition 3.3 *We consider the system of integral equations associated with the transport equation (1)*

$$\begin{aligned} \phi_m(x) &= \int_V \alpha_m(v) e^{-\Sigma d(x,v)} g(x - d(x,v)v, v)\,dv \\ &\quad + \int_\Omega \frac{e^{-\Sigma|x-y|}}{|x-y|} \alpha_m(\widehat{x-y}) \sum_{n=1}^{M} \sigma_n \alpha_n(\widehat{x-y}) \phi_n(y)\,dy, \end{aligned} \tag{12}$$

for $1 \le m \le M$, where $\hat{y} = |y|^{-1} y$. We denote by $\Phi = (\phi_1, \ldots, \phi_M)$. With obvious notation, we recast the system of equations (12) in the more compact form

$$\Phi(x) = T[\Phi](x) + H(x), \tag{13}$$

where T is a matrix of integral operators and $H = (h_1, \cdots, h_M)$ where

$$h_m(x) = \int_V \alpha_m(v) e^{-\Sigma d(x,v)} g(x - d(x,v)v, v)\,dv. \tag{14}$$

We consider two cases:

(i) If $g(x,v) \in L^\infty(\Gamma_-)$ or $g(x,v) = g(x)\delta_{v_0}(v)$ where $g(x) \in L^\infty(\partial\Omega)$, then there exists a unique solution Φ to (13) that can be written as the sum of two terms

$$\Phi = \Phi^0 + \Phi^s$$

where the $\Phi^0 = H(x) \in L^\infty(\Omega)^M$ is the ballistic part and $\Phi^s(x) = T[\Phi](x) \in (C^0(\bar{\Omega}))^M$.

(ii) If $g(x,v) = \delta_{x_0}(x)\delta_{v_0}(v)$, where $n(x_0)\cdot v_0 < 0$, then there exists a unique solution Φ to (13) that can be written as the sum of three terms

$$\Phi = \Phi^0 + \Phi^{\log} + \Phi^s,$$

where $\Phi^0 = H$, given explicitly in (20) below, is supported on the segment

$$l(x_0, v_0) = \{y \in \bar{\Omega} \mid y = x_0 + tv_0 \text{ for some } t \in I\!R\}$$

and the restriction of Φ^0 to the segment $l(x_0, v_0)$ is regular; where $\Phi^{\log}$ given explicitly in (21) below, is a continuous function except on $l(x_0, v_0)$; and where $\Phi^s(x) \in (C^0(\bar{\Omega})^M)$. When $x \in \partial\Omega$ is close to x_0, $\Phi^{\log}$ is given by

$$\phi_m^{\log}(x) = c_0 \ln |(x - x_0)\cdot v_0^{\perp}|^{-1} \alpha_m(-v_0)\Sigma k(v_0, -v_0) + O(1), \tag{15}$$

where $c_0 = |v_0 \cdot v(x_0)|$.

We deduce from this proposition that the moments ϕ_m are continuous in the vicinity of x_0 in the case of a boundary condition of the form $g(x,v) = g(x)\delta_{v_0}(v)$ where $g(x) \in L^\infty(\partial\Omega)$. So the reconstruction of Σ follows from (9) assuming that we know the coefficients $\Sigma\sigma_m$. We now focus on the reconstruction of these coefficients.

We deduce from Proposition 3.3 that for a boundary condition of the form $g(x,v) = \delta_{x_0}(x)\delta_{v_0}(v)$, the functions $\phi_m(x)$ can be written as the sum of two components. The first one is bounded and continuous, except maybe at $x = x_0$, and the second one, denoted by $\phi_m^{\log}(x)$, has a logarithmic blow up close to the point x_0 for most incident directions v_0. Further we have

$$\alpha_m(v(x))\phi_m^{\log}(x) = c_0 \ln |(x - x_0)\cdot v_0^{\perp}|^{-1} \alpha_m(-v_0)\alpha_m(v(x_0))\Sigma k(-v_0, v_0) + \varphi_m(x),$$

where $c_0 = |v_0 \cdot v(x_0)|$ and $\varphi_m(x)$ for $x \in \partial\Omega$ in a neighborhood of x_0 is continuous except may be at $x = x_0$. By assumption on the scattering kernel, $k(-v_0, v_0) > 0$ for every v_0.

Consider a direction v_0 such that $\alpha_m(v_0) \neq 0$ for every $1 \leq m \leq M$. By assumption on α_m, almost all points $v \in S^1$ satisfy this hypothesis. Consider now a boundary condition of the form

$$g(x,v) = \sum_{i=1}^{M} \delta_{y_i}(x)\delta_{v_0}(v)$$

where the points $y_i \in \partial\Omega$ are different from one another, such that $n(y_i)\cdot v_0 < 0$ and such that the family $\alpha_m(v(y_i))$ is independent; i.e., the $M \times M$ matrix A with coefficients $A_{mi} = \alpha_m(v(y_i))$ is invertible. This implies that the functions $\alpha_m(v(x))\phi_m(x)$ are linearly independent since their singular (logarithmic) parts are linearly independent. Moreover, these functions are continuous except at the points y_i. Therefore, we can find M points x_i different from the points y_k such that the vectors $(\alpha_m(v(x_i))\phi_m(x_i))_{1\leq m\leq M}$ are linearly independent and the functions $\alpha_m(v(x))\phi_m(x)$ are continuous in the vicinity of those points x_i. Consequently $T_{im} = \alpha_m(v(x_i))\phi_m(x_i)$ forms an invertible matrix and (10) holds. $\square$

Before giving a proof of Proposition 3.3, we need the following two lemmas. First, we denote by $C^{0,\alpha}(\Omega)$ the space of Hölder-continuous functions that bounded for the norm

$$\|\phi\|_{C^{0,\alpha}} = \sup_{x\in\Omega} |\phi(x)| + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|\phi(x)-\phi(y)|}{|x-y|^\alpha}.$$

Lemma 3.4 *Consider the integral equation (13) with weakly singular integral kernel T obtained from (12). Assume that H is given by (14), where $g \in L^\infty(\Gamma_-)$. Then, there exists a unique bounded solution $\Phi \in (L^\infty(\Omega))^M$ to (13). If $H \in (C^{0,\alpha}(\bar{\Omega}))^M$ for $0 \le \alpha < 1$, then $\Phi \in (C^{0,\alpha}(\bar{\Omega}))^M$.*

Proof. The integral equation (13) is equivalent to the integral equation for $u(x,v)$

$$u(x,v) = e^{-\Sigma d(x,v)} g(x - d(x,v)v, v) + \int_0^{d(x,v)} e^{-\Sigma t} \Sigma K[u](x - vt, v)dt. \tag{16}$$

Here,

$$K[u](x,v) = \int_V k(v,v')u(x,v')d\mu(v'), \tag{17}$$

where $k(v,v')$ is given by (4) and the moments $\phi_m(x)$ and the function $u(x,v)$ are related by (8) and (11). It is clear that the two equations are equivalent in the L^∞ sense. Let us now prove that there exists a unique solution to (16) satisfying

$$\|u\|_{L^\infty(\Omega\times V)} \le \|g\|_{L^\infty(\Gamma_-)}.$$

The proof is adapted from [8] in which (3) is a strict inequality. Let us denote by $u^0(x,v) = e^{-\Sigma d(x,v)} g(x - d(x,v)v, v)$ and by induction

$$u^{k+1}(x,v) = e^{-\Sigma d(x,v)} g(x - d(x,v)v, v) + \int_0^{d(x,v)} e^{-\Sigma t} \Sigma K[u^k](x - vt, v)dt.$$

The functions u^{k+1} are well-defined in $L^\infty(\Omega \times V)$. Let us denote by $\delta(\Omega) < \infty$ the diameter of Ω. Assume that $\|u^k\|_{L^\infty(\Omega\times V)} \le \|g\|_{L^\infty(\Gamma_-)}$. According to (3), we obtain that

$$\|K[u^k]\|_{L^\infty(\Omega\times V)} \le \|u^k\|_{L^\infty(\Omega\times V)}$$

Therefore we have

$$\|u^{k+1}\|_{L^\infty(\Omega\times V)} \le e^{-\Sigma\delta(\Omega)} \|g\|_{L^\infty(\Gamma_-)} + (1 - e^{-\Sigma\delta(\Omega)}) \|u^k\|_{L^\infty(\Omega\times V)},$$

hence $\|u^{k+1}\|_{L^\infty(\Omega\times V)} \le \|g\|_{L^\infty(\Gamma_-)}$. Using the same estimates, we deduce that

$$\|u^{k+2} - u^{k+1}\|_{L^\infty(\Omega\times V)} \le (1 - e^{-\Sigma\delta(\Omega)}) \|u^{k+1} - u^k\|_{L^\infty(\Omega\times V)}.$$

This implies the convergence of the sequence u^k to $u \in L^\infty(\Omega \times V)$, which obviously solves (16). Moreover $|u|$ is bounded by $\|g\|_{L^\infty(\Gamma_-)}$. By linearity, we also obtain that the difference of two solutions is bounded by 0; hence the solution is unique in $L^\infty(\Omega \times V)$. Using (8), we obtain a unique solution for the moments ϕ_m is $L^\infty(\Omega)$. Using Lemma 3.5 below, we obtain that Φ is Hölder continuous provided H is. □

Lemma 3.5 *Let Ω be a bounded convex domain in $\mathbb{R}^2$. Let $G(x)$ be Lipschitz on $\mathbb{R}^2$ and α be Lipschitz on the unit circle S^1. Let*

$$f(x) = \int_\Omega \frac{G(x-y)\alpha(\widehat{x-y})}{|x-y|} g(y)dy \quad \text{on} \quad \Omega. \tag{18}$$

If $g \in L^p(\Omega)$ for $2 < p < \infty$, then $f \in C^{0,1-2/p}(\bar\Omega)$ and

$$\|f\|_{C^{0,1-2/p}(\bar\Omega)} \leq C\|g\|_{L^p(\Omega)},$$

where C is a constant independent of g.

Proof. We want to estimate $f(x) - f(z)$ for $x, z \in \bar\Omega$. We split Ω into two parts $B = B(x, 2|z-x|) \cap \Omega$, where $B(x,\rho)$ is the unit disc of center x and radius ρ, and $B^C = \Omega \backslash B$, its complementary in Ω.

Assume first that $y \in B^C$. We have

$$\frac{G(x-y)\alpha(\widehat{x-y})}{|x-y|} - \frac{G(z-y)\alpha(\widehat{z-y})}{|z-y|} = \alpha(\widehat{z-y})G(z-y)\Big(\frac{1}{|x-y|} - \frac{1}{|z-y|}\Big) + \frac{(\alpha(\widehat{x-y}) - \alpha(\widehat{z-y}))G(x-y) + \alpha(\widehat{z-y})(G(x-y) - G(z-y))}{|x-y|}.$$

Since $y \in B^C$, we obtain by the triangle inequality that

$$\Big|\frac{1}{|x-y|} - \frac{1}{|z-y|}\Big| = \Big|\frac{|z-y| - |x-y|}{|x-y|\,|z-y|}\Big| \leq 2\frac{|x-z|}{|x-y|^2}.$$

We also have by the same triangle inequality that

$$\big|\widehat{x-y} - \widehat{z-y}\big| = \Big|\frac{x-z}{|x-y|} + (y-z)\frac{|z-y| - |x-y|}{|x-y|\,|z-y|}\Big| \leq 2\frac{|x-z|}{|x-y|}.$$

Therefore, since α and G are Lipschitz and $|x-z|$ is bounded, we have

$$\Big|\frac{G(x-y)\alpha(\widehat{x-y})}{|x-y|} - \frac{G(z-y)\alpha(\widehat{z-y})}{|z-y|}\Big| \leq C\frac{|x-z|}{|x-y|^2}.$$

For $y \in B$, we have

$$\Big|\frac{G(x-y)\alpha(\widehat{x-y})}{|x-y|} - \frac{G(z-y)\alpha(\widehat{z-y})}{|z-y|}\Big| \leq C\Big(\frac{1}{|x-y|} + \frac{1}{|z-y|}\Big).$$

We obtain that

$$|f(x) - f(z)| \leq C\int_{B(0,2\rho)} \frac{|g(y-x)|}{|y|}dy + \int_{x+B^C} \frac{C|x-z|}{|y|^2}|g(y-x)|dy.$$

We estimate those two terms by Hölder's inequality. Denoting by $(p')^{-1} + p^{-1} = 1$, we have

$$\Big(\int_{B(0,\rho)} \frac{dy}{|y|^{p'}}\Big)^{1/p'} = C\rho^{2/p'-1} = C\rho^{1-2/p}.$$

On the other hand,

$$\Big(\int_{B^C(0,\rho)} \frac{dy}{|y|^{2p'}}\Big)^{1/p'} \leq \Big(\int_\rho^C r^{1-2p'}dr\Big)^{1/p'} \leq C\rho^{2(1-p')/p'} = C\rho^{-2/p}.$$

As a consequence, we get the uniform bound

$$|f(x)-f(z)| \leq C\|g\|_{L^p(\Omega)}\Big(\rho^{1-2/p}+|x-z|\rho^{-2/p}\Big) \leq C\|g\|_{L^p(\Omega)}|x-z|^{1-2/p}.$$

This concludes the proof of the lemma. □

Proof of Proposition 3.3. Let us first derive the system of integral transport equations (12). Integrating (11) multiplied by $\alpha_m(v)$ over V yields

$$\begin{aligned}\phi_m(x) &= \int_V \alpha_m(v)e^{-\Sigma d(x,v)}g(x-d(x,v)v,v)dv \\ &\quad + \int_V\int_0^{d(x,v)} e^{-\Sigma t}\alpha_m(v)\sum_{n=1}^M \sigma_n\alpha_n(v)\phi_n(x-vt)\,dv\,dt.\end{aligned}$$

By the change of variables $y=x-vt$, hence $v=\widehat{y-x}$, we obtain (12). So far we have obtained a system of equation for the moments $\phi_m(x)$ from the equation for $u(x,v)$. Once the moments $\phi_m(x)$ are known, the solution $u(x,v)$ can be found through (11), and the two formulations are equivalent.

The system of equations (13) is a weakly singular Fredholm equation of the second kind. The term 'weak' stems from the singularity of the integral kernel of order $|x-y|^{-1}$, which is unbounded but integrable. The theory of weakly singular integral operators is well-established [12, 13, 14]. For instance, we get from [13, Theorem 3.1] that

$$T[\Phi]\in(C^{0,\alpha}(\bar\Omega))^M \quad \text{if} \quad \Phi\in(L^\infty(\Omega))^M \quad \text{for all } \alpha\in(0,1). \tag{19}$$

We need a similar lemma showing that $T[\Phi]$ is also Hölder continuous when $\Phi\in(L^p(\Omega))^M$ for certain values of p. This is Lemma 3.5. Because of the boundary $\partial\Omega$, the operator T does not regularize smooth functions (see [13]). However, it regularizes singular functions as we now see.

We readily deduce from (14) that $H(x)\in(L^\infty(\Omega))^M$ for $g(x,v)\in L^\infty(\Gamma_-)$ or $g(x,v)=g(x)\delta_{v_0}(v)$ where $g(x)\in L^\infty(\partial\Omega)$. So for such boundary conditions, we deduce from Lemma 3.4 the existence of a unique solution $\Phi\in(L^\infty(\Omega))^M$ to (13). Since $\Phi=H+T[\Phi]$, case 1 of the theorem follows from (19).

Case 2 is more involved because H is not bounded. Its components are given by

$$h_m(x)=\alpha_m(v_0)e^{-\Sigma d(x,v_0)}\delta^\sigma_{x_0}(\bar x(x,v_0)), \tag{20}$$

where $\delta^\sigma_{x_0}$ denotes the Dirac measure on the boundary $\partial\Omega$. We denote by $\Phi^0=H$. We verify that its support is on the segment $l(x_0,v_0)$ and that its restriction to its support is smooth. Let us define $\Phi^1=T[\Phi^0]$. Its components are given by

$$\begin{aligned}\phi^1_m(x) &= \int_\Omega \frac{e^{-\Sigma|x-y|}}{|x-y|}\alpha_m(\widehat{x-y})\sum_{n=1}^M \Sigma\sigma_n\alpha_n(v_0)\alpha_n(\widehat{x-y})e^{-\Sigma d(y,v_0)}\delta^\sigma_{x_0}(\bar x(y,v_0))dy \\ &= c_0\int_0^{d(x_0,-v_0)}\frac{e^{-\Sigma r(t)}}{r(t)}\alpha_m(\theta(t))\sum_{n=1}^M \Sigma\sigma_n\alpha_n(v_0)\alpha_n(\theta(t))e^{-\Sigma t}dt,\end{aligned}$$

where $c_0=|v_0\cdot v(x_0)|$, $r(t)=|x-x_0-tv_0|$, and $\theta(t)=\widehat{x-x_0-tv_0}$. Let us write $x=x_0+x_1v_0+x_2v_0^\perp$, where $x_1=(x-x_0)\cdot v_0$ and $x_2=(x-x_0)\cdot v_0^\perp$. We assume that $x_2>0$. The case $x_2<0$ is treated similarly. Then, $x-y=x-x_0-tv_0=(x_1-t)v_0+x_2v_0^\perp$.

We want to extract the singular part of ϕ^1_m. We first get rid of the exponential terms. Denoting by $r = |x-y| = \sqrt{(t-x_1)^2 + x_2^2}$, we introduce

$$e^{-\Sigma r}e^{-\Sigma t} = e^{-\Sigma x_1}\Big(e^{-\Sigma r}e^{-\Sigma(t-x_1)} - 1\Big) + e^{-\Sigma x_1}.$$

We have that $r \geq (1/2)[r + |t-x_1|]$. Therefore,

$$\frac{e^{-\Sigma r}e^{-\Sigma(t-x_1)} - 1}{r} \leq 2\frac{e^{-\Sigma r}e^{-\Sigma(t-x_1)} - 1}{r + |t-x_1|}.$$

This expression is uniformly bounded since r and $t-x_1$ take bounded values and since it is bounded when both r and $t-x_1$ go to 0. The integration of a bounded function on a bounded interval being continuous, we decompose

$$\Phi^1 = \Phi^{\log} + \Phi^{1c},$$

where Φ^{1c} is continuous and the components of $\Phi^{\log}$ are given by

$$\phi^{\log}_m(x) = c_0 e^{-\Sigma x_1}\int_0^{d(x_0,-v_0)} \frac{\alpha_m(\theta(t))}{r(t)}\sum_{n=1}^M \Sigma\sigma_n\alpha_n(v_0)\alpha_n(\theta(t))dt.$$

Consider the change of variables $\theta = \theta(t) = \widehat{x - x_0 - tv_0}$. We identify the vector $\theta \in S^1$ with its representation in $[0, 2\pi)$. Remarking that $\theta = v_0 + \arctan x_2/(x_1 - t)$, we obtain

$$t = x_1 - \frac{x_2}{\tan(\theta - v_0)}, \quad x_2 = r\sin(\theta - v_0), \quad dt = \frac{x_2\, d\theta}{\sin^2(\theta - v_0)}, \quad \frac{dt}{r} = \frac{d\theta}{\sin(\theta - v_0)}.$$

Therefore, we have

$$\phi^{\log}_m(x) = c_0 e^{-\Sigma x_1}\int_{\theta_0}^{\theta_1} \frac{\alpha_m(\theta)}{\sin(\theta - v_0)}\sum_{n=1}^M \Sigma\sigma_n\alpha_n(v_0)\alpha_n(\theta)d\theta, \tag{21}$$

where $\theta_0 = v_0 + \arctan(x_2/x_1)$ and $\theta_1 = v_0 + \arctan(x_2/(x_1 - d(x, -v_0)))$. This function is continuous at those points x satisfying $x_2 \neq 0$. Let us fix x_1 and send $x_2 \to 0$. We want to extract the asymptotic behavior of the above expression as $x_2 \to 0$. We obtain it by differentiating the expression w.r.t. x_2. Denoting by $f(\theta)$ the integrand in the above expression, we get

$$\frac{\partial}{\partial x_2}\int_{\theta_0}^{\theta_1} f(\theta)d\theta = \frac{\partial\theta_1}{\partial x_2}f(\theta_1) - \frac{\partial\theta_0}{\partial x_2}f(\theta_0).$$

We have

$$\frac{\partial\theta_0}{\partial x_2} = \frac{x_1}{x_1^2 + x_2^2}, \quad \text{and} \quad \frac{\partial\theta_1}{\partial x_2} = \frac{x_1 - d(x_0,-v_0)}{(x_1 - d(x_0,-v_0))^2 + x_2^2}.$$

Assume first that the point x is inside the domain, and that $|x_2| << |x_1|$ and $|x_2| << |d(x_0,-v_0) - x_1|$. Then we verify that

$$\begin{aligned}
&\frac{\partial\theta_0}{\partial x_2} = \frac{1}{x_1} + o(x_2), \quad \frac{1}{\sin(\theta_0 - v_0)} = \frac{x_1}{x_2} + o(1),\\
&\theta_0 = v_0 + O(x_2),\\
&f(\theta_0) = \frac{x_1}{x_2}\alpha_m(v_0)\sum_{n=1}^M \Sigma\sigma_n\alpha_n(v_0)\alpha_n(v_0) + O(1).
\end{aligned}$$

Here, $o(x)$ stands for an expression such that $o(x)/|x| \to 0$ as $x \to 0$ and $O(x)$ stands for an expression such that $|O(x)|/|x|$ remains bounded for x small enough. We have similar results for θ_1 and obtain that

$$\frac{\partial}{\partial x_2}\int_{\theta_0}^{\theta_1} f(\theta)d\theta = \frac{1}{x_2}\sum_{n=1}^{M}\Sigma\sigma_n\alpha_n(v_0)[\alpha_m(-v_0)\alpha_n(-v_0)+\alpha_m(v_0)\alpha_n(v_0)] + O(1).$$

Integrating this expression w.r.t. x_2, we get

$$\phi_m^{\log}(x) = c_0 e^{-\Sigma x_1}\ln|x_2|^{-1}[G_m(v_0)+G_m(-v_0)] + O(1), \tag{22}$$

where

$$G_m(v) = \alpha_m(v)\sum_{n=1}^{M}\Sigma\sigma_n\alpha_n(v_0)\alpha_n(v) = \alpha_m(v)\Sigma k(v,v_0). \tag{23}$$

We recall that x_1 and x_2 are the coordinates of x in $(x_0, v_0, v_0^\perp)$; i.e., $x = x_0 + x_1 v_0 + x_2 v_0^\perp$.

Assume now that the point $x \in \partial\Omega$ is close to x_0. Then we have approximately that $x_1 = \alpha x_2$ for some constant α measuring the angle between $v(x_0)$ and v_0. When $x_2 \to 0$, $f(\theta_0)$ remains bounded. Therefore, we have

$$\phi_m^{\log}(x) = c_0\ln|x_2|^{-1}G_m(-v_0) + O(1). \tag{24}$$

This is equation (15). Notice that when $x \in \partial\Omega$ is close to $x_0 + d(x,-v_0)v_0$, it is $f(\theta_1)$ that remains bounded and we get

$$\phi_m^{\log}(x) = c_0 e^{-\Sigma d(x,-v_0)}\ln|x_2|^{-1}G_m(v_0) + O(1), \tag{25}$$

which describes the logarithmic blow-up of single scattering on the other side of the domain. This concludes our treatment of the single scattering component Φ^1. It remains to deal with the higher order scattering terms and show that they are all continuous. The initial equation for Φ is

$$\Phi = T[\Phi] + H.$$

Moreover, $\Phi^0 = H$ and $\Phi^1 = T[\Phi^0]$. Therefore

$$\Phi^{ms} = \Phi - \Phi^0 - \Phi^1 = T[\Phi^{ms}] + T[\Phi^1].$$

So showing that $T[\Phi^1] \in (C^0(\bar\Omega))^M$ suffices to deduce from Lemma 3.4 that $\Phi^{ms} \in (C^0(\bar\Omega))^M$, which concludes the proof of the theorem. We know that Φ^1 is the sum of a continuous function and $\Phi^{\log}$. Moreover $\Phi^{\log}$ blows up logarithmically in the vicinity of $l(x_0, v_0)$. Therefore, $\Phi^{\log}$ and Φ^1 are in every $L^p(\Omega)$ for $1 \le p < \infty$. We deduce from Lemma 3.5 that $T[\Phi^1] \in (C^0(\bar\Omega))^M$ since every component of the operator matrix T obviously satisfies the hypotheses of this lemma. □

4. Boundary with corners

The analysis of the previous subsection can be somewhat simplified when the boundary of Ω has corners. To simplify the presentation, we consider here the case where Ω is a square. Let x be a corner of Ω. We denote by $V(x)$ the following subset of the unit circle S^1

$$V(x) = \{v \in S^1 \quad \text{such that} \quad x + tv \notin \Omega \text{ for all } |t| > 0\}. \tag{26}$$

When the boundary is smooth and strictly convex in the neighborhood of x, the set $V(x) = \{v(x)\}$. In the case of domains with corners, we have access to the outgoing flux in the vicinity of x for all directions in $V(x)$; i.e., an infinite number. Therefore, the knowledge of the outgoing flux in the vicinity of a corner should be sufficient to obtain the coefficients σ_m and the total cross section Σ.

The analog to (7) is now

$$\sum_{m=1}^{M} \Sigma\sigma_m\alpha_m(v)\phi_m(x) = \lim_{\delta\to 0} \frac{u(\tilde{x}(\delta, v), v)}{\delta}, \tag{27}$$

where $v \in V(x)$ and $\tilde{x}(\delta, v)$ is the closest point to x such that the line $\{\tilde{x}(\delta, v)+tv,\ t \in I\!\!R\}$ intersects Ω on an interval of size δ. Consider M directions $v_i \in V(x)$ such that $\alpha_m(v_i)$ is a linearly independent family. Denoting by $T_{im} = \alpha_m(v_i)$ and

$$b_i = \lim_{\delta\to 0} \frac{u(\tilde{x}(\delta, v_i), v_i)}{\delta},$$

we obtain that

$$\Sigma\sigma_m\phi_m(x) = \sum_{i=1}^{M} T^{-1}_{mi} b_i. \tag{28}$$

To complete the reconstruction of the coefficients $\Sigma\sigma_m$, we need an experiment for which $\phi_m(x) \neq 0$ for $1 \le m \le M$. Let us emphasize that the results of Proposition 3.3 still hold in the presence of corners. The geometry of Ω, except for the requirement that it be convex, does not appear in its proof. The ballistic part Φ^0 has its support concentrated on the bounded interval $l(x_0, v_0)$ and the first scattering has a logarithmic blow up in the vicinity of that interval. So if we choose x_0 close enough to x and v_0 such that $n(x_0) \cdot v_0 < 0$ and $\alpha_m(-v_0) \neq 0$ for $1 \le m \le M$, (15) holds and $\phi_m(x) \neq 0$ for $1 \le m \le M$. Indeed, when x_0 is close enough to x, the logarithmic part dominates the rest of the solution. So we can write

$$\Sigma\sigma_m = \frac{1}{\phi_m(x)} \sum_{i=1}^{M} T^{-1}_{mi} b_i. \tag{29}$$

The analog to (9) is

$$\Sigma = \lim_{\delta\to 0} \frac{1 - u(\tilde{x}(\delta, v), v)}{\delta} + \sum_{m=1}^{M} \Sigma\sigma_m\alpha_m(v)\phi_m(x) \tag{30}$$

for every $v \in V(x)$. Since we know $\Sigma\sigma_m$ from (29), we can deduce Σ from (30). This concludes the reconstruction of the coefficients σ_m and Σ.

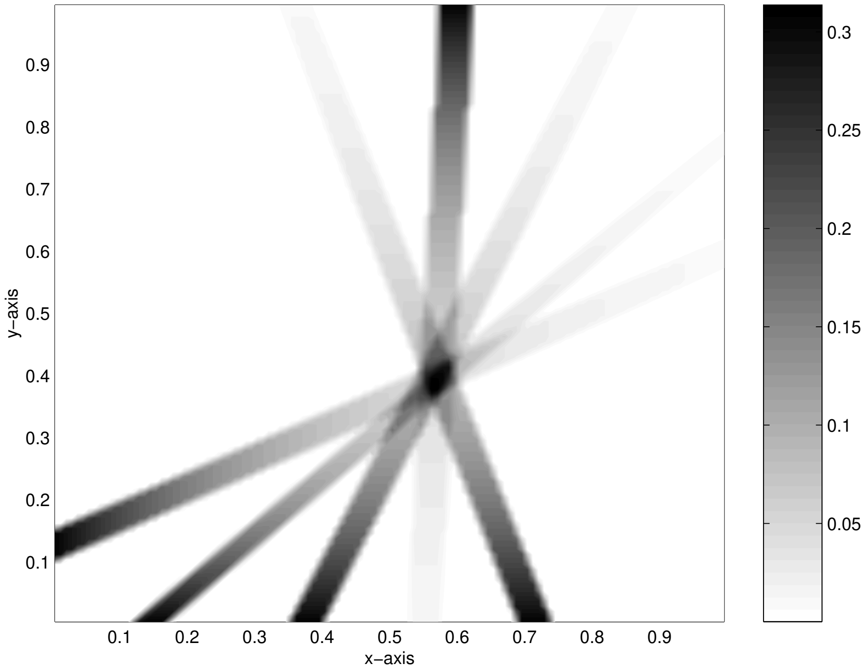

Figure 2. Ballistic part $u^0(x, v)$.

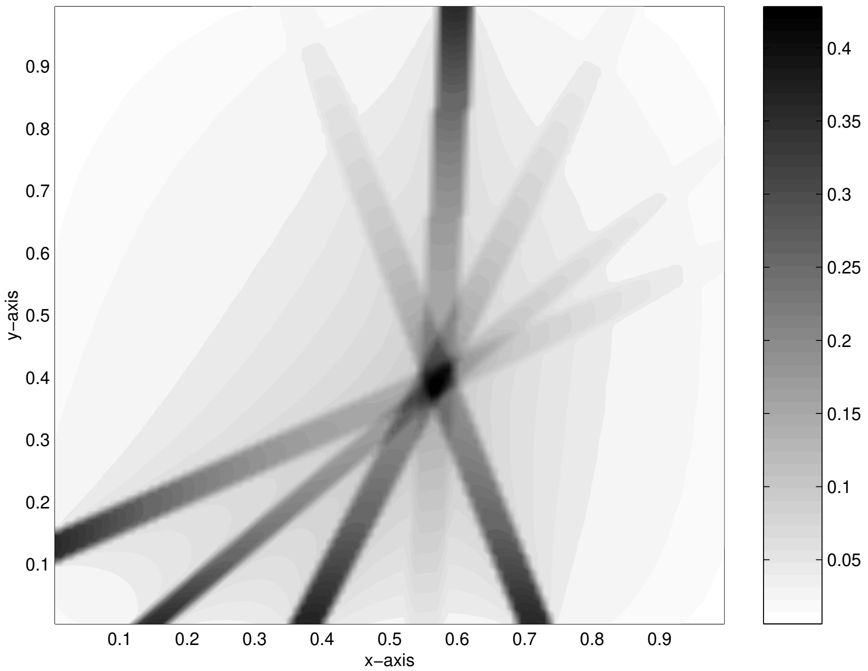

Figure 3. Transport solution $u(x, v)$.

5. Numerical Application

We consider a numerical application showing the independence of the moments of the flux at the boundary of the domain claimed in the previous section. Since it is difficult to deal with smooth boundaries numerically, we only consider here the case of Ω being a square. We also show that the moments do not vanish at the corners. This information is used to reconstruct the scattering operator in a simplified setting.

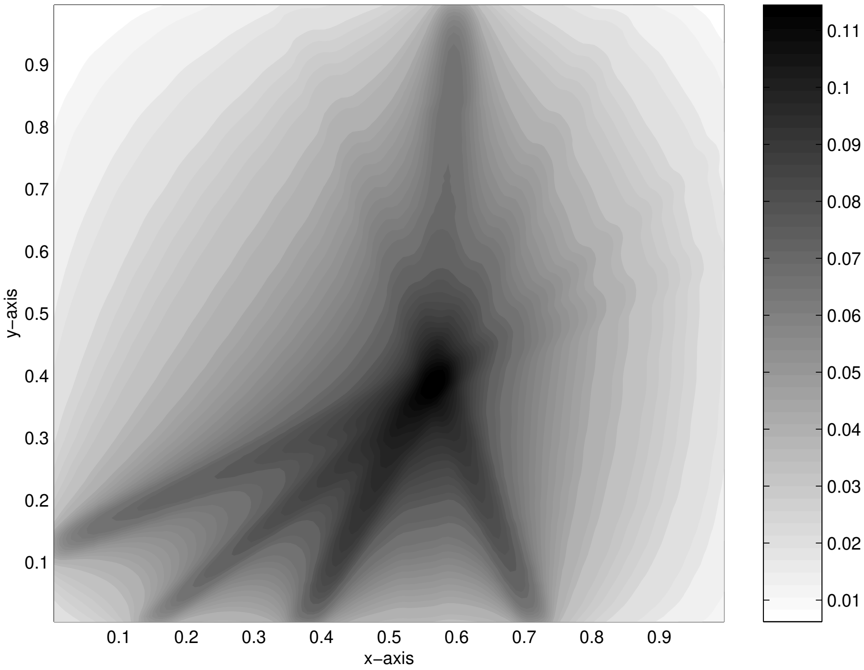

Figure 4. Multiple scattering, given by $u^s(x,v) = u(x,v) - u^0(x,v)$.

Our numerical setting is a two-dimensional caricature of the propagation of beams inside a medium. We consider five converging beams emitted from different edges of $\Omega = (0,1)^2$. The width of each beam at the boundary is $1/20$. The cross section is $\Sigma = 2$. The basis functions α_m are given by $\alpha_{2m}(\theta) = \cos m\theta$ and $\alpha_{2m+1} = \sin m\theta$ for $m \in I\!N$. The coefficients σ_m are given by

$$\sigma_0 = 1, \qquad \sigma_2 = \sigma_3 = 0, \qquad \sigma_4 = \frac{1}{3}, \qquad \sigma_5 = \frac{-1}{3}, \tag{31}$$

which can be seen as a two-dimensional version of Rayleigh scattering.

We have used a classical discrete ordinates method with $4N = 64$ directions to discretize the velocity variable [15]. The spatial discretization is a uniform 120×120 rectangular mesh discretization. The numerical scheme has been tailored so as to compute rays propagating through the medium without distorting them. The method is explained elsewhere [16]. A two-dimensional plot of the ballistic solution

$$u^0(x,v) = e^{-\Sigma d(x,v)} g(\bar{x}(x,v), v)$$

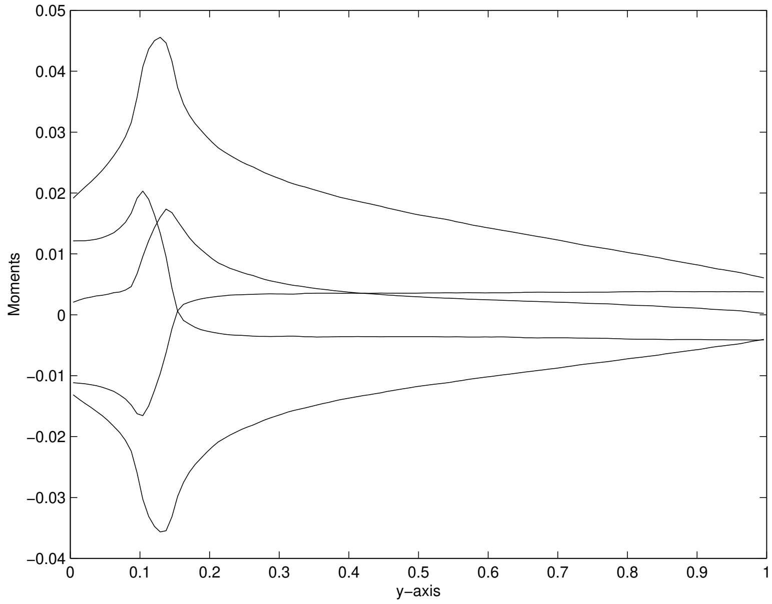

Figure 5. Trace of the moments on the left edge of Ω. On the left of the figure ($y=0$), from top to bottom: ϕ_1, ϕ_5, ϕ_4, ϕ_3, and ϕ_2.

is given in Fig. 2. The full solution $u(x,v)$ to (1) is shown in Fig. 3. The difference of these two solutions, describing the scattering off the main directions of the beamlets, is given in Fig. 4.

We have represented in Fig. 5 the trace of the moments ϕ_m at the boundary $\partial\Omega$. Even though the logarithmic blow up is very difficult to observe numerically, we see a clear hump of the moments close to the location of the beam. We can check the linear independence of the moments ϕ_m. Also the moments do not vanish in the two corners of the left edge. Hence these values can be used to reconstruct the coefficients $\Sigma\sigma_m$.

Let x be a corner of Ω. The numerical value of the right side of (27) is obtained by looking at the outgoing fluxes at the edge of the discretization cell to which x belongs. It depends on the chosen spatial discretization, but can be traced back to the value of the source term on that cell in every direction $v_n \in V(x)$ of the angular discretization

$$\Sigma\sigma_m\alpha_m(v_n)\phi_m(x).$$

Let us mention that when the number of outgoing directions $2N$ is larger than the number of non vanishing coefficients σ_m, the problem is actually overdetermined. Denoting by

$$b_i = \lim_{\delta\to 0}\frac{u(\tilde{x}(\delta,v_i),v_i)}{\delta},$$

for every chosen direction v_i, $1\le i\le 2N$, we have

$$A(\Sigma\sigma\phi(x)) = b, \tag{32}$$

where A is a $2N\times M$ matrix of coefficients $A_{im} = \alpha_m(v_i)$, b is the N-vector of components b_i and $(\Sigma\sigma\phi(x))$ is the M-vector of components $\Sigma\sigma_m\phi_m(x)$. Since we have an overdetermined problem, we seek the solution $\Sigma\sigma\phi(x)$ of (32) that minimizes the residual $\|b - A(\Sigma\sigma\phi(x))\|_2$, where $\|\cdot\|_2$ is the classical matrix norm defined as the square root of the sum of the coefficients squared. This solution is obtained by solving the system of normal equations

$$\Sigma\sigma\phi(x) = (A^*A)^{-1}A^*b. \tag{33}$$

The matrix $P = (A^*A)^{-1}A^*$ is the pseudo-inverse of A [17]. Numerical simulations of (33) with $4N = 64$ and $M = 5$ allows us to reconstruct $\Sigma\sigma_m$, where the coefficients σ_m are given by (31). The agreement with (31) is perfect for data without noise and not reproduced here.

When the measured data have a noisy component b_n, assuming that this error is uncorrelated with the measurement of the moments $\phi_m(x)$, which is physically realistic since the measurement involves the directions that come from the interior of Ω, the structure of the pseudo-inverse P becomes important. Let us write the noisy part of (33) as

$$x^n = Pb^n.$$

The variance of the components of x^n is related to the variance σ^2 of the components of b^n (assuming that σ^2 is independent of the component of b^n and the different components are uncorrelated) by

$$\langle x_i^n x_i^n \rangle = (PP^t)_{ii}\sigma^2.$$

So the maximal value on the diagonal of PP^t governs the accuracy of the reconstruction. We compare the diagonal of PP^t for two cases. In the first one, all directions are considered and the pseudo inverse P is a 5×32 matrix. In the second case, only half of the directions are considered and the pseudo-inverse $P_{1/2}$ is a 5×16 matrix. The diagonal coefficients are given in table 1. These results show that the variance of the

Table 1. Diagonal coefficients of the matrices PP^t and $P_{1/2}P^t_{1/2}$.

Index	(1,1)	(2,2)	(3,3)	(4,4)	(5,5)
PP^t	0.17	0.11	0.11	0.06	0.33
$P_{1/2}P^t_{1/2}$	2560	2470	2470	15	403

noisy data is amplified by a factor of the order 10^3 when only half of the directions are used for the reconstruction, whereas the same variance is decreased by a factor 3 when all measurements are taken into account. So even in the presence of small noise, a large number of directions of measurement is necessary to obtain a satisfactory reconstruction.

Acknowledgment

This work was supported in part by the ASCI Flash Center at the University of Chicago under DOE contract B341495 and in part by the MRSEC Program of the National Science Foundation under Award Number DMR-9808595. The author would like to thank the referees for their comments and suggestions.

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