Wave transport for a scalar model of the Love waves

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Abstract

We study transport for a scalar model of Love waves. These waves arise in the propagation of seismic waves whose energy is concentrated in the vicinity of the earth surface. We derive radiative transfer equations from first principles for the angularly resolved energy density of the Love waves in a simplified acoustic model. We consider a rough top surface with weak fluctuations at the scale of the wavelength. The transport equation accounts for the multiple scattering of the Love waves and their scattering into volume waves. We also analyze a diffusive regime when energy is universally distributed over various modes of the Love waves.

1 Introduction

Propagation of wave energy in random media in the regime of weak random fluctuations and long propagation distances relative to the wavelength has been analyzed using radiative transfer equations for more than thirty years [5, 6, 7, 8, 14]. A systematic and efficient way to obtain transport equations is presented in [16, 18]. The mathematics involved in this process is explored in [9, 19]. We also refer to the recent collection of papers on various aspects of wave propagation in random media [11].

The aforementioned works deal with the propagation of volume waves, when the underlying background is either slowly varying, or periodic on the fast scale [2]. Multiple scattering of surface waves on a randomly inhomogeneous surface was treated recently in [3] for scalar waves propagating on a flat surface with randomly varying negative impedance. The present paper addresses the transport regime for a scalar wave modeling the propagation of Love waves. Love waves are seismic waves whose energy density is primarily concentrated in a wave guide layer (the Earth crust) of relatively small thickness below the earth surface. Their energy density decays exponentially beneath this layer. Along with Rayleigh waves, they account for a large part of the energy of seismic waves measured at the earth surface after an earthquake. We refer to [1, 10, 15] for a more detailed description.

We consider here a scalar model to simplify the derivation of the transport equations. This model can also be used to describe the propagation of acoustic waves. The generalization to the full elasticity equations can be obtained as in [18] in a straightforward though tedious manner. The inhomogeneities leading to multiple scattering of the Love waves can be of several types. Here we assume that both the material compressibility and density are uniform to concentrate on the effects of the roughness of the upper surface (between the layer and the atmosphere), which has small variations about a flat surface with spatial frequency of oscillations comparable to that of the waves. Other sources of randomness can be treated by similar techniques.

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Love waves are dispersive, and there may exist several modes at the same frequency but with different moduli of the wavenumber. In a uniform layer with flat boundaries, these modes propagate independently from each other. In the presence of a rough surface, they may couple through surface scattering. This is the main effect of the rough surface on wave propagation. To analyze this scattering, we pass to the high frequency limit and obtain a family of two-dimensional coupled transport equations for angularly resolved energy densities $w_j(\mathbf{x}, \phi_{\mathbf{k}})$ at position $\mathbf{x} \in I\!R^2$ and in direction $\phi_{\mathbf{k}} \in (0, 2\pi)$. The index j labels the modes of the Love waves. The system of transport equations takes the form

$$\Omega'(k_j)\hat{\mathbf{k}} \cdot \nabla w_j(\mathbf{x}, \phi_{\mathbf{k}}) = \sum_m \int_0^{2\pi} \frac{d\phi_{\mathbf{p}}}{2\pi} \sigma_{jm}(\phi_{\mathbf{k}}, \phi_{\mathbf{p}})[w_m(\mathbf{x}, \phi_{\mathbf{p}}) - w_j(\mathbf{x}, \phi_{\mathbf{k}})] - M_j w_j(\mathbf{x}, \phi_{\mathbf{k}}). \tag{1}$$

The first term on the right side of (1) describes scattering from and into other modes of the Love waves and the second one accounts for the leakage of Love waves into volume waves at the lower interface. Volume waves do not return to the interface because the lower half-space is uniform. This produces a loss of energy of Love waves. The differential scattering cross-sections σ_{jm} and the leakage coefficients M_j are given explicitly by (24) and (25) below in terms of the power spectrum of the random surface. The function $\Omega(\omega, k)$ determines the dispersion relation for the Love waves and is given explicitly by (13). The wave numbers of the different modes are solutions of $\Omega(\omega, k) = 0$. We denoted $\Omega' = \partial\Omega/\partial k$. The total number of modes increases with the depth of the boundary layer.

We identify spectra of the random surface that nearly suppress leakage into volume waves. Then, propagation over long distances leads to a universal energy distribution among the modes independent of the roughness. More precisely, at a fixed frequency ω the ratio of energies of two modes corresponding to wave numbers k_j and k_m is given by

$$\frac{\mathcal{E}_j(\mathbf{x}; \omega)}{\mathcal{E}_m(\mathbf{x}; \omega)} = \frac{k_j|\Omega'(k_j)|}{k_m|\Omega'(k_m)|} \tag{2}$$

if the modes j and m are coupled. This is similar to the P-to-S energy equipartition of elastic waves studied in [17, 18, 20]. Identifying equipartition ratios for seismic waves is important since experimental seismic measurements are mostly done near the surface of the Earth. Therefore the role of the surface and the presence of Rayleigh and Love waves cannot be ignored. This is confirmed by a recent experimental study [13] that did not observe the bulk P-to-S ratio but still confirms the universality of the P-to-S energy ratio.

Equipartition (2) applies to modes that are strongly coupled by scattering. We analyze the diffusion limit when the modes are split into several groups. Modes inside each group are strongly coupled to each other and weakly coupled to the modes of the other groups. Inside each group, we have the energy equipartition (2). Moreover, energy densities become approximately independent of the direction of propagation. Energy exchanges among the groups are described by the system of diffusion equations (37) for the energy densities of each group.

The paper is organized as follows. Equations for the Love waves and the dispersion relation are discussed in section 2. Transport equations are derived in section 3. Diffusion limit and equipartition are discussed in section 4.

2 Equations for the Love waves

2.1 The dispersion relation

As a first approximation, we replace the equations of elasticity by the acoustics equations for the pressure field

$$\kappa \frac{\partial^2 \phi}{\partial t^2} - \nabla \cdot \frac{1}{\rho} \nabla \phi = 0, \tag{3}$$

with appropriate initial and boundary conditions. Here κ is the compressibility and ρ the density of the underlying medium. The Love waves (in the acoustic case) arise as follows. Let us assume that there exists a layer $0 \le z \le H$, with H the coordinate of the Earth surface, where the density and the compressibility take the (constant) values κ_B and ρ_B, respectively, different from the (also constant) values κ and ρ taken in the half space $z \le 0$. The propagation velocities in the layer and the lower half space are then given by

$$c_B = \frac{1}{\sqrt{\kappa_B \rho_B}} \quad \text{and} \quad c = \frac{1}{\sqrt{\kappa \rho}}, \tag{4}$$

respectively. The time-harmonic version of (3) with Neumann boundary conditions at the upper surface is obtained by replacing ϕ by $\exp(i\omega t)\phi$. It is given by

$$\begin{aligned}
&\Delta \phi^B + \frac{\omega^2}{c_B^2} \phi^B = 0, && 0 < z < H, \\
&\Delta \phi + \frac{\omega^2}{c^2} \phi = 0, && z < 0, \\
&\phi = \phi^B, \quad \mu \frac{\partial \phi}{\partial z} = \mu_B \frac{\partial \phi^B}{\partial z}, && z = 0 \\
&\frac{\partial \phi^B}{\partial n} = 0, && z = H
\end{aligned} \tag{5}$$

with $\mu = \rho^{-1}$. We denote the horizontal variables by $\mathbf{x} = (x, y)$ and the vertical variable by z.

The planar Love waves are solutions of (5) of the form

$$\phi = e^{i\mathbf{k}\cdot\mathbf{x}} e^{k_z z}, \qquad \phi^B = e^{i\mathbf{k}^B\cdot\mathbf{x}} (A \cos(k_z^B z) + B \sin(k_z^B z)). \tag{6}$$

Here $k_z > 0$ so that the wave energy decays exponentially as $z \to -\infty$. Such a solution corresponds to a wave trapped in the layer $0 \le z \le H$. Non-trivial modes exist for certain values of the physical parameters. We insert expressions (6) into (5) and get

$$\begin{aligned}
&\mathbf{k} = \mathbf{k}^B \\
&A = 1 \qquad \text{and} \qquad B = \tan(k_z^B) \\
&\mu_B k_z^B \tan(k_z^B H) = \mu k_z \\
&\omega^2 = (k^2 - k_z^2) c^2 = ((k_z^B)^2 + k^2) c_B^2.
\end{aligned} \tag{7}$$

Here, we have defined $k = |\mathbf{k}|$. We deduce from the last relation in (7) the dispersion relation for the Love waves

$$\Big(\frac{1}{c_B^2} - \frac{1}{c^2}\Big)\omega^2 = \left(k_z^B\right)^2 \left(1 + \frac{\mu_B^2}{\mu^2}(\tan^2\left(k_z^B H\right))\right), \quad (k_z^B)^2 + k^2 = \frac{\omega^2}{c_B^2}. \tag{8}$$

This implies that $c > c_B$ is necessary for the existence of a solution to (7). We assume from now on that this constraint holds. Depending on the value of H, one or more solutions k_z^B to the above equation can be found. The number of solutions is however always finite. Once k_z^B has been obtained, k_z and k can be calculated from (7). Notice that k as a function of ω, or inversely ω as a function of k, is nonlinear. Therefore the Love waves are dispersive. In the limit of small values of H however, the dispersion relation is $\omega \approx ck$. The waves are almost non-dispersive, but their support is large as the decay rate k_z is of order H. In the limit of large H, we easily obtain (see also section 4) that the number of modes is proportional to H. The solutions of (8) are represented graphically in Figure 1 for different values of H and μ_B/μ. The layer thickness H mostly determines the number of modes

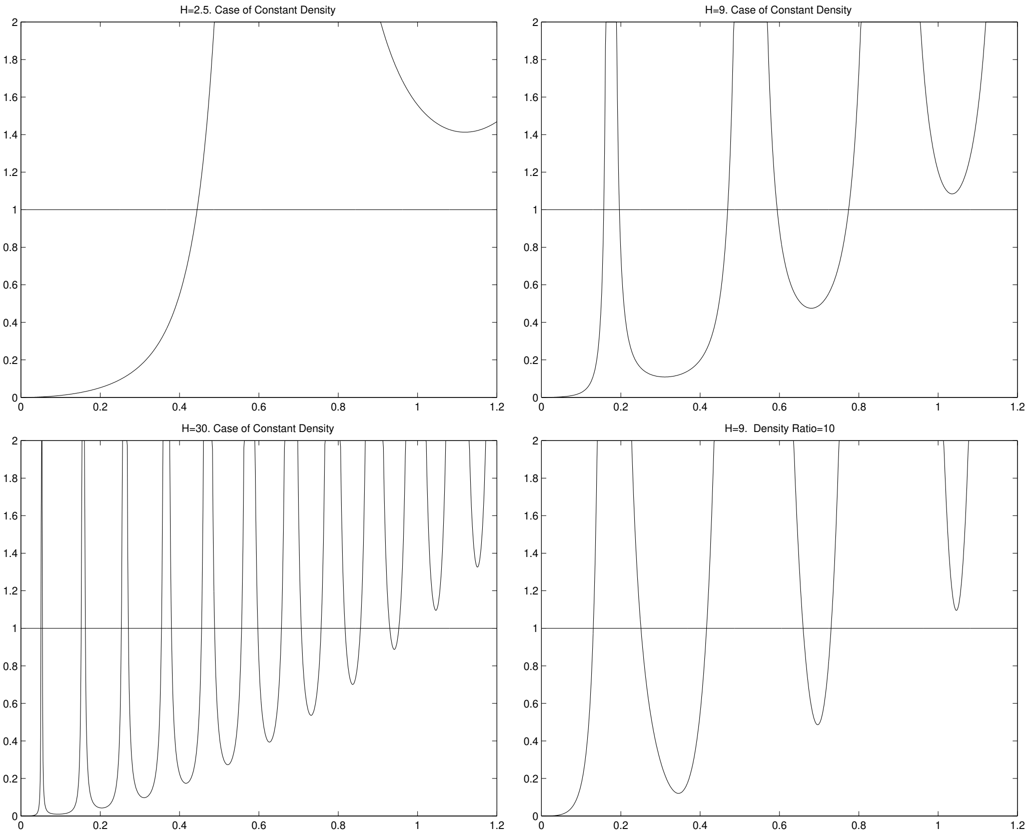

Figure 1: Graphical representation of the solutions of (8) for the frequency such that $(c_B^{-2} - c^{-2})\omega^2 = 1$. The solutions are represented as the intersections of the straight line with the curved line. Upper left: $H = 2.5$ and $\mu_B = \mu$. Upper right: $H = 9$ and $\mu_B = \mu$. Lower left: $H = 30$ and $\mu_B = \mu$. Lower right: $H = 9$ and $\mu_B = 10\mu$.

in the waveguide for a fixed frequency ω and wave speeds c and c_B. In particular only one mode (all propagating waves have the same modulus of the wavenumber) exists in the waveguide for H sufficiently small. The ratio μ_B/μ mostly effects not the number of solutions of (8) but rather their wavenumber.

2.2 Equation for the Love wave under a rough surface

Assume now that the upper surface is weakly rough: $H \to H + \sqrt{\varepsilon} h(\mathbf{x})$, with $\varepsilon \ll 1$. Here $h(\mathbf{x})$ is a spatially homogeneous mean zero random process with correlation function $R(\mathbf{x})$ and power spectrum $\hat{R}(\mathbf{k})$:

$$\langle h(\mathbf{x}) \rangle = 0, \quad \langle h(\mathbf{x}) h(\mathbf{y}) \rangle = R(\mathbf{x} - \mathbf{y}), \quad \langle \hat{h}(\mathbf{p}) \hat{h}(\mathbf{q}) \rangle = (2\pi)^2 \hat{R}(\mathbf{p}) \delta(\mathbf{p} + \mathbf{q}).$$

The effects of the weak roughness will be of order one only for distances much larger than the correlation length of $h(\mathbf{x})$. Therefore we rescale the spatial variables: $\mathbf{x} \to \mathbf{x}/\varepsilon$, $z \to z/\varepsilon$. The problem (5) becomes then

$$
\begin{aligned}
&\varepsilon^2 \Delta \phi_\varepsilon^B + \frac{\omega^2}{c_B^2} \phi_\varepsilon^B = 0, && 0 < z < \varepsilon H + \varepsilon^{3/2} h(\frac{\mathbf{x}}{\varepsilon}), \\
&\varepsilon^2 \Delta \phi_\varepsilon + \frac{\omega^2}{c^2} \phi_\varepsilon = 0, && z < 0, \\
&\phi_\varepsilon = \phi_\varepsilon^B, \quad \mu \frac{\partial \phi_\varepsilon}{\partial z} = \mu_B \frac{\partial \phi_\varepsilon^B}{\partial z}, && z = 0 \\
&\frac{\partial \phi_\varepsilon^B}{\partial n} = 0, && z = \varepsilon H + \varepsilon^{3/2} h(\frac{\mathbf{x}}{\varepsilon}).
\end{aligned}
\tag{9}
$$

The Neumann condition at the rough surface in (9) can be recast more explicitly as

$$
\frac{\partial \phi_\varepsilon^B}{\partial z} - \sqrt{\varepsilon} \nabla h(\frac{\mathbf{x}}{\varepsilon}) \cdot \nabla_{\mathbf{x}} \phi_\varepsilon^B = 0, \quad z = \varepsilon H + \varepsilon^{3/2} h(\frac{\mathbf{x}}{\varepsilon}).
$$

Let us now introduce the plane wave decomposition inside the wave guide

$$
\phi_\varepsilon^B(\mathbf{x}, z) = \int \frac{d\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{x}/\varepsilon} \left[A_\varepsilon(\mathbf{k}) e^{ik_n^B z/\varepsilon} + B_\varepsilon(\mathbf{k}) e^{-ik_n^B z/\varepsilon} \right], \quad 0 \le z \le \varepsilon H + \varepsilon^{3/2} h(\frac{\mathbf{x}}{\varepsilon}) \tag{10}
$$

with

$$
k_n^B = \left(\frac{\omega^2}{c_B^2} - k^2 \right)^{1/2}.
$$

In the half space below the wave guide we write

$$
\phi_\varepsilon(\mathbf{x}, z) = \int \frac{d\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{x}/\varepsilon} F_\varepsilon(\mathbf{k}) e^{-ik_n z/\varepsilon}, \quad z \le 0 \tag{11}
$$

with

$$
k_n = \pm \left(\frac{\omega^2}{c^2} - k^2 \right)^{1/2}.
$$

Notice that k_n^B and k_n can be real or purely imaginary depending on the wavenumber k. The sign of k_n is chosen so that there are no incoming waves from below and no unbounded modes. More precisely, k_n is positive when it is real, and $\mathrm{Im} k_n$ is positive when it is purely imaginary. This decomposition is similar to (6), except that k_z used for the evanescent modes in the absence of surface fluctuations has been replaced by $-ik_n$. Our objective in this section is to obtain a closed integral equation for the function $\phi_\varepsilon(\mathbf{x}, H)$ on the averaged top surface.

The boundary conditions (9) at the interface $z = 0$ imply that

$$
\begin{aligned}
&A_\varepsilon(\mathbf{k}) + B_\varepsilon(\mathbf{k}) = F_\varepsilon(\mathbf{k}), \\
&\mu_B k_n^B [A_\varepsilon(\mathbf{k}) - B_\varepsilon(\mathbf{k})] = -\mu k_n F_\varepsilon(\mathbf{k})
\end{aligned}
$$

and therefore

$$
\begin{aligned}
A_\varepsilon(\mathbf{k}) &= \frac{1}{2} (1 - m(\mathbf{k})) F_\varepsilon(\mathbf{k}) \\
B_\varepsilon(\mathbf{k}) &= \frac{1}{2} (1 + m(\mathbf{k})) F_\varepsilon(\mathbf{k})
\end{aligned}
$$

with

$$m(\mathbf{k}) = \frac{\mu k_n}{\mu_B k_n^B}.$$

The Neumann boundary conditions in (5) now take the form

$$0 = \frac{\partial \phi_\varepsilon^B}{\partial z} - \sqrt{\varepsilon}\nabla h(\frac{\mathbf{x}}{\varepsilon}) \cdot \nabla_{\mathbf{x}} \phi_\varepsilon^B = \int \frac{d\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{x}/\varepsilon + ik_n^B H + i\sqrt{\varepsilon}k_n^B h(\mathbf{x}/\varepsilon)} \frac{ik_n^B}{\varepsilon} A_\varepsilon(\mathbf{k})$$
$$- \int \frac{d\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{x}/\varepsilon - ik_n^B H - i\sqrt{\varepsilon}k_n^B h(\mathbf{x}/\varepsilon)} \frac{ik_n^B}{\varepsilon} B_\varepsilon(\mathbf{k})$$
$$-\sqrt{\varepsilon}\frac{\partial h}{\partial x_j}\left(\frac{\mathbf{x}}{\varepsilon}\right) \int \frac{d\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{x}/\varepsilon + ik_n^B H + i\sqrt{\varepsilon}k_n^B h(\mathbf{x}/\varepsilon)} \frac{ik_j}{\varepsilon} A_\varepsilon(\mathbf{k})$$
$$-\sqrt{\varepsilon}\frac{\partial h}{\partial x_j}\left(\frac{\mathbf{x}}{\varepsilon}\right) \int \frac{d\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{x}/\varepsilon - ik_n^B H - i\sqrt{\varepsilon}k_n^B h(\mathbf{x}/\varepsilon)} \frac{ik_j}{\varepsilon} B_\varepsilon(\mathbf{k}).$$

We expand the exponent keeping terms up to the order $O(\varepsilon)$. This is the highest order that will appear in the transport equation. This yields

$$0 = \int \frac{d\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{x}/\varepsilon + ik_n^B H} \left(1 + i\sqrt{\varepsilon}k_n^B h\left(\frac{\mathbf{x}}{\varepsilon}\right) - \frac{\varepsilon}{2}(k_n^B)^2 h^2\left(\frac{\mathbf{x}}{\varepsilon}\right)\right) k_n^B A_\varepsilon(\mathbf{k})$$
$$-\sqrt{\varepsilon}\int \frac{d\mathbf{p}d\mathbf{k}}{(2\pi)^4} e^{i\mathbf{p}\cdot\mathbf{x}/\varepsilon + i\mathbf{k}\cdot\mathbf{x}/\varepsilon + ik_n^B H} \left(1 + i\sqrt{\varepsilon}k_n^B h\left(\frac{\mathbf{x}}{\varepsilon}\right)\right) ip_j \hat{h}(\mathbf{p}) k_j A_\varepsilon(\mathbf{k})$$
$$- \int \frac{d\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{x}/\varepsilon - ik_n^B H} \left(1 - i\sqrt{\varepsilon}k_n^B h\left(\frac{\mathbf{x}}{\varepsilon}\right) - \frac{\varepsilon}{2}(k_n^B)^2 h^2\left(\frac{\mathbf{x}}{\varepsilon}\right)\right) k_n^B B_\varepsilon(\mathbf{k})$$
$$-\sqrt{\varepsilon}\int \frac{d\mathbf{p}d\mathbf{k}}{(2\pi)^4} e^{i\mathbf{p}\cdot\mathbf{x}/\varepsilon + i\mathbf{k}\cdot\mathbf{x}/\varepsilon - ik_n^B H} \left(1 - i\sqrt{\varepsilon}k_n^B h\left(\frac{\mathbf{x}}{\varepsilon}\right)\right) ip_j \hat{h}(\mathbf{p}) k_j B_\varepsilon(\mathbf{k}).$$

We rewrite the above expression collecting powers of ε as

$$0 = \int \frac{d\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{x}/\varepsilon} k_n^B \left[A_\varepsilon(\mathbf{k}) e^{ik_n^B H} - B_\varepsilon(\mathbf{k}) e^{-ik_n^B H}\right]$$
$$+\sqrt{\varepsilon}\int \frac{d\mathbf{p}d\mathbf{k}}{(2\pi)^4} e^{i\mathbf{p}\cdot\mathbf{x}/\varepsilon + i\mathbf{k}\cdot\mathbf{x}/\varepsilon} \hat{h}(\mathbf{p}) \Big\{ i(k_n^B)^2 A_\varepsilon(\mathbf{k}) e^{ik_n^B H} - i(\mathbf{p}\cdot\mathbf{k}) A_\varepsilon(\mathbf{k}) e^{ik_n^B H}$$
$$+ i(k_n^B)^2 B_\varepsilon(\mathbf{k}) e^{-ik_n^B H} - i(\mathbf{p}\cdot\mathbf{k}) B_\varepsilon(\mathbf{k}) e^{-ik_n^B H} \Big\}$$
$$+\varepsilon\int \frac{d\mathbf{p}d\mathbf{k}d\mathbf{q}}{(2\pi)^6} e^{i\mathbf{p}\cdot\mathbf{x}/\varepsilon + i\mathbf{k}\cdot\mathbf{x}/\varepsilon + i\mathbf{q}\cdot\mathbf{x}/\varepsilon} \hat{h}(\mathbf{p})\hat{h}(\mathbf{q}) \Big\{ -\frac{1}{2}(k_n^B)^3 A_\varepsilon(\mathbf{k}) e^{ik_n^B H} + (\mathbf{p}\cdot\mathbf{k}) k_n^B A_\varepsilon(\mathbf{k}) e^{ik_n^B H}$$
$$+ \frac{1}{2}(k_n^B)^3 B_\varepsilon(\mathbf{k}) e^{-ik_n^B H} - (\mathbf{p}\cdot\mathbf{k}) k_n^B B_\varepsilon(\mathbf{k}) e^{-ik_n^B H} \Big\}.$$

We insert the expressions for A_ε and B_ε in terms of F_ε in the above equation and obtain

$$k_n^B[-i\sin(k_n^B H) + m(\mathbf{k})\cos(k_n^B H)] F_\varepsilon(\mathbf{k})$$
$$+\varepsilon\int \frac{d\mathbf{p}d\mathbf{q}}{(2\pi)^4} \hat{h}(\mathbf{k}-\mathbf{p})\hat{h}(\mathbf{p}-\mathbf{q}) q_n^B \left[-\frac{1}{2}(q_n^B)^2 + (\mathbf{q}\cdot(\mathbf{k}-\mathbf{p}))\right] [m(\mathbf{q})\cos(q_n^B H) - i\sin(q_n^B H)] F_\varepsilon(\mathbf{q})$$
$$= i\sqrt{\varepsilon}\int \frac{d\mathbf{p}}{(2\pi)^2} \hat{h}(\mathbf{k}-\mathbf{p}) \alpha(\mathbf{k}\cdot\mathbf{p}) \left(\cos(p_n^B H) - im(\mathbf{p})\sin(p_n^B H)\right) F_\varepsilon(\mathbf{p}),$$

where $\alpha(\mathbf{k}\cdot\mathbf{p})$ is defined as

$$\alpha(\mathbf{k}\cdot\mathbf{p}) = \frac{\omega^2}{c_B^2} - (\mathbf{k}\cdot\mathbf{p}). \tag{12}$$

We replace the order ε term on the second line in the above equation by its average because we anticipate that only the average of the term in front of $\varepsilon F_\varepsilon$ will appear in the radiative transport equation as an effective absorption. We will later see that actually even its average does not effect the energy propagation in the transport limit. We then obtain the equation

$$\begin{aligned}
&\left[k_n^B + \varepsilon \tilde{Q}(\mathbf{k})\right]\left[m(\mathbf{k})\cos(k_n^B H) - i\sin(k_n^B H)\right] F_\varepsilon(\mathbf{k}) \\
&= i\sqrt{\varepsilon}\int \frac{d\mathbf{p}}{(2\pi)^2}\hat{h}(\mathbf{k}-\mathbf{p})\alpha(\mathbf{k}\cdot\mathbf{p})\left(\cos(p_n^B H) - im(\mathbf{p})\sin(p_n^B H)\right) F_\varepsilon(\mathbf{p})
\end{aligned}$$

with

$$\tilde{Q}(\mathbf{k}) = \int \frac{d\mathbf{p}}{(2\pi)^2}\hat{R}(\mathbf{p})k_n^B\left[-\frac{1}{2}(k_n^B)^2 + (\mathbf{k}\cdot\mathbf{p})\right]$$

Next we define

$$\begin{aligned}
\hat{D}_\varepsilon(\mathbf{k}) &= [\cos(k_n^B H) - im(\mathbf{k})\sin(k_n^B H)]F_\varepsilon(\mathbf{k}), \qquad (13)\\
\Omega(\mathbf{k}) &= -k_n^B\frac{\sin(k_n^B H) + im(\mathbf{k})\cos(k_n^B H)}{\cos(k_n^B H) - im(\mathbf{k})\sin(k_n^B H)} = -\frac{\mu_B k_n^B\sin(k_n^B H) + i\mu k_n\cos(k_n^B H)}{\mu_B\cos(k_n^B H) - i\mu k_n\sin(k_n^B H)/k_n^B},\\
Q(\mathbf{k}) &= \frac{\tilde{Q}(\mathbf{k})}{k_n^B} = \int d\mathbf{p}\hat{R}(\mathbf{p})\left[-\frac{1}{2}(k_n^B)^2 + (\mathbf{k}\cdot\mathbf{p})\right].
\end{aligned}$$

Finally we get the following equation for $\hat{D}_\varepsilon$:

$$[1+\varepsilon Q(\mathbf{k})]\,\Omega(\mathbf{k})\hat{D}_\varepsilon(\mathbf{k}) = \sqrt{\varepsilon}\int\frac{d\mathbf{p}}{(2\pi)^2}\hat{h}(\mathbf{k}-\mathbf{p})\alpha(\mathbf{k}\cdot\mathbf{p})\hat{D}_\varepsilon(\mathbf{p}). \qquad (14)$$

It is straightforward to verify that

$$\phi_\varepsilon^B(\mathbf{x}, H) = \int\frac{d\mathbf{k}}{(2\pi)^2}e^{i\mathbf{k}\cdot\mathbf{x}/\varepsilon}\hat{D}_\varepsilon(\mathbf{k})$$

is the pressure on the averaged top surface. Therefore we have

$$\hat{\phi}_\varepsilon(\mathbf{k}, H) = \varepsilon^2\hat{D}_\varepsilon(\varepsilon\mathbf{k}).$$

Here hat denotes the Fourier transform only in the horizontal variable $\mathbf{x}$. In the absence of inhomogeneities when $\varepsilon = 0$, corresponding to the case where the top surface is flat, we obtain that $\Omega(\mathbf{k})\hat{D}_0(\mathbf{k}) = 0$. This implies that the support of $D_0(\mathbf{k})$ is restricted to the set $\mathcal{S} = \{\mathbf{k} : \ \Omega(\mathbf{k}) = 0\}$. One can verify that this equation describes the same dispersion relation as the one obtained in (7). Equation (14) generalizes the problem to account for the rough upper boundary.

3 Derivation of the transport equation

3.1 Equation for the Wigner distribution

Solving the integral equation (14) on the top surface remains complicated because it still depends on the detailed structure of the boundary h. The advantage of transport, or radiative transfer, theory, is that it provides homogenized equations that only depend on the power spectrum of the random field h. We now derive radiative transport equations for surface waves starting from the Wigner distribution (see [18]) for the surface wave function $\psi_\varepsilon(\mathbf{x}) = \phi_\varepsilon^B(\mathbf{x}, H)$:

$$W_\varepsilon(\mathbf{x},\mathbf{k}) = \int\frac{d\mathbf{y}}{(2\pi)^2}e^{i\mathbf{k}\cdot\mathbf{y}}\psi_\varepsilon\left(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2}\right)\bar{\psi}_\varepsilon\left(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2}\right) = \varepsilon^2\int\frac{d\mathbf{q}}{(2\pi)^4}e^{i\mathbf{q}\cdot\mathbf{x}}\hat{D}_\varepsilon\left(\mathbf{k}+\frac{\varepsilon\mathbf{q}}{2}\right)\overline{\hat{D}_\varepsilon}\left(\mathbf{k}-\frac{\varepsilon\mathbf{q}}{2}\right).$$

The Wigner distribution has many useful properties. It is real, its weak limit as $\varepsilon \to 0$ is a non-negative distribution, and energy and flux density may be expressed as

$$
\begin{aligned}
|\psi_\varepsilon(\mathbf{x})|^2 &= \int d\mathbf{k} W_\varepsilon(\mathbf{x},\mathbf{k}),\\
\frac{i\varepsilon}{2}[\psi_\varepsilon \nabla \bar{\psi}_\varepsilon - \bar{\psi}_\varepsilon \nabla \psi_\varepsilon] &= \int d\mathbf{k}\, \mathbf{k} W_\varepsilon(\mathbf{x},\mathbf{k}),
\end{aligned}
$$

respectively. We can interpret the limit of $W_\varepsilon(\mathbf{x},\mathbf{k})$ as $\varepsilon \to 0$ (the high frequency limit) as an energy density in the phase space. Detailed mathematical properties of the Wigner distribution may be found in [12, 16, 18] and references within. In time-harmonic acoustics, the volume density of kinetic energy and the volume density of deformation energy are equal and the total energy is given by $\mathcal{E}_\varepsilon(\mathbf{x}) = \kappa|\phi_\varepsilon|^2(\mathbf{x})$. Thus W_ε defined above in the limit $\varepsilon \to 0$ is proportional to the angularly resolved volume energy density of the surface waves:

$$
\mathcal{E}_\varepsilon(\mathbf{x}) = \kappa \int d\mathbf{k} W_\varepsilon(\mathbf{x},\mathbf{k}).
$$

Using the definition of the Wigner transform, we have

$$
\begin{aligned}
&\varepsilon^{-2}\int d\mathbf{y} e^{i\mathbf{k}\cdot\mathbf{y}}\left[\hat{\Omega}(\mathbf{y}) + \varepsilon\widehat{Q\Omega}(\mathbf{y})\right] W_\varepsilon\left(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2},\mathbf{k}\right)\\
&= \int d\mathbf{y} e^{i\mathbf{k}\cdot\mathbf{y}}\left[\hat{\Omega}(\mathbf{y}) + \varepsilon\widehat{Q\Omega}(\mathbf{y})\right]\int \frac{d\mathbf{q}}{(2\pi)^4} e^{i\mathbf{q}\cdot(\mathbf{x}+\varepsilon\mathbf{y}/2)}\hat{D}_\varepsilon\left(\mathbf{k}+\frac{\varepsilon\mathbf{q}}{2}\right)\overline{\hat{D}_\varepsilon}\left(\mathbf{k}-\frac{\varepsilon\mathbf{q}}{2}\right)\\
&= \int \frac{d\mathbf{q}}{(2\pi)^2} e^{i\mathbf{q}\cdot\mathbf{x}}\Omega\left(\mathbf{k}+\frac{\varepsilon\mathbf{q}}{2}\right)\left[1+\varepsilon Q\left(\mathbf{k}+\frac{\varepsilon\mathbf{q}}{2}\right)\right]\hat{D}_\varepsilon\left(\mathbf{k}+\frac{\varepsilon\mathbf{q}}{2}\right)\overline{\hat{D}_\varepsilon}\left(\mathbf{k}-\frac{\varepsilon\mathbf{q}}{2}\right)\\
&= \sqrt{\varepsilon}\int \frac{d\mathbf{q}d\mathbf{p}}{(2\pi)^4} e^{i\mathbf{q}\cdot\mathbf{x}}\hat{h}\left(\mathbf{k}+\frac{\varepsilon\mathbf{q}}{2}-\mathbf{p}\right)\alpha((\mathbf{k}+\frac{\varepsilon\mathbf{q}}{2})\cdot\mathbf{p})\hat{D}_\varepsilon(\mathbf{p})\overline{\hat{D}_\varepsilon}\left(\mathbf{k}-\frac{\varepsilon\mathbf{q}}{2}\right)\\
&= \frac{\sqrt{\varepsilon}}{\varepsilon^2}\int \frac{d\mathbf{q}d\mathbf{p}d\mathbf{y}}{(2\pi)^2} e^{i\mathbf{q}\cdot\mathbf{x}-i\mathbf{y}\cdot(\mathbf{p}-\mathbf{k}+\frac{\varepsilon\mathbf{q}}{2})/\varepsilon}\hat{h}\left(\mathbf{k}+\frac{\varepsilon\mathbf{q}}{2}-\mathbf{p}\right)\alpha((\mathbf{k}+\frac{\varepsilon\mathbf{q}}{2})\cdot\mathbf{p})W_\varepsilon\left(\mathbf{y},\frac{\mathbf{k}+\mathbf{p}}{2}-\frac{\varepsilon\mathbf{q}}{4}\right).
\end{aligned}
$$

Still using its definition, we then obtain the following closed equation for W_ε:

$$
\begin{aligned}
&\int d\mathbf{y} e^{i\mathbf{k}\cdot\mathbf{y}}\left[\hat{\Omega}(\mathbf{y}) + \varepsilon\widehat{\Omega Q}(\mathbf{y})\right] W_\varepsilon\left(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2},\mathbf{k}\right)\\
&= \sqrt{\varepsilon}\int \frac{d\mathbf{q}d\mathbf{p}d\mathbf{y}}{(2\pi)^2} e^{i\mathbf{q}\cdot\mathbf{x}/\varepsilon-i\mathbf{y}\cdot(\mathbf{p}+\frac{\mathbf{q}}{2\varepsilon})}\hat{h}\left(-\varepsilon\mathbf{p}+\frac{\mathbf{q}}{2}\right)\alpha((\mathbf{k}+\frac{\mathbf{q}}{2})\cdot(\mathbf{k}+\varepsilon\mathbf{p}))W_\varepsilon\left(\mathbf{y},\mathbf{k}+\frac{\varepsilon\mathbf{p}}{2}-\frac{\mathbf{q}}{4}\right)\\
&= \sqrt{\varepsilon}\int \frac{d\mathbf{q}d\mathbf{p}d\mathbf{y}}{(2\pi\varepsilon)^2} e^{i\mathbf{q}\cdot\mathbf{x}/\varepsilon-i\mathbf{y}\cdot(\mathbf{q}-\mathbf{p})/\varepsilon}\hat{h}(\mathbf{p})\alpha((\mathbf{k}+\frac{\mathbf{q}}{2})\cdot(\mathbf{k}-\mathbf{p}+\frac{\mathbf{q}}{2}))W_\varepsilon\left(\mathbf{y},\mathbf{k}-\frac{\mathbf{p}}{2}\right)\\
&= \sqrt{\varepsilon}\int \frac{d\mathbf{q}d\mathbf{p}d\mathbf{y}}{(2\pi)^2} e^{i\mathbf{p}\cdot\mathbf{x}/\varepsilon+i\mathbf{y}\cdot(\mathbf{q}-\mathbf{p})}\hat{h}(\mathbf{p})\alpha((\mathbf{k}+\frac{\mathbf{q}}{2})\cdot(\mathbf{k}-\mathbf{p}+\frac{\mathbf{q}}{2}))W_\varepsilon\left(\mathbf{x}-\varepsilon\mathbf{y},\mathbf{k}-\frac{\mathbf{p}}{2}\right).
\end{aligned}
$$

Thus the equation for the Wigner distribution is

$$
\begin{aligned}
&\int d\mathbf{y} e^{i\mathbf{k}\cdot\mathbf{y}}\left[\hat{\Omega}(\mathbf{y}) + \varepsilon\widehat{Q\Omega}(\mathbf{y})\right] W_\varepsilon\left(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2},\mathbf{k}\right) \qquad (15)\\
&= \sqrt{\varepsilon}\int \frac{d\mathbf{q}d\mathbf{p}d\mathbf{y}}{(2\pi)^2} e^{i\mathbf{p}\cdot\mathbf{x}/\varepsilon+i\mathbf{y}\cdot(\mathbf{q}-\mathbf{p})}\hat{h}(\mathbf{p})\alpha((\mathbf{k}+\frac{\mathbf{q}}{2})\cdot(\mathbf{k}-\mathbf{p}+\frac{\mathbf{q}}{2}))W_\varepsilon\left(\mathbf{x}-\varepsilon\mathbf{y},\mathbf{k}-\frac{\mathbf{p}}{2}\right).
\end{aligned}
$$

3.2 The high frequency limit

The equation for W_ε still depends on the details of the random field h and is therefore not simpler to solve than the initial wave equation. Only in the limit $\varepsilon \to 0$, that is to say in the high frequency limit, will the equation somewhat simplify and involve only little statistical information about h (namely, its power spectrum). To perform this high frequency analysis, we follow [18] and introduce the following two-scale asymptotic expansion

$$W_\varepsilon(\mathbf{x},\mathbf{k}) = W_0(\mathbf{x},\mathbf{k}) + \sqrt{\varepsilon} W_1(\mathbf{x},\mathbf{z},\mathbf{k}) + \varepsilon W_2(\mathbf{x},\mathbf{z},\mathbf{k}) + \ldots, \quad \mathbf{z} = \frac{\mathbf{x}}{\varepsilon}$$

with the leading order term independent of the fast variable $\mathbf{z}$. We now insert this expression into (15) and equate like powers of ε. We get in the leading order that

$$\Omega(k) W_0(\mathbf{x},\mathbf{k}) = 0. \tag{16}$$

Therefore $W_0(\mathbf{x},\mathbf{k})$ is supported on the set $\mathcal{S} = \{\mathbf{k}: \ \Omega(k) = 0\}$. Recall that $\Omega(k) = 0$ is nothing but the dispersion relation (8). This means that in the leading order of the high frequency limit, all the energy is concentrated in the waveguide modes. This is because we have assumed that there are no incoming waves from below. Therefore we can decompose W_0 over the waveguide modes as

$$W_0(\mathbf{x},\mathbf{k}) = \sum_j w_j(\mathbf{x},\phi)|\Omega'(k_j)|\delta(k-k_j). \tag{17}$$

Here the k_j are the solutions of $\Omega(k) = 0$ and ϕ is the angular coordinate of $\mathbf{k} = (k\cos\phi, k\sin\phi)$. It is straightforward to verify that

$$\frac{\omega}{c} \le k_j \le \frac{\omega}{c_B}. \tag{18}$$

The wave energy may be now represented as

$$\mathcal{E}(\mathbf{x}) = \sum_j \mathcal{E}_j(\mathbf{x}), \qquad \mathcal{E}_j(\mathbf{x}) = k_j|\Omega'(k_j)| \int_0^{2\pi} d\phi w_j(\mathbf{x},\phi) \tag{19}$$

with $\mathcal{E}_j(\mathbf{x})$ being the physical space energy density of the j-th waveguide mode.

The next order term in the expansion of (15) yields

$$\int d\mathbf{y} e^{i\mathbf{k}\cdot\mathbf{y}} \hat{\Omega}(\mathbf{y}) W_1(\mathbf{x},\mathbf{z}+\frac{\mathbf{y}}{2},\mathbf{k}) = \int \frac{d\mathbf{p}d\mathbf{q}d\mathbf{y}}{(2\pi)^2} e^{i\mathbf{p}\cdot\mathbf{z}+i\mathbf{y}\cdot(\mathbf{q}-\mathbf{p})} \hat{h}(\mathbf{p})\alpha((\mathbf{k}+\frac{\mathbf{q}}{2})\cdot(\mathbf{k}-\mathbf{p}+\frac{\mathbf{q}}{2}))$$
$$\times \quad W_0(\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2}) = \int d\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{z}} \hat{h}(\mathbf{p})\alpha((\mathbf{k}+\frac{\mathbf{p}}{2})\cdot(\mathbf{k}-\frac{\mathbf{p}}{2})) W_0(\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2})$$

and thus after taking the Fourier transform in $\mathbf{z}$ we obtain

$$\Omega(\mathbf{k}+\frac{\mathbf{p}}{2})\hat{W}_1(\mathbf{x},\mathbf{p},\mathbf{k}) = \hat{h}(\mathbf{p})\alpha((\mathbf{k}+\frac{\mathbf{p}}{2})\cdot(\mathbf{k}-\frac{\mathbf{p}}{2}))W_0(\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2}). \tag{20}$$

A symmetrized version of (20) is

$$\hat{W}_1(\mathbf{x},\mathbf{p},\mathbf{k}) = \frac{\hat{h}(\mathbf{p})[\alpha((\mathbf{k}+\frac{\mathbf{p}}{2})\cdot(\mathbf{k}-\frac{\mathbf{p}}{2}))W_0(\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2}) - \alpha((\mathbf{k}-\frac{\mathbf{p}}{2})\cdot(\mathbf{k}+\frac{\mathbf{p}}{2}))W_0(\mathbf{x},\mathbf{k}+\frac{\mathbf{p}}{2})]}{\Omega(\mathbf{k}+\frac{\mathbf{p}}{2}) - \bar{\Omega}(\mathbf{k}-\frac{\mathbf{p}}{2}) - i\theta}. \tag{21}$$

Here θ is a regularization parameter as in [18]. In the next order, we obtain

$$\begin{aligned}
&\int d\mathbf{y} e^{i\mathbf{k}\cdot\mathbf{y}}\hat{\Omega}(\mathbf{y})W_2(\mathbf{x},\mathbf{z}+\frac{\mathbf{y}}{2},\mathbf{k})+\frac{1}{2}\int d\mathbf{y} e^{i\mathbf{k}\cdot\mathbf{y}}\hat{\Omega}(\mathbf{y})\mathbf{y}\cdot\nabla_{\mathbf{x}}W_0(\mathbf{x},\mathbf{k}) \qquad (22)\\
&=\int \frac{d\mathbf{p}d\mathbf{q}d\mathbf{y}}{(2\pi)^2}e^{i\mathbf{p}\cdot\mathbf{z}+i\mathbf{y}\cdot(\mathbf{q}-\mathbf{p})}\hat{h}(\mathbf{p})\alpha((\mathbf{k}+\frac{\mathbf{q}}{2})\cdot(\mathbf{k}-\mathbf{p}+\frac{\mathbf{q}}{2}))W_1(\mathbf{x},\mathbf{z}-\mathbf{y},\mathbf{k}-\frac{\mathbf{p}}{2}).
\end{aligned}$$

We can check from the definition of $\Omega(\mathbf{k})$ that it is real for $\mathbf{k}\in\mathcal{D}:=\{\mathbf{k}:\ \omega/c\le k\le\omega/c_B\}$. However, $W_0(\mathbf{x},\mathbf{k})$ is supported inside the set $\mathcal{D}$ and so we are interested only in these values of $\mathbf{k}$. Therefore we get

$$\begin{aligned}
&\mathrm{Im}\langle\int d\mathbf{y}e^{i\mathbf{k}\cdot\mathbf{y}}\hat{\Omega}(\mathbf{y})W_2(\mathbf{x},\mathbf{z}+\frac{\mathbf{y}}{2},\mathbf{k})\rangle\\
&=\frac{1}{2i}\int d\mathbf{z}d\mathbf{y}\left[e^{i\mathbf{k}\cdot\mathbf{y}}\hat{\Omega}(\mathbf{y})W_2(\mathbf{x},\mathbf{z}+\frac{\mathbf{y}}{2},\mathbf{k})-e^{-i\mathbf{k}\cdot\mathbf{y}}\overline{\hat{\Omega}}(\mathbf{y})W_2(\mathbf{x},\mathbf{z}+\frac{\mathbf{y}}{2},\mathbf{k})\right]\\
&=\mathrm{Im}\left\{\Omega(\mathbf{k})\right\}\int d\mathbf{z}W_2(\mathbf{x},\mathbf{z},\mathbf{k})=0
\end{aligned}$$

for $\mathbf{k}\in\mathcal{D}$. Here $\langle\cdot\rangle$ denotes statistical average, and we assume that it is also equal to spatial average by ergodicity. Therefore we obtain from (22) for $\mathbf{k}\in\mathcal{D}$ that

$$\begin{aligned}
\nabla\Omega(\mathbf{k})\cdot\nabla_{\mathbf{x}}W_0 = i\Big\langle\int&\frac{d\mathbf{q}d\mathbf{p}d\mathbf{y}}{(2\pi)^4}e^{i\mathbf{p}\cdot\mathbf{z}+i\mathbf{y}\cdot(\mathbf{q}-\mathbf{p})}\Big[\alpha((\mathbf{k}+\frac{\mathbf{q}}{2})\cdot(\mathbf{k}-\mathbf{p}+\frac{\mathbf{q}}{2}))W_1(\mathbf{x},\mathbf{z}-\mathbf{y},\mathbf{k}-\frac{\mathbf{p}}{2})\\
&-\alpha((\mathbf{k}-\frac{\mathbf{q}}{2})\cdot(\mathbf{k}+\mathbf{p}-\frac{\mathbf{q}}{2}))W_1(\mathbf{x},\mathbf{z}-\mathbf{y},\mathbf{k}+\frac{\mathbf{p}}{2})\Big]\hat{h}(\mathbf{p})\Big\rangle = I+I^*.
\end{aligned}$$

We use (21) to rewrite the term on the right-hand side as

$$\begin{aligned}
I&=i\Big\langle\int\frac{d\mathbf{q}d\mathbf{p}d\mathbf{y}d\mathbf{s}}{(2\pi)^6}e^{i\mathbf{p}\cdot\mathbf{z}+i\mathbf{y}\cdot(\mathbf{q}-\mathbf{p})+i\mathbf{s}\cdot(\mathbf{z}-\mathbf{y})}\hat{h}(\mathbf{p})\alpha((\mathbf{k}+\frac{\mathbf{q}}{2})\cdot(\mathbf{k}-\mathbf{p}+\frac{\mathbf{q}}{2}))\hat{W}_1(\mathbf{x},\mathbf{s},\mathbf{k}-\frac{\mathbf{p}}{2})\Big\rangle\\
&=i\Big\langle\int\frac{d\mathbf{q}d\mathbf{p}}{(2\pi)^4}e^{i\mathbf{q}\cdot\mathbf{z}}\hat{h}(\mathbf{p})\alpha((\mathbf{k}+\frac{\mathbf{q}}{2})\cdot(\mathbf{k}-\mathbf{p}+\frac{\mathbf{q}}{2}))\hat{W}_1(\mathbf{x},\mathbf{q}-\mathbf{p},\mathbf{k}-\frac{\mathbf{p}}{2})\Big\rangle\\
&=i\Big\langle\int\frac{d\mathbf{q}d\mathbf{p}}{(2\pi)^4}e^{i\mathbf{q}\cdot\mathbf{z}}\hat{h}(\mathbf{p})\alpha((\mathbf{k}+\frac{\mathbf{q}}{2})\cdot(\mathbf{k}-\mathbf{p}+\frac{\mathbf{q}}{2}))\hat{h}(\mathbf{q}-\mathbf{p})\Big\rangle\\
&\times\frac{[\alpha((\mathbf{k}-\mathbf{p}+\mathbf{q}/2)\cdot(\mathbf{k}-\mathbf{q}/2))W_0(\mathbf{x},\mathbf{k}-\mathbf{q}/2)-\alpha((\mathbf{k}-\mathbf{q}/2)\cdot(\mathbf{k}-\mathbf{p}+\mathbf{q}/2))W_0(\mathbf{k}-\mathbf{p}+\mathbf{q}/2)]}{\Omega(\mathbf{k}-\mathbf{p}+\mathbf{q}/2)-\bar{\Omega}(\mathbf{k}-\mathbf{q}/2)+i\theta}\\
&=i\int\frac{d\mathbf{p}}{(2\pi)^2}\hat{R}(\mathbf{p})\alpha(\mathbf{k}\cdot(\mathbf{k}-\mathbf{p}))\frac{\alpha((\mathbf{k}-\mathbf{p})\cdot\mathbf{k})W_0(\mathbf{x},\mathbf{k})-\alpha(\mathbf{k}\cdot(\mathbf{k}-\mathbf{p}))W_0(\mathbf{x},\mathbf{k}-\mathbf{p})}{\Omega(\mathbf{k}-\mathbf{p})-\bar{\Omega}(\mathbf{k})-i\theta}.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
I+I^*=i\int\frac{d\mathbf{p}}{(2\pi)^2}\hat{R}(\mathbf{k}-\mathbf{p})&\Big[\frac{|\alpha(\mathbf{k}\cdot\mathbf{p})|^2W_0(\mathbf{x},\mathbf{k})-|\alpha(\mathbf{k}\cdot\mathbf{p})|^2W_0(\mathbf{x},\mathbf{p})}{\Omega(\mathbf{p})-\bar{\Omega}(\mathbf{k})-i\theta}\\
&-\frac{|\alpha(\mathbf{k}\cdot\mathbf{p})|^2W_0(\mathbf{x},\mathbf{k})-|\alpha(\mathbf{k}\cdot\mathbf{p})|^2W_0(\mathbf{x},\mathbf{p})}{\bar{\Omega}(\mathbf{p})-\Omega(\mathbf{k})+i\theta}\Big].
\end{aligned}$$

We are interested only in $\mathbf{k}\in\mathcal{D}$, where W_0 is supported in $\mathcal{S}$ and $\Omega(\mathbf{k})$ is real. We split the above integral into three regions:

$$\mathcal{D}_1=\{|\mathbf{p}|\le\omega/c\},\quad \mathcal{D}=\{\omega/c\le|\mathbf{p}|\le\omega/c_B\},\quad \mathcal{D}_2=\{|\mathbf{p}|\ge\omega/c_B\}.$$

Note that $\text{Im }\Omega(\mathbf{p})=0$ in $\mathcal{D}\cup\mathcal{D}_2$. Therefore we have

$$i\int_{\mathcal{D}_1}\frac{d\mathbf{p}}{(2\pi)^2}\hat{R}(\mathbf{k}-\mathbf{p})|\alpha(\mathbf{k}\cdot\mathbf{p})|^2\left[\frac{W_0(\mathbf{x},\mathbf{k})-W_0(\mathbf{x},\mathbf{p})}{\Omega(\mathbf{p})-\bar{\Omega}(\mathbf{k})-i\theta}-\frac{W_0(\mathbf{x},\mathbf{k})-W_0(\mathbf{x},\mathbf{p})}{\bar{\Omega}(\mathbf{p})-\Omega(\mathbf{k})+i\theta}\right]$$
$$=-2\int_{\mathcal{D}_1}\frac{d\mathbf{p}}{(2\pi)^2}\hat{R}(\mathbf{k}-\mathbf{p})|\alpha(\mathbf{k}\cdot\mathbf{p})|^2W_0(\mathbf{x},\mathbf{k})\text{Im}\left[\frac{1}{\Omega(\mathbf{p})-\Omega(\mathbf{k})}\right].$$

The second term is given by

$$i\int_{\mathcal{D}}\frac{d\mathbf{p}}{(2\pi)^2}\hat{R}(\mathbf{k}-\mathbf{p})|\alpha(\mathbf{k}\cdot\mathbf{p})|^2\left[\frac{W_0(\mathbf{x},\mathbf{k})-W_0(\mathbf{x},\mathbf{p})}{\Omega(\mathbf{p})-\bar{\Omega}(\mathbf{k})-i\theta}-\frac{W_0(\mathbf{x},\mathbf{k})-W_0(\mathbf{x},\mathbf{p})}{\bar{\Omega}(\mathbf{p})-\Omega(\mathbf{k})+i\theta}\right]$$
$$\to\int_{\mathcal{D}}\frac{d\mathbf{p}}{2\pi}\hat{R}(\mathbf{k}-\mathbf{p})|\alpha(\mathbf{k}\cdot\mathbf{p})|^2\left[W_0(\mathbf{x},\mathbf{p})-W_0(\mathbf{x},\mathbf{k})\right]\delta(\Omega(\mathbf{p})-\Omega(\mathbf{k}))$$

and the integral over the last region is

$$\int_{\mathcal{D}_2}d\mathbf{p}\hat{R}(\mathbf{k}-\mathbf{p})|\alpha(\mathbf{k}\cdot\mathbf{p})|^2\left[\frac{[W_0(\mathbf{x},\mathbf{k})-W_0(\mathbf{x},\mathbf{p})}{\Omega(\mathbf{p})-\Omega(\mathbf{k})-i\theta}-\frac{W_0(\mathbf{x},\mathbf{k})-W_0(\mathbf{x},\mathbf{p})}{\Omega(\mathbf{p})-\Omega(\mathbf{k})+i\theta}\right]$$
$$\to-\frac{2\pi}{i}\int_{\mathcal{D}_2}d\mathbf{p}\hat{R}(\mathbf{k}-\mathbf{p})|\alpha(\mathbf{k}\cdot\mathbf{p})|^2\delta(\Omega(\mathbf{k})-\Omega(\mathbf{p}))W_0(\mathbf{x},\mathbf{k})=0$$

because for $\mathbf{k}\in\mathcal{S}$ (the support of W_0) and $\mathbf{p}\in\mathcal{D}_2$ we have $\Omega(\mathbf{k})=0$, $\Omega(\mathbf{p})>0$.

Putting together integrals over all three domains $\mathcal{D}$, $\mathcal{D}_1$ and $\mathcal{D}_2$, we obtain the transport equation

$$\nabla_{\mathbf{k}}\Omega(\mathbf{k})\cdot\nabla_{\mathbf{x}}W_0(\mathbf{x},\mathbf{k})=\int_{\mathcal{D}}\frac{d\mathbf{p}}{2\pi}\hat{R}(\mathbf{k}-\mathbf{p})|\alpha(\mathbf{k}\cdot\mathbf{p})|^2\left[W_0(\mathbf{x},\mathbf{p})-W_0(\mathbf{x},\mathbf{k})\right]\delta(\Omega(\mathbf{p})-\Omega(\mathbf{k}))$$
$$-2\int\limits_{\mathcal{D}_1}\frac{d\mathbf{p}}{(2\pi)^2}\hat{R}(\mathbf{k}-\mathbf{p})|\alpha(\mathbf{k}\cdot\mathbf{p})|^2\text{Im}\left[\frac{1}{\Omega(\mathbf{p})-\Omega(\mathbf{k})}\right]W_0(\mathbf{x},\mathbf{k}).\quad(23)$$

This equation is the main result of this paper. The first term on the right side of (23) corresponds to waveguide-waveguide scattering, while the second one describes leakage of Love waves into volume waves, which propagate away from the interface in the lower half-space. Because the half space is assumed to be homogeneous, the volume waves generated in this process do not return to the interface and there is no transformation of volume waves into Love waves. Consequently, the waveguide-volume scattering results in a net loss of Love wave energy density. It is straightforward to verify that

$$\text{Im}\left[\frac{1}{\Omega(\mathbf{p})-\Omega(\mathbf{k})}\right]>0$$

for $\mathbf{p}\in\mathcal{D}_1$, so this term indeed corresponds to leakage. Notice also that on the support of $W_0(\mathbf{x},\mathbf{k})$, $\Omega(\mathbf{k})=0$. So only $\text{Im}[\Omega^{-1}(\mathbf{p})]>0$ shows up. We may rewrite the transport equation (23) in a more explicit form using the decomposition (17) of W_0 over wave guide modes:

$$\Omega'(k_j)\hat{\mathbf{k}}\cdot\nabla w_j(\mathbf{x},\phi_{\mathbf{k}})=\sum_m\int\limits_0^{2\pi}\frac{d\phi_{\mathbf{p}}}{2\pi}\hat{R}(\mathbf{k}-\mathbf{p})|\alpha(\mathbf{k}\cdot\mathbf{p})|^2k_m|\Omega'(k_m)|[w_m(\mathbf{x},\phi_{\mathbf{p}})-w_j(\mathbf{x},\phi_{\mathbf{k}})]$$
$$-M_jw_j(\mathbf{x},\phi_{\mathbf{k}})\quad(24)$$

with $\mathbf{k}=k_j(\cos\phi_{\mathbf{k}},\sin\phi_{\mathbf{k}})$ and $\mathbf{p}=k_m(\cos\phi_{\mathbf{p}},\sin\phi_{\mathbf{p}})$. We denoted here by $\Omega'(k)=\frac{d\Omega}{dk}$. The attenuation coefficients are given by

$$M_j=2\int\limits_{\mathcal{D}_1}\frac{d\mathbf{p}}{(2\pi)^2}\hat{R}(\mathbf{k}-\mathbf{p})|\alpha(\mathbf{k}\cdot\mathbf{p})|^2\text{Im}\frac{1}{\Omega(\mathbf{p})},\qquad|\mathbf{k}|=k_j\quad(25)$$

and are always non-negative.

4 Energy equipartition and diffusive regime

4.1 Leakage suppressing spectra

The system of transport equations (24) is more general than the transport equation for surface waves on a surface with random negative impedance derived in [3]. In addition to leakage from Love waves into volume waves, we have mode coupling between waveguide modes. Nevertheless the analysis of the influence of the power spectrum on the scattering and absorption kernels in (23) still holds. Depending on the structure of the power spectrum $\hat{R}$, scattering and absorption can be made arbitrarily small. We refer to [3] for a detailed analysis. We concentrate here on a regime where several wave guide modes (possibly all of them) form a closed sub-system of (24) but their leakage into volume is small. That was impossible in the problem treated in [3] since there was only one surface mode. As we show below, in the limit of propagation distances large compared to the mean free path, the system of transport equations (24) can be approximated by a lower-dimensional system of diffusion equations. As a result of multiple scattering, energy is partitioned among the modes in a universal way independent of the power spectrum $\hat{R}(\mathbf{p})$. This is similar to the P-to-S energy equipartition in a random medium [17, 20].

First we establish the possibility of a regime where the modes with wavenumber close to ω/c_B decouple from those with wavenumber close to ω/c. We consider a situation where the thickness H of the layer is large. More precisely, assume that

$$H\omega/c_B \gg 1$$

while

$$\frac{c-c_B}{c} = O(1), \quad \frac{c-c_B}{c_B} = O(1).$$

We also assume that the power spectrum $\hat{R}(\mathbf{p})$ is supported in a compact set of size between $c_B/(\omega H^2)$ and $1/H$:

$$\hat{R}(\mathbf{p}) = 0 \text{ for } p \geq k_c \text{ with } \frac{c_B}{\omega H^2} \ll k_c \ll \frac{1}{H}. \tag{26}$$

The wave numbers k_j that solve $\Omega(k_j) = 0$ are given by

$$k_j = \sqrt{\frac{\omega^2}{c_B^2} - \frac{\xi_j^2}{H^2}}$$

with ξ_j being solutions of

$$\omega^2 H^2 \left(\frac{1}{c_B^2} - \frac{1}{c^2}\right) = \xi_j^2 \left(1 + \frac{\mu_B^2}{\mu^2}\tan^2 \xi_j\right). \tag{27}$$

There is always a root ξ_1 in the interval $(0, \pi/2)$, and there are two solutions of (27) in each interval $(2j-1)\pi/2 < \xi < (2j+1)\pi/2$ if

$$\omega H \left(\frac{1}{c_B^2} - \frac{1}{c^2}\right)^{1/2} > j\pi.$$

The number N of solutions of (27) is thus at least $2J+1$ with the integer J determined by

$$J \leq \frac{\omega H}{\pi}\left(\frac{1}{c_B^2} - \frac{1}{c^2}\right)^{1/2} \leq J+1. \tag{28}$$

Therefore the total number of modes is

$$N \approx \frac{2\omega H}{\pi}\left(\frac{1}{c_B^2}-\frac{1}{c^2}\right)^{1/2}+1 \gg 1.$$

The level separation for $j \ll H\omega/c_B$ may be estimated using the above description of the location of the roots ξ_j by

$$|k_j - k_{j-1}| = \left|\sqrt{\frac{\omega^2}{c_B^2}-\frac{\xi_j^2}{H^2}} - \sqrt{\frac{\omega^2}{c_B^2}-\frac{\xi_{j-1}^2}{H^2}}\right| \approx \frac{\omega}{c_B}\left(\frac{\xi_j^2 c_B^2}{2\omega^2 H^2}-\frac{\xi_{j-1}^2 c_B^2}{2\omega^2 H^2}\right) \le \frac{c_B \pi^2 (2j+1)}{\omega H^2}. \tag{29}$$

Next we look at the separation between the lowest mode

$$k_N = \sqrt{\frac{\omega^2}{c_B^2}-\frac{\xi_N^2}{H^2}}$$

and the volume waves $|k| \le \omega/c$. Note that $\xi_N \le (N\pi/2)$ (N is generically odd, except for values of H in a set of measure zero in $I\!R^+$) and therefore we have

$$k_N \ge \sqrt{\frac{\omega^2}{c_B^2}-\frac{N^2\pi^2}{4H^2}}.$$

Let us assume that H is such that in addition to (28) with $N = 2J+1$ we have

$$J+1-\varepsilon \le \frac{\omega H}{\pi}\left(\frac{1}{c_B^2}-\frac{1}{c^2}\right)^{1/2} \le J+1 \tag{30}$$

with $\varepsilon < 1/3$. Then we have

$$\begin{aligned}
k_N &\ge \sqrt{\frac{\omega^2}{c_B^2}-\frac{(2J+1)^2\pi^2}{4H^2}} \ge \sqrt{\frac{\omega^2}{c_B^2}-\frac{\pi^2}{H^2}\left[\frac{\omega H}{\pi}\left(\frac{1}{c_B^2}-\frac{1}{c^2}\right)^{1/2}+\varepsilon-\frac{1}{2}\right]^2} \\
&= \sqrt{\frac{\omega^2}{c^2}+\frac{\pi\omega}{H}\left(\frac{1}{c_B^2}-\frac{1}{c^2}\right)^{1/2}(1-2\varepsilon)-\frac{\pi^2}{H^2}\left(\varepsilon-\frac{1}{2}\right)^2} \\
&\approx \frac{\omega}{c}\left(1+\frac{\pi c^2}{2H\omega}(1-2\varepsilon)\left(\frac{1}{c_B^2}-\frac{1}{c^2}\right)^{1/2}\right) \ge \frac{\omega}{c}+\frac{\pi c}{6H}\left(\frac{1}{c_B^2}-\frac{1}{c^2}\right)^{1/2}
\end{aligned}$$

and therefore

$$k_N - \frac{\omega}{c} \ge \frac{\pi c}{6H}\left(\frac{1}{c_B^2}-\frac{1}{c^2}\right)^{1/2}. \tag{31}$$

Therefore if we choose the power spectrum as in (26) we will have

$$\hat{R}(\mathbf{k}-\mathbf{p}) = 0$$

for all $\mathbf{k}$ such that $|\mathbf{k}| = k_j$ for some j and $|\mathbf{p}| \le \omega/c$. Thus leakage into volume is suppressed for such power spectra. The same calculations actually show that $|k_j - k_{j-2}| = O(H^{-1})$ for $j/N = O(1)$, that is to say for the modes of lower wavenumber. There is therefore no coupling between the modes k_j and k_{j-2} for such values of j. We can conclude the existence of a critical value N_c such that all the modes $1 \le j \le N_c$ are coupled through scattering. These modes are uncoupled from those for which $j > N_c$.

4.2 The diffusion limit

The results of the previous section show that the leakage of Love waves into volume waves may be suppressed or made arbitrarily small. More generally, one may construct power spectra so that all modes are split into several groups. Interaction between modes in the same group is strong while scattering between modes from different groups and leakage into volume are much weaker. This leads to a non-trivial long-time behavior that we treat in this section. We assume that the power spectrum $\hat{R}(\mathbf{p})$ may be decomposed as $\hat{R}(\mathbf{p}) = \hat{R}_1(\mathbf{p}) + \delta^2 \hat{R}_2(\mathbf{p})$. The first term $\hat{R}_1$ is such as described in the previous section. That is to say, the waveguide modes are split into groups $\Gamma_1 = \{1, 2, \ldots, N_1\}$, $\Gamma_2 = \{N_1+1, N_2+2, \ldots, N_2\}$, $\ldots$ $\Gamma_M = \{N_{M-1}+1, N_{M-1}+2, \ldots, N_M = N\}$ so that $\hat{R}_1(\mathbf{k}-\mathbf{p}) = 0$ for $|\mathbf{k}| = k_j$, $|\mathbf{p}| = k_m$ with k and m not in the same group Γ_J. This term produces neither the leakage of Love waves into the volume nor the coupling between modes in different groups Γ_J. We use the notation $\Gamma(j)$ to denote the group Γ_J, to which j belongs. The second term $\hat{R}_2$ is assumed to be nearly constant for $|\mathbf{k}| \le 2k_N$. It introduces a weak coupling of all modes and a weak leakage into volume waves. Physically the small parameter $\delta \ll 1$ measures the relative strength of short-range versus long-range correlations. We are interested in large propagation distances over which leakage and mode coupling in other groups have order one effects and rescale accordingly $\mathbf{x} \to \mathbf{x}/\delta$. The densities $w_1, w_2, \ldots, w_N$ form a system of equations

$$\delta\, \Omega'(k_j)\hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} w_j(\mathbf{x}, \phi_{\mathbf{k}}) = \sum_{m \in \Gamma(j)} \int_0^{2\pi} \frac{d\phi_{\mathbf{p}}}{2\pi} \sigma_{jm}(\phi_{\mathbf{k}}, \phi_{\mathbf{p}})[w_m(\mathbf{x}, \phi_{\mathbf{p}}) - w_j(\mathbf{x}, \phi_{\mathbf{k}})] \tag{32}$$

$$+\delta^2 \sum_{m \notin \Gamma(j)} \int_0^{2\pi} \frac{d\phi_{\mathbf{p}}}{2\pi} \sigma_{jm}(\phi_{\mathbf{k}}, \phi_{\mathbf{p}})[w_m(\mathbf{x}, \phi_{\mathbf{p}}) - w_j(\mathbf{x}, \phi_{\mathbf{k}})] - \delta^2 M_j(\phi_{\mathbf{k}}) w_j(\mathbf{x}, \phi_{\mathbf{k}}) + \delta^2 S_j(\mathbf{x}, \phi_{\mathbf{k}})$$

for $1 < j \le N$. Here $\mathbf{k} = k_j(\cos\phi_{\mathbf{k}}, \sin\phi_{\mathbf{k}})$ and $\mathbf{p} = k_m(\cos\phi_{\mathbf{p}}, \sin\phi_{\mathbf{p}})$. The differential scattering cross-sections σ_{jm} for j and m in the same group are given by

$$\sigma_{jm}(\phi_{\mathbf{k}}, \phi_{\mathbf{p}}) = \left[\hat{R}_1(\mathbf{k}-\mathbf{p}) + \delta^2 \hat{R}_2(\mathbf{k}-\mathbf{p})\right] |\alpha(\mathbf{k}\cdot\mathbf{p})|^2 k_m |\Omega'(k_m)| \quad \text{if } \Gamma(j) = \Gamma(m)$$

with $\mathbf{k}$ and $\mathbf{p}$ as above, and by

$$\sigma_{jm}(\phi_{\mathbf{k}}, \phi_{\mathbf{p}}) = \hat{R}_2(\mathbf{k}-\mathbf{p}) |\alpha(\mathbf{k}\cdot\mathbf{p})|^2 k_m |\Omega'(k_m)|, \quad \text{if } \Gamma(j) \ne \Gamma(m)$$

for j and m in two different groups of modes. We have added on the right side of (32) a source term that has some effect only over the large distances of propagation we consider in this section. Using techniques similar to those in [4], we can show that the system of transport equations (32) admits a unique solution of bounded energy (the $|w_j|$ are of bounded integral) independent of the parameter δ (because intrinsic absorption $M_j > 0$).

We consider the asymptotic expansion

$$w_j(\mathbf{x}, \phi) = w_j^0(\mathbf{x}, \phi) + \delta\, w_j^1(\mathbf{x}, \phi) + \delta^2 w_j^2(\mathbf{x}, \phi) + \ldots$$

and insert it into (32). The leading order term $O(1)$ in (32) implies that for each group Γ_J, $J = 1, \ldots, M$ there exists a function $\bar{w}_0^J(\mathbf{x})$ such that

$$w_m^0(\mathbf{x}, \phi) = \bar{w}_0^J(\mathbf{x}), \quad \text{if } m \in \Gamma_J. \tag{33}$$

Thus the leading order term is independent of the angular variable $\phi_{\mathbf{k}}$ and is the same for all modes in each group. The next order term in (32) implies that for $j \in \Gamma_J$ we have

$$\Omega'(k_j)\hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} \bar{w}_0^J(\mathbf{x}) = \sum_{m \in \Gamma_J} \int_0^{2\pi} \frac{d\phi_{\mathbf{p}}}{2\pi} \sigma_{jm}(\phi_{\mathbf{k}}, \phi_{\mathbf{p}})[w_m^1(\mathbf{x}, \phi_{\mathbf{p}}) - w_j^1(\mathbf{x}, \phi_{\mathbf{k}})]. \tag{34}$$

Note that for each j and m in the same group Γ_J there exists a constant λ_{jm} such that

$$\lambda_{jm}\hat{\mathbf{k}} = \int_0^{2\pi} \frac{d\phi_{\mathbf{p}}}{2\pi} \sigma_{jm}(\phi_{\mathbf{k}}, \phi_{\mathbf{p}})\hat{\mathbf{p}}.$$

The function $\hat{\mathbf{k}}$ is an eigenfunction of these operators because σ_{jm} depend only on the inner product $\mathbf{k} \cdot \mathbf{p}$, and thus $\sigma(\phi_{\mathbf{k}} + \gamma, \phi_{\mathbf{p}} + \gamma) = \sigma(\phi_{\mathbf{k}}, \phi_{\mathbf{p}})$. Moreover $\sigma(\phi_{\mathbf{k}}, 0)$ depends only on $\cos\phi_{\mathbf{k}}$. This yields

$$\int_0^{2\pi} \frac{d\phi_{\mathbf{p}}}{2\pi} \sigma_{jm}(\phi_{\mathbf{k}}, \phi_{\mathbf{p}}) \begin{pmatrix} \cos\phi_{\mathbf{p}} \\ \sin\phi_{\mathbf{p}} \end{pmatrix} = \int_0^{2\pi} \frac{d\phi_{\mathbf{p}}}{2\pi} \sigma_{jm}(\phi_{\mathbf{k}} - \phi_{\mathbf{p}}, 0) \begin{pmatrix} \cos\phi_{\mathbf{p}} \\ \sin\phi_{\mathbf{p}} \end{pmatrix}$$
$$= \int_0^{2\pi} \frac{d\phi_{\mathbf{p}}}{2\pi} \sigma_{jm}(\phi_{\mathbf{p}}, 0) \begin{pmatrix} \cos(\phi_{\mathbf{k}} - \phi_{\mathbf{p}}) \\ \sin(\phi_{\mathbf{k}} - \phi_{\mathbf{p}}) \end{pmatrix} = \int_0^{2\pi} \frac{d\phi_{\mathbf{p}}}{2\pi} \sigma_{jm}(\phi_{\mathbf{p}}, 0) \cos\phi_{\mathbf{p}} \begin{pmatrix} \cos\phi_{\mathbf{k}} \\ \sin\phi_{\mathbf{k}} \end{pmatrix}$$

because

$$\int_0^{2\pi} \frac{d\phi_{\mathbf{p}}}{2\pi} \sigma_{jm}(\phi_{\mathbf{p}}, 0) \sin\phi_{\mathbf{p}} = 0.$$

Thus λ_{jm} are given explicitly by

$$\lambda_{jm} = \int_0^{2\pi} \frac{d\phi_{\mathbf{p}}}{2\pi} \sigma_{jm}(\phi_{\mathbf{p}}, 0) \cos\phi_{\mathbf{p}}$$

and may be interpreted as the average cosine of the angle between the incident wave in mode m and the scattered wave in mode j. Therefore we have

$$w_j^1(\mathbf{x}, \phi) = -c_j \hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} \bar{w}_0^J(\mathbf{x}) + w_j^{10}(\mathbf{x}), \quad j \in \Gamma_J$$

with an arbitrary function $w_j^{10}(\mathbf{x})$ that may determined from solvability conditions for higher order terms and whose knowledge is not necessary to derive the diffusion limit. The constants c_j are determined from the following linear system of equations

$$\Omega'(k_j) = Z_j c_j - \sum_{m \in \Gamma_J} \lambda_{jm} c_m, \quad j \in \Gamma_J. \tag{35}$$

Here Z_j is the total scattering cross-section for the j-th mode:

$$Z_j = \sum_{m \in \Gamma_J} \int_0^{2\pi} \frac{d\phi_{\mathbf{p}}}{2\pi} \sigma_{jm}(\phi_{\mathbf{k}}, \phi_{\mathbf{p}}), \quad j \in \Gamma_J.$$

The linear system (35) is solvable. Indeed, we may symmetrize it as

$$k_j|\Omega'(k_j)|\Omega'(k_j) = \tilde{Z}_j c_j - \sum_{m\in\Gamma_J} \tilde{\lambda}_{jm} c_m, \quad j \in \Gamma_J$$

with $\tilde{Z}_j = k_j|\Omega'(k_j)|Z_j$ and $\tilde{\lambda}_{jm} = k_j|\Omega'(k_j)|\lambda_{jm}$. Let us introduce the matrices $\tilde{Z} = \mathrm{diag}(\tilde{Z}_j)$ and $\tilde{\Lambda} = \left(\tilde{\lambda}_{jm}\right)$. The matrix $\tilde{\Lambda}$ is symmetric and $\tilde{Z}_j > \sum_{m\in\Gamma_J}\left|\tilde{\lambda}_{jm}\right|$ for all $j \in \Gamma_J$. Therefore the matrix $\tilde{Z} - \tilde{\Lambda}$ is positive-definite by the Gershgorin theorem and the system (35) has a unique solution.

The order $O(\delta^2)$ term in (32) implies that for $j \in \Gamma_J$ we have

$$\begin{aligned}\Omega'(k_j)\hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}} w_j^1(\mathbf{x}) &= \sum_{m\in\Gamma_J}\int_0^{2\pi}\frac{d\phi_{\mathbf{p}}}{2\pi}\sigma_{jm}(\phi_{\mathbf{k}},\phi_{\mathbf{p}})[w_m^2(\mathbf{x},\phi_{\mathbf{p}}) - w_j^2(\mathbf{x},\phi_{\mathbf{k}})] \\ &+ \sum_{m\notin\Gamma_J}\int_0^{2\pi}\frac{d\phi_{\mathbf{p}}}{2\pi}\sigma_{jm}(\phi_{\mathbf{k}},\phi_{\mathbf{p}})[\bar{w}_m^0(\mathbf{x}) - \bar{w}_0^J(\mathbf{x})] - M_j\bar{w}_0^J + S_j(\mathbf{x},\phi_{\mathbf{k}}).\end{aligned} \tag{36}$$

The solvability condition for (36) is

$$\begin{aligned}&\sum_{j\in\Gamma_J}|\Omega'(k_j)|\Omega'(k_j)k_j\int_0^{2\pi}\frac{d\phi_{\mathbf{k}}}{2\pi}\hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}} w_j^1(\mathbf{x}) \\ &= \sum_{j\in\Gamma_J}\sum_{m\notin\Gamma_J}|\Omega'(k_j)|k_j\int_0^{2\pi}\frac{d\phi_{\mathbf{p}}}{2\pi}\sigma_{jm}(\phi_{\mathbf{k}},\phi_{\mathbf{p}})[\bar{w}_m^0(\mathbf{x}) - \bar{w}_0^J(\mathbf{x})] \\ &- \sum_{j\in\Gamma_J}|\Omega'(k_j)|k_j\int_0^{2\pi}\frac{d\phi_{\mathbf{p}}}{2\pi}M_j(\phi_{\mathbf{k}})w_0(\mathbf{x}) + \sum_{j\in\Gamma_J}|\Omega'(k_j)|k_j\int_0^{2\pi}d\phi_{\mathbf{k}}S_j(\mathbf{x},\phi_{\mathbf{k}}).\end{aligned}$$

This leads to the following system of diffusion equations for $w_0(\mathbf{x})$, $\mathbf{x}\in I\!R^2$:

$$-D_J\Delta\bar{w}_0^J + (K_J + \bar{M}_J)\bar{w}_0^J = \sum_{P\neq J}K_{JP}\bar{w}_0^P + \bar{S}_J, \quad J = 1,\ldots,M \tag{37}$$

with the summation on the right side being over groups other than Γ_J. Here the diffusion coefficients are

$$D_J = \sum_{j\in\Gamma_J}\frac{|\Omega'(k_j)|\Omega'(k_j)c_jk_j}{2}.$$

The diffusion coefficients are all positive since they may be written as

$$D_J = \frac{1}{2}\left(\left(\tilde{Z} - \tilde{\Lambda}\right)\mathbf{c}\cdot\mathbf{c}\right) > 0, \quad \mathbf{c} = (c_j),\ j\in\Gamma_J$$

because the matrix $\tilde{Z} - \tilde{\Lambda}$ is positive-definite. Here c_j solve the linear system (35). The effective source term is

$$\bar{S}_J(\mathbf{x}) = \sum_{j\in\Gamma_J}|\Omega'(k_j)|k_j\int_0^{2\pi}d\phi_{\mathbf{k}}S_j(\mathbf{x},\phi_{\mathbf{k}}).$$

The constants K_{JP} and K_J are given by

$$K_{JP} = \sum_{j\in\Gamma_J}\sum_{m\in\Gamma_P} |\Omega'(k_j)|k_j \int_0^{2\pi} \frac{d\phi_{\mathbf{p}}}{2\pi}\sigma_{jm}(\phi_{\mathbf{k}},\phi_{\mathbf{p}}), \qquad K_J = \sum_{P\neq J} K_{JP},$$

and the attenuation coefficients by

$$\bar{M}_J = \sum_{j\in\Gamma_J} |\Omega'(k_j)|k_j \int_0^{2\pi} \frac{d\phi_{\mathbf{p}}}{2\pi} M_j(\phi_{\mathbf{k}}).$$

Energy propagation in the diffusive regime is described by the system of diffusion equations (37). Because the intrinsic absorption coefficients $\bar{M}_J > 0$, the system (37) is well-posed.

4.3 Energy equipartition

Expression (33) implies that in the diffusive limit the distribution of energy between various modes of the Love waves inside each group becomes universal and independent of the power spectrum $\hat{R}(\mathbf{k})$ of the rough surface. However, the power spectrum determines which modes are strongly coupled and to which group modes belong. Indeed, according to (19), the energy contained in the j-th mode is equal to

$$\mathcal{E}_j = 2\pi k_j|\Omega'(k_j)|\bar{w}_0^J(\mathbf{x}), \quad j\in\Gamma_J.$$

Therefore the ratios

$$\frac{\mathcal{E}_j}{\mathcal{E}_m} = \frac{k_j|\Omega'(k_j)|}{k_m|\Omega'(k_m)|} \tag{38}$$

for j and m in the same group Γ_J depend only on the background problem in the homogeneous medium that determines the levels k_j and the function $\Omega(k)$. This is very similar to the P-to-S energy equipartition of the compressional and shear elastic waves analyzed in [17, 18, 20]. In both cases, equipartition of energy is caused by multiple scattering.

5 Conclusions

We have analyzed multiple scattering for the acoustic approximation of Love waves propagating in a layer with rough upper boundary in the radiative transport regime. Wave energy propagation is described by a coupled system of radiative transport equations (24) for energy densities of different waveguide modes. These equations have terms that account both for scattering into and from other waveguide modes as well as effective absorption terms caused by leakage into volume waves. This leakage may be suppressed for certain types of power spectra. Then, the energy contained in each mode is partitioned in a universal way (38) that is independent of the rough surface. The power spectrum determines only which group modes belong to. Inside each group, energy is partitioned in a universal way. Energy propagation in such situation is described by the system of diffusion equations (37). When leakage is not weak, the total energy decays exponentially over large distances of propagation.

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