Diffusive Energy Scattering from Weakly Random Surfaces

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Abstract

We derive transport theoretic boundary conditions for acoustic wave reflection at a weakly rough boundary in an inhomogeneous half space. We use the Wigner distribution to go from waves to energy transport in the high frequency limit. We generalize known results on the reflection of acoustic plane waves in a homogeneous medium. We analyze higher order corrections, which include a enhanced backscattering effect in the back direction.

1 Introduction

Wave propagation in weakly fluctuating random media over distances large compared to the wavelength can be described by incoherent energy transport. This is the radiative transport regime [3, 14]. Near boundaries and interfaces, waves undergo coherent or partially coherent reflection. Angularly resolved energy reflection and transmission in homogeneous media in average, has been studied extensively in the past [4, 6, 12, 18]. Recently, the problem has been revisited in the transport theoretic context using a plane wave decomposition; see [2]. There, the boundary conditions for the transport equations in a domain with rough boundaries of small amplitude and homogeneous background are derived. In this paper, we derive boundary conditions in the case of inhomogeneous domains. We systematically use the Wigner transform to study the reflection of angularly resolved acoustic energy density from a Dirichlet surface. The scale of the volume inhomogeneities

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is large compared to the wave length. The boundary conditions are a direct generalization of those obtained in the case of a homogeneous medium, with a reflection operator depending upon the position at the boundary. Our main ingredient is a perturbation analysis around the flat boundary case studied in [15].

1.1 Transport equations for the energy density

As we recall in Section 2, the phase space acoustic energy density $\mu(\mathbf{x},\mathbf{k})$ satisfies the transport equation

$$\nabla_{\mathbf{k}}\omega_+ \cdot \nabla_{\mathbf{x}}\mu - \nabla_{\mathbf{x}}\omega_+ \cdot \nabla_{\mathbf{k}}\mu = 0. \tag{1.1}$$

Here $\omega_+(\mathbf{x},\mathbf{k})$ is the eigenfrequency of the acoustic waves given by

$$\omega_+ = c|\mathbf{k}|, \quad c = \frac{1}{\sqrt{\kappa\rho}}, \tag{1.2}$$

where $\rho(\mathbf{x})$ is the density and $\kappa(\mathbf{x})$ is the compressibility of the background medium. These equations hold in the high frequency regime for monochromatic waves, when the wavelength is much smaller than the variations of the density and compressibility. They were derived in domains without boundaries or interfaces by several authors in the context of geometrical optics (see for instance [8]).

1.2 Scattering from a rough boundary in a homogeneous medium

In the previous section, the free transport equation (1.1) is posed in the whole space, with no boundary. In realistic applications, it is important to consider problems in bounded domains with correct boundary conditions.

We assume that the domain is given by $H_\varepsilon = \left\{\mathbf{x} \in R^n : x_n > \varepsilon\eta h\left(\frac{\mathbf{x}'}{\varepsilon}\right)\right\}$. The function $h(\mathbf{x}')$ is a mean zero stationary random process with covariance function $R(\mathbf{y}')$ defined by

$$\langle h(\mathbf{x}'+\mathbf{y}')h(\mathbf{x}')\rangle = R(\mathbf{y}'). \tag{1.3}$$

The power spectrum $\hat{R}(\mathbf{k}')$ is the Fourier transform of $R(\mathbf{y}')$:

$$R(\mathbf{y}') = \int \frac{d\mathbf{p}}{(2\pi)^{d-1}} e^{i\mathbf{p}'\cdot\mathbf{y}'}\hat{R}(\mathbf{p}'). \tag{1.4}$$

The boundary of our domain is varying on the scale of the wavelength ε, which gives rise to scattering of incoherent, or diffuse, energy from the boundary. The small parameter $\eta \ll 1$ is measuring the height of the surface relative to the wavelength.

One finds by a direct computation using a plane wave decomposition that the average energy of reflected waves going in the direction $\mathbf{k}^+(\mathbf{k}')$ is given, for Dirichlet boundary conditions, by

$$\mu_{out}(\mathbf{x}',\mathbf{k}') = \mu_{out}^{spec} + \mu_{out}^{diff} = \left(1 - \eta^2 \int \frac{d\mathbf{p}'}{(2\pi)^{n-1}} \hat{R}(\mathbf{k}' - \mathbf{p}') p_n^+ k_n^+ \right) \mu_{in}(\mathbf{x}',\mathbf{k}') \quad (1.5)$$
$$+\eta^2 \int \frac{d\mathbf{p}'}{(2\pi)^{n-1}} \hat{R}(\mathbf{k}' - \mathbf{p}') (p_n^+)^2 \mu_{in}(\mathbf{x}',\mathbf{p}').$$

Here we have defined for every horizontal wave vector $\mathbf{k}'$:

$$\mathbf{k}^\pm(\mathbf{k}') = (\mathbf{k}', k_n^\pm), \quad |\mathbf{k}'|^2 + (k_n^\pm)^2 = K^2 = \frac{\omega_+^2}{c^2}. \quad (1.6)$$

Notice that $\mathbf{k}^+$ corresponds to outgoing waves, and $\mathbf{k}^-$ corresponds to incoming waves.

A detailed computation both for the Dirichlet and Neumann problems can be found, for instance, in references [4, 6, 12, 18]. The first term in (1.5) represents the specular reflection including a correction due to surface roughness. The second term is produced by the diffuse scattering from the rough boundary.

The plane wave decomposition used in this calculation is not available in non homogeneous media. Thus one cannot directly generalize the above calculations to variable media. We have recently derived in [15] transport boundary conditions for the energy in inhomogeneous media with boundaries which vary on a large scale compared to the wavelength. We treat here the Dirichlet problem in an inhomogeneous domain with rough boundary for $\eta \ll 1$ as a perturbation of the smooth boundary considered in [15]. We show that the results of [15] allow us to compute the higher corrections in η of the reflected energy, which coincide with (1.5) in the case of a homogeneous medium.

It is known that the Born expansion we use in this paper diverges for the Neumann and impedance problems at grazing angles and we do not consider them here. However, our method can be adapted to incorporate the smoothing method [2, 17, 19]. Then, we can reproduce the results obtained in [9, 13] for uniform media including the coherent backscattering effect, now in the setup of a variable media.

The paper is organized as follows. In Section 2, we review the results of [15] and some basic facts from the Wigner distribution theory. We state our results for rough boundaries in Section 3. The perturbation expansion is treated in Section 4 up to second order in η. Finally, we investigate higher order corrections in Section 5.

2 Energy propagation in a half space

2.1 The Wigner distribution

One way to describe scattering of phase space resolved energy is to consider the Wigner distribution matrix of the family $\mathbf{w}_\varepsilon$ of the solutions of (2.6). A detailed exposition to the theory of Wigner distributions can be found in [7, 15]. Given a family of vector-valued functions $\mathbf{u}_\varepsilon(\mathbf{x})$ bounded in $L^2(R^n)$, its Wigner transform matrix is defined by

$$\mathbf{W}_\varepsilon(\mathbf{x},\mathbf{k}) = \int \frac{d\mathbf{y}}{(2\pi)^n} e^{i\mathbf{k}\cdot\mathbf{y}} \mathbf{u}_\varepsilon(\mathbf{x} - \frac{\varepsilon\mathbf{y}}{2})\mathbf{u}_\varepsilon^*(\mathbf{x} + \frac{\varepsilon\mathbf{y}}{2}). \tag{2.1}$$

The family $\mathbf{W}_\varepsilon$, possibly after extracting a subsequence, has a weak limit as $\varepsilon \to 0$ in the sense of Schwartz distributions. The limit matrix $\mathbf{W}(\mathbf{x},\mathbf{k})$, called the Wigner distribution, has a number of important properties, which we summarize in the following proposition.

Proposition 1 *The matrix* $\mathbf{W}(\mathbf{x},\mathbf{k})$ *is self-adjoint and non-negative. Given a bounded continuous function* $\theta(\mathbf{x})$*, the Wigner distribution of the family* $g_\varepsilon(\mathbf{x}) = \theta(\mathbf{x})f_\varepsilon(\mathbf{x})$ *is*

$$\mathbf{W}[g_\varepsilon](\mathbf{x},\mathbf{k}) = \theta(\mathbf{x})\mathbf{W}[f_\varepsilon](\mathbf{x},\mathbf{k})\theta^*(\mathbf{x}). \tag{2.2}$$

Given a differential operator $L(\mathbf{x}, D)$ *with smooth coefficients, the Wigner matrix of the family* $p_\varepsilon(\mathbf{x}) = L(\mathbf{x}, \varepsilon D)f_\varepsilon(\mathbf{x})$ *is*

$$\mathbf{W}[p_\varepsilon](\mathbf{x},\mathbf{k}) = L(\mathbf{x}, i\mathbf{k})\mathbf{W}[f_\varepsilon](\mathbf{x},\mathbf{k})[L(\mathbf{x}, i\mathbf{k})]^*. \tag{2.3}$$

If the family f_ε *is* ε*–oscillatory [7], then*

$$Tr \int \mathbf{W}(\mathbf{x}, d\mathbf{k}) = \lim_{\varepsilon\to 0} |\mathbf{u}_\varepsilon(\mathbf{x})|^2. \tag{2.4}$$

The property (2.2) allows one to consider the Wigner distributions of families bounded in $L^2_{loc}(R^n)$. This is important in dealing with time-harmonic solutions of the acoustic equations. The property (2.3) allows one to deal with high frequency waves in variable media. The last property shows that the Wigner matrix captures the energy of high frequency waves, which oscillate at a frequency of order ε^{-1} at most.

One can also consider the Wigner matrix of two different families $\mathbf{u}_\varepsilon$, $\mathbf{v}_\varepsilon$, defined as the limit of

$$\mathbf{W}[\mathbf{u}_\varepsilon, \mathbf{v}_\varepsilon](\mathbf{x},\mathbf{k}) = \int \frac{d\mathbf{y}}{(2\pi)^n} e^{ik\cdot\mathbf{y}} \mathbf{u}_\varepsilon(\mathbf{x} - \frac{\varepsilon\mathbf{y}}{2})\mathbf{v}_\varepsilon^*(\mathbf{x} + \frac{\varepsilon\mathbf{y}}{2}).$$

It has similar properties [7].

2.2 Acoustic wave transport

The time harmonic acoustic equations for the acoustic velocity $\mathbf{v} = (v_1, \ldots, v_n)$, $n = 2, 3$, and pressure p, in the high frequency regime, are

$$\begin{aligned} \varepsilon \nabla p_\varepsilon &= i\omega\rho(\mathbf{x})\mathbf{v}_\varepsilon \\ \varepsilon \nabla \cdot \mathbf{v}_\varepsilon &= i\omega\kappa(\mathbf{x})p_\varepsilon . \end{aligned} \tag{2.5}$$

Here ρ and κ are the density and compressibility of the medium, and the parameter $\varepsilon \ll 1$ is the ratio of the wave length to the typical scale of the variations of $\rho(\mathbf{x})$ and $\kappa(\mathbf{x})$. Equations (2.5) can be rewritten as a reduced symmetric hyperbolic system

$$\sum_{j=1}^{n} \varepsilon D^j \frac{\partial \mathbf{w}_\varepsilon}{\partial x_j} - i\omega A(\mathbf{x})\mathbf{w}_\varepsilon = 0 \tag{2.6}$$

for the vector $\mathbf{w}_\varepsilon = (\mathbf{v}_\varepsilon, p_\varepsilon) \in C^{n+1}$. Here the diagonal matrix $A(\mathbf{x}) = \mathrm{diag}(\rho, \rho, \rho, \kappa)$, for $n = 3$, and the symmetric matrices D^j are defined appropriately from (2.5).

Let $\mathbf{w}_\varepsilon$ be an ε-oscillatory family of solutions of (2.6) bounded in $L^2_{loc}(R^n)$. Then the limit Wigner matrix $\mathbf{W}(\mathbf{x}, \mathbf{k})$ is described as follows. The dispersion matrix of the system (2.6) is (in three dimensions)

$$L = A^{-1} \sum_{j=1}^{n} k_j D^j = \begin{pmatrix} 0 & 0 & 0 & k_1/\rho \\ 0 & 0 & 0 & k_2/\rho \\ 0 & 0 & 0 & k_3/\rho \\ k_1/\kappa & k_2/\kappa & k_3/\kappa & 0 \end{pmatrix}. \tag{2.7}$$

The eigenvector $\mathbf{b}$ of the matrix L that corresponds to forward propagating waves is

$$\mathbf{b} = \left(\frac{\hat{\mathbf{k}}}{\sqrt{2\rho(\mathbf{x})}}, \frac{1}{\sqrt{2\kappa(\mathbf{x})}} \right), \tag{2.8}$$

where $\hat{\mathbf{k}} = \dfrac{\mathbf{k}}{|\mathbf{k}|}$. The corresponding eigenfrequency is given by

$$\omega_+ = c(\mathbf{x})|\mathbf{k}|, \quad c(\mathbf{x}) = \frac{1}{\sqrt{\kappa(\mathbf{x})\rho(\mathbf{x})}}. \tag{2.9}$$

The solutions of (2.6) have frequency ω, thus we introduce the resonant wave number $K(\mathbf{x}) = \omega\sqrt{\kappa(\mathbf{x})\rho(\mathbf{x})}$. Then, the positive definite matrix $\mathbf{W}(\mathbf{x}, \mathbf{k})$ has the form

$$\mathbf{W}(\mathbf{x}, \mathbf{k}) = \mu(\mathbf{x}, \mathbf{k})\mathbf{b}(\mathbf{x}, \mathbf{k})\mathbf{b}^*(\mathbf{x}, \mathbf{k}).$$

Here the scalar measure $\mu(\mathbf{x}, \mathbf{k})$ is supported on the set

$$\mathcal{S} = \{(\mathbf{x}, \mathbf{k}) : \ |\mathbf{k}| = K(\mathbf{x})\} . \tag{2.10}$$

This statement is a generalization of the eikonal equation of geometrical optics. The measure μ satisfies the transport equation (1.1) [7, 14].

The scalar measure $\mu(\mathbf{x}, \mathbf{k})$ can be considered as the phase space resolved energy density of acoustic waves in the high frequency limit. Namely, the high frequency limit of the physical space energy density is given by [14]

$$\lim_{\varepsilon \to 0} \mathcal{E}_\varepsilon(\mathbf{x}) = \int \mu(\mathbf{x}, d\mathbf{k}).$$

Here the acoustic energy density is

$$\mathcal{E}_\varepsilon(\mathbf{x}) = \frac{\rho|\mathbf{v}_\varepsilon|^2}{2} + \frac{\kappa|p_\varepsilon|^2}{2}.$$

The limit energy flux $\mathcal{F} = \dfrac{1}{2}\left(p_\varepsilon \bar{\mathbf{v}}_\varepsilon + \bar{p}_\varepsilon \mathbf{v}_\varepsilon\right)$ is expressed via μ by

$$\lim_{\varepsilon \to 0} \mathcal{F}_\varepsilon(\mathbf{x}) = c(\mathbf{x}) \int \hat{\mathbf{k}} \mu(\mathbf{x}, d\mathbf{k}).$$

2.3 Acoustic energy transport in a half space

We review here briefly the results of [15]. Let $\mathbf{w}_\varepsilon$ be an ε-oscillatory family of solutions of the acoustic equations (2.6) in the half space $x_n > 0$ (without imposing any boundary conditions yet). We assume that the family $\mathbf{w}_\varepsilon(\mathbf{x})$ and the family

$$\mathbf{r}_\varepsilon(\mathbf{x}') = \mathbf{w}_\varepsilon(\mathbf{x}', 0) \tag{2.11}$$

of its boundary traces are bounded in $L^2_{loc}(R^n)$ and $L^2_{loc}(R^{n-1})$, respectively. Here and below we use the notation $\mathbf{x}'$, $\mathbf{k}'$ for the position on the boundary and the tangential component of the wave vector, respectively. In addition we assume that the measure of the resonant set $B_{cr} = \{(\mathbf{x}', \mathbf{k}') \in R^{n-1} \times R^{n-1} : \ c(\mathbf{x}')|\mathbf{k}'| = \omega\}$ of grazing rays with respect to the Wigner measure $\boldsymbol{\nu}(\mathbf{x}', \mathbf{k}')$ of the family $\mathbf{r}_\varepsilon$ is zero. The grazing rays in the phase space energy context were studied recently in [10, 11] and we avoid these technical complications. Physically, our assumption means that grazing rays are not charged. Then we have the following proposition.

Proposition 2 *Under the above mentioned assumptions the following holds:*

(i) The Wigner matrix $\boldsymbol{\nu}(\mathbf{x}',\mathbf{k})$ of the family $\mathbf{r}_\varepsilon(\mathbf{x}')$ has the form

$$\boldsymbol{\nu} = \nu_\alpha \mathbf{b}(\mathbf{k}_-)\mathbf{b}^*(\mathbf{k}_-) + \nu_\beta \mathbf{b}(\mathbf{k}_+)\mathbf{b}^*(\mathbf{k}_+) + \nu_{\alpha\beta}\mathbf{b}(\mathbf{k}_-)\mathbf{b}^*(\mathbf{k}_+) + \bar{\nu}_{\alpha\beta}\mathbf{b}(\mathbf{k}_+)\mathbf{b}^*(\mathbf{k}_-) \tag{2.12}$$

with $\nu_{\alpha,\beta,\alpha\beta}$ being distributions so that the matrix $\begin{pmatrix} \nu_\alpha & \nu_{\alpha\beta} \\ \bar{\nu}_{\alpha\beta} & \nu_\beta \end{pmatrix}$ is positive definite.

(ii) The Wigner matrix $\mathbf{W}(\mathbf{x},\mathbf{k})$ of the family $\mathbf{w}_\varepsilon(\mathbf{x})$ has the form

$$\mathbf{W}(\mathbf{x},\mathbf{k}) = \mu(\mathbf{x},\mathbf{k})\mathbf{b}(\mathbf{x},\mathbf{k})\mathbf{b}^*(\mathbf{x},\mathbf{k}). \tag{2.13}$$

The scalar measure μ is supported on the set (2.10) and satisfies weakly the transport equation

$$\nabla_{\mathbf{k}}\omega_+ \cdot \nabla_{\mathbf{x}}\mu - \nabla_{\mathbf{x}}\omega_+ \cdot \nabla_{\mathbf{k}}\mu = c\hat{k}_n(\mu_{in}\delta(k_n - k_n^-) + \mu_{out}\delta(k_n - k_n^+)). \tag{2.14}$$

Here the measures μ_{in} and μ_{out} are given by

$$\mu_{in}(\mathbf{x}',\mathbf{k}') = \nu_\alpha(\mathbf{x}',\mathbf{k}'), \qquad \mu_{out}(\mathbf{x}',\mathbf{k}') = \nu_\beta(\mathbf{x}',\mathbf{k}'). \tag{2.15}$$

with ν_α, ν_β defined by (2.12), $\omega_+(\mathbf{x},\mathbf{k}) = c(\mathbf{x})|\mathbf{k}|$ is given by (1.2), and the wave vector $\mathbf{k}^\pm$ is defined by (1.6) with $K = K(\mathbf{x}') = \omega\sqrt{\kappa(\mathbf{x}')\rho(\mathbf{x}')}$.

Note also that if the measure μ is continuous up to the boundary $x_n = 0$, then the weak form (2.14) is equivalent to the boundary value problem:

$$\nabla_{\mathbf{k}}\omega_+ \cdot \nabla_{\mathbf{x}}\mu - \nabla_{\mathbf{x}}\omega_+ \cdot \nabla_{\mathbf{k}}\mu = 0 \tag{2.16}$$

$$\mu(\mathbf{x}',0,\mathbf{k}) = \mu_{in}\delta(k_n - k_n^-) + \mu_{out}\delta(k_n - k_n^+). \tag{2.17}$$

Thus we can interpret ν_α as the phase space resolved energy of the incoming waves at the boundary, and ν_β as the energy of the outgoing waves.

2.4 Boundary conditions for the transport equation

The boundary conditions for the transport equation (2.16) can be obtained from those for the acoustic equations (2.6), by using the relations (2.15) and (2.17). Consider the Dirichlet boundary condition in the upper half space

$$w_{\varepsilon,n+1}(\mathbf{x}',0) = 0. \tag{2.18}$$

Then the (n+1)-row and column of the matrix $\boldsymbol{\nu}$ vanish. Using (2.12), the explicit form (2.8) of the eigenvectors $\mathbf{b}(\mathbf{x}, \mathbf{k})$, and (1.6), we get

$$\nu_\alpha = \nu_\beta = -\nu_{\alpha\beta}. \tag{2.19}$$

Then (2.17) implies that

$$\mu(\mathbf{x}', 0, \mathbf{k}', k_n) = \mu(\mathbf{x}', 0, \mathbf{k}', -k_n), \tag{2.20}$$

which is the boundary condition for (2.16).

The Neumann boundary condition

$$w_{\varepsilon,n}(\mathbf{x}', 0) = 0 \tag{2.21}$$

implies that the n-th row and column of the matrix $\boldsymbol{\nu}$ vanish. Then we obtain, similarly to (2.19), that

$$\nu_\alpha = \nu_\beta = \nu_{\alpha\beta}, \tag{2.22}$$

so that (2.20) still holds. A convenient way to rewrite (2.20) is

$$\mu_{out}(\mathbf{x}', \mathbf{k}') = \mu_{in}(\mathbf{x}', \mathbf{k}'),$$

so that all the energy is reflected specularly as expected.

3 Energy reflection at a rough surface

We consider scattering of acoustic waves described by (2.6) from a rough surface. The surface ∂H_ε is described by the equation $x_n = \varepsilon\eta h(\frac{\mathbf{x}'}{\varepsilon})$. The small parameter η is the ratio of the height of the surface to the wavelength. The height of the surface is varying on the scale of the wavelength ε. Recall that the random process $h(\mathbf{y}')$ has mean zero, and is stationary, with covariance function $R(\mathbf{y}')$ and with power spectrum $\hat{R}(\mathbf{k}')$, given by (1.3) and (1.4) respectively. We assume that $h(\mathbf{y}') \in C^1(R^{n-1})$ a.s. so that solutions of the Dirichlet problem exist for every positive ε. The Dirichlet boundary condition for (2.6), corresponding to a vanishing pressure, is given by

$$w_{\varepsilon,n+1}\left(\mathbf{x}', \varepsilon\eta h(\frac{\mathbf{x}'}{\varepsilon})\right) = 0. \tag{3.1}$$

We seek the solution $\mathbf{w}_\varepsilon$ of (2.6) as a power series in the parameter η:

$$\mathbf{w}_\varepsilon(\mathbf{x}) = \mathbf{w}_\varepsilon^0(\mathbf{x}) + \eta\mathbf{w}_\varepsilon^1(\mathbf{x}) + \eta^2\mathbf{w}_\varepsilon^2(\mathbf{x}) + \dots \tag{3.2}$$

with all the terms bounded in $L^2_{loc}(R^n)$ and their boundary values $\mathbf{r}_\varepsilon^j(\mathbf{x}') = \mathbf{w}_\varepsilon^j(\mathbf{x}', 0)$ bounded in $L^2_{loc}(R^{n-1})$. Our main assumption is that such an expansion exists, i.e. that the rest in (3.2) is bounded uniformly in η and ε by $o(\eta^2)$. We also assume that the medium is homogeneous above a certain height $x_n = L$ with L arbitrarily large. This assumption is purely technical and allows us to formulate an outgoing condition at infinity. Namely, for $x_n > L$, the plane wave decomposition is valid, and we assume then that the amplitudes of the incoming waves $\alpha_\varepsilon(\mathbf{k}')$ are deterministic and given. The correctors $\mathbf{w}_\varepsilon^j$ are all outgoing in that region, so that $\alpha_\varepsilon^j(\mathbf{k}') = 0$ for $j \geq 1$. The characteristics of the transport equation (2.16) are given by

$$\begin{aligned}\frac{d\mathbf{x}}{ds} &= \nabla_{\mathbf{k}}\omega_+(\mathbf{x}, \mathbf{k})\\ \frac{d\mathbf{k}}{ds} &= -\nabla_{\mathbf{x}}\omega_+(\mathbf{x}, \mathbf{k}).\end{aligned}$$

We assume that the characteristics that leave the surface $x_n = 0$ reach the level $x_n = L$ and vice versa. Moreover, we assume that these characteristics do not come back. Then we have the following theorem.

Theorem 1 *Under the above assumptions, the Wigner matrix $\mathbf{W}(\mathbf{x}, \mathbf{k})$ of the family $\mathbf{w}_\varepsilon$ of solutions of the Dirichlet problem (3.1) is supported on the set $\mathcal{S} = \{(\mathbf{x}, \mathbf{k}) :\ c(\mathbf{x})|\mathbf{k}| = \omega\}$, and has the form (2.13). The scalar measure μ has the form $\mu(\mathbf{x}, \mathbf{k}) = \mu'(\mathbf{x}, \mathbf{k}) + o(\eta^2)$. The measure $\mu'(\mathbf{x}, \mathbf{k})$ is a solution of the transport equation (2.14). The average measure $\langle\mu'_{out}(\mathbf{x}', \mathbf{k}')\rangle$ is given in terms of $\mu'_{in}(\mathbf{x}', \mathbf{k}')$ by*

$$\begin{aligned}\langle\mu'_{out}(\mathbf{x}', \mathbf{k}')\rangle =& \left[1 - 4\eta^2\int\frac{d\mathbf{p}'}{(2\pi)^{n-1}}\hat{R}(\mathbf{k}' - \mathbf{p}')p_n^+(\mathbf{x}', \mathbf{p})k_n^+(\mathbf{x}', \mathbf{k}')\right]\mu'_{in}(\mathbf{x}', \mathbf{k}')\\ &+4\eta^2\int\frac{d\mathbf{p}'}{(2\pi)^{n-1}}\hat{R}(\mathbf{k}' - \mathbf{p}')(p_n^+(\mathbf{p}'))^2\mu'_{in}(\mathbf{x}', \mathbf{p}').\end{aligned} \tag{3.3}$$

with $\mathbf{k}^+$ and $\mathbf{p}^+$ given by (1.6) with $K = K(\mathbf{x}')$. Here $K(\mathbf{x}')$, determined by $c(\mathbf{x}')K(\mathbf{x}') = \omega$, is the radius of the sphere $\mathcal{S}$ of wave vectors, on which the measure μ is supported.

The first term in (3.3) corresponds to the specular reflection and provides the correction to the reflection coefficient. The second term is the result of the diffuse scattering at the surface. It appears because the boundary is varying on the scale of the wavelength. Note that the total mean flux across the boundary $x_n = 0$ vanishes:

$$\begin{aligned}\langle \mathcal{F}_n(\mathbf{x}',0)\rangle =& c(\mathbf{x}')\int d\mathbf{k}\hat{k}_n\langle \mu(\mathbf{x}',0,\mathbf{k})\rangle = c(\mathbf{x}')\int d\mathbf{k}'[\hat{k}_n^+ + \hat{k}_n^-]\mu_{in}(\mathbf{x}',\mathbf{k}')\\ &+4\eta^2 c(\mathbf{x}')\int d\mathbf{k}'d\mathbf{p}'\hat{R}(\mathbf{k}'-\mathbf{p}')\left\{(p_n^+)^2\hat{k}_n^+\mu_{in}(\mathbf{x}',\mathbf{k}') - p_n^+k_n^+\hat{k}_n^+\mu_{in}(\mathbf{x}',\mathbf{p}')\right\} = 0.\end{aligned}$$

The expression (3.3) reduces in a homogeneous medium to (1.5) as one would expect.

The statement regarding the support and form of the matrix $\mathbf{W}(\mathbf{x},\mathbf{k})$ is proved exactly as in [15], so we will not repeat it here. We derive (3.3) in the following sections. Note that we can treat more general interface problems in a similar way, at least for incident energy fluxes away from grazing angle. The result is then a direct generalization of the formulae given in [2] for homogeneous media.

4 The perturbation analysis

Now, we show how the diffusive scattering is obtained when the wave equation satisfies Dirichlet boundary conditions. We derive (3.3) in three steps. First we analyze the asymptotic expansion (3.2) in Section 4.1. Then the diffuse part of the scattered energy is computed in Section 4.2. At last, we derive the correction to the reflection coefficient in Section 4.3.

4.1 The asymptotic expansion

We note that since the series (3.2) is asymptotic in $L^2_{loc}(R^n)$, the Wigner matrix $\mathbf{W}$ is approximated by the Wigner matrix of the sum of the first N terms up to order $o(\eta^N)$. Thus we can compute the Wigner measure of the first three terms in the expansion (3.2) in order to approximate $\mathbf{W}$ up to $o(\eta^2)$. The terms $\mathbf{w}_\varepsilon^j$, $j = 0,1,2$ solve the acoustic equations (2.6) in the upper half space with the following boundary conditions:

$$w^0_{\varepsilon,n+1}(\mathbf{x}',0) = 0 \tag{4.1}$$

$$w^1_{\varepsilon,n+1}(\mathbf{x}',0) = -h(\frac{\mathbf{x}'}{\varepsilon})\varepsilon\frac{\partial w^0_{\varepsilon,n+1}}{\partial x_n}(\mathbf{x}',0) \tag{4.2}$$

$$w^2_{\varepsilon,n+1}(\mathbf{x}',0) = -\frac{1}{2}h^2(\frac{\mathbf{x}'}{\varepsilon})\varepsilon^2\frac{\partial^2 w^0_{\varepsilon,n+1}}{\partial x_n^2}(\mathbf{x}',0) - h(\frac{\mathbf{x}'}{\varepsilon})\varepsilon\frac{\partial w^1_{\varepsilon,n+1}}{\partial x_n}(\mathbf{x}',0). \tag{4.3}$$

These boundary conditions are constructed so as to satisfy the Dirichlet boundary conditions (3.1) on the rough surface ∂H_ε up to order η^2. The boundary conditions at infinity, where the medium is homogeneous above the level $x_n = L$, are as described above: $\mathbf{w}_\varepsilon^0$ has prescribed incoming flux, and $\mathbf{w}_\varepsilon^{1,2}$ are outgoing for $x_n > L$.

The boundary conditions (4.2) and (4.3) can be rewritten using the acoustic equations for $\mathbf{w}_\varepsilon^0$ and $\mathbf{w}_\varepsilon^1$ and the Dirichlet boundary condition (4.1) as

$$r_{\varepsilon,n+1}^1(\mathbf{x}') = -i\omega\rho(\mathbf{x}')h(\frac{\mathbf{x}'}{\varepsilon})r_{\varepsilon,n}^0(\mathbf{x}') \tag{4.4}$$

and

$$r_{\varepsilon,n+1}^2(\mathbf{x}') = -\frac{i\varepsilon\omega}{2}\frac{\partial\rho}{\partial x_n}(\mathbf{x}',0)h^2(\frac{\mathbf{x}'}{\varepsilon})r_{\varepsilon,n}^0(\mathbf{x}') - i\omega\rho(\mathbf{x}')h(\frac{\mathbf{x}'}{\varepsilon})r_{\varepsilon,n}^1(\mathbf{x}'). \tag{4.5}$$

Here $\mathbf{r}_\varepsilon^j(\mathbf{x}')$ is the value of $\mathbf{w}_\varepsilon^j$ at the boundary. The Wigner matrix of the sum of the first three terms in (3.2) has the form

$$\mathbf{W} = \mathbf{W}_0 + \eta(\mathbf{W}_{01} + \mathbf{W}_{10}) + \eta^2(\mathbf{W}_1 + \mathbf{W}_{02} + \mathbf{W}_{20}) + o(\eta^2), \tag{4.6}$$

where $\mathbf{W}_0 = \mathbf{W}[\mathbf{w}_\varepsilon^0]$, $\mathbf{W}_1 = \mathbf{W}[\mathbf{w}_\varepsilon^1]$, $\mathbf{W}_{01} = \mathbf{W}[\mathbf{w}_\varepsilon^0, \mathbf{w}_\varepsilon^1]$, etc. The leading order term in the expansion (4.6) has the form

$$\mathbf{W}_0 = \mu_0(\mathbf{x},\mathbf{k})\mathbf{b}(\mathbf{x},\mathbf{k})\mathbf{b}^*(\mathbf{x},\mathbf{k}) \tag{4.7}$$

since $\mathbf{w}_\varepsilon^0$ solves the acoustic equations. The Dirichlet boundary conditions (4.1) for $\mathbf{w}_\varepsilon^0$ imply that the Wigner measure $\boldsymbol{\nu}^0$ of $\mathbf{r}_\varepsilon^0$ has the form (2.12) with the coefficients $\nu_{\alpha,\beta,\alpha\beta}$ related by (2.19):

$$\boldsymbol{\nu}^0(\mathbf{x}',\mathbf{k}') = \nu_\alpha^0(\mathbf{x}'\mathbf{k}')\left[\mathbf{b}(\mathbf{k}_+)\mathbf{b}^*(\mathbf{k}_+) + \mathbf{b}(\mathbf{k}_-)\mathbf{b}^*(\mathbf{k}_-) - \mathbf{b}(\mathbf{k}_+)\mathbf{b}^*(\mathbf{k}_-) - \mathbf{b}(\mathbf{k}_-)\mathbf{b}^*(\mathbf{k}_+)\right] \tag{4.8}$$

Then the outgoing Wigner measure μ_{out}^0 is

$$\mu_{out}^0 = \mu_{in} \tag{4.9}$$

where the measure μ_{in} is known. Notice that $\langle \mathbf{w}_\varepsilon^1(\mathbf{x})\rangle = 0$, and since $\mathbf{w}_\varepsilon^0$ is deterministic, we get

$$\langle \mathbf{W}_{01}\rangle = \langle \mathbf{W}_{10}\rangle = 0. \tag{4.10}$$

Thus we have to compute only $\langle \mathbf{W}_1\rangle$ and $\langle \mathbf{W}_{02} + \mathbf{W}_{20}\rangle$. The first term gives rise to the diffuse scattering and is treated in the following section. The second term produces the correction to the energy reflection coefficient and is considered in Section 4.3.

4.2 The diffuse reflected wave

The matrix $\mathbf{W}^1$, being the Wigner matrix of a family of solutions $\mathbf{w}_\varepsilon^1$ of the acoustic equations, has the form

$$\mathbf{W}_1 = \mu_1(\mathbf{x},\mathbf{k})\mathbf{b}(\mathbf{x},\mathbf{k})\mathbf{b}^*(\mathbf{x},\mathbf{k}).$$

The solution $\mathbf{w}_\varepsilon^1$ is outgoing at infinity. Since we assumed that characteristics do not come back to the boundary, it is equivalent to being outgoing at the boundary. Then we have $\nu_\alpha^1 = \mu_{in}^1 = 0$. Since the matrix $\begin{pmatrix} \nu_\alpha^1 & \nu_{\alpha\beta}^1 \\ \bar{\nu}_{\alpha\beta}^1 & \nu_\beta^1 \end{pmatrix}$ is non-negative and definite, we also have $\nu_{\alpha\beta}^1 = 0$. Then the boundary Wigner measure $\boldsymbol{\nu}^1$ of $\mathbf{r}_\varepsilon^1$ has the form

$$\boldsymbol{\nu}^1 = \nu_\beta^1 \mathbf{b}(\mathbf{k}_+)\mathbf{b}^*(\mathbf{k}_+) \tag{4.11}$$

and the boundary value of the measure μ_1 is

$$\mu_1(\mathbf{x}',0,\mathbf{k}) = \nu_\beta^1(\mathbf{x}',\mathbf{k}')\delta(k_n - k_n^+),$$

hence $\mu_{out}^1 = \nu_\beta^1(\mathbf{x}',\mathbf{k}')$. The measure ν_β^1 is determined as follows. We have from (4.11),

$$\nu_{n+1,n+1}^1 = \frac{1}{2\kappa(\mathbf{x}')}\nu_\beta^1, \tag{4.12}$$

where the left side is the $(n+1, n+1)$ entry of the matrix $\boldsymbol{\nu}^1(\mathbf{x}',\mathbf{k}')$. This entry is the Wigner measure of $r_{\varepsilon,n+1}^1(\mathbf{x}')$, which is explicitly given by the boundary condition (4.4). Therefore,

$$\nu_{n+1,n+1}^1 = \omega^2\rho^2(\mathbf{x}')\nu[h(\frac{\mathbf{x}'}{\varepsilon})r_{\varepsilon,n}^0]$$

and so the average $\langle \nu_{n+1,n+1}^1 \rangle$ is given by

$$\langle \nu_{n+1,n+1}^1 \rangle = \omega^2\rho^2(\mathbf{x}')\int \frac{d\mathbf{p}'}{(2\pi)^{n-1}}\hat{R}(\mathbf{k}'-\mathbf{p}')\nu_{n,n}^0(\mathbf{x}',\mathbf{p}'). \tag{4.13}$$

We use expression (4.8) for $\boldsymbol{\nu}^0$ to get

$$\langle \nu_{n+1,n+1}^1 \rangle(\mathbf{x}',\mathbf{k}') = \omega^2\rho^2(\mathbf{x}')\int \frac{d\mathbf{p}'}{(2\pi)^{n-1}}\hat{R}(\mathbf{k}'-\mathbf{p}')\frac{4\nu_\alpha^0(\mathbf{p}')p_n^{+2}}{2\rho(\mathbf{x}')|\mathbf{p}^+|^2}.$$

Insert this into (4.12) and use the relations $|\mathbf{p}^+|^2 = \dfrac{\omega^2}{c^2(\mathbf{x}')} = \omega^2\kappa(\mathbf{x}')\rho(\mathbf{x}')$ and $\mu_{in} = \nu_\alpha^0$ yields

$$\langle \mu_{out}^1 \rangle = \langle \nu_\beta^1 \rangle(\mathbf{x}',\mathbf{k}') = 4\int \frac{d\mathbf{p}'}{(2\pi)^{n-1}}\hat{R}(\mathbf{k}'-\mathbf{p}')p_n^{+2}\mu_{in}(\mathbf{x}',\mathbf{p}'). \tag{4.14}$$

This is the second term in (3.3).

4.3 Correction to the reflection coefficient

The correction to the coherent, or specular reflection coefficient arises from the term $\langle \mathbf{W}_{02} + \mathbf{W}_{20} \rangle$ in (4.6). Our analysis of this term proceeds in several steps. In Section 4.3.1 we reduce the computation to evaluating the average $\langle \boldsymbol{\nu}[\mathbf{r}^0, \mathbf{u}^1] \rangle$ of the cross Wigner distribution of $\mathbf{r}_\varepsilon^0$ and the conjugated wave function $\mathbf{u}_\varepsilon^1$ defined by (4.20). This Wigner distribution is described by Lemma 1. We use this lemma in Section 4.3.2 to derive (3.3). Finally we prove Lemma 1 in Section 4.3.3.

4.3.1 The reduction to the conjugated wave functions

The functions $\mathbf{w}_\varepsilon^0$, $\mathbf{w}_\varepsilon^2$ and $\mathbf{w}_\varepsilon^0 + \mathbf{w}_\varepsilon^2$ are solutions of the acoustic equations, so the Wigner matrices $\mathbf{W}[\mathbf{w}^0]$, $\mathbf{W}[\mathbf{w}^2]$ and $\mathbf{W}[\mathbf{w}^0 + \mathbf{w}^2]$ are all of the form (2.13), and then

$$\begin{aligned}\mathbf{W}_{02} + \mathbf{W}_{20} &= \mathbf{W}[\mathbf{w}^0 + \mathbf{w}^2] - \mathbf{W}[\mathbf{w}^0] - \mathbf{W}[\mathbf{w}^2] = (\mu[\mathbf{w}^0 + \mathbf{w}^2] - \mu[\mathbf{w}^0] - \mu[\mathbf{w}^2])\mathbf{b}(\mathbf{k})\mathbf{b}^*(\mathbf{k}) \\ &= \mu_{02}\mathbf{b}(\mathbf{k})\mathbf{b}^*(\mathbf{k}), \qquad (4.15)\end{aligned}$$

so that $\mathbf{W}_{02} + \mathbf{W}_{20}$ has the same form as $\mathbf{W}_0$ and $\mathbf{W}_1$. The value of μ_{02} on the boundary is

$$\begin{aligned}\mu_{02}(\mathbf{x}', 0, \mathbf{k}) &= (\mu[\mathbf{w}^0 + \mathbf{w}^2] - \mu[\mathbf{w}^0] - \mu[\mathbf{w}^2])(\mathbf{x}', 0, \mathbf{k}) \qquad (4.16) \\ &= (\nu_\beta[\mathbf{r}^0 + \mathbf{r}^2] - \nu_\beta[\mathbf{r}^0] - \nu_\beta[\mathbf{r}^2])(\mathbf{x}', \mathbf{k}')\delta(k_n - k_n^+) \\ &= (\nu_\beta[\mathbf{r}^0, \mathbf{r}^2] + \nu_\beta[\mathbf{r}^2, \mathbf{r}^0])\delta(k_n - k_n^+) = \nu_\beta^{02}\delta(k_n - k_n^+).\end{aligned}$$

The terms involving the incoming wave vector $\mathbf{k}_-$ cancel out since $\mathbf{w}^2$ is outgoing at the boundary and thus $\mu_{in}[\mathbf{w}^2] = 0$, and $\mu_{in}[\mathbf{w}^0 + \mathbf{w}^2] = \mu_{in}[\mathbf{w}^0]$. Thus we have to find $\nu_\beta^{02} = \nu_\beta[\mathbf{r}^0, \mathbf{r}^2] + \nu_\beta[\mathbf{r}^2, \mathbf{r}^0]$ to finish the computation. We note that the matrix Wigner measure $\boldsymbol{\nu}[\mathbf{r}^0, \mathbf{r}^2] + \boldsymbol{\nu}[\mathbf{r}^2, \mathbf{r}^0]$ has the form

$$\begin{aligned}\boldsymbol{\nu}[\mathbf{r}^0, \mathbf{r}^2] + \boldsymbol{\nu}[\mathbf{r}^2, \mathbf{r}^0] =& (\nu_\beta[\mathbf{r}^0, \mathbf{r}^2] + \nu_\beta[\mathbf{r}^2, \mathbf{r}^0])\mathbf{b}(\mathbf{k}_+)\mathbf{b}^*(\mathbf{k}_+) \qquad (4.17) \\ &+ \nu_{\alpha\beta}^{02}\mathbf{b}(\mathbf{k}_-)\mathbf{b}^*(\mathbf{k}_+) + \bar{\nu}_{\alpha\beta}^{02}\mathbf{b}(\mathbf{k}_+)\mathbf{b}^*(\mathbf{k}_-)\end{aligned}$$

since $\mathbf{r}_\varepsilon^2$ is outgoing. Evaluating the entry $(n+1, n+1)$ of both sides of (4.17), we get that ν_β^{02} is real. Moreover, evaluating the entry $(n, n+1)$ of the two sides of (4.17) we obtain

$$\mu_{out}^{02} = \nu_\beta^{02} = \frac{2\sqrt{\kappa\rho}}{\hat{k}_n^+}\mathrm{Re}\nu[r_n^0, r_{n+1}^2]. \qquad (4.18)$$

Thus we have reduced our problem to computing $\nu[r_n^0, r_{n+1}^2]$. Recall that $r_{\varepsilon,n+1}^2(\mathbf{x}')$ is given by (4.5) and observe that the first term in (4.5) vanishes in the limit $\varepsilon \to 0$ in L^2_{loc}. Then we have :

$$\begin{aligned}\nu[r_n^0, r_{n+1}^2] &= \nu[r_n^0(\mathbf{x}'), -h(\frac{\mathbf{x}'}{\varepsilon})i\omega\rho(\mathbf{x}')r_{\varepsilon n}^1(\mathbf{x}')] \\ &= i\omega\rho(\mathbf{x}')\int \frac{d\mathbf{p}'}{(2\pi)^{n-1}}\hat{h}(\mathbf{p}')\nu_{nn}[\mathbf{r}^0, \mathbf{u}^1](\mathbf{x}', \mathbf{k}', \mathbf{p}').\end{aligned} \tag{4.19}$$

Here we have introduced the conjugated wave functions $\mathbf{u}_\varepsilon^1$:

$$x\mathbf{u}_\varepsilon^1(\mathbf{x}', \mathbf{p}') = \mathbf{r}_\varepsilon(\mathbf{x}')e^{-i\mathbf{p}'\cdot\mathbf{x}'/\varepsilon}, \tag{4.20}$$

where the vector $\mathbf{p}'$ plays the role of a parameter. The Wigner matrix $\boldsymbol{\nu}[\mathbf{r}^0, \mathbf{u}^1]$ is described by the following Lemma.

Lemma 1 *The Wigner distribution matrix $\boldsymbol{\nu}[\mathbf{r}^0, \mathbf{u}^1]$ has the form*

$$\boldsymbol{\nu}[\mathbf{r}^0, \mathbf{u}^1] = \tilde{\nu}_\alpha \mathbf{b}(\mathbf{k}^+)\mathbf{b}^*((\mathbf{k}+\mathbf{p})^+) - \tilde{\nu}_\alpha \mathbf{b}(\mathbf{k}^-)\mathbf{b}^*((\mathbf{k}+\mathbf{p})^+), \tag{4.21}$$

where $\tilde{\nu}_\alpha$ is some distribution.

We prove Lemma 1 in Section 4.3.3.

4.3.2 The average reflection coefficient

We use Lemma 1 to evaluate $\nu_{nn}[\mathbf{r}^0, \mathbf{u}^1]$ in (4.19) and get

$$\langle \nu[r_n^0, h(\frac{\mathbf{x}'}{\varepsilon})r_{\varepsilon n}^1]\rangle = \int \frac{d\mathbf{p}'}{(2\pi)^{n-1}}\langle \hat{h}(\mathbf{p}')\tilde{\nu}_\alpha(\mathbf{x}', \mathbf{k}', \mathbf{p}')\rangle \frac{k_n^+(\mathbf{k}+\mathbf{p})^+}{|\mathbf{k}^+||(\mathbf{k}+\mathbf{p})^+|\rho(\mathbf{x})}. \tag{4.22}$$

The average $\langle \hat{h}(\mathbf{p})\tilde{\nu}_\alpha\rangle$ is evaluated as follows. We have from the boundary conditions (4.4)

$$u_{\varepsilon,n+1}^1(\mathbf{x}') = e^{-i\mathbf{p}'\cdot\mathbf{x}'/\varepsilon}r_{\varepsilon,n+1}^1(\mathbf{x}') = -e^{-i\mathbf{p}'\cdot\mathbf{x}'/\varepsilon}i\omega\rho(\mathbf{x}')h(\frac{\mathbf{x}'}{\varepsilon})r_{\varepsilon,n}^0(\mathbf{x}')$$

and hence

$$\langle h(\mathbf{p}')\nu_{n,n+1}[\mathbf{r}^0, \mathbf{u}^1]\rangle = i\omega\rho(\mathbf{x})\hat{R}(\mathbf{p}')\nu_{nn}^0(\mathbf{x}', \mathbf{k}'). \tag{4.23}$$

But we also have from the representation (4.21)

$$\langle \hat{h}(\mathbf{p}')\nu_{n,n+1}[\mathbf{r}^0, \mathbf{u}^1]\rangle = \langle \hat{h}(\mathbf{p}')\tilde{\nu}_\alpha\rangle \frac{k_n^+}{|\mathbf{k}^+|\sqrt{\rho\kappa}},$$

so that (4.23) implies

$$\langle \hat{h}(\mathbf{p}')\tilde{\nu}_\alpha\rangle \frac{k_n^+}{|\mathbf{k}^+|\sqrt{\rho}} = \sqrt{\kappa}\hat{R}(\mathbf{p}')i\omega\rho(\mathbf{x}')\nu_{nn}^0.$$

We insert this into (4.22) and get

$$\begin{aligned}\langle \nu[r_{\varepsilon n}^0, h(\frac{\mathbf{x}'}{\varepsilon})r_{\varepsilon n}^1]\rangle &= \int \frac{d\mathbf{p}'}{(2\pi)^{n-1}}\hat{R}(\mathbf{p}')\sqrt{\kappa\rho}i\omega\nu_{nn}^0(\mathbf{x}',\mathbf{k}')\frac{(\mathbf{k}+\mathbf{p})_n^+}{|(\mathbf{k}+\mathbf{p})^+|} \\ &= \int \frac{d\mathbf{p}'}{(2\pi)^{n-1}}\hat{R}(\mathbf{k}'-\mathbf{p}')\sqrt{\kappa\rho}i\omega\hat{p}_n^+\nu_{nn}^0(\mathbf{x}',\mathbf{k}').\end{aligned} \tag{4.24}$$

Then we insert (4.24) into (4.19) and obtain

$$\langle \nu[r_n^0, r_{n+1}^2]\rangle = -\omega^2\rho(\mathbf{x}')\sqrt{\kappa(\mathbf{x}')\rho(\mathbf{x}')}\int \frac{d\mathbf{p}'}{(2\pi)^{n-1}}\hat{R}(\mathbf{k}'-\mathbf{p}')\frac{p_n^+}{|\mathbf{p}_+|}\nu_{nn}^0(\mathbf{x}',\mathbf{k}').$$

Recall that we have from (4.8)

$$\nu_{nn}^0(\mathbf{x}',\mathbf{k}') = \frac{4\nu_\alpha^0(\hat{k}_n^+)^2}{2\rho}.$$

Then we get from (4.18) and (4.24)

$$\begin{aligned}\langle \mu_{out}^{02}\rangle = \langle \nu_\beta^{02}\rangle &= -\frac{2\sqrt{\kappa\rho}}{\hat{k}_n^+}\frac{\omega^2\rho}{c}\int \frac{d\mathbf{p}'}{(2\pi)^{n-1}}\hat{R}(\mathbf{k}'-\mathbf{p}')\hat{p}_n^+\frac{4\nu_\alpha^0(\hat{k}_n^+)^2}{2\rho} \\ &= -4\nu_\alpha^0(\mathbf{x}',\mathbf{k}')\int \frac{d\mathbf{p}'}{(2\pi)^{n-1}}\hat{R}(\mathbf{k}'-\mathbf{p}')p_n^+k_n^+\end{aligned} \tag{4.25}$$

because $|\mathbf{k}_+| = |\mathbf{p}_+| = \dfrac{\omega}{c}$.

Putting together (4.9), (4.14) and (4.25), we get (3.3).

4.3.3 Proof of Lemma 1

The proof of Lemma 1 is similar to that of the first part of Theorem 2, which is given in [15]. The functions $\mathbf{u}_\varepsilon^1(\mathbf{x}')$ satisfy the system

$$\sum_{j=1}^{n-1}\varepsilon D^j\frac{\partial \mathbf{u}_\varepsilon^1}{\partial x_j} + i\sum_{j=1}^{n-1}p_jD^j\mathbf{u}_\varepsilon^1 - i\omega A(\mathbf{x}')\mathbf{u}_\varepsilon^1 = -\varepsilon D^n\frac{\partial \mathbf{w}_\varepsilon^1}{\partial x_n}(\mathbf{x}',0), \tag{4.26}$$

while $\mathbf{r}_\varepsilon^0(\mathbf{x}') = \mathbf{w}_\varepsilon^0(\mathbf{x}',0)$ satisfy the system

$$\sum_{j=1}^{n-1}\varepsilon D^j\frac{\partial \mathbf{r}_\varepsilon^0}{\partial x_j} - i\omega A(\mathbf{x}')\mathbf{r}_\varepsilon^0 = -\varepsilon D^n\frac{\partial \mathbf{w}_\varepsilon^0}{\partial x_n}(\mathbf{x}',0). \tag{4.27}$$

Let us define the reduced dispersion matrix

$$L'(\mathbf{x},\mathbf{k}) = \sum_{j=1}^{n-1} k_j D^j - \omega A(\mathbf{x}).$$

Let P be any matrix such that $PD^n = 0$, then (4.26) implies that

$$PL'(\mathbf{k}+\mathbf{p})\boldsymbol{\nu}[\mathbf{u}^1,\mathbf{r}^0] = 0$$

and (4.27) implies that

$$PL'(\mathbf{k})\boldsymbol{\nu}[\mathbf{r}^0,\mathbf{u}^1] = 0.$$

Then the Wigner matrix $\boldsymbol{\nu}[\mathbf{r}^0,\mathbf{u}^1]$ has the specific form:

$$\begin{aligned}\boldsymbol{\nu}[\mathbf{r}^0,\mathbf{u}^1] =& \tilde{\nu}_\alpha \mathbf{b}(\mathbf{k}^+)\mathbf{b}^*((\mathbf{k}+\mathbf{p})^+) + \tilde{\nu}_\beta \mathbf{b}(\mathbf{k}^-)\mathbf{b}^*((\mathbf{k}+\mathbf{p})^-) \\ &+ \tilde{\nu}_{\alpha\beta} \mathbf{b}(\mathbf{k}^+)\mathbf{b}^*((\mathbf{k}+\mathbf{p})^-) + \tilde{\nu}_{\beta\alpha} \mathbf{b}(\mathbf{k}^-)\mathbf{b}^*((\mathbf{k}+\mathbf{p})^+).\end{aligned} \tag{4.28}$$

Recall that $\mathbf{u}^1_\varepsilon$ is outgoing, so (4.28) reduces to

$$\boldsymbol{\nu}[\mathbf{r}^0,\mathbf{u}^1] = \tilde{\nu}_\alpha \mathbf{b}(\mathbf{k}^+)\mathbf{b}^*((\mathbf{k}+\mathbf{p})^+) + \tilde{\nu}_{\beta\alpha} \mathbf{b}(\mathbf{k}^-)\mathbf{b}^*((\mathbf{k}+\mathbf{p})^+). \tag{4.29}$$

The Dirichlet boundary conditions for $\mathbf{w}^0_\varepsilon(\mathbf{x})$ imply that the fourth row of the matrix $\boldsymbol{\nu}[\mathbf{r}^0,\mathbf{u}^1]$ vanishes. Then $\tilde{\nu}_{\alpha\beta} = -\tilde{\nu}_\alpha$, and (4.29) becomes

$$\boldsymbol{\nu}[\mathbf{r}^0,\mathbf{u}^1] = \tilde{\nu}_\alpha \mathbf{b}(\mathbf{k}^+)\mathbf{b}^*((\mathbf{k}+\mathbf{p})^+) - \tilde{\nu}_\alpha \mathbf{b}(\mathbf{k}^-)\mathbf{b}^*((\mathbf{k}+\mathbf{p})^+), \tag{4.30}$$

which is (4.21).

5 Higher order anisotropic effects

The scattering cross-sections in the correctors of order η^2 in (3.3) depend upon the outgoing direction only through the power spectrum $\hat{R}(\mathbf{k}'-\mathbf{p}')$. We show in this section that some anisotropy of the scattered field is captured in the higher order terms. We obtain, in particular, a peak in the backscattering direction. This peak is similar to the coherent backscattering effect in the Neumann and impedance problems in homogeneous media studied in [5, 9, 13, 16], although much broader for Dirichlet boundary conditions. The nature of this peak is also the constructive interference between direct and reverse paths of scattered waves. The singular behavior of the reflection operator at grazing angles in the Neumann and impedance problems makes this effect much more pronounced. The study of the coherent backscattering effect for these cases in an inhomogeneous medium requires appropriate modifications and the smoothing method [1].

5.1 The backscattering for the Dirichlet problem

The angularly resolved energy density of the second order corrector in the Dirichlet problem is described by the following proposition.

Proposition 3 *Assume that $h(\mathbf{y})$ is a mean zero Gaussian process with covariance matrix $R(\mathbf{y})$. Then the Wigner matrix $\mathbf{W}_2$ of the second order term $\mathbf{w}_\varepsilon^2$ in the asymptotic expansion (3.2) of the solution of the Dirichlet problem has the form $\mathbf{W}_2 = \mu_2(\mathbf{x},\mathbf{k})\mathbf{b}(\mathbf{x},\mathbf{k})\mathbf{b}^*(\mathbf{x},\mathbf{k})$. The scalar measure $\mu_2(\mathbf{x},\mathbf{k})$ is supported on the sphere $\mathcal{S}(\mathbf{x}) = \left\{\mathbf{k} : |\mathbf{k}| = \frac{\omega}{c(\mathbf{x})}\right\}$ and satisfies the transport equation (2.14) with $\mu_2^{in} = 0$ and*

$$\begin{aligned}\langle\mu_2^{out}(\mathbf{x}',\mathbf{k}')\rangle = &\, 4\int \frac{d\mathbf{p}'d\mathbf{q}'}{(2\pi)^{2d-2}}\hat{R}(\mathbf{k}'-\mathbf{p}')\hat{R}(\mathbf{p}'-\mathbf{q}')q_n^2 p_n\left[p_n + (\mathbf{q}'+\mathbf{k}'-\mathbf{p}')_n\right]\mu_{in}'(\mathbf{q}') \\ &+4\int \frac{d\mathbf{p}'d\mathbf{q}'}{(2\pi)^{2d-2}}\hat{R}(\mathbf{k}'-\mathbf{p}')\hat{R}(\mathbf{k}'-\mathbf{q}')p_n q_n k_n^2 \mu_{in}'(\mathbf{k}'). \end{aligned} \tag{5.1}$$

Here $k_n(\mathbf{x},\mathbf{k}') = \sqrt{\dfrac{\omega^2}{c^2(\mathbf{x})}}$ is the normal component of the outgoing wave vector in $\mathcal{S}$, which has horizontal component $\mathbf{k}'$ and is pointing upwards, and μ_{in}' is as in Theorem 1.

The first term in (5.1) corresponds to the diffusive scattering. The second term provides a correction to the reflection coefficient. The differential scattering cross-section in (5.1) is no longer isotropic, and is centered in the backscattered direction. This can be seen as follows. Let us assume that the incident energy density has the form $\mu_{in}'(\mathbf{q}') = C(\mathbf{x}')\delta(\mathbf{q}'-\mathbf{q}_0')$, so that waves are coming from a single direction $\mathbf{q}_0$. Then the diffusive scattering is maximal in the direction with tangential component $\mathbf{k}' = -\mathbf{q}_0'$ because both terms in the diffusive scattering cross-section are the same. The second term in this cross-section is smaller in other directions, because when $\mathbf{k}+\mathbf{q}_0 \neq 0$, then the integration in $\mathbf{p}'$ in that term is carried over the region where both $\mathbf{p}'$ and $\mathbf{k}'+\mathbf{q}_0'-\mathbf{p}'$ lie in the disk of radius $K = \dfrac{\omega}{c}$. This region is shrinking as $\mathbf{k}$ moves away from $-\mathbf{q}_0$, and so the contribution of this term diminishes. In particular, if $\mathbf{q}_0'$ is close to the boundary of the disk, and the incident wave is close to the grazing angle, then the contribution of this term in the forward direction $\mathbf{k}' = \mathbf{q}_0'$ vanishes. This can be interpreted as an enhanced backscattering phenomenon, since the contribution of this term in (5.1) corresponds to the interference of the direct and reverse paths, as will be seen in the derivation of (5.1).

5.2 Derivation of the scattering cross-section

The statements regarding the form of the Wigner matrix $\mathbf{W}_2$ and the support of the measure μ_2 in Proposition 3 follow immediately from Proposition 2, and from the fact that $\mathbf{w}_\varepsilon^2$ is outgoing at infinity, together with our assumption that characteristics do not come back to the boundary. Thus, the only part we have to verify is the expression (5.1) for the measure μ_2 at the boundary. Let $\mathbf{r}_\varepsilon^2(\mathbf{x}') = \mathbf{w}_\varepsilon^2(\mathbf{x}', 0)$ be the boundary value of $\mathbf{w}_2$. Then we have, using (4.5)

$$\mu_2^{out}(\mathbf{x}', \mathbf{k}') = 2\kappa(\mathbf{x}')\nu[r_{\varepsilon,n+1}^2] = 2\kappa(\mathbf{x}')\omega^2\rho^2(\mathbf{x}')\nu[h\left(\frac{\mathbf{x}'}{\varepsilon}\right) r_{\varepsilon,n}^1]. \tag{5.2}$$

This can be rewritten with the help of the conjugated wave functions (4.20):

$$\nu[h\left(\frac{\mathbf{x}'}{\varepsilon}\right) r_{\varepsilon,n}^1] = \int \frac{d\mathbf{p}'d\mathbf{q}'}{(2\pi)^{2d-2}} \hat{h}(\mathbf{p}')\hat{h}(\mathbf{q}')\nu[u_{\varepsilon,n}^1(\mathbf{x}', -\mathbf{p}'), u_{\varepsilon,n}^1(\mathbf{x}', \mathbf{q}')].$$

It is easy to check that, similarly to Lemma 1 we have for any $\mathbf{p}_1'$, $\mathbf{p}_2'$:

$$\boldsymbol{\nu}[\mathbf{u}_\varepsilon^1(\mathbf{p}_1'), \mathbf{u}_\varepsilon^1(\mathbf{p}_2')](\mathbf{x}', \mathbf{k}') = \theta(\mathbf{x}', \mathbf{k}', \mathbf{p}_1', \mathbf{p}_2')\mathbf{b}(\mathbf{k}' + \mathbf{p}_1')\mathbf{b}^*(\mathbf{k}' + \mathbf{p}_2') \tag{5.3}$$

with θ being some unknown distribution. Thus we have

$$\langle\nu[h\left(\frac{\mathbf{x}'}{\varepsilon}\right) r_{\varepsilon,n}^1]\rangle = \int \frac{d\mathbf{p}'d\mathbf{q}'}{(2\pi)^{2d-2}} \langle\hat{h}(\mathbf{p}')\hat{h}(\mathbf{q}')\theta(\mathbf{x}', \mathbf{k}', -\mathbf{p}', \mathbf{q}')\rangle b_n(\mathbf{k}' - \mathbf{p}')b_n(\mathbf{k}' + \mathbf{q}') \tag{5.4}$$

and we need to evaluate the average inside the integral. Expression (5.3) implies that

$$\langle\hat{h}(\mathbf{p}')\hat{h}(\mathbf{q}')\theta(\mathbf{x}', \mathbf{k}', -\mathbf{p}', \mathbf{q}')\rangle = 2\kappa\langle\hat{h}(\mathbf{p}')\hat{h}(\mathbf{q}')\nu[u_{\varepsilon,n+1}^1(-\mathbf{p}'), u_{\varepsilon,n+1}^1(\mathbf{q}')](\mathbf{x}', \mathbf{k}')\rangle. \tag{5.5}$$

The average on the right side can be computed using the expression (4.4) for $r_{\varepsilon,n+1}^1$ in terms of $r_{\varepsilon,n}^0$, and the assumption that $h(\mathbf{y})$ is a Gaussian random process so that

$$\begin{aligned}\langle\hat{h}(\mathbf{p})\hat{h}(\mathbf{q})\hat{h}(\mathbf{p}_2)\hat{h}(\mathbf{q}_2)\rangle =& \langle\hat{h}(\mathbf{p})\hat{h}(\mathbf{q})\rangle\langle\hat{h}(\mathbf{p}_2)\hat{h}(\mathbf{q}_2)\rangle + \langle\hat{h}(\mathbf{p})\hat{h}(\mathbf{q}_2)\rangle\langle\hat{h}(\mathbf{q})\hat{h}(\mathbf{p}_2)\rangle \\ &+\langle\hat{h}(\mathbf{p})\hat{h}(\mathbf{p}_2)\rangle\langle\hat{h}(\mathbf{q})\hat{h}(\mathbf{q}_2)\rangle.\end{aligned} \tag{5.6}$$

Then we get

$$\langle\hat{h}(\mathbf{p}')\hat{h}(\mathbf{q}')\nu[u_{\varepsilon,n+1}^1(-\mathbf{p}'), u_{\varepsilon,n+1}^1(\mathbf{q}')](\mathbf{x}', \mathbf{k}')\rangle = I + II + III, \tag{5.7}$$

where

$$\begin{aligned}I =& \omega^2\rho^2(\mathbf{x}')\hat{R}(\mathbf{p}')\delta(\mathbf{p}' + \mathbf{q}') \int d\mathbf{p}_1'\hat{R}(\mathbf{p}_1')\nu_{nn}^0(\mathbf{k}' - \mathbf{p}' - \mathbf{p}_1'), \\ II =& \omega^2\rho^2(\mathbf{x}')\hat{R}(\mathbf{p}')\hat{R}(\mathbf{q}')\nu_{nn}^0(\mathbf{k}' + \mathbf{q}' - \mathbf{p}')\end{aligned}$$

and

$$III = \omega^2 \rho^2(\mathbf{x}')\hat{R}(\mathbf{p}')\hat{R}(\mathbf{q}')\nu^0_{nn}(\mathbf{k}').$$

The three terms in (5.7) have a natural interpretation in terms of wave scattering from a collection of discrete random scatterers [16]. The first term in (5.7) comes from the first term in (5.6) and corresponds to the interaction of a path with itself. It produces a scattering cross-section that is essentially isotropic. The second term in (5.7) comes from the second term in (5.6) and corresponds to the interaction of a path and its reverse one in the discrete picture. It has a peak in the backscattering direction as explained in the previous section. The last term arises from paths scattering twice on the same scatterer. It contributes to the specular reflection coefficient.

Finally we note that $\nu^0_{nn}(\mathbf{k}') = \dfrac{k_n^2}{2\rho|\mathbf{k}|^2} 4\mu'_{in}(\mathbf{x}', \mathbf{k}')$, and putting this together with (5.2), (5.4), (5.5), we obtain (5.1). This completes the proof of Proposition 3.

6 Conclusions

We have derived the boundary conditions for the transport equation for the phase space resolved energy density in an inhomogeneous medium. Our derivation is based on the assumption that the asymptotic expansion (3.2) holds. These boundary conditions can be used for the radiative transport equation for acoustic waves [3, 14] when randomness of the medium is independent of the randomness of the surface. Moreover, one can use similar boundary conditions for more general radiative transport equations [14] for electromagnetic, elastic and other waves in domains with rough boundaries. Our result may also be generalized to reflection and transmission at interfaces between two inhomogeneous media. The results are then a generalization of the diffuse energy reflection and transmission at a rough interface considered in [2].

The analysis of the Neumann problem in an inhomogeneous medium with a rough boundary requires the smoothing method or any equivalent regularization technique. We plan to address this in a separate note [1]. This allows one to incorporate the coherent backscattering effect into the boundary conditions for the radiative transport equation.

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