Inverse problems for homogeneous transport equations.
Part I: One-dimensional case.

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Abstract. This paper and a companion paper [1] are part I and II of a series dealing with the reconstruction from boundary measurements of the scattering operator of homogeneous linear transport equations. This first part addresses the one-dimensional half space setting, both for time-dependent and steady state problems. We propose a reconstruction procedure based on a Riccati equation for the response operator. In the steady state case, we show that the reconstruction is a well-posed inverse problem provided the scattering operator satisfies enough symmetry constraints. The problem becomes ill-posed when some of these constraints are dropped. We show that ill-posed problems in the steady state case become well-posed in the time-dependent setting with time independent scattering operators.

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1. Introduction

This paper considers the reconstruction from boundary measurements of the scattering operator of homogeneous one-dimensional linear transport equations in half space. The same inverse problem in convex bounded domains in higher dimensions is treated in a companion paper [1].

Linear transport equations are well-understood mathematically and physically [2, 3, 4]. Even though most physical applications require two or three dimensional settings, some problems can be satisfactorily described by one-dimensional transport equations, like the propagation of light through turbulent atmospheres [5].

Inverse problems related to transport theory have also been studied for many years. We refer to the recent surveys [6, 7] and references within for a complete account. Inverse problems aim at reconstructing the scattering operator of the transport equation that describes the interaction of the quantity of interest (e.g., light energy density) with the underlying medium though which it propagates (e.g., turbulent atmosphere) from boundary measurements (e.g., light energy density at the Earth's surface). More precisely, the scattering operator describes the probability that some energy density at position x with velocity v interacting with the medium is scattered into a new direction v'. Reconstruction of the scattering operator gives therefore valuable structural information on the medium via its scattering properties. The more general inverse problem consists in considering space dependent, velocity dependent scattering operators. We suppose here that the scattering operator is *independent* of the position x; i.e., the medium is *homogeneous*.

Our reconstruction procedure is based on the invariant imbedding technique and the Riccati equation for the response operator. Since Chandrasekhar's work [5], invariant imbedding techniques have been often applied to transport equations [8, 9, 10, 11]. They have also been used very intensively to deal with inverse problems in wave theory [12].

In general, the inverse problem is ill-posed; i.e. the boundary measurement of the outgoing distribution for all incoming distributions does not allow for a unique reconstruction of the scattering operator. We show that the inverse problem is still ill-posed in the case of symmetric scattering operators –see the constraint (5) below– although the reconstruction can be theoretically worked out for degenerate scattering kernels. This corroborates results obtained by a different method in [13], where the ill-posedness is shown to be equivalent to the classical ill-posedness arising in the inversion of integral equations of the first kind. However, the additional constraint that the probabilities of scattering into the directions v' and $-v'$ are equal renders the inverse problem well-posed and stable. This additional symmetry amounts to saying that forward and backward scattering are equally probable. We underscore the stability of the procedure in this 'super-symmetric' case with some numerical simulations. Finally, we obtain that the inverse problem without the latter constraint becomes well-posed in the case of time dependent transport equations with time independent scattering operator.

The outline of the paper is as follows. Sections 2 and 3 present the steady-state forward and inverse problems. The 'super-symmetric' case is treated in section 4 and some numerical simulations are given in section 5. Section 6 comes back to the symmetric, but not 'super-symmetric' case. Section 7 deals with the time dependent problem.

2. One-dimensional steady problem

The one-dimensional monoenergetic steady-state half-space transport equation in homogeneous medium is given by

$$\begin{aligned} &\mu\frac{\partial}{\partial x}u(x,\mu)+\Sigma u(x,\mu)=\Sigma K[u(x,\cdot)](\mu) \qquad \text{in} \quad X,\\ &u(0,\mu)=g(\mu), \qquad \text{for} \quad \mu>0. \end{aligned} \tag{1}$$

Here, X is the phase space

$$X=\mathbb{R}^{+}\times[-1,1], \tag{2}$$

Σ is the total cross section, $K[u]$ the scattering operator, defined as

$$K[u](\mu)=\int_{-1}^{1}k(\mu,\mu')u(\mu')d\mu', \qquad \text{for} \quad \mu\in[-1,1], \tag{3}$$

and $g(\mu)$ is a known boundary condition. Provided that the physical parameters are suitably chosen, this transport problem is known to be well-posed [3, 4].

The monogroup, or monoenergetic, problem is a good approximation in radiative transfer theory, where the energy densities with different frequencies satisfy uncoupled equations [5, 14]. In neutron transport, multigroup equations are generally necessary. The technique presented here can readily be generalized to the multigroup case, even though one-dimensional neutron transport equations are of weak physical interest. Of greater physical interest is the case of transport equations accounting for polarization effects [5, 14]. In the one-dimensional setting, these equations take the form of a multi-group transport equation [5], to which the theory presented here applies.

3. Invariant imbedding and inverse problem

The transport solution is regular enough that we can define the outgoing distribution $u(0,\mu)$ for $\mu<0$. The response operator R maps the incoming boundary condition $g(\mu)=u(0,\mu)$ for $\mu>0$ to the outgoing distribution $u(0,-\mu)$ for $\mu>0$:

$$R[g](\mu)=u(0,-\mu) \qquad \text{for} \quad \mu>0. \tag{4}$$

The inverse problem consists in reconstructing the scattering operator K from boundary measurement, that is from the operator R.

The first question is whether general scattering operators K can be reconstructed from the knowledge of $R[g]$ for all possible incoming distributions g. The answer is clearly negative as we shall see, and some restrictions have to be imposed on the scattering kernel.

Let us assume that the scattering kernel has the form

$$k(\mu,\mu') = \sum_{m=0}^{\infty} \sigma_m \alpha_m(\mu)\alpha_m(\mu'), \tag{5}$$

where the functions α_m form an orthonormal basis on $L^2(-1,1)$ such that α_{2m} is even, α_{2m+1} is odd for $m \in I\!N$ and $\alpha_0 = 1/\sqrt{2}$. This condition amounts to saying that $k(\mu,\mu') = k(\mu',\mu)$, which is a reciprocity relation, and that $k(-\mu,-\mu') = k(\mu,\mu')$, which is an isotropy condition. An important example in practice is when the probability of scattering from a direction Ω into another direction Ω', where Ω and Ω' are unit vectors in $I\!R^3$, depends only on the angle between the two directions $\Omega\cdot\Omega'$. In this case, it is known [3] that the scattering kernel in slab geometry has the form (5), where the α_m are the Legendre polynomials on $(-1,1)$ normalized so that $\int_{-1}^{1}\alpha_m(\mu)\alpha_n(\mu)d\mu = \delta_{mn}$.

Again, one can ask whether such a symmetry is sufficient to allow for the reconstruction of the scattering operator K. The answer is still negative unless additional restrictions are imposed. One way to overcome the ill-posedness is to require that

$$k(\mu,\mu') = k(-\mu,\mu') \qquad \text{for all} \quad \mu,\,\mu' \in [-1,1]. \tag{6}$$

This additional symmetry amounts to selecting the terms in (5) that satisfy $\alpha_m(\mu) = \alpha_m(-\mu)$, i.e., to selecting the even basis functions, indexed by $m = 2n$ even. Physically, the symmetry means that forward and backward scattering are equally probable. This is an important restriction, although it is satisfied by many scattering kernels of practical interest [14, 15], for instance when the probability of scattering from one direction to another only depends on the absolute value of the angle between the two directions. In this context, the inverse problem is well-posed and we can reconstruct the coefficients α_{2m} from boundary measurements. Section 4 is devoted to the analysis of this problem. We have postponed the analysis of the problem where the symmetry (6) is not assumed to section 6.

To handle this inverse problem, we use the invariant imbedding method. For half space problems, the invariant imbedding comes to remarking that the response operator is translation invariant. To be more specific, the response operator defined by (4) is not modified if we look for the outgoing distribution $u(x,-\mu)$ for $x \geq 0$ and $\mu > 0$ knowing the incoming distribution $u(x,\mu)$ at the same point x for $\mu > 0$; that is

$$u(x,-\mu) = R[u(x,\cdot)](\mu) \qquad \text{independently of } x.$$

Notice that this property only holds for homogeneous media. For an introduction to invariant imbedding and its application to transport theory, we refer to [3, 5, 8].

We will more impose rigorous mathematical assumptions later. For the moment, let us focus on the formal derivation of the central result of invariant imbedding for our purpose: the Riccati equation for the response operator. We first remark that by the change of variables $x \to \Sigma x$, the transport equation (1) remains unchanged except for the constant Σ that is replaced by 1. Moreover, this change of variables obviously leaves the response operator unmodified. Thus, the response operator R cannot be used to measure the total cross section Σ, and without loss of generality, we set $\Sigma = 1$. Let us introduce the forward and backward propagating densities for all $x \in \mathbb{R}^+$,

$$u^+(x,\mu) = u(x,\mu) \qquad \text{and} \qquad u^-(x,\mu) = u(x,-\mu) \quad \text{for} \quad \mu > 0. \quad (7)$$

We then rewrite the transport equation as the system of coupled equations

$$\begin{aligned} \mu\partial_x u^+ + u^+ &= K_+^+[u^+] + K_-^+[u^-] \\ -\mu\partial_x u^- + u^- &= K_+^-[u^+] + K_-^-[u^-], \end{aligned} \qquad (8)$$

where we have used the notation $\partial_x = \frac{\partial}{\partial x}$. The operators K_+^+, K_-^+, K_+^-, and K_-^- are integral operators defined by

$$K_{\tau'}^{\tau}[u](\mu) = \int_0^1 k(\tau\mu, \tau'\mu')u(\mu')d\mu', \qquad \text{for} \quad \mu \in [0,1], \quad \tau, \tau' \in \{+,-\}. (9)$$

The outgoing distribution satisfies $u^-(x,\mu) = R[u^+(x,\cdot)](\mu)$ and R does not depend on x so the second equation in (8) is

$$-\mu R[\partial_x u^+] + R[u^+] = K_+^-[u^+] + K_-^- R[u^+]. \qquad (10)$$

Let us multiply the first equation in (8) by μ^{-1}, apply the response operator R, and multiply by μ. We obtain

$$\mu R[\partial_x u^+] + \mu R\Big[\frac{u^+}{\mu}\Big] = \mu R\Big[\frac{1}{\mu}K_+^+[u^+]\Big] + \mu R\Big[\frac{1}{\mu}K_-^+ R[u^+]\Big]. \qquad (11)$$

Let us define the operator R^* as

$$R^* = \mu R \frac{1}{\mu}. \qquad (12)$$

We will see that the operator R^* is the adjoint to R for the duality product $(f,g) = \int_0^1 f(\mu)g(\mu)d\mu$ in the case of symmetric scattering kernels, and anticipate this notation. Adding (10) to (11) yields

$$R + R^* = K_+^- + R^* K_+^+ + K_-^- R + R^* K_-^+ R. \qquad (13)$$

This is the Riccati equation for the operator R (since R^* is given by (12)).

Let us now show why the inverse problem is ill-posed in general. Consider a general scattering operator K_0; for instance the isotropic operator with kernel $k_0(\mu,\mu') \equiv 1/2$. The response operator R_0 exists and satisfies (13). Consider now a new scattering operator K_1 defined by

$$K_+^- = R_0 + R_0^*, \qquad K_+^+ = 0, \quad K_-^- = 0, \quad K_-^+ = 0.$$

We will see that R_0 can be written as an integral operator $R_0[g](\mu) = \int_0^1 r_0(\mu,\mu')g(\mu')d\mu'$. Therefore, the operator K_1 takes the form (3) with $k_1(-\mu,\mu') = r_0(\mu,\mu')(1+\mu/\mu')$ for $0 \le \mu, \mu' \le 1$ and $k_1 = 0$ otherwise. The corresponding response operator R_1 also satisfies (13); hence $R_0 + R_0^* = R_1 + R_1^*$ and therefore $R_0 = R_1$. So two clearly different scattering operators yield the same response operator. We deduce that the knowledge of the response operator is not sufficient in general to recover the scattering kernel.

Let us turn to the case of symmetric scattering kernels given by (5). The assumption is satisfied for most physical problems, with the notable exception of moving media (for instance the inverse Compton scattering; see [15]), for which other symmetries can be considered. Following (5), we define

$$K_+ \equiv K_+^+ = K_-^-, \qquad K_- \equiv K_-^+ = K_+^-. \tag{14}$$

The operators K_+ and K_- do not need to be equal. We have the new Riccati equation

$$R + R^* = K_- + R^* K_- R + R^* K_+ + K_+ R. \tag{15}$$

We will see that R can be recast as an integral operator

$$R[g](\mu) = \int_0^1 r(\mu,\mu')g(\mu')d\mu'. \tag{16}$$

Then (15) can be seen as an equation for the kernel $r(\mu,\nu)$:

$$\begin{aligned}\mu\frac{r(\mu,\nu)}{\nu} + r(\mu,\nu) = \int_0^1 \mu\frac{r(\mu,\mu')}{\mu'}k(\mu',\nu)d\mu' + \int_0^1 k(\mu,\mu')r(\mu',\nu)d\mu' \\ +k(\mu,-\nu) + \int_0^1\int_0^1 \mu\frac{r(\mu,\mu')}{\mu'}k(\mu',-\mu'')r(\mu'',\nu)d\mu'd\mu''.\end{aligned} \tag{17}$$

Denoting by $s(\mu,\nu) = \nu^{-1}r(\mu,\nu)$, we have

$$\begin{aligned}(\mu+\nu)s(\mu,\nu) = (\mu+\nu)\int_0^1 k(\mu,\mu')s(\mu',\nu)d\mu' + k(\mu,-\nu) \\ + \int_0^1\int_0^1 \mu\,\nu\, s(\mu,\mu')k(\mu',-\mu'')s(\mu'',\nu)d\mu'd\mu'',\end{aligned}$$

since $k(\mu,\mu') = k(\mu',\mu)$. Therefore, $s(\mu,\nu) = s(\nu,\mu)$, and R^* is the adjoint operator to R, with kernel $r^*(\mu,\nu) = r(\nu,\mu)$.

Let us consider the following example showing that the knowledge of R is not sufficient to reconstruct the scattering operator K. Let

$$k(\mu,\mu') = \frac{1}{2} + \frac{3}{2}\sigma_1\mu\mu'. \tag{18}$$

We assume that σ_1 is small enough that $k(\mu,\mu')$ is uniformly bounded from below by a positive constant. As we shall see below this condition ensures the well-posedness of the transport equation and the existence of the response operator R in $\mathcal{L}(L^1_\mu(0,1))$. Here we use the following notation: $\mathcal{L}(Y,Z)$ is the set of bounded linear operators from the Banach space Y to the Banach space Z and $\mathcal{L}(Y) = \mathcal{L}(Y,Y)$. We define the norm

$$\|g\|_\mu \equiv \|g\|_{L^1_\mu(0,1)} = \int_0^1 \mu|g(\mu)|d\mu. \tag{19}$$

We also define the spaces $L^p_\mu(0,1)$ of functions f such that $\mu f^p \in L^1(0,1)$ and the spaces $L^p_{1/\mu}(0,1)$ of functions f such that $f^p/\mu \in L^1(0,1)$. We will show that for every $g \in L^1_\mu(0,1)$, $R[g]$ exists and is in $L^1_\mu(0,1)$. By inspection we notice that $R[1] = 1$ and

$R^*[\mu] = \mu$. Moreover, the Riccati equation with the choice of scattering kernel (18) yields

$$\begin{aligned}(R^*+R)[u] &= \frac{1}{2}\Big[(R^*[1]+1)\Big(\int_0^1 u(\mu')d\mu' + \int_0^1 R[u](\mu')d\mu'\Big)\Big] \\ &+ \sigma_1\Big[(R^*[\mu]-\mu)\Big(\int_0^1 \mu u(\mu)d\mu - \int_0^1 \mu' R[u](\mu')d\mu'\Big)\Big].\end{aligned} \tag{20}$$

Since $R^*[\mu] = \mu$, the term involving σ_1 cancels. So the response operator R does not depend on σ_1. It is therefore obvious that the knowledge of R does not suffice to reconstruct the whole scattering operator K. Notice also that the range of $R + R^*$ is of rank 1 and spanned by $R^*[1] + 1$.

4. Kernels satisfying (5) and (6)

We shall come back later to the case of kernels satisfying only (5). Let us now assume that the scattering operator K satisfies *both* (5) and (6). A consequence is $K_+ = K_- = \tilde{K}$. So the Riccati equation simplifies to

$$R + R^* = (I + R^*)\tilde{K}(I + R), \tag{21}$$

where I is the identity operator, and $\tilde{K}$ is the operator with kernel $k(\mu,\mu')$ on $(0,1)^2$. According to (5) and (6), we can write $\tilde{K}$ as

$$\tilde{K} = \sum_{m=0}^{\infty} \sigma_{2m}\alpha_{2m} \otimes \alpha_{2m} = \alpha_e^t \Sigma_e \alpha_e \tag{22}$$

where we have denoted by $\alpha_e = (\alpha_0, \cdots, \alpha_{2k}, \cdots)^t$ and $\Sigma_e = Diag(\sigma_0, \cdots, \sigma_{2k}, \cdots)$. The notation means that $\tilde{K}[g](\mu) = \sum_{m=0}^{\infty} \sigma_{2m}\alpha_{2m}(\mu) \int_0^1 \alpha_{2m}(\mu')g(\mu')d\mu'$.

So in vector notation, we have

$$R + R^* = H^t\Sigma \otimes H, \qquad \text{where} \qquad H = (R^* + Id)[\alpha_e]. \tag{23}$$

The vector H collects the celebrated Chandrasekhar's functions [5]. We state the main result of this section.

Theorem 4.1 *We assume that the scattering operator K, given by its kernel $k(\mu,\mu')$ satisfies the following assumptions*

(i) K is bounded in $\mathcal{L}(L^2(-1,1)) \cap \mathcal{L}(L^\infty(-1,1))$ and satisfies (5).

(ii) $k(\mu,\mu')$ is uniformly bounded from below by a positive constant.

(iii) The medium is conservative or dissipative: $0 < \sigma_0 \le 1$.

Then the transport equation (1) is well-posed in $L^\infty(I\!R^+ \times [-1,1])$ for all $\Sigma > 0$. The response operator R defined by (4) is independent of Σ, and belongs to $\mathcal{L}(L^\infty(0,1)) \cap \mathcal{L}(L^1_\mu(0,1))$. The adjoint operator R^* defined by (12) belongs to $\mathcal{L}(L^\infty_{1/\mu}(0,1)) \cap \mathcal{L}(L^1(0,1))$.

Assume that K satisfies the additional symmetry condition (6). The inverse problem consisting in recovering the coefficients σ_m in (22) from the knowledge of the response *operator R is well-posed. The Riccati equation is well-posed in $\mathcal{L}(L^\infty(0,1), L^1(0,1))$. The coefficients σ_{2k} are given explicitly by*

$$\sigma_{2k} = 4\int_0^1 \alpha_{2k}(\mu)\Big((I+R^*)^{-1}(R+R^*)(I+R)^{-1}\Big)[\alpha_{2k}](\mu)d\mu. \tag{24}$$

Proof of Theorem 4.1. The proof of existence in the dissipative case when $\sigma_0 < 1$ is classical; see for instance [4]. The proof of existence when $\sigma_0 = 1$ can be seen in [16, 17, 18, 19]. We deduce from [16] that the transport problem is well-posed in $L^\infty(I\!R^+\times[-1,1])$ and the response operator is well-defined in $\mathcal{L}(L^\infty(0,1))$. The response operator is then extended as an operator in $\mathcal{L}(L^p_\mu(0,1))$ for $1 \le p < \infty$ [19].

The operator $R^* = \mu R \mu^{-1}$ is well-defined in $\mathcal{L}(L^1(0,1))$. Indeed, for $g \in L^1(0,1)$, we have $\int_0^1 |\mu R[\mu^{-1}g]|d\mu < \infty$ since $R \in \mathcal{L}(L^1_\mu(0,1))$. We also check that

$$\int_0^1 \frac{1}{\mu}R^*[f](\mu)g(\mu)d\mu = \int_0^1 \frac{1}{\mu}f(\mu)R^*[g](\mu)d\mu.$$

So R^* belongs to $\mathcal{L}(L^\infty_{1/\mu}(0,1))$.

The response operator is obviously translation invariant. This justifies the derivation of the Riccati equation (21) in $\mathcal{L}(L^\infty(0,1), L^1(0,1))$. To obtain $\tilde{K}$ it remains to show that $I + R^*$ and $I + R$ have bounded inverse operators in $\mathcal{L}(L^1(0,1))$ and $\mathcal{L}(L^\infty(0,1))$, respectively. This is done in the following lemma, whose proof is postponed to the end of this section.

Lemma 4.2 *Assume that Hypotheses $1-3$ of Theorem 4.1 hold. Then the operators $I+R^*$ and $I+R$ are invertible in $\mathcal{L}(L^1(0,1))$ and $\mathcal{L}(L^\infty(0,1))$, respectively, with bounded inverse operators.*

So, we get from (21) in $\mathcal{L}(L^\infty(0,1), L^1(0,1))$ that

$$\tilde{K} = (I+R^*)^{-1}(R+R^*)(I+R)^{-1}.$$

We deduce from the diagonalization (22) that

$$\tilde{K}[\alpha_{2k}](\mu) = \sigma_{2k}\alpha_{2k}(\mu),$$

since $\int_0^1 \alpha_{2k}(\mu)\alpha_{2l}(\mu)d\mu = (1/2)\delta_{(2k)(2l)}$ as the basis functions of even index also form an orthogonal family on $(0,1)$. The latter equality also yields (24). $\square$

Before giving a proof of Lemma 4.2, we need to introduce the so-called Doeblin condition [19, 20, 21], which we now state and prove.

Lemma 4.3 *There exists a positive constant $\delta_0 > 0$ such that for all positive functions $h \in L^1_\mu[0,1]$ we have*

$$R[h] \ge \delta_0 \int_0^1 \mu h(\mu)d\mu \qquad in\ [-1,0]. \tag{25}$$

Proof. We use the integral formulation of the transport equation (1), given by

$$\begin{aligned} u(x,\mu) &= \int_0^x ds\frac{1}{\mu}e^{-s/\mu}\int_{-1}^1 d\mu' k(\mu,\mu')u(x-s,\mu') + e^{-x/\mu}g(0,\mu) \quad && \mu > 0, \\ &= \int_0^\infty ds\frac{1}{|\mu|}e^{-s/|\mu|}\int_{-1}^1 d\mu' k(\mu,\mu')u(x+s,\mu') && \mu < 0. \end{aligned}$$

We define the zero scattering term

$$u_0(x,\mu) = e^{-x/\mu} g(0,\mu) \qquad \text{for } \mu > 0,\ x > 0$$

and $u_0(x,\mu) = 0$ elsewhere. The single scattering distribution is defined by

$$\begin{aligned} u_1(x,\mu) &= \int_0^x ds \frac{1}{\mu} e^{-s/\mu} \int_{-1}^1 d\mu' k(\mu,\mu') u_0(x-s,\mu') + e^{-x/\mu} g(0,\mu) \quad && \mu > 0, \\ &= \int_0^\infty ds \frac{1}{|\mu|} e^{-s/|\mu|} \int_{-1}^1 d\mu' k(\mu,\mu') u_0(x+s,\mu') && \mu < 0. \end{aligned}$$

Then we have that $u \geq u_1 \geq u_0$ in $\mathbb{R}^+ \times [-1,1]$ since $g(0,\mu) \geq 0$. Further, we get for $\mu < 0$,

$$\begin{aligned} u_1(0,\mu) &= \int_0^\infty ds \frac{1}{|\mu|} e^{-s/|\mu|} \int_0^1 d\mu' k(\mu,\mu') e^{-s/\mu'} g(0,\mu') \\ &= \int_0^1 d\mu' k(\mu,\mu') \frac{\mu'}{|\mu|+\mu'} g(0,\mu') \geq \frac{\gamma_1}{2} ||g||_{L^1_\mu[0,1]}, \end{aligned} \tag{26}$$

with γ_1 obtained from Hypothesis 2 of Theorem 4.1. This proves the Doeblin condition (25) for the response operator. □

Proof of Lemma 4.2. Consider $\sigma_0 = 1$, which corresponds to the conservative case. As in the proof of the theorem, we obtain that R and R^* are in $\mathcal{L}(L^\infty(0,1))$. The solution of (1) converges exponentially to a constant when $x \to \infty$ [16, 19]. So integrating (1) over X yields the conservation of flux

$$\int_0^1 \mu R[g](\mu) d\mu = \int_0^1 \mu g(\mu) d\mu.$$

Splitting g into positive and negative parts, $g = g^+ - g^-$, where $g^+ = \sup(0,g)$ and $g^- = \sup(0,-g)$, and recalling that $\|g\|_\mu = \|g^+\|_\mu + \|g^-\|_\mu$, we see that $\|R\|_\mu \leq 1$ and actually $\|R\|_\mu = 1$ since $R[1] = 1$. When $\sigma_0 < 1$, the outgoing flux is less than the incoming flux due to absorption and $\|R\|_\mu \leq 1$ too. It is then not difficult to prove that $\|R\|_\mu < 1$, even though we do not need such a result here. We now need the following additional information on the response operator: the Doeblin condition (25) given in Lemma 4.3. We then write

$$I + R = (I + R_0) + (R - R_0) = (I + R_0)[I + (I + R_0)^{-1}(R - R_0)], \tag{27}$$

where $R_0 = \delta_0 \int \mu\cdot$. Here δ_0 is given by Lemma 4.3, or is chosen small enough that $I + R_0$ is invertible. We easily deduce from $(I + R_0)^{-1} = \sum_{m=0}^\infty (-1)^m R_0^m$, that

$$(I + R_0)^{-1} h = h - \frac{\delta_0}{1 + \delta_0/2} \int_0^1 \mu h(\mu) d\mu, \qquad \text{hence} \qquad \|(I + R_0)^{-1} h\|_\mu \leq \frac{1}{1 + \delta_0/2} \|h\|_\mu,$$

since it is clearly true for $h \geq 0$, hence for every $h = h^+ - h^-$. We deduce from Lemma 4.3 that $(R - R_0)$ maps positive functions to positive functions. Therefore, if h^+ is nonnegative, $(R - R_0)h^+$ is also nonnegative, and

$$\begin{aligned} &(1 + \frac{\delta_0}{2}) \|(I + R_0)^{-1}(R - R_0)h^+\|_\mu \leq \|(R - R_0)h^+\|_\mu \\ &= \int_0^1 \mu(R - R_0)h^+ = \int_0^1 \mu R[h^+] - \frac{\delta_0}{2} \int_0^1 \mu h^+ = (1 - \frac{\delta_0}{2}) \|h^+\|_\mu. \end{aligned}$$

So for $h \geq 0$, $\|(I+R_0)^{-1}(R-R_0)h\|_\mu \leq (1-\varepsilon)\|h\|_\mu$ for some $\varepsilon > 0$ independent of h. For any function $h \in L^1_\mu(0,1)$, we write $h = h^+ - h^-$. We deduce from the triangle inequality that

$$\|(I+R_0)^{-1}(R-R_0)h\|_\mu < (1-\epsilon)\|h\|_\mu.$$

This shows that $I + (I+R_0)^{-1}(R-R_0)$ is invertible in $\mathcal{L}(L^1_\mu(0,1))$. Since $I+R_0$ is also invertible, we deduce from (27) that $I+R$ is invertible in $\mathcal{L}(L^1_\mu(0,1))$.

For the duality product $\langle f, g\rangle = \int_0^1 \mu f(\mu) g(\mu) d\mu$, we have $\langle R[f], g\rangle = \langle f, R[g]\rangle$. So the adjoint for this duality product of $(I+R)^{-1} \in \mathcal{L}(L^1_\mu(0,1))$ is $(I+R)^{-1} \in \mathcal{L}(L^\infty(0,1))$. This shows that $(I+R)$ has bounded inverse operator in $\mathcal{L}(L^\infty(0,1))$. It remains to check that $(I+R^*)^{-1} = \mu^{-1}(I+R)^{-1}\mu$, which is well-defined in $\mathcal{L}(L^1(0,1))$. □

5. Numerical application

We consider in this section numerical simulations for the reconstruction (24) in the case of the unpolarized approximation to Rayleigh scattering, or more specifically to the mixing of Rayleigh and isotropic scattering. This model is motivated by the study of the scattering by anisotropic particles [5, §18].

We assume that the medium is conservative, so $\sigma_0 = 1$. Rayleigh scattering is characterized by the next term $\sigma_2 = 1/10$. All higher orders σ_{2k} for $k \geq 2$ vanish.

The first Legendre polynomials are given by

$$\alpha_0(\mu) = \frac{1}{\sqrt{2}}, \qquad \alpha_1(\mu) = \sqrt{\frac{3}{2}}\,\mu, \qquad \alpha_2(\mu) = \sqrt{\frac{5}{8}}\,(3\mu^2 - 1). \tag{28}$$

We then deduce from (5) that Rayleigh scattering is given by

$$k(\mu, \mu') = \alpha_0(\mu)\alpha_0(\mu') + \frac{1}{10}\alpha_2(\mu)\alpha_2(\mu'), \tag{29}$$

i.e., $\sigma_0 = 1$ and $\sigma_2 = 1/10$. The more general scattering operator we consider is θ *Rayleigh* $+ (1-\theta)$ *isotrope*, given by

$$\sigma_0 = 1, \qquad \sigma_2 = \frac{\theta}{10}, \tag{30}$$

for $0 \leq \theta \leq 1$.

Using (23), we find the following system for H_0 and H_2:

$$\begin{aligned} H_0(\mu) &= \alpha_0 + \int_0^1 \frac{\mu}{\mu+\nu}\alpha_0[H_0(\mu)H_0(\nu) + \sigma_2 H_2(\mu)H_2(\nu)]d\nu, \\ H_2(\mu) &= \alpha_2(\mu) + \int_0^1 \frac{\mu}{\mu+\nu}\alpha_2(\nu)[H_0(\mu)H_0(\nu) + \sigma_2 H_2(\mu)H_2(\nu)]d\nu. \end{aligned} \tag{31}$$

We solve this system numerically, replacing the integration by a numerical summation based on a uniform discretization of $(0,1)$. Once the functions H_0 and H_2 are known, we obtain the full response operator from (23).

Let us introduce the adjoint Chandrasekhar functions

$$H^*_{2k}(\mu) = (I+R)^{-1}[\alpha_{2k}](\mu). \tag{32}$$

Since $(I + R)$ is invertible, these functions are well-defined. Moreover, we readily show that

$$2\int_0^1 H_{2k}^*(\mu)H_{2l}(\mu) = \delta_{kl}.$$

We recast (24) as

$$\sigma_{2k} = 4\int_0^1 H_{2k}^*(\mu)(R+R^*)[H_{2k}^*](\mu)d\mu = 4(H_{2k}^*, (R+R^*)[H_{2k}^*]), \qquad (33)$$

where $(\cdot,\cdot)$ is the L^2 scalar product on $(0,1)$.

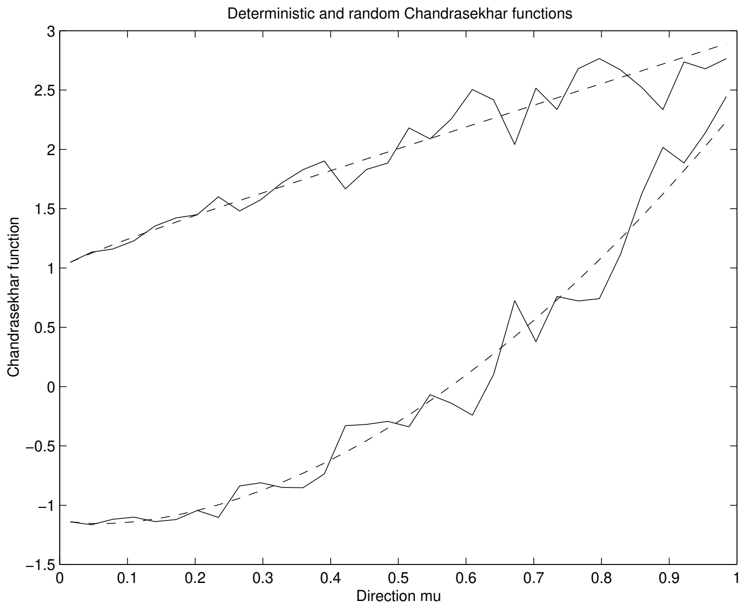

Figure 1. Deterministic (dashed lines) and random (solid lines; one realization) Chandrasekhar functions $\sqrt{2}H_0$ (upper curves) and $\sqrt{2}H_2$ (lower curves).

The polynomial α_4 is given by $\alpha_4(\mu) = 105/(8\sqrt{2})(\mu^4 - (6/7)\mu^2 + 3/35)$. So we can compute H_4^* and check that $\sigma_4 = 0$. Considering a uniform discretization of $(0,1)$ of $N = 32$ points, we have obtained in the case $\theta = 0.5$ the numerical reconstruction

$$\sigma_0^r = 1, \qquad \sigma_2^r = 4.99\,10^{-2}, \qquad \sigma_4^r = 2\,10^{-6}, \qquad \theta^r = 0.499.$$

This numerical simulation merely measures the accuracy of the utilized quadrature rule. Still, we observe that the inversion is accurate. More interesting is the behavior of the inversion when noise is added. In the absence of noise, the inversion consists in L^2 projecting the response operator onto the set spanned by the exact functions $H_0 \otimes H_0$ and $H_2 \otimes H_2$. When additive noise is turned on, the algorithm projects the response operator onto the set spanned by the noisy functions $H_0^n \otimes H_0^n$ and $H_2^n \otimes H_2^n$. The algorithm is clearly non linear in the noise; but remains stable numerically.

The discretized response operator R is a $N \times N$ matrix. We add an additive noise of the form $W_{ij}(\omega) = \tau_n w(\omega)/N$, where $w(\omega)$ is a random variable uniformly distributed on $(-1/2, 1/2)$, and τ_n is the amplitude of the noise:

$$R_n = R + \tau_n W. \qquad (34)$$

The noise could be added in a symmetric way so that $\mu^{-1}R_n$ remains symmetric. We do not impose this constraint. Once R_n is constructed, we compute the noisy functions $H_k^n(\mu)$ using (23) and the adjoint functions H_k^{n*} using (32), where R is replaced by R_n. We then use (33) to reconstruct the scattering cross sections.

We consider the numerical inversion with a rate of noise $\tau_n = 0.5$. This corresponds to a fairly large amount of noise since the coefficients of $\tau_n W$ are of the same order as those of R. With this amplitude of noise, many coefficients of the matrix R_n are negative. We do not consider any filtering of the noisy data here before applying the algorithm. The results are collected in Table 1. The Chandrasekhar functions (multiplied by $\sqrt{2}$ to

Table 1. Mean and standard deviation of the reconstructed coefficients σ_0, σ_2, and σ_4 for a level of noise $\tau_n = 0.5$

Reconstruction	σ_0	σ_2	σ_4	θ
exact	1	0.05	0	0.5
mean	1.0018	0.0514	0.0016	0.5136
standard deviation	0.0310	0.0046	0.0022	0.0460

be consistent with reference [5]) of the deterministic case and of one realization of the noisy problem are given in Fig. 1. We see that the function H_k^n are still fairly noisy, as are the adjoint functions H_k^{n*} (not represented). Nonetheless, the reconstruction of the scattering coefficients σ_k is good and stable. Notice that since the scattering operator is given explicitly by $\tilde{K} = (I+R^*)^{-1}(R+R^*)(I+R)^{-1}$ careful analysis of the reconstructed kernel with adapted filtering of the noisy data can be performed. We do not dwell on this issue here.

6. Kernels satisfying only the symmetry (5)

Let us return to the case of symmetric kernels that do not satisfy (6). According to (5) we have

$$K = \sum_{m=0}^{\infty} \sigma_m \alpha_m \otimes \alpha_m = \alpha^t \Sigma \otimes \alpha, \tag{35}$$

where $\alpha = (\alpha_0, \ldots, \alpha_k, \ldots)$ and $\Sigma = (\sigma_0, \ldots, \sigma_k, \ldots)$. The basis functions satisfy that α_{2m} is even and α_{2m+1} is odd for every $m \geq 0$. Therefore, we can decompose the scattering operator as

$$K = K_e + K_o,$$

where

$$\begin{aligned} K_e &= \sum_{m=0}^{\infty} \sigma_{2m}\alpha_{2m}\otimes\alpha_{2m} = \alpha_e^t\Sigma_e\alpha_e \\ K_o &= \sum_{m=0}^{\infty} \sigma_{2m+1}\alpha_{2m+1}\otimes\alpha_{2m+1} = \alpha_o^t\Sigma_o\alpha_o. \end{aligned} \tag{36}$$

With this notation we check that

$$K_+ = K_e + K_o \qquad \text{and} \qquad K_- = K_e - K_o.$$

So we can rearrange the Riccati equation (15) to obtain

$$R + R^* = (R^*+I)K_e(R+I) - (R^*-I)K_o(R-I). \tag{37}$$

We have seen that the inverse problem, consisting in recovering K from R, is not always well-posed. In the conservative case ($\sigma_0 = 1$), it is not possible to recover σ_1 since it does not appear in the Riccati equation. This counter-example however is deceptive and peculiar since μ is the only eigenvector such that $R^*[\mu] = \mu$ (the Doeblin condition in Lemma 4.3 can be used to show that 1 is the only possible eigenvector such that $R[h] = h$).

Nevertheless, the ill-posedness goes beyond the non recovery of σ_1. In general situations, the inverse problem (37) does not admit a unique solution. Indeed, let K_e and K_o be given and non trivial, yielding (37). Since (R^*+I) and $(R+I)$ are invertible, a possible solution to (37) is then

$$\tilde{K}_e = (R^*+I)^{-1}(R+R^*)(R+I)^{-1}, \qquad \text{and} \qquad \tilde{K}_o = 0. \tag{38}$$

The choice of these scattering kernels will yield the same Riccati equation and the same response operator R as K_e and K_o. In the dissipative case, where it is known that R and R^* are strict contractions, (R^*-I) and $(R-I)$ are invertible. So a third solution would be

$$\hat{K}_o = -(R^*-I)^{-1}(R+R^*)(R-I)^{-1}, \qquad \text{and} \qquad \hat{K}_e = 0. \tag{39}$$

The latter kernel $\hat{K}_o$ is obviously not positive and could be ruled out for that reason. However $\tilde{K}_e$ in (38) is positive (in the sense that $g \geq 0$ implies $\tilde{K}_e[g] \geq 0$) and can be extended as an operator on $(-1,1)$ that satisfies (5). These scattering operators may not belong to $\mathcal{L}(L^\infty(0,1))$, but they are well-defined and symmetric so (36) is satisfied, possibly with a different basis $\{\alpha_m\}$ than for K. Further, the problem of the space of definition of these operators becomes irrelevant once the problems are discretized. So every discretized inverse problem admits these three solutions, two of them being positive hence admissible operators.

We have to impose some restrictions on the scattering operators to have a chance to solve the inverse problem. Let us introduce the Chandrasekhar functions

$$X_{em} = (R^*+I)[\alpha_{2m|(0,1)}], \qquad \text{and} \qquad X_{om} = (R^*-I)[\alpha_{2m+1|(0,1)}]. \tag{40}$$

The Riccati equation (37) has the form

$$R + R^* = X_e^t\,\Sigma_e \otimes X_e - X_o^t\,\Sigma_o \otimes X_o. \tag{41}$$

We have seen that we cannot reconstruct the scattering operator uniquely in general. However, if we assume that the number of non vanishing terms in Σ_e and Σ_o are finite, we can expect to invert the problem. Let us make the following hypotheses: Let M be such that $\sigma_{em} = 0$ and $\sigma_{om} = 0$ for $m > M$ and let $X_M = \{X_{em}, X_{om};\ 0 \leq m \leq M\}$ be a family of *linearly independent* functions.

Then we can construct an adjoint family X_M^* such that for $x_m^* \in X_M^*$,

$$\int_0^1 x_m^* x_n d\mu = \delta_{mn} \qquad \text{for every} \qquad x_n \in X_M.$$

Defining now $\Sigma = \{\Sigma_e, -\Sigma_o\}$, we obtain that

$$\sigma_m = \int_0^1 x_m^*(\mu)(R + R^*)[x_m^*](\mu) d\mu \qquad \text{for every} \qquad x_n \in X_M. \tag{42}$$

It is clear that the inverse problem is well-posed. Since X_M is a family of finitely many linearly independent functions, the calculation of the adjoint family X_M^* is a well-posed problem. Notice however that this problem becomes more and more ill-conditioned as the number of terms to recover increases.

7. One-dimensional time dependent problem

In this section we consider the time dependent one-dimensional homogeneous half-space transport problem. Our model problem is given in $X \times [0, T]$ by

$$\begin{aligned}
&\frac{\partial}{\partial t} u(x, \mu, t) + \mu \frac{\partial}{\partial x} u(x, \mu, t) + \Sigma u(x, \mu, t) = \Sigma K[u(x, \cdot, t)](\mu), \\
&u(0, \mu, t) = g(\mu)\delta_0(t), \qquad \text{for} \quad \mu > 0, \quad t \geq 0, \\
&u(x, \mu, 0) = 0, \qquad \text{for} \quad x > 0, \quad \mu > 0.
\end{aligned} \tag{43}$$

The response operator, still denoted by R, now maps the incoming boundary condition at time $t = 0$ given by $g(\mu)$ for $\mu > 0$ to the outgoing distribution $u(0, -\mu, t)$ for $\mu > 0$ and $t \geq 0$:

$$R[g](\mu, t) = u(0, -\mu, t) \qquad \text{for} \quad \mu > 0 \quad \text{and } t \geq 0. \tag{44}$$

We deal with symmetric scattering kernels of the form (35) to simplify. The inverse problem still consists in reconstructing the coefficients in (35) from the knowledge of the response operator R. Since we assume that the medium is time independent, we now have an overdetermined problem. We will see that the inverse problem is now well-posed independently of the symmetry condition (6) and that we can also reconstruct the total cross section Σ. Assuming that the scattering kernel is positive, the problem (43) is well-posed [4].

We will again use the invariant imbedding technique. The response is still invariant by translation in space, and now also invariant by translation in time. Thus, the response at time t of a delta function at time 0 is the same as the response at time $t + \tau$ of a delta function at time τ for every real number τ. With these two symmetries, we can use the same techniques as in the previous sections to obtain a closed time dependent

Riccati equation for the response operator R. We shall proceed in a slightly different manner and work in a dual space by first applying the Laplace transform to the transport equation.

The Laplace transform is given by

$$\hat{u}(s) = \int_0^\infty e^{-ts} u(t)dt. \tag{45}$$

An obvious property is that

$$\widehat{u'}(s) = s\hat{u}(s) + u(0). \tag{46}$$

Applying the Laplace transform to (43) yields

$$(s+\Sigma)\hat{u}(x,\mu,s) + \mu\frac{\partial}{\partial x}\hat{u}(x,\mu,s) = \Sigma K[\hat{u}(x,\cdot,s)](\mu), \tag{47}$$

on the domain $\mathbb{R}\times[-1,1]\times\mathbb{R}^+$. Applying further the Laplace transform to the boundary conditions yields the boundary value problem

$$\begin{aligned} &(s+\Sigma)\hat{u}(x,\mu,s) + \mu\frac{\partial}{\partial x}\hat{u}(x,\mu,s) = \Sigma K[\hat{u}(x,\cdot,s)](\mu), \\ &\hat{u}(0,\mu,s) = g(\mu), \qquad \text{for} \quad \mu > 0. \end{aligned} \tag{48}$$

So the time dependent transport problem is replaced by the family of transport problems (48) indexed by the dual variable s. The inversion of this problem by inverse Laplace transform is notoriously ill-posed. However we will not need such an inversion here.

It is straightforward to check that the Laplace transform of the response operator (44) is given by the response operator associated with the transport equation (48). Therefore, we introduce the family of response operators R_s indexed by the dual variable s that map $g(\mu)$ to the outgoing distribution $\hat{u}(0,-\mu,s)$ of the transport problem (48). Proceeding as in the previous section, we readily obtain the following generalization to (15):

$$\frac{\Sigma+s}{\Sigma}(R_s + R_s^*) = K_- + R_s^* K_- R_s + R_s^* K_+ + K_+ R_s. \tag{49}$$

A classical technique to show that the scattering operator can be reconstructed from boundary measurements in the steady-state case consists in splitting the solution into three parts: ballistic part, single scattering, and multiple scattering [22]. In one-dimensional half space geometry, the ballistic part cannot be measured as it propagates away from the boundary. In addition, single scattering, double scattering, and higher order scattering cannot be differentiated from each other according to their regularity as singularities, which help differentiate them in three-dimensional geometries, also propagate away from the boundary. However, the time dependent information allows us to discriminate among the different orders of scattering. The reason is that single scattering is dominant for early times, double scattering is a first order correction, and so on. The early time information is contained in the modes of large dual variable s. So we perform an asymptotic expansion of (49) as $s \to \infty$. At time $t = 0$, the response of the medium is zero, so we consider the Ansatz

$$sR_s = R_1 + \frac{1}{s}R_2 + o\Big(\frac{1}{s}\Big). \tag{50}$$

It is straightforward to check that in the time domain, R_1 corresponds to single scattering and R_2 corresponds to double scattering. Plugging this Ansatz into (49) and equating like powers of s^{-1} yields

$$\begin{aligned} R_1 + R_1^* &= \Sigma K_- \\ \Sigma(R_1 + R_1^*) + (R_2 + R_2^*) &= R_1^* \Sigma K_+ + \Sigma K_+ R_1. \end{aligned} \tag{51}$$

The first equation allows us to obtain ΣK_-. Notice that it is equivalent to the expression of single scattering obtained in (26). According to (5) and (9), the operator ΣK_- is expressed in terms of the coefficients σ_m as

$$K_- = \sum_{m=0}^{\infty} (-1)^m \sigma_m \alpha_m \otimes \alpha_m. \tag{52}$$

So we can reconstruct $\Sigma\sigma_m$ as follows. The functions $\alpha_m(\mu)$ are orthogonal on $(-1,1)$ and not on $(0,1)$. So the α_m do not form an orthogonal basis of $L^2(0,1)$. We assume that they do form a basis of $L^2(0,1)$, which is true, for instance, when the functions α_m are the Legendre polynomials. Thus there exists a dual basis α_m^* such that

$$\int_0^1 \alpha_m^*(\mu)\alpha_n(\mu)d\mu = \delta_{mn}.$$

Since R_1 and R_1^* are adjoint for the L^2 scalar product, we obtain that

$$(-1)^m \Sigma\sigma_m = 2\int_0^1 \alpha_m^*(\mu) R_1[\alpha_m^*](\mu)d\mu. \tag{53}$$

We deduce from the second equation in (51) that

$$\int_0^1 \alpha_m^*(\mu) R_2[\alpha_m^*](\mu)d\mu = \Sigma(\sigma_m - 1)\int_0^1 \alpha_m^*(\mu) R_1[\alpha_m^*](\mu)d\mu. \tag{54}$$

This allows us to obtain Σ since $\Sigma\sigma_m$ is known. We state the following result.

Theorem 7.1 *Assume that the Hypotheses 1-3 of Theorem 4.1 hold and that the functions α_m form a basis of $L^2(0,1)$. Then the equation (43) admits a unique solution in $C^1(0,T;L^2(X))$. Its Laplace transform satisfies the equation (48). The results of Theorem 4.1 hold for the response operator R_s associated with the Laplace transform $\hat{u}(\cdot,\cdot,s)$. The response operators satisfy the asymptotic expansion (50).*

The inverse problem consisting in reconstructing the coefficients σ_m in (5) and the total cross section Σ from boundary measurements is well-posed. Moreover, denoting by

$$I_k = \int_0^1 R_k[1]d\mu \qquad \textit{for} \qquad k = 1, 2.$$

we have $I_1 > 0$ and

$$\Sigma = 2I_1 - \frac{I_2}{I_1}. \tag{55}$$

The coefficients σ_m are given by

$$\sigma_m = \frac{2(-1)^m}{\Sigma}\int_0^1 \alpha_m^*(\mu) R_1[\alpha_m^*](\mu)d\mu. \tag{56}$$

Proof. The existence result can be found in [4]. The asymptotic expansion (50) can be checked by hand, as was done for (26). Since the scattering kernel is assumed to be positive everywhere, the coefficient σ_0 is positive. We deduce from (53) that $\Sigma\sigma_0 = 2I_1$ which is also positive. The formula (55) is then easily deduced from (54). Thus, (53) can be recast as (56). □

It is not surprising that time dependent problems yield more information than steady state problems. It is also not surprising that we need to have recourse to the small time behavior of the response operator to reconstruct the scattering operator since the response operator for the steady state problem is ill-posed when only (5) is satisfied. We have seen however that the reconstruction is a well-posed problem when the additional symmetry (6) holds. In this case the small time behavior should not be necessary to reconstruct the scattering operator. This is confirmed by the following result

Corollary 7.2 *Assume that the hypotheses of Theorem 4.1 are satisfied. Then the results of Theorem 4.1 hold for the response operator R_s and the coefficients σ_{2k} satisfy for every $s \geq 0$*

$$\frac{\Sigma\sigma_{2k}}{\Sigma+s} = \frac{1}{4}\int_0^1 \alpha_{2k}(\mu)\Big((I+R_s^*)^{-1}(R_s+R_s^*)(I+R_s)^{-1}\Big)[\alpha_{2k}](\mu)d\mu. \qquad (57)$$

Therefore the coefficients σ_{2k} and the total cross-section Σ can be reconstructed from the knowledge of R_{s_1} and R_{s_2} for every pair of different positive numbers s_1 and s_2.

The proof is the same as for Theorem 4.1 using (49) with $K_- = K_+$ instead of (21).

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