Homogenization of a spectral equation with drift in linear transport

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Abstract

This paper deals with the homogenization of a spectral equation in linear transport theory posed in a periodic heterogeneous domain. This equation models the stability of nuclear reactor cores. The neutron density at equilibrium is given by the unique normalized positive eigenvector of this spectral equation. The corresponding eigenvalue indicates the amount of fission necessary to reach this equilibrium. When the physical parameters satisfy certain symmetries explained below, it is known that the eigenvectors of this equation can be factored as the product of two term asymptotically as the number of periodicity cells increases. The first term solves a local transport spectral equation posed in the periodicity cell. The second term solves a homogeneous spectral diffusion equation posed in the entire core. Furthermore, the eigenvalues of the spectral equation in the core are approximately given by the leading eigenvalue of the equation posed in the periodicity cell with periodic boundary conditions, with a correction proportional to the eigenvalues of the diffusion equation. The new contribution of this paper is to extend these results to the case where the symmetry mentioned above is broken. We show that the factorization remains valid, except that the diffusion equation is now replaced by a convection-diffusion equation with large drift. One consequence is that the particle density decays exponentially in the direction of this drift. Moreover, the limit of the eigenvalues of the problem in the core is no longer given by that of the problem on the periodicity cell with periodic boundary conditions, but is smaller and obtained by solving a spectral equation on the periodicity cell with suitably chosen boundary conditions. The techniques of this paper can also be applied to the time-dependent linear transport equation with drift that appears in radiative transfer theory, which models the propagation of acoustic, electromagnetic, or elastic waves in heterogeneous media.

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1 Introduction

The power distribution in a nuclear reactor is determined by solving a linear transport spectral equation for the density of neutrons in the core. This equation expresses the

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balance between the production of neutrons by fission and its absorption in the reactor core and leakage at the boundary. The unknowns are the neutron density $\phi(x,v)$ in phase space, where x stands for position and v for velocity, and the constant $k_{\rm eff}$, which measures how much fission is necessary to reach an equilibrium. The couple $(k_{\rm eff}, \phi(x,v))$ is the largest eigenvalue and the corresponding positive eigenvector of the equation

$$\begin{aligned} &v \cdot \nabla\phi + \Sigma(x,v)\phi = \int_V f(x,v',v)\phi(x,v')dv' + \frac{1}{k_{\rm eff}}\int_V \sigma(x,v',v)\phi(x,v')dv', \\ &\phi(x,v) = 0 \quad \text{on} \quad \Gamma_- = \{(x,v) \in \partial\Omega \times V, \ \text{s.t. } v\cdot n(x) < 0\}, \end{aligned} \tag{1.1}$$

posed in a convex bounded open set $\Omega \subset \mathbb{R}^n$ and with a compact velocity space $V \subset \mathbb{R}^n$. This spectral equation is also called the criticality eigenvalue problem. The vacuum boundary conditions are not very realistic but are assumed here for simplicity. The core is characterized by the absorption coefficient Σ, the differential scattering coefficient f and the fission coefficient σ. These coefficients are in practice highly heterogeneous and accurate numerical solutions of (1.1) still prohibitive even with modern computers. As a first approximation, these coefficients can be assumed periodic. The problem lends then itself naturally to the theory of homogenization, which consists in averaging out the local periodic oscillations and obtaining an equation that only involves the global scale. The numerical solution of this homogenized equation is then significantly less expensive. The homogenization of transport problems has been widely addressed in the past both in the physical [7, 11, 16] and mathematical [9, 22, 23, 27, 29] literature. We [2, 4, 5] have recently revisited it and obtained new properties of the spectral equation (1.1).

The homogenization technique proceeds as follows. The density $\phi(x,v)$ is factored as a product of two terms

$$\phi(x,v) \;=\; \psi(x,v)u(x). \tag{1.2}$$

Here, $\psi(x,v)$ is a periodic solution of the same spectral transport equation (1.1) but posed in an infinite medium (or equivalently with periodic boundary conditions at the boundary of the periodicity cell; see (2.3) below). It characterizes the local oscillations of the density. The function $u(x)$ is the solution of a homogenized diffusion equation posed in the domain Ω. This term gives the global behavior of the density.

In the works mentioned above, the physical coefficients are required to satisfy an additional condition. Let us denote by $\psi^*(x,v)$ the solution of the adjoint transport problem to that defining $\psi(x,v)$ (see equation (2.5) below). Then it is assumed that no intrinsic drift takes place on the periodicity cell Y, which is expressed by the relation

$$J \;\equiv\; \int_Y\int_V v\psi(x,v)\psi^*(x,v)dv\; dx \;=\; 0. \tag{1.3}$$

This condition is satisfied in most practical applications. For instance, the current J vanishes when the physical parameters satisfy a cubic symmetry in Y and the velocity space verifies $V = -V$. It also vanishes when the cross sections do not depend on the velocity variables v and v'. We refer to [2] for additional details.

It was shown in [25] however that J does not vanish for specific anisotropic cells. When this condition is not fulfilled, the particle density is quite different. In [25], where time-dependent problems are considered, the drift is compensated by an exponential increase or decrease of the density in time. A comparable result was formulated in [8] for the heat equation. Here, we want to compensate for the drift by modifying the amount of fission. In other words, we want to obtain the fission coefficient and the corresponding criticality constant k_{eff} that ensures equilibrium. Using an appropriate change of variables, we will show that the eigenvectors of (1.1) are asymptotically given as the product of a local transport solution and of a large scale solution of a homogeneous diffusion problem with fast exponential decay. Moreover, the criticality constant no longer converges to that of the spectral problem with periodic conditions as in the case with zero drift. Because of an $O(1)$ leak at the boundary of the domain, more fission is required to reach an equilibrium than in the symmetric case and the criticality constant is lowered by the presence of a drift.

Drift effects have also been analyzed in the homogenization of other equations. In the homogenization of heterogeneous diffusion, parabolic, and transport equations, large drifts are also absent when the physical media satisfy suitable symmetry conditions [8, 9]. For more general configurations however, large drift effects may have to be removed first to obtain a limiting homogeneous diffusion equation. Time-dependent and steady-state diffusion equations with drift have been studied in this context [8, 12, 13, 17, 20]. This paper focuses on the criticality spectral equation (1.1). However, drift effects appear in other linear transport equations. In radiative transfer, the energy density of waves (acoustic, electromagnetic, or seismic waves for instance) propagating in highly heterogeneous media is modeled by a time dependent linear transport equation [28], which after Laplace transform in time has the form

$$\begin{aligned} &v \cdot \nabla\phi + \Sigma(x,v)\phi = \int_V f(x,v',v)\phi(x,v',\lambda)dv' + \lambda\phi, \\ &\phi(x,v,\lambda) = 0 \quad \text{on} \quad \Gamma_- = \{(x,v) \in \partial\Omega \times V, \ \text{s.t.} \ v\cdot n(x) < 0\}. \end{aligned} \tag{1.4}$$

The techniques of this paper can be applied to the analysis of this spectral equation with asymmetric differential scattering function $f(x,v',v)$. See [6] for an application in seismic wave propagation in the absence of drift.

Our framework is the following. We assume that the domain Ω is composed of the order of ε^{-n} periodicity cells $\varepsilon Y = (0,\varepsilon)^n$, where the unit periodicity cell is $Y = (0,1)^n$. Homogenization consists in finding the limit of the particle density and multiplication factor as the number of cells tends to infinity, or equivalently as ε tends to 0. Physically, the mean free path, the average distance between two interactions of a particle with the underlying medium, is independent of the number of cells in a domain. It is therefore of order ε in our modeling. The physical parameters f, σ, and Σ, counting a number of interactions by unit length, are also on the order of ε^{-1}. Introducing $\lambda_\varepsilon = k_{\text{eff}}^{-1}$, the equation (1.1) is replaced by the following sequence of criticality problems in $\Omega \times V$

$$\begin{aligned} &\varepsilon v \cdot \nabla\phi_\varepsilon + \Sigma^\varepsilon(x,v)\phi_\varepsilon = \int_V f^\varepsilon(x,v',v)\phi_\varepsilon(x,v')dv' + \lambda_\varepsilon \int_V \sigma^\varepsilon(x,v',v)\phi_\varepsilon(x,v')dv', \\ &\phi_\varepsilon = 0 \qquad \text{on} \quad \Gamma_-. \end{aligned} \tag{1.5}$$

Here, the cross sections are periodic functions in Ω and are given by

$$\Sigma^\varepsilon(x,v) = \Sigma(\frac{x}{\varepsilon},v),\ f^\varepsilon(x,v',v) = f(\frac{x}{\varepsilon},v',v),\ \sigma^\varepsilon(x,v',v) = \sigma(\frac{x}{\varepsilon},v',v), \qquad \text{for } x \in \Omega,$$

where Σ, f, and σ are positive Y-periodic functions of the space variable and are independent of ε. For simplicity, we consider here a vacuum boundary condition for (1.5). This means that no particles enter the domain at its boundary.

An outline of this paper is as follows. Section 2 recalls and extends some results on the spectral equation in finite domain and with periodic boundary conditions. Section 3 is devoted to our main results. Using an appropriate change of variables described in Theorem 3.1, we show how to replace (1.5) by an equivalent spectral equation with zero drift. The classical homogenization theory of [2] then applies and yields Theorem 3.2. The proof of Theorem 3.1 is based on the properties of exponentially decaying solutions of spectral equations in an infinite medium. It is given in section 4.

2 Existence Results for the Spectral Equation

This section states some results of existence and regularity of the solutions of the spectral equation both in finite and periodic domains. Several basic results, such as Theorems 2.1 and 2.2 and Proposition 2.3, have already appeared in the literature [2, 3, 4, 15]. We recall them here for completeness. Theorem 2.4 seems however to be new. This important theorem in the sequel gives an optimal estimate for the smallest eigenvalue of the spectral equation (2.8) below in terms of the absorption function Σ. The main hypotheses on the physical parameters are the following.

(H1) The domain Ω is a convex bounded open subset of $\mathbb{R}^n$.

(H2) The velocity space V is the closure of a bounded open subset of $\mathbb{R}^n$ that does not contain 0 and such that its n-dimensional measure is normalized to have $|V| = 1$.

(H3) Denote by $Y = (0,1)^n$ the periodicity cell. The functions $\Sigma(y,v)$, $f(y,v',v)$, and $\sigma(y,v',v)$ are Y-periodic with respect to the space variable y. They are positive bounded functions of their arguments and there exists a positive constant $C > 0$ such that, for a.e. $(y,v',v) \in Y \times V \times V$,

$$\begin{aligned} &\sigma(y,v',v) \geq C, \\ &\Sigma(y,v) - \int_V f(y,v',v)dv' \geq C, \\ &\Sigma(y,v) - \int_V f(y,v,v')dv' \geq C. \end{aligned} \tag{2.1}$$

These assumptions are natural and quite general. They are commented in detail in [2]. Denote by $W^p(\Omega \times V)$, $1 \leq p \leq \infty$, the following Banach spaces (see [15])

$$W^p(\Omega \times V) = \{u \in L^p(\Omega \times V) \text{ s.t. } v \cdot \nabla u \in L^p(\Omega \times V)\}. \tag{2.2}$$

An existence result for the spectral problem (1.5) is then as follows [2, 3, 4, 15].

Theorem 2.1 *The spectral problem (1.5) has at most a countable number of eigenvalues and of associated eigenvectors in $W^2(\Omega \times V)$. Furthermore, there exists a real and positive eigenvalue, of smallest modulus, with multiplicity one, such that its associated eigenvector is the unique normalized positive eigenvector of (1.5).*

The positive solution of the spectral eigenvalue problem posed in an infinite medium governs the fast scale behavior of the solution in finite domain. We denote by λ_∞ and ψ the first eigenvalue and eigenvector of the following spectral equations

$$\begin{aligned} &T\psi = K\psi + \lambda_\infty F\psi \\ &y \mapsto \psi(y,v) \quad \text{is} \quad Y-\text{periodic}, \end{aligned} \tag{2.3}$$

where we have defined

$$\begin{aligned} T\phi(y,v) &= v\cdot\nabla_y\phi(y,v) + \Sigma(y,v)\phi(y,v) \\ K\phi(y,v) &= \int_V f(y,v',v)\phi(y,v')dv' \\ F\phi(y,v) &= \int_V \sigma(y,v',v)\phi(y,v')dv'. \end{aligned} \tag{2.4}$$

Introducing the adjoint functions $f^*(y,v',v) = f(y,v,v')$ and $\sigma^*(y,v',v) = \sigma(y,v,v')$, an adjoint problem to (2.3) is defined as

$$\begin{aligned} &T^*\psi^* = K^*\psi^* + \lambda_\infty F^*\psi^*, \\ &y \mapsto \psi^*(y,v) \;\; \text{is } Y-\text{periodic}, \end{aligned} \tag{2.5}$$

where the adjoint operators are defined as

$$\begin{aligned} T^*\phi(y,v) &= -v\cdot\nabla_y\phi(y,v) + \Sigma(y,v)\phi(y,v) \\ K^*\phi(y,v) &= \int_V f^*(y,v',v)\phi(y,v')dv' \\ F^*\phi(y,v) &= \int_V \sigma^*(y,v',v)\phi(y,v')dv'. \end{aligned} \tag{2.6}$$

An existence result for these problems is as follows [2, 3, 4].

Theorem 2.2 *There exists an eigenvalue λ_∞ common to both problems (2.3) and (2.5), which is real, positive, of smallest modulus, and with multiplicity one. Its associated eigenvectors ψ and ψ^* are positive elements of $W^2(Y\times V)$. Up to renormalization, ψ and ψ^* are the unique positive solutions to (2.3) and (2.5), respectively. Furthermore, there exist two positive constants $C' \geq C > 0$ such that*

$$0 < C \leq \psi(y,v) \leq C' \quad \textit{and} \quad 0 < C \leq \psi^*(y,v) \leq C' \quad \textit{a.e. in } Y\times V.$$

More generally every eigenvector of (2.3) or (2.5) in $L^2(Y\times V)$ is also an element of $L^\infty(Y\times V)$.

We also have the Fredholm alternative:

Proposition 2.3 *Let λ_∞ and ψ be the first eigenvalue and eigenvector of (2.3). Let $S(y,v)$ be a source term in $L^2(Y\times V)$. There exists a solution $\varphi(y,v)\in W^2(Y\times V)$ of*

$$\begin{aligned} &T\varphi = K\varphi + \lambda_\infty F\varphi + S \\ &y \mapsto \varphi(y,v) \quad \textit{is} \quad Y-\textit{periodic} \end{aligned} \tag{2.7}$$

if and only if S is orthogonal to the first eigenvector ψ^* of (2.5), i.e. S satisfies the compatibility condition

$$\int_Y\int_V S(y,v)\psi^*(y,v)dydv \;=\; 0.$$

Furthermore, if it exists, the solution φ is unique up to the addition of a multiple of ψ.

The last two inequalities in (H3) imply that absorption is uniformly greater than scattering; hence multiplication (fission) is necessary to ensure that an equilibrium is reached. This hypothesis on the absorption function is not optimal and we will need a finer result in the sequel. We consider a slightly simpler problem than (2.3), which can be interpreted as a case where the multiplication operator F is proportional to the scattering operator K,

$$\begin{array}{l} T\varphi = \tilde{\mu}\tilde{K}\varphi \\ y \mapsto \varphi(y,v) \quad \text{is} \quad Y-\text{periodic.} \end{array} \tag{2.8}$$

The operator $\tilde{K}$ is defined as $\tilde{K}\phi(x,v) = \int_V \tilde{f}(x,v',v)\phi(x,v')dv'$. We want optimal conditions on the absorption function Σ that ensure the well-posedness of this spectral equation and want to obtain an estimate for the smallest eigenvalue $\tilde{\mu}$. Let us introduce some notation. For every $(y,v,s) \in Y \times V \times \mathbb{R}$ we define the optical length $\alpha(y, y-sv)$ as

$$\alpha(y, y-sv) = \int_0^s \Sigma(y-tv, v)dt,$$

and the following line integrals

$$\Pi(\Sigma)(y,v) = \int_0^\infty \exp(-\alpha(y, y-sv))ds. \tag{2.9}$$

We easily check that $\Pi(\Sigma)$ is Y-periodic in the variable y. We denote by $L^p_\#(Y \times V)$, $1 \le p \le \infty$, the spaces of periodic functions in $\mathbb{R}^n \times V$ whose restrictions to $Y \times V$ are elements of $L^p(Y \times V)$.

Hypothesis (H3) in this context is then replaced by

(H3') The functions $\Sigma(y,v)$ and $\tilde{f}(y,v',v)$ are Y-periodic with respect to the space variable y. They are bounded functions of their arguments and there exists a positive constant $f_0 > 0$ such that, for a.e. $(y,v',v) \in Y \times V \times V$,

$$\tilde{f}(y,v',v) \ge f_0.$$

(H4) We assume that

$$\sup_{(y,v)\in Y\times V} \Pi(\Sigma)(y,v) < \infty.$$

Theorem 2.4 *Assume that (H1), (H2), (H3'), and (H4) hold. Problem (2.8) admits a unique positive eigenvector up to renormalization, associated with the smallest eigenvalue $\tilde{\mu}$, which is real and positive. Moreover, we have the following estimate*

$$0 < \tilde{\mu} \le \frac{C}{f_0 \|\Pi(\Sigma)\|^{1/2}_{L^1_\#(Y\times V)}}, \tag{2.10}$$

where C is independent of Σ and f_0.

Proof. The functions defined in $Y \times V$ are extended to $\mathbb{R}^n \times V$ by $Y-$periodicity. We obtain that

$$T^{-1}(\phi) = \int_0^\infty \exp(-\alpha(y, y - sv))\phi(y - sv, v)ds.$$

Therefore T^{-1} is positive and bounded in $L^\infty_\#(Y \times V)$ since (H3') is satisfied. The same results hold for the adjoint operator T^* and therefore T^{-1} is bounded in $L^1_\#(Y \times V)$ and by interpolation [10] in any $L^p_\#(Y \times V)$ for $1 \le p \le \infty$. The operator $\tilde{K}$ is also bounded and positive, and it is a classical result that $T^{-1}\tilde{K}$ is then compact (see [18, 26]). This is sufficient to obtain the same results as in Theorem 2.2 (see [2, 3, 4]).

Let φ be the positive solution of $T^{-1}\tilde{K}\varphi = \tilde{\mu}^{-1}\varphi$. We obviously have $(T^{-1}\tilde{K})^2\varphi = \tilde{\mu}^{-2}\varphi$. By assumption (H2) the interior of V is non-empty. In particular, there exists a velocity v_0 and a parameter $\delta > 0$ such that the following angular sector $\mathcal{S}$ is included in V

$$\mathcal{S} = \left\{ v \in \mathbb{R}^N \text{ s.t. } 1 - \delta \le \frac{|v|}{|v_0|} \le 1 + \delta,\ \frac{v}{|v|}.\frac{v_0}{|v_0|} \ge 1 - \delta \right\} \subset V.$$

For $y \in \mathbb{R}^n$, we define the infinite cone of origin y as $\{y + sv \text{ s.t. } s \in \mathbb{R}^+, v \in \mathcal{S}\}$. By virtue of hypotheses (H2) and (H3'), we denote by $\Sigma_{\max}$ and $v_{\max}$ the maximum of $|\Sigma(x, v)|$ on $\Omega \times V$ and $|v|$ on V, respectively. We then obtain that

$$\begin{aligned} \tilde{K}T^{-1}\tilde{K}(\varphi)(y) &\ge f_0^2 \int_V dv \int_0^\infty ds \exp(-\alpha(y, y - sv)) \int_V dv' \varphi(y - sv, v') \\ &\ge C \int_{C_\mathcal{S}} dz \frac{\exp\left(-\frac{\Sigma_{\max}}{v_{\max}}|z - y|\right)}{|z - y|^{n-1}} \int_V dv' \varphi(z, v'), \end{aligned}$$

owing to the change of variables $z = y - sv$ from polar to Cartesian coordinates. Clearly the cone $C_\mathcal{S}$ contains at least one periodicity cell Y and therefore

$$\tilde{K}T^{-1}\tilde{K}(\varphi)(y) \ \ge\ C \int_Y \int_V \varphi(z, v')dv'dz = C\|\varphi\|_{L^1(Y \times V)},$$

since φ is positive. Finally we obtain that

$$\begin{aligned} \tilde{\mu}^{-2}\|\varphi\|_{L^1(Y\times V)} &= \|T^{-1}\tilde{K}T^{-1}\tilde{K}(\varphi)\|_{L^1(Y\times V)} \ge C\|\varphi\|_{L^1(Y\times V)}\|T^{-1}(1)\|_{L^1(Y\times V)} \\ &\ge C\|\varphi\|_{L^1(Y\times V)}\|\Pi(\Sigma)\|_{L^1(Y\times V)}. \end{aligned}$$

This concludes the proof of the estimate. □

The estimate for $\tilde{\mu}$ given in the theorem may not be optimal (the factor 1/2 may not be optimal). However, we see that $\tilde{\mu}$ tends to 0 when $\Pi(\Sigma)(y, v)$ goes to infinity on a fixed set of positive measure. This proves that hypothesis (H4) is optimal in this sense since the smallest eigenvalue of (2.8) cannot be positive when (H4) is violated.

3 Homogenization of the spectral equation with drift

This section presents the main results of this paper stated in Theorems 3.1 and 3.2 below. We begin with a formal derivation of the diffusion equation that yields the correct result in the absence of drift, that is to say when the physical parameters satisfy enough symmetry constraints. We utilize the classical *two-scale* expansion, which has often been used in the past to obtain the asymptotic limit of heterogeneous transport equations with periodic coefficients [2, 4, 7, 9, 15, 22, 23, 24].

The homogenization of the transport equation without drift proceeds as follows. We define

$$\psi_\varepsilon(x,v) = \psi(\frac{x}{\varepsilon}, v), \tag{3.1}$$

where ψ is the positive solution of the local spectral equation (2.3). Since the medium is periodic, we can expect that the solutions of (1.5) and (2.3) share the same fast-scale oscillations. Since ψ_ε is a positive function, we introduce the factored density u_ε

$$u_\varepsilon(x,v) = \frac{\phi_\varepsilon(x,v)}{\psi_\varepsilon(x,v)}, \tag{3.2}$$

where ϕ_ε is an eigenvector of (1.5). After some algebra, the equation for u_ε is found to be

$$\begin{aligned} &\frac{1}{\varepsilon} v\cdot\nabla u_\varepsilon + \frac{1}{\varepsilon^2}\int_V \sigma_\infty^\varepsilon(x,v',v)\frac{\psi_\varepsilon(x,v')}{\psi_\varepsilon(x,v)}[u_\varepsilon(x,v) - u_\varepsilon(x,v')]dv' \\ = \; &\frac{\lambda_\varepsilon - \lambda_\infty}{\varepsilon^2}\int_V \sigma^\varepsilon(x,v',v)\frac{\psi_\varepsilon(x,v')}{\psi_\varepsilon(x,v)} u_\varepsilon(x,v')dv', \end{aligned} \tag{3.3}$$

where $\sigma_\infty^\varepsilon = f^\varepsilon + \lambda_\infty \sigma^\varepsilon$. The two-scale method consists then in assuming the following expansion,

$$\begin{aligned} u_\varepsilon(x,v) &= u_0(x,\frac{x}{\varepsilon},v) + \varepsilon u_1(x,\frac{x}{\varepsilon},v) + \varepsilon^2 u_2(x,\frac{x}{\varepsilon},v) + \ldots \\ \lambda_\varepsilon &= \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \ldots, \end{aligned} \tag{3.4}$$

where the functions $y \mapsto u_k(x,y,v)$ are $Y-$periodic for $k = 0,\ldots,2$. Here, the variable x corresponds to the global scale whereas $y = \frac{x}{\varepsilon}$ describes the local fluctuations of the particle density. Plugging this expansion (3.4) into the transport equation (3.3) and equating like powers of ε yields the following three equations

$$v\cdot\nabla_y u_0 + Q(u_0) = (\lambda_0 - \lambda_\infty)\tilde{F}(u_0) \tag{3.5}$$

$$v\cdot\nabla_y u_1 + Q(u_1) = -v\cdot\nabla_x u_0 + \lambda_1\tilde{F}(u_0) \tag{3.6}$$

$$v\cdot\nabla_y u_2 + Q(u_2) = -v\cdot\nabla_x u_1 + \lambda_1\tilde{F}(u_1) + \lambda_2\tilde{F}(u_0). \tag{3.7}$$

The scattering operator Q and fission operator $\tilde{F}$ are defined as

$$\begin{aligned} Q(u)(y,v) &= \int_V \sigma_\infty(y,v',v)\frac{\psi(y,v')}{\psi(y,v)}[u(y,v) - u(y,v')]dv' \\ \tilde{F}(u)(y,v) &= \int_V \sigma(y,v',v)\frac{\psi(y,v')}{\psi(y,v)} u(y,v')dv', \end{aligned} \tag{3.8}$$

where $\sigma_\infty = f + \lambda_\infty\sigma$. We now want to solve the three equations (3.5), (3.6), and (3.7) in turn. The fact that one of these equations does not admit any solution will be an indication that the expansion (3.4) does not hold or becomes trivial. The latter is precisely what happens in the presence of drift.

We first deduce from (3.5), Theorem 2.2, and Proposition 2.3 that $\lambda_0 = \lambda_\infty$ and $u_0 \equiv u_0(x)$. As expected, the fast oscillations of ϕ_ε are captured by the local eigenfunction ψ_ε and u_ε converges (strongly) to a function that does not depend on the fast scale. We obtain from the second equation (3.6) and the same Fredholm alternative in Proposition 2.3 that

$$\int_Y \int_V (-v \cdot \nabla_x u_0 + \lambda_1 \tilde{F}(u_0))\psi(y,v)\psi^*(y,v)dydv \;=\; 0.$$

Since u_0 in independent of y and v, the latter is equivalent to

$$J \cdot \nabla u_0 \;=\; \lambda_1 \tilde{\sigma}\; u_0, \tag{3.9}$$

where $\tilde{\sigma} = \int_Y \int_V \tilde{F}(1)\psi(y,v)\psi^*(y,v)dydv$ is a positive constant and

$$J = \int_Y \int_V v\psi(y,v)\psi^*(y,v)dydv.$$

When $J = 0$, we obtain that $\lambda_1 = 0$. Then solving (3.6) for u_1 and plugging its expression into (3.7) gives a compatibility condition which is a spectral diffusion equation for u_0. This equation is similar to (3.14) below. Although only formal here, this procedure can be justified rigorously [4, 15] provided the domain is sufficiently regular. The regularity constraint can be circumvented by using the method of *two-scale convergence* [1, 2]. The final result can be read off Theorem 3.2 by setting $\nu = 0$.

When $J \neq 0$, we obtain that u_0 is exponentially increasing or decreasing in the direction J. However the boundary conditions impose that $u_0 = 0$ at the boundary. Therefore $u_0 \equiv 0$ and the asymptotic expansion fails to give the right answer. It is proved in [2] that the solution u_ε of an associated source problem indeed converges strongly to 0 when $J \neq 0$. Notice that the asymptotic expansion (3.4) is still valid. We will see that the solution u_ε is actually exponentially small away from the boundary $\partial\Omega$. Therefore, all the terms in (3.4) ought to vanish, as we have demonstrated for u_0.

We deduce from the previous asymptotic expansion that $\psi(y,v)$ does not reproduce the local behavior of the solution ϕ_ε since the expansion for u_ε yields a trivial result. Also, the expression in (3.9) shows that the leading term wants to decay exponentially. These observations lead us to the following modifications. We want to impose an exponential decay of the solution by introducing the following change of variables

$$\varphi_{\varepsilon,\nu} = \exp(\frac{\nu \cdot x}{\varepsilon})\; \phi_\varepsilon, \tag{3.10}$$

where ν is a vector in $\mathbb{R}^n$. We readily verify that the equation for $\varphi_{\varepsilon,\nu}$ is still (1.5), where Σ^ε has been replaced by $\Sigma^\varepsilon - v \cdot \nu$. The objective is to find a vector ν that allows us to use the known results for vanishing drift. Let us introduce $\lambda_{\infty,\nu}$, ψ_ν, and ψ_ν^*, the smallest eigenvalue and corresponding positive eigenvectors of the two following spectral equations

$$\begin{aligned} &T\psi_\nu - v \cdot \nu\; \psi_\nu = K\psi_\nu + \lambda_{\infty,\nu} F\psi_\nu \\ &y \mapsto \psi_\nu(y,v) \quad \text{is} \quad Y-\text{periodic} \end{aligned} \tag{3.11}$$

and

$$\begin{array}{l} T^*\psi_\nu^* - v\cdot\nu\ \psi_\nu^* = K^*\psi_\nu^* + \lambda_{\infty,\nu}F^*\psi_\nu^* \\ y \mapsto \psi_\nu^*(y,v) \quad \text{is} \quad Y-\text{periodic}. \end{array} \tag{3.12}$$

We want to find ν such that

$$J_\nu \ = \ \int_Y\int_V v\ \psi_\nu(y,v)\psi_\nu^*(y,v)dy\,dv \ = \ 0. \tag{3.13}$$

Notice that similar techniques have been used for parabolic problems [8, 21] or in the homogenization of heterogeneous elliptic spectral equations [12, 13]. That such a vector ν exists is stated in our main result:

Theorem 3.1 *There exists a unique vector $\nu \in \mathbb{R}^n$ such that the spectral equations (3.11) and (3.12) are well-posed (in the sense that the conclusions of Theorem 2.2 apply), such that (3.13) holds, and such that $\lambda_{\infty,\nu} \geq 0$. Moreover, if $J \neq 0$, the leading eigenvalue $\lambda_{\infty,\nu}$ satisfies*

$$\lambda_{\infty,\nu} \ > \ \lambda_\infty = \lambda_{\infty,0}.$$

This theorem is proved in section 4. Thanks to this theorem, we have the right change of variables (3.10) that allows us to replace a problem with large drift by a problem with vanishing drift. We can then apply known results on the homogenization of the spectral transport equation without drift. Let us define the εY-periodic function $\psi_{\varepsilon,\nu}$ as

$$\psi_{\varepsilon,\nu}(x,v) \ = \ \psi_\nu(\frac{x}{\varepsilon},v),$$

where ψ_ν is the positive eigenvector of (3.11). In our context, Theorem 3.1 in [2] now reads:

Theorem 3.2 *Let the vector ν be given as in Theorem 3.1. Let λ_ε^k be the k^{th} eigenvalue of (1.5) and let ϕ_ε^k be a normalized associated eigenvector. Then*

$$\lim_{\varepsilon\to 0}\frac{\lambda_\varepsilon^k - \lambda_{\infty,\nu}}{\varepsilon^2} \ = \ \rho^k,$$

where $\lambda_{\infty,\nu}$ is the first eigenvalue of (3.11) and ρ^k is the k^{th} eigenvalue of the homogenized diffusion problem

$$\begin{array}{rcll} -\nabla\cdot D\nabla u^k(x) & = & \rho^k\overline{\sigma}u^k(x) & \text{in } \Omega \\ u^k(x) & = & 0 & \text{on } \partial\Omega. \end{array} \tag{3.14}$$

Furthermore, up to a subsequence,

$$\frac{\phi_\varepsilon^k(x,v)\exp(\dfrac{\nu\cdot x}{\varepsilon})}{\psi_{\varepsilon,\nu}(x,v)} \to u^k(x) \ \text{strongly in } L^2(\Omega\times V),$$

where u^k is an eigenvector associated with the eigenvalue ρ^k. The homogenized fission cross section is given by

$$\overline{\sigma} \ = \ \int_Y\int_V\int_V \sigma(y,v',v)\psi_\nu^*(y,v')\psi_\nu(y,v)dy\,dv\,dv', \tag{3.15}$$

while the homogenized diffusion tensor $D = (D_{ij})_{1\le i,j\le N}$ *in (3.14) is defined by*

$$D_{ij} \;=\; -\int_Y\int_V v_j\psi_\nu(y,v)\psi_\nu^*(y,v)\theta_\nu^i(y,v)dv\,dy, \tag{3.16}$$

and the functions $(\theta_\nu^i)_{1\le i\le N}$ *are the solutions of the cell problems*

$$\begin{array}{l} v\cdot\nabla_y\theta_\nu^i \;+\; Q_\nu(\theta_\nu^i) \;=\; -v_i \quad \textit{ in } Y\times V \\ y\mapsto \theta_\nu^i(y,v) \quad \textit{is } Y - \textit{periodic}, \end{array} \tag{3.17}$$

where the local scattering operator Q_ν *is defined by*

$$Q_\nu(\theta) = \int_V \frac{\psi_\nu(y,v')}{\psi_\nu(y,v)}\sigma_{\infty,\nu}(y,v',v)[\theta(y,v)-\theta(y,v')]dv'$$

with the notation $\sigma_{\infty,\nu}(y,v',v) = f(y,v',v)+\lambda_{\infty,\nu}\sigma(y,v',v)$.

The labeling of the eigenvalues of (1.5) are made by increasing order of their real part. These eigenvalues may be complex; however in the limit of small mean free path, they are all real. The eigenvalues of the diffusion equation (3.14) are real and labeled by increasing order.

We can recover the separation of scales (1.2) given in the introduction. Indeed, as a corollary of the previous theorem, we easily show that

$$\phi_\varepsilon^k(x,v) = \psi_{\varepsilon,\nu}(x,v)u_\varepsilon^k(x) + \text{ lower order terms} \tag{3.18}$$

where u_ε^k is a solution of the following convection-diffusion equation

$$\begin{array}{rcl} -\nabla\cdot D\nabla u_\varepsilon^k(x) + \dfrac{2}{\varepsilon}\nu\cdot(D\nabla u_\varepsilon^k)(x) &=& \rho^k\overline{\sigma}u_\varepsilon^k(x) \quad \text{in } \Omega \\ u_\varepsilon^k(x) &=& 0 \quad \text{on } \partial\Omega. \end{array} \tag{3.19}$$

Therefore in a certain sense, there is still separation of scales. The solution of the transport equation is approximated by the product of a fast oscillating function and of a smooth function, which now decays exponentially in the fast scale. Notice that the local behavior of the particle density is governed by $\psi_{\varepsilon,\nu}(x,v)$, and no longer by $\psi_{\varepsilon,0}(x,v)$ as in the vanishing drift case. Therefore, the expansion for u_ε defined in (3.3) had no chance to lead to the right answer. The problem with drift is typical of a homogenization procedure where the asymptotic expansion *cannot* yield the correct result.

Let us also give a qualitative remark concerning the density at the boundary of the domain. Notice that the outgoing density at the boundary of the domain Ω has not been estimated. The Dirichlet condition for the convection-diffusion approximation and the relation (3.18) yield a vanishing exiting flux which is not physical. The size of the exiting flux is known in the case of a similar source problem [4] with vanishing drift. In this case, the exiting flux is shown to be of order ε when the L^2 norm of the solution of the transport equation is of order 1. Up to the addition of a boundary layer that we do not consider here, the transport solution is approximated by a diffusion term w_ε satisfying an equation in Ω similar to (3.14) with Robin boundary condition

$$w_\varepsilon + \varepsilon L\frac{\partial w_\varepsilon}{\partial n} = 0,$$

where the extrapolation length L is positive. Hence, the diffusion term is of order ε at the boundary of the domain. The solution of the transport equation is given by

$$\phi_\varepsilon(x,v) = \psi_{\varepsilon,\nu}(x,v)\exp(\frac{\nu\cdot x}{\varepsilon})w_\varepsilon(x) + \text{ boundary layer } + \text{ lower order terms.}$$

Assume that $\Omega = (0,1)^n$ and to simplify that the components of ν are non-positive, $\nu_i < 0$ for $1 \le i \le n$. Then the drift directs the particles towards the sides of Ω for which $x_i = 0$. In order to have a normalized solution ϕ_ε of norm 1 in $L^2(\Omega\times V)$, the L^2 norm of w_ε must be $O(\varepsilon^{-1})$. Therefore we obtain that the exiting distribution of particles $\phi_\varepsilon(x,v)$ is of order 1 on the sides $x_i = 0$ and not of order ε as when the drift vanishes.

As a consequence, the condition of vanishing incoming flux of particles implies that a boundary layer of $O(1)$ is unavoidable. Indeed, the transport solution is very anisotropic at the vicinity of the boundary. It is of order one on the sides $x_i = 0$ for the exiting directions, and vanishing for the incoming directions. Therefore the computation of the boundary layer derived in [4] is necessary to accurately describe the exiting density. Notice also that the leakage of particles at the boundary of the domain is of $O(1)$ because the exiting flux is of $O(1)$. This is consistent with $\lambda_{\infty,\nu} > \lambda_\infty$. More fission is needed to compensate for leakage. The criticality constant k_{eff} in (1.1) is therefore lowered when some drift is generated by the anisotropy of the physical coefficients.

4 Exponentially decaying solutions in infinite medium

This section is devoted to the proof of Theorem 3.1. Our objective is to derive exponentially decaying solutions of the spectral equation in the infinite periodic domain for which the drift term (3.13) vanishes. The parameter at our disposal is the vector ν pointing in the direction in which the solution decays. We are therefore looking for the smallest eigenvalue and the associated positive eigenvector of the following sequence of problems parameterized by ν:

$$\begin{aligned} &T\phi_\nu = K\phi_\nu + \lambda_{\infty,\nu}F\phi_\nu \\ &y \mapsto \exp(\nu\cdot y)\phi_\nu(y,v) \qquad \text{is} \quad Y-\text{periodic.} \end{aligned} \tag{4.1}$$

Notice that this family of periodic problems is similar to the one defined in [8, §4.5.8] where the heat equation is analyzed. Unlike the results of this reference, we have not been able to show that the function $\nu \mapsto -\lambda_{\infty,\nu}$ is strictly convex. This is consistent with the results in [13]. However, this convexity property is not necessary to ensure uniqueness of the vector ν such that the eigenvalue $\lambda_{\infty,\nu}$ is positive and the drift term (3.13) vanishes. Introducing the change of variables $\psi_\nu(y,v) = \exp(\nu\cdot y)\phi_\nu(y,v)$, we recast (4.1) as

$$\begin{aligned} &T\psi_\nu - v\cdot\nu\psi_\nu = K\psi_\nu + \lambda_{\infty,\nu}\psi_\nu \\ &y \mapsto \psi_\nu(y,v) \qquad \text{is} \quad Y-\text{periodic.} \end{aligned} \tag{4.2}$$

Before analyzing (4.2), we consider an auxiliary simpler problem where scattering is absent. Introduce the following sequence of eigenvalue problems

$$\begin{aligned} &T\varphi_\nu - v\cdot\nu\varphi_\nu = \tilde\mu_\nu\tilde K\varphi_\nu \\ &y \mapsto \varphi_\nu(y,v) \qquad \text{is} \quad Y-\text{periodic.} \end{aligned} \tag{4.3}$$

The usual sequence of adjoint eigenvalue problems is given by

$$\begin{aligned} &T^*\varphi^*_\nu - v\cdot\nu\varphi^*_\nu = \tilde\mu_\nu \tilde K^*\varphi^*_\nu \\ &y \mapsto \varphi^*_\nu(y,v) \qquad \text{is} \quad Y-\text{periodic.} \end{aligned} \tag{4.4}$$

The set of admissible values of ν is such that the absorption $\Sigma - v\cdot\nu$ remains positive on average since we are interested only in positive values of $\tilde\mu_\nu$. We consequently define the set

$$\Lambda = \{\nu \in \mathbb{R}^n, \text{ s.t. } (\Sigma - v\cdot\nu)(y,v) \text{ satisfies (H4)}\}. \tag{4.5}$$

We now state the main result on the function $\tilde\mu_\nu$.

Proposition 4.1 *Assume that hypotheses (H1), (H2), and (H3') hold. We define the function $\nu \mapsto \tilde\mu_\nu$ with domain of definition Λ as the smallest (positive) eigenvalue of (4.3). We denote by φ_ν and φ^*_ν the corresponding positive normalized eigenvectors of (4.3) and (4.4), respectively.*

Then, we have that $\nu \mapsto \tilde\mu_\nu$ is twice continuously differentiable and attains its maximal value at a unique point ν_0. We further have that

$$\frac{\partial\tilde\mu_\nu}{\partial\nu}(\nu_0) = 0 \qquad \textit{and} \qquad -\frac{\partial^2\tilde\mu_\nu}{\partial\nu_i\partial\nu_j}(\nu_0) \quad \textit{is a positive definite } n\times n \textit{ matrix.}$$

Moreover, the drift term vanishes

$$\tilde J_{\nu_0} = \int_Y\int_V v\,\varphi_{\nu_0}(y,v)\varphi^*_{\nu_0}(y,v)dy\,dv = 0.$$

Proof. The first step is to show some regularity properties of the function $\nu \mapsto (\tilde\mu_\nu, \varphi_\nu)$. The general theory of [14, 19] can be utilized to obtain most of the results of regularity presented here. Since the proof of local convexity requires the same calculations, we present them in detail now. Notice that analogous results have been obtained in [8] in the framework of second-order elliptic equations.

We first deduce from Theorem 2.4 the existence of $(\tilde\mu_\nu, \varphi_\nu)$ for $\nu \in \Lambda$. Let now $\nu \in \Lambda$ and $\delta\nu \in \mathbb{R}^n$ such that $\nu + \delta\nu \in \Lambda$. For a sequence of functions θ_ν, we define $\delta\theta_\nu = \theta_{\nu+\delta\nu} - \theta_\nu$. We deduce from (4.3) that

$$(T - v\cdot\nu - \tilde\mu_\nu\tilde K)\delta\varphi_\nu = v\cdot\delta\nu\varphi_{\nu+\delta\nu} + \delta\tilde\mu_\nu\tilde K\varphi_{\nu+\delta\nu}. \tag{4.6}$$

From the Fredholm alternative of Proposition 2.3, we have

$$\delta\tilde\mu_\nu(\tilde K\varphi_{\nu+\delta\nu}, \varphi^*_\nu) = -\delta\nu\cdot(v\varphi_{\nu+\delta\nu}, \varphi^*_\nu). \tag{4.7}$$

Since both $\varphi_{\nu+\delta\nu}$ and φ^*_ν are positive functions, we deduce that

$$|\tilde\mu_{\nu+\delta\nu} - \tilde\mu_\nu| \le C|\delta\nu|,$$

where C is independent of $\delta\nu$; hence the continuity of $\nu \mapsto \tilde\mu_\nu$. We then deduce from equation (4.6) and Proposition 2.3 that

$$\delta\varphi_\nu = C\varphi_\nu + O(|\delta\nu|).$$

Since φ_ν and $\varphi_{\nu+\delta\nu}$ are normalized, we deduce that $C = O(|\delta\nu|)$ and the continuity of $\nu \mapsto \varphi_\nu$. Again owing to (4.7), (4.6), and the continuity of $\nu \mapsto (\tilde{\mu}_\nu, \varphi_\nu)$, we obtain that this map also differentiable. The equation for $\frac{\partial}{\partial\nu}\varphi_\nu$ is defined up to a constant. Upon differentiating the normalization equation $\int_Y \int_V (\varphi_\nu)^2 dydv = 1$, we easily deduce that this constant vanishes. We thus obtain that

$$\frac{\partial}{\partial\xi}\tilde{\mu}_\nu = -\bar{\sigma}_\nu^{-1}\ \xi \cdot \tilde{J}_\nu,$$

where $\bar{\sigma}_\nu = \int_Y \int_V K(\varphi_\nu)\varphi_\nu^* dydv > 0$ and the drift current $\tilde{J}_\nu$ is defined as

$$\tilde{J}_\nu \ = \ \int_Y \int_V v\varphi_\nu(y,v)\varphi_\nu^*(y,v)dydv.$$

For $\xi \in \mathbb{R}^n$, we define

$$\varphi_{\nu,\xi} = \frac{\partial}{\partial\xi}\varphi_\nu \quad \text{and} \quad \tilde{\mu}_{\nu,\xi} = \frac{\partial}{\partial\xi}\tilde{\mu}_\nu.$$

We deduce from (4.6) that

$$(T - v\cdot\nu - \tilde{\mu}_\nu \tilde{K})\varphi_{\nu,\xi} = v\cdot\xi\varphi_\nu + \tilde{\mu}_{\nu,\xi}\tilde{K}\varphi_\nu. \tag{4.8}$$

Therefore, we have

$$\begin{aligned}(T - v\cdot\nu - \tilde{\mu}_\nu\tilde{K})(\delta\varphi_{\nu,\xi}) \ &= \ v\cdot(\delta\nu\, \varphi_{\nu+\delta\nu,\xi} + \xi\delta\varphi_\nu) \\ &+ \ \tilde{K}(\varphi_{\nu+\delta\nu,\xi}\,\delta\tilde{\mu}_\nu + \tilde{\mu}_{\nu,\xi}\,\delta\varphi_\nu + \varphi_\nu\,\delta\tilde{\mu}_{\nu,\xi}).\end{aligned} \tag{4.9}$$

Using the same Fredholm alternative as above, we deduce that $\delta\tilde{\mu}_{\nu,\xi}$, and then $\delta\phi_{\nu,\xi}$ are of order $O(|\delta\nu|)$. Therefore, $\tilde{\mu}_\nu$ is twice differentiable and

$$\bar{\sigma}_\nu \frac{\partial^2}{\partial\zeta\partial\xi}\tilde{\mu}_\nu = -\left(v\cdot[\zeta\varphi_{\nu,\xi} + \xi\varphi_{\nu,\zeta}] + \tilde{K}(\tilde{\mu}_{\nu,\zeta}\varphi_{\nu,\xi} + \tilde{\mu}_{\nu,\zeta}\varphi_{\nu,\xi}), \varphi_\nu^*\right). \tag{4.10}$$

Let us now define

$$D_{ij}(\nu) = -\frac{\partial^2\tilde{\mu}_\nu}{\partial\nu_i\partial\nu_j}. \tag{4.11}$$

This tensor is clearly symmetric. Iterating the above technique one more time, we also show that it depends continuously on $\nu \in \Lambda$.

Since $0 \in \Lambda$ and $\tilde{\mu}_0 > 0$, and since $\tilde{\mu}_\nu$ tends to 0 as ν approaches the boundary of Λ thanks to Theorem 2.4, we deduce that the continuous function $\nu \to \tilde{\mu}_\nu$ reaches its maximum in the interior of Λ, say at ν_0. We do not have uniqueness of the point ν_0 yet. However, we have that $\tilde{\mu}_{\nu_0,\xi} = 0$. Let us show that this implies the positive definiteness of $D_{ij}(\nu_0)$. Upon multiplying (4.8) by $\varphi_{\nu,\xi}\varphi_\nu^*(\varphi_\nu)^{-1}$at $\nu = \nu_0$, we deduce from (4.10) and (4.11) that

$$\bar{\sigma}_{\nu_0} D(\nu_0)\xi\cdot\xi = 2\int_{Y\times V}(T\varphi_{\nu_0,\xi} - v\cdot\nu_0\varphi_{\nu_0,\xi} - \tilde{\mu}_{\nu_0}\tilde{K}\varphi_{\nu_0,\xi})\frac{\varphi_{\nu_0,\xi}}{\varphi_{\nu_0}}\varphi_{\nu_0}^* dydv. \tag{4.12}$$

Notice that this equality holds at every point ν such that $\tilde{\mu}_{\nu,\xi} = 0$. Keeping this in mind, we drop the index 0 and define $\theta = \frac{\varphi_{\nu,\xi}}{\varphi_\nu}$. Since φ_ν is a positive solution of (4.3), we obtain that

$$\bar{\sigma}_\nu D(\nu)\xi\cdot\xi \;=\; 2\int_{Y\times V}(v\cdot\nabla_y\theta + \tilde{\mu}_\nu[\frac{\theta}{\varphi_\nu}\tilde{K}(\varphi_\nu) - \frac{1}{\varphi_\nu}\tilde{K}(\varphi_\nu\theta)])\theta\varphi_\nu\varphi_\nu^* dydv. \qquad (4.13)$$

We deduce from (4.3) and its adjoint equation that

$$v\cdot\nabla(\varphi_\nu\varphi_\nu^*) \;=\; \tilde{\mu}_\nu(\varphi_\nu^*\tilde{K}(\varphi_\nu) - \varphi_\nu\tilde{K}^*(\varphi_\nu^*)).$$

Integrating (4.13) by parts using the above relation yields after some algebra that

$$\bar{\sigma}_\nu D(\nu)\xi\cdot\xi \;=\; \int_{Y\times V}\int_Y f(y,v',v)\varphi_\nu^*(y,v)\varphi_\nu(y,v')|\theta(y,v') - \theta(y,v)|^2 dv'dydv. \qquad (4.14)$$

The same techniques are used in [2, 4] to obtain crucial a priori estimates that are used to deal with the spectral transport equation in the absence of drift. Therefore, $D(\nu)$ is positive. Assume now that $D\xi\cdot\xi = 0$. Then from (4.14), θ is independent of v. We deduce then from (4.8) that

$$v\cdot\nabla_y\theta = v\cdot\xi.$$

Upon integrating this equality over the cell Y, we get that $\xi\cdot v = 0$ for every $v\in V$, which, owing to hypothesis (H2), yields $\xi = 0$. Therefore, $D(\nu)$ is positive definite at every point ν such that $\tilde{\mu}_{\nu,\xi} = 0$. By continuity, we obtain the local convexity of the function $\nu\to-\tilde{\mu}_\nu$ in the vicinity of such points. This actually implies the uniqueness of the point ν_0 such that $\tilde{\mu}_{\nu,\xi} = 0$. Indeed, the domain Λ is clearly connected and star-like by construction. Assume now that $\nu\to\tilde{\mu}_\nu$ has two local maxima. Since $\tilde{\mu}_\nu$ is a C^2 function that vanishes at the boundary of Λ, there exists one point where its gradient vanishes and its Hessian matrix cannot be negative definite (saddle point or local minimum). However, $\tilde{\mu}_{\nu_m,\xi} = 0$ and then $D(\nu_m)$ is positive definite, which is a contradiction. We deduce that there exists a unique point ν_0 such that $\tilde{\mu}_{\nu_0,\xi} = 0$ and equivalently such that $\tilde{J}_{\nu_0} = 0$. This concludes the proof of this proposition. □

We are now in a position to prove Theorem 3.1. To use the results of the above proposition, we introduce the following two-parameter family of eigenvalue problems

$$\begin{array}{l} T\psi(\lambda,\nu) - v\cdot\nu\psi(\lambda,\nu) = \mu(\lambda,\nu)(K+\lambda F)\psi(\lambda,\nu) \\ y\mapsto\psi_\nu(y,v) \qquad \text{is} \quad Y-\text{periodic}. \end{array} \qquad (4.15)$$

The smallest (positive) eigenvalue $\mu(\lambda,\nu)$ exists for $(\lambda,\mu)\in\mathbb{R}^+\times\Lambda$ and we can associate the positive eigenvector $\psi(\lambda,\nu)$. We proceed as in the proof of Proposition 4.1 to show that $\mu(\lambda,\nu)$ and $\psi(\lambda,\nu)$ are twice continuously differentiable. Moreover the partial derivative $\partial_\lambda\mu(\lambda,\nu) < 0$. Indeed, differentiating (4.15) with respect to λ, multiplying by the adjoint function $\psi^*(\lambda,\nu)$ and integrating over $Y\times V$ yields

$$\partial_\lambda\mu(\lambda,\nu) = -\frac{\mu(\lambda,\nu)(F\psi(\lambda,\nu),\psi^*(\lambda,\nu))}{((K+\lambda F)\psi(\lambda,\nu),\psi^*(\lambda,\nu))},$$

which is clearly negative as the functions ψ and ψ^* are positive. Define now the function $\bar{\mu}(\lambda) = \mu(\lambda_\mu, \nu_0(\lambda))$, where ν_0 is the unique point obtained in Proposition 4.1 with $\tilde{K} = K + \lambda F$. We have that $\bar{\mu}(\lambda)$ is decreasing. Indeed, for $\delta\lambda > 0$ sufficiently small,

$$\bar{\mu}(\lambda + \delta\lambda) = \mu(\lambda + \delta\lambda, \nu_0(\lambda + \delta\lambda)) \leq \mu(\lambda + \delta\lambda, \nu_0(\lambda)),$$

since $\nu_0(\lambda + \delta\lambda)$ is a maximum for $\nu_0 \mapsto \mu(\lambda + \delta\lambda, \nu_0)$ at $\lambda + \delta\lambda$ fixed. Since $\partial_\lambda \mu < 0$, we also have $\mu(\lambda + \delta\lambda, \nu_0(\lambda)) < \mu(\lambda, \nu_0(\lambda)) = \bar{\mu}(\lambda)$. Thus, $\partial_\lambda \bar{\mu}(\lambda) < 0$. Now from hypotheses (H3), $\bar{\mu}(0) > 1$ since total absorption is greater than scattering. Furthermore, $\bar{\mu}(\lambda) \to 0$ as $\lambda \to \infty$ thanks to Theorem 2.4 with $\tilde{K} = K + \lambda F$. By continuity, there exists a unique value of λ_0 such that $\bar{\mu}(\lambda_0) = 1$. We denote by $\lambda_{\infty,\nu}$ this value, where $\nu = \nu_0(\lambda_0)$. Clearly, (3.11) and (3.12) are satisfied and $J_\nu = \tilde{J}_\nu = 0$ in (3.13). When $J = J_0 \neq 0$, we deduce from the extremality of ν that $\lambda_{\infty,\nu} > \lambda_{\infty,0}$. This concludes the proof of Theorem 3.1.

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