Mathematical Models and Methods in Applied Sciences

SPATIALLY VARYING DISCRETE ORDINATES METHODS IN XY- GEOMETRY

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We consider the coupling of angular discretizations of the two-dimensional linear transport equation. We show the well-posedness of the coupled problem and give an error estimate. The angular discretization is based on the Discrete Ordinates Method. It involves a quadrature rule specifically designed for our coupling. The theory uses the integral formulation of transport and will be demonstrated on simplified geometries.

Keywords: neutron transport, integral formulation, discrete ordinates methods, polynomial interpolation, coupling of equations, error estimates

1. Introduction

The neutron transport theory was developed to describe the behavior of neutrons in nuclear reactors. Transport equations are physically and mathematically well understood. However, their numerical simulation is highly demanding; therefore, transport equations have been rarely implemented in industrial settings. The recent advances of modern computers in terms of computation time and memory capacities enable us to solve transport equations of practical interest. Nevertheless, realistic whole-core calculations cannot be performed accurately today and probably will not be in the near future, unless diffusion equations are considered[19,11,2,8,9]. Approximations have to be made and fine transport resolution, needed to quantify the impact of local accidents such as depressurizations, has to be implemented locally. To deal with such problems we consider in this paper the coupling of two different Discrete Ordinates Methods used for the angular discretization of the transport equation on non-overlapping subdomains.

Discrete ordinates methods have been widely studied in the past. The first works[22,26] are based on the theory of collectively compact operators developed by Anselone[3]. The framework is the space of continuous functions except on a suitable set of null measure. More recently the analysis has been extended to the setting of L^p spaces, which allow us to obtain more accurate and sometimes optimal error estimates[24,18,4,5,6]. The technical results obtained by Johnson and Pitkäranta[18] will be of important use here. In these works, the number of discrete ordinates was not allowed to depend on the spatial position. We present here a model where two

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different angular discretizations are used on two identical squares sharing a common side. This geometry is aimed at simplifying the presentation of the theory.

The stationary one-velocity neutron transport problem in a domain surrounded by vacuum can be formulated as follows. Given a source term q, a total cross section Σ and a scattering cross section f, find the angular flux $\psi = \psi(\mathbf{x}, \boldsymbol{\mu})$ such that for $(\mathbf{x}, \boldsymbol{\mu}) \in \Omega \times S^2$:

$$\begin{aligned} &\boldsymbol{\mu} \cdot \boldsymbol{\nabla} \psi(\mathbf{x}, \boldsymbol{\mu}) + \Sigma(\mathbf{x})\psi(\mathbf{x}, \boldsymbol{\mu}) = \int_{S^2} f(\mathbf{x}, \boldsymbol{\mu}', \boldsymbol{\mu})\psi(\mathbf{x}, \boldsymbol{\mu}')d\boldsymbol{\mu}' + q(\mathbf{x}, \boldsymbol{\mu}) \\ &\psi(\mathbf{x}, \boldsymbol{\mu}) = 0, \qquad \text{for } \mathbf{x} \in \Gamma^-_{\boldsymbol{\mu}}. \end{aligned} \tag{1.1}$$

Here Ω is a bounded domain in $I\!R^3$ with boundary Γ, $\boldsymbol{\mu} \cdot \boldsymbol{\nabla} = \sum_{i=1}^{3} \mu_i \frac{\partial}{\partial x_i}$ and

$$S^2 = \{\boldsymbol{\mu} \in I\!R^3, |\boldsymbol{\mu}| = 1\}, \qquad \Gamma^-_{\boldsymbol{\mu}} = \{\mathbf{x} \in \Gamma, \boldsymbol{\mu} \cdot \boldsymbol{n}(\mathbf{x}) < 0\},$$

where $\boldsymbol{n}(\mathbf{x})$ is the outward unit normal to Γ at $\mathbf{x} \in \Gamma$. We refer to Duderstadt and Martin[17] for further details. This problem has been studied mathematically by e.g., Vladimirov[27] and Dautray-Lions[15], Chapter 21. It admits a unique solution when the physical data are sufficiently regular and the neutron production by scattering or fission is smaller than the neutron absorption.

Because $\Omega \in I\!R^3$ and S^2 is two-dimensional, there are five independent variables in this problem. So even moderately accurate discretizations yield discrete problems with many unknowns. Industrial applications generally involve cylindrical geometries for which the spatial dimension is reduced to two. The relevant angular domain is then the unit disc in $I\!R^2$, defined as the projection of the unit sphere on a plane. The analysis of the coupling of two discrete ordinates methods in this setting is complicated. To simplify the presentation we consider a model problem, where the angular domain has been replaced by the unit circle. This problem is of poor physical interest. However, it exhibits the main mathematical difficulties of the real 2D problem[4,18]. The main new mathematical tool in the context of the discretization of transport equations is the crucial use of L^2 interpolation results –see (2.17) below. They have been introduced recently in the theory of spectral approximations by Bernardi and Maday[13,14]. The extension to the realistic 2D problem is presented in the author's Ph.D. thesis[7]. The coupling operators presented here have been tested numerically[7,10]. The results given in these works show the efficiency of the coupling, in terms of robustness and accuracy, on Benchmarks of industrial interest.

The paper is organized as follows. In section 2 we present the model problem and reformulate it as a Fredholm integral equation for the scalar flux, defined as the integration of the angular flux over the unit circle. We also present the angular discretization based on the discrete ordinates method and design a well-suited quadrature rule for our coupling. Section 3 is devoted to the derivation of the coupled problem in integro-differential and integral form. It contains the definition of the coupling operators and gives our convergence and error estimate results. The proof of these results is postponed to section 4. In section 5 we state our conclusions.

2. A Model problem

As mentioned in the introduction, the analysis of the full problem (1.1) in cylindrical geometry is complicated. Therefore, we consider the unit circle as our velocity space instead of the unit sphere. The model is not physically relevant, since the neutrons are confined in a plane. However, it allows for spatially varying discretizations of the velocity space, which is our main interest in this paper. We also consider homogeneous and isotropic cross sections f and Σ, i.e., independent of the position $\mathbf{x}$ and the velocity variables $\boldsymbol{\mu}$ and $\boldsymbol{\mu}'$. The same coupling can be defined for non isotropic and spatially inhomogeneous cross sections. However, its theoretical analysis would be more involved technically. Provided that the physical data are sufficient regular, the proofs in this paper can be extended to the more general setting of spatially varying coefficients without major difficulty. Our model is the following transport problem. For λ and $q(\mathbf{x})$ given, we want to find $\psi(\mathbf{x},\boldsymbol{\mu})$ satisfying

$$\begin{aligned} &\boldsymbol{\mu}\cdot\boldsymbol{\nabla}\psi(\mathbf{x},\boldsymbol{\mu})+\psi(\mathbf{x},\boldsymbol{\mu})=\lambda\int_S\psi(\mathbf{x},\boldsymbol{\mu}')d\boldsymbol{\mu}'+q(\mathbf{x}), \qquad (\mathbf{x},\boldsymbol{\mu})\in\Omega\times S,\\ &\psi(\mathbf{x},\boldsymbol{\mu})=0 \qquad \text{on } \Gamma^-_{\boldsymbol{\mu}}, \end{aligned} \tag{2.2}$$

where Ω with boundary Γ is the reunion of two identical squares defined in a direct orthogonal basis $\{\boldsymbol{e}_x,\boldsymbol{e}_y\}$ of $I\!R^2$ as $\Omega_1=(-L,0)\times(0,L)$ and $\Omega_2=(0,L)\times(0,L)$ with boundary Γ_1 and Γ_2 respectively, and where

$$S=\{\boldsymbol{\mu}\in I\!R^2, |\boldsymbol{\mu}|=1\}, \qquad \Gamma^-_{\boldsymbol{\mu}}=\{\mathbf{x}\in\Gamma, \boldsymbol{\mu}\cdot\boldsymbol{n}(\mathbf{x})<0\}.$$

Here, $\boldsymbol{n}(\mathbf{x})$ is the outward unit normal to Γ at $\mathbf{x}\in\Gamma$. We denote by γ the interface between Ω_1 and Ω_2, and by $\boldsymbol{n}_\gamma(\mathbf{x})$ its outward unit normal with respect to Ω_1 at point $\mathbf{x}\in\gamma$ (see Fig.1).

The convergence theory developed in this paper is based on the integral formulation of the neutron transport. It amounts to inverting the free transport operator by the usual method of characteristics; therefore, its derivation depends on the geometry of the domain. To focus only on the mathematical difficulties of the coupling, we restrict ourselves to the simplified geometry mentioned above. More complicated geometries have been considered numerically[7,10].

2.1. *Integral formulation of neutron transport*

We first derive some properties of the free transport solution; i.e. when there is no production term. The free transport problem reads: Given $g\in L^2(\Omega)$, find w such that

$$\begin{aligned} &\boldsymbol{\mu}\cdot\boldsymbol{\nabla}w+w=g \qquad \text{in } \Omega,\\ &w=0 \qquad \text{on } \Gamma^-_{\boldsymbol{\mu}}. \end{aligned} \tag{2.3}$$

We define the operator $T_{\boldsymbol{\mu}}$ as $T_{\boldsymbol{\mu}}g=w$, where w is the solution to (2.3). We check that

$$T_{\boldsymbol{\mu}}g(\mathbf{x})=\int_0^{d(\mathbf{x},\boldsymbol{\mu})}e^{-s}g(\mathbf{x}-s\boldsymbol{\mu})ds, \tag{2.4}$$

where $d(\mathbf{x},\boldsymbol{\mu})$ is the distance from $\mathbf{x}\in\Omega$ to Γ in direction $-\boldsymbol{\mu}$; i.e., $d(\mathbf{x},\boldsymbol{\mu})=\overline{s}$ where $\mathbf{x}-\overline{s}\boldsymbol{\mu}\in\Gamma$, $\overline{s}>0$. Let us introduce similar notation. We define $d_\gamma(\mathbf{x},\boldsymbol{\mu})$ as the distance from $\mathbf{x}\in\Omega$ to γ for the points $(\mathbf{x},\boldsymbol{\mu})\in\Omega\times S$ such that the set $\{\mathbf{x}-s\boldsymbol{\mu}, s>0\}$ crosses the interface γ. Replacing Ω by Ω_k and Γ by Γ_k in (2.3), we also define the operators $T_{\boldsymbol{\mu}}^{\Omega_k}$ and the distances $d_k(\mathbf{x},\boldsymbol{\mu})$, for $k=1,2$. We can show[18] that $T_{\boldsymbol{\mu}}$ is bounded in the following sense

$$\|\boldsymbol{\mu}\cdot\nabla T_{\boldsymbol{\mu}}g\|+\|T_{\boldsymbol{\mu}}g\|+\Big(\int_\Gamma (T_{\boldsymbol{\mu}}g)^2\,|\boldsymbol{n}\cdot\boldsymbol{\mu}|\,ds\Big)^{1/2}\le C\,\|g\|, \tag{2.5}$$

where C is a constant independent of g and $\|\cdot\|$ denotes the $L^2(\Omega)$-norm. Let us now introduce the *scalar flux* ϕ and the *integral operator* T defined by

$$\phi(\mathbf{x})=\int_S \psi(\mathbf{x},\boldsymbol{\mu})\,d\boldsymbol{\mu},\qquad T=\int_S T_{\boldsymbol{\mu}}d\boldsymbol{\mu}. \tag{2.6}$$

Then problem (2.2) can be formulated as $\psi(\mathbf{x},\boldsymbol{\mu})=T_{\boldsymbol{\mu}}(\lambda\phi+q)(\mathbf{x})$. Integrating this equation over S yields the following integral equation for the scalar flux

$$(I-\lambda T)\phi=Tq. \tag{2.7}$$

By change of variables from polar to Cartesian coordinates, we have

$$Tg(\mathbf{x})=\int_\Omega \frac{e^{-|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}g(\mathbf{y})\,d\mathbf{y}. \tag{2.8}$$

The operator T is positive, self adjoint, and weakly singular (see Mikhlin[21], Chapter 6); hence, it maps $L^2(\Omega)$ into $H^1(\Omega)$ and is compact as a linear operator from $L^2(\Omega)$ to $L^2(\Omega)$. Consequently, (2.7) is a Fredholm equation of the second kind.

Assume that $|\lambda|<\lambda_{max}$ where λ_{max}^{-1} is the largest eigenvalue of T in absolute value. Then according to the Fredholm alternative, $(I-\lambda T)^{-1}$ exists and is bounded as a linear map from $L^2(\Omega)$ to $L^2(\Omega)$. This implies that (2.7) admits a unique solution $\phi=(I-\lambda T)^{-1}Tq$, and that there exists a constant C such that

$$\|(I-\lambda T)V\|\ge C\|V\|,\qquad \forall V\in L^2(\Omega). \tag{2.9}$$

2.2. *Discrete Ordinates Method*

Several numerical methods have been developed to solve the integral equation (2.7) instead of (2.2) (see e.g. Lewis and Miller[20]). Although only the domain Ω needs to be discretized, these methods are generally quite expensive and can be used only on small domains. In order to solve (2.2), Discrete Ordinates Methods combined with spatial discretizations such as nodal methods have been widely used in engineering computations of neutron transport problems[18,20]. In this paper we focus on the discretization of the angular variable. Let us introduce the following quadrature rule:

$$\int_S \varphi(\boldsymbol{\mu})\,d\boldsymbol{\mu}\sim\sum_{\boldsymbol{\mu}\in Q}\omega_{\boldsymbol{\mu}}\varphi(\boldsymbol{\mu}), \tag{2.10}$$

where $Q = Q_N = \{\boldsymbol{\mu}^1, \dots, \boldsymbol{\mu}^N\}$ is a discrete set of quadrature points $\boldsymbol{\mu}^i \in S$ with positive weights $\omega_{\boldsymbol{\mu}^i}$. The discrete ordinates method consists of replacing the integral term in (2.2) using (2.10) and solving the following coupled system of equations:

$$\begin{aligned} \psi_N(\mathbf{x}, \boldsymbol{\mu}) &= T_{\boldsymbol{\mu}}(\lambda\phi_N + q)(\mathbf{x}), \qquad \text{for } (\mathbf{x}, \boldsymbol{\mu}) \in \Omega \times Q, \\ \phi_N(\mathbf{x}) &= \sum_{\boldsymbol{\mu} \in Q} \omega_{\boldsymbol{\mu}}\, \psi_N(\mathbf{x}, \boldsymbol{\mu}). \end{aligned} \tag{2.11}$$

Summing over the directions of Q yields the following integral equation (with a singular kernel, since the spatial integration is performed only over the characteristics oriented by the directions of Q):

$$(I - \lambda T_N)\phi_N = T_N q, \qquad \text{where} \quad T_N = \sum_{\boldsymbol{\mu} \in Q} \omega_{\boldsymbol{\mu}}\, T_{\boldsymbol{\mu}}. \tag{2.12}$$

Johnson and Pitkäranta[18] proved that problem (2.12) admits a unique solution for a large variety of quadrature rules. The framework and convergence analysis of the sequence of operators T_N to T given in their paper will be the starting point of our coupling analysis. Let us also mention that optimal error estimates have been derived by M. Asadzadeh et al.[6] in $L^1(\Omega)$.

2.3. *A well-suited quadrature rule*

For the purpose of the coupling, we impose several constraints on the quadrature rule, which define it almost uniquely, as can be seen from the list of properties below. Assume that the number of directions of the angular discretization equals $4N_k$ on Ω_k, for $k = 1, 2$. Let us denote by $\delta = (N_1, N_2)$ and

$$N_{max} = \max(N_1, N_2), \qquad N_{min} = \min(N_1, N_2).$$

We now define the quadrature rules $Q_k = \{\boldsymbol{\nu}_i^k, |i| = 1, \dots, 2N_k\}$, $\boldsymbol{\nu}_i^k \in S$, which consist of $4N_k$ different directions and positive weights ω_i^k. From now onwards, we use bold type for a vector $\boldsymbol{\nu} \in S$ (where S is the unit circle); its polar representation, i.e. the angle between $\boldsymbol{\nu}$ and $\boldsymbol{e}_x$, element of $(-\pi, \pi)$ will be denoted by ν. Thus, $\boldsymbol{\nu} = (\cos\nu, \sin\nu)$. For any function F depending on ν or $\boldsymbol{\nu}$, we write $F(\nu) = F(\boldsymbol{\nu})$ identically. With this notation, we distribute the quadrature points on each quarter of circle as follows:

$$\begin{array}{ll} \nu_i^k \in SW = (-\pi, -\pi/2) & \text{for } i \in I_k^{SW} = \{-2N_k, \dots, -N_k - 1\} \\ \nu_i^k \in NW = (\pi/2, \pi) & \text{for } i \in I_k^{NW} = \{-N_k, \dots, -1\} \\ \nu_i^k \in NE = (0, \pi/2) & \text{for } i \in I_k^{NE} = \{1, \dots, N_k\} \\ \nu_i^k \in SE = (-\pi/2, 0) & \text{for } i \in I_k^{SE} = \{N_k + 1, \dots, 2N_k\}. \end{array} \tag{2.13}$$

In order to define the points uniquely, we assume that $\nu_i^k \to i$ is an counter-clockwise increasing function (ranking the points is crucial for defining the coupling operators). Let us also introduce the following weights on each quarter of circle:

$$\begin{aligned} w_{NE}(\theta) &= (\frac{\pi}{2}-\theta), \quad \theta \in NE, & w_{SE}(\theta) &= (\theta+\frac{\pi}{2}), \quad \theta\in SE, \\ w_{SW}(\theta) &= (-\frac{\pi}{2}-\theta), \quad \theta\in SW, & w_{NW}(\theta) &= (\theta-\frac{\pi}{2}), \quad \theta \in NW. \end{aligned} \tag{2.14}$$

We define the quadrature rule Q_k as the *Gauss-Legendre quadrature rule* of order N_k for each quarter of circle $D \in \{NE, SE, SW, NW\}$ (see Szegö[25]). It satisfies the following properties:

- For each polynomial φ of order less than $2N_k - 1$, we have

$$\int_D \varphi = \sum_{i\in I_D^k} \omega_i^k \varphi(\nu_i^k). \tag{2.15}$$

- For each sufficiently regular function φ, we have the interpolation estimate

$$\Big|\int_S \varphi - \sum_{|i|=1}^{2N_k} \omega_i^k \varphi(\nu_i^k)\Big| \le \frac{C}{N_k}\int_0^{2\pi}\Big|\frac{d\varphi}{d\nu}\Big| d\nu, \tag{2.16}$$

- and for each quarter of circle D the crucial L^2-error estimate †

$$\left(\int_D \left(\mathcal{J}_D^{N_k-1}\varphi-\varphi\right)^2 w_D\right)^{\frac12} \le \frac{C}{N_k^s}\,\|\varphi\|_{s,D}\,, \tag{2.17}$$

for any $s > 1/2$, where $\|.\|_{s,D}$ stands for the usual $H^s(D)$-norm[1].

- The distribution of points and weights satisfies the following relations

$$\sum_{|i|=1}^{2N_k}\omega_i^k = 2\pi, \qquad \sum_{|i|=1}^{2N_k}\frac{\omega_i^k}{|\nu_i^k|} \le C(1+\log N_k), \tag{2.18}$$

where C is a constant independent of N_k.

- Let $\varepsilon > 0$. We define

$$\begin{aligned} I'_{\varepsilon,k} &= \left\{(i,j), (\boldsymbol{\nu}_i^k,\boldsymbol{\nu}_j^k)\in Q_k^2 \text{ s.t.} \min\left(\xi(\boldsymbol{\nu}_i^k,\boldsymbol{\nu}_j^k),\xi(\boldsymbol{\nu}_i^k,\boldsymbol{t}),\xi(\boldsymbol{\nu}_j^k,\boldsymbol{t})\right)\ge\varepsilon, \boldsymbol{t}\in\{\boldsymbol{e}_x,\boldsymbol{e}_y\}\right\} \\ I''_{\varepsilon,k} &= \left\{(i,j),(\boldsymbol{\nu}_i^k,\boldsymbol{\nu}_j^k)\in Q_k^2 \text{ s.t. } (i,j)\notin I'_{\varepsilon,k}\right\} \\ J'_{\varepsilon,k,k'} &= \Big\{(i,j),(\boldsymbol{\nu}_i^k,\boldsymbol{\nu}_j^{k'})\in Q_k\times Q_{k'} \text{ s.t.} \\ &\qquad \min\left(\xi(\boldsymbol{\nu}_i^k,\boldsymbol{\nu}_j^{k'}),\xi(\boldsymbol{\nu}_i^k,\boldsymbol{t}),\xi(\boldsymbol{\nu}_j^{k'},\boldsymbol{t})\right)\ge\varepsilon,\boldsymbol{t}\in\{\boldsymbol{e}_x,\boldsymbol{e}_y\}\Big\}, \\ J''_{\varepsilon,k,k'} &= \left\{(i,j),(\boldsymbol{\nu}_i^k,\boldsymbol{\nu}_j^{k'})\in Q_k\times Q_{k'} \text{ s.t. } (i,j)\notin J'_{\varepsilon,k,k'}\right\}, \end{aligned} \tag{2.19}$$

where $\xi(\boldsymbol{\mu},\boldsymbol{\nu}) = \sin\alpha(\boldsymbol{\mu},\boldsymbol{\nu})$ and $\alpha(\boldsymbol{\mu},\boldsymbol{\nu})$ is the smallest angle between $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$. Let us also denote by

$$\sigma(\varepsilon,N_k,N_{k'}) = \sup\Big(\sum_{(i,j)\in J''_{\varepsilon,k,k'}}\omega_i^k\omega_j^{k'}\,,\ \sum_{(i,j)\in I''_{\varepsilon,k}}\omega_i^k\omega_j^k\Big). \tag{2.20}$$

† $\mathcal{J}_D^{N_k-1}\varphi$ is the polynomial interpolation on D of the function φ at the points ν_i^k of the quadrature rule. It is defined precisely in (3.22) below.

Then we have

$$\sigma(\varepsilon, N_k, N_{k'}) \to 0, \quad \text{when } \varepsilon \to 0,\ N_k \to \infty \text{ and } N_{k'} \to \infty. \tag{2.21}$$

In this definition, $I'_{\varepsilon,k}$ corresponds to the set of pairs of directions that are not close to those of the boundary nor close to one another. The set $J'_{\varepsilon,k,k'}$ plays the same role when the directions belong to two different quadrature rules.

Remark 1 Properties (2.15,2.16,2.18) are classical in the theory of approximation. They ensure that the discrete integration is performed accurately. Inequality (2.17) is a stronger result. Not only does it assert that the integration is accurate, but also it ensures that the polynomial interpolation is close to the exact function in the sense of the L^2-norm. This forms a key result in the analysis of spectral methods for elliptic problems. Its derivation has been obtained recently for Gaussian quadrature rules[13,14]. Condition (2.21) is natural. The measure of the set of couples $(\boldsymbol{\mu}, \boldsymbol{\nu}) \in S \times S$ close to one another or close to one of the directions of the boundary $\boldsymbol{e}_x$ or $\boldsymbol{e}_y$ goes to zero with the measure of this closeness. This behavior is also to be expected from the quadrature rules as the numbers of directions N_k and $N_{k'}$ go to infinity.

Remark 2 The quadrature rule satisfies Hypotheses (4.1a) and (4.1b) given by Johnson and Pitkäranta[18]. Therefore we can apply the results on the discrete ordinates methods given in this reference. Namely problem (2.12) admits a unique solution provided N is sufficiently large.

Proof. Now we prove that the Gaussian quadrature rule indeed satisfies these properties. We refer to other works[14,13] for the most important results on the quadrature, essentially (2.15) and (2.17) (Theorem 13.2 knowing that w_D is bounded). The proof of (2.16) is given by de Vore and Scott[16] and the proof of (2.18) by Pitkäranta and Scott, Lemma 3.1[24] (we have by [p.930 1)][24] that $(\nu_1^k)^{-1} \leq C N_k^2$). It remains to check (2.21). This is a consequence of the result given by Szegö[25], Section 3.41:

$$\sum_{i=k}^{k+p} \omega_i \leq |\nu_{k+p+1} - \nu_{k-1}|.$$

This inequality means that the sum of the weights associated with the directions that belong to a given connected subdomain of S corresponds asymptotically to the measure of this subdomain (see also the distribution of the points of the Gauss Legendre quadrature in section 6.6[25]). We deduce that there exists a constant C independent of N_k and ε such that the sum of the weights associated with the directions close up to ε to a given direction is bounded by $C(\varepsilon + 1/N)$. Treating all contributions of the sum (2.20) separately yields (2.21). $\square$

3. Coupled Problem and convergence results

We use the quadrature rule defined in the previous section to obtain an angular discretization of equation (2.2). Assume that we want to obtain a more accurate approximation of the solution on Ω_1, say, than on Ω_2. Then we use different discretizations of the unit circle S on both subdomains. More precisely S is discretized by using $4N_k$ angular directions on the subdomain Ω_k, for $k = 1, 2$, with $N_1 \geq N_2$.

3.1. *Integro-differential and integral coupled problems*

In both situations $N_1 = N_2$ and $N_1 \neq N_2$, the scattering kernel (integration over the velocity space V) is replaced by a well-suited discrete integration on each subdomain. When the number of directions on Ω_1 and Ω_2 differ, it remains to find out some compatibility conditions at the interface γ. More precisely, we have to transform an outgoing flux defined by its value at $2N_1$ points into an incoming flux defined by its value at $2N_2$ points, and conversely. To define a flux in a given direction from its values on a set of other directions, we perform an interpolation. We introduce the polynomial interpolation of the flux to deal with functions defined on the whole angular domain.

Let $D \in \{NE, SE, NW, SW\}$ and $\mathbb{P}_N(D)$ be the set of polynomials on D of degree at most N. For each continuous function f on D we define the polynomial interpolation $\mathcal{J}_D^{N_k-1} f \in \mathbb{P}_{N_k-1}(D)$ by

$$\left(\mathcal{J}_D^{N_k-1} f\right)(\nu_i^k) = f(\nu_i^k), \qquad \forall\, i \in I_k^D, \tag{3.22}$$

where $k = 1, 2$.

Using the polynomial interpolation formulation we define the coupled problem as follows: Find the pair of functions $(\psi_1(\mathbf{x}, \nu), \psi_2(\mathbf{x}, \nu))$ such that for $k = 1, 2$ and for each given $\mathbf{x} \in \Omega_k$, $\psi_k(\mathbf{x}, \nu)$ is a polynomial of degree $N_k - 1$ with respect to ν on each quarter of circle $D \in \{NE, SE, NW, SW\}$, satisfying the following equation for $(k, k') = (1, 2)$ and $(k, k') = (2, 1)$:

$$\left\{\begin{array}{l} \boldsymbol{\nu}_i^k \cdot \boldsymbol{\nabla} \psi_k(\mathbf{x}, \nu_i^k) + \psi_k(\mathbf{x}, \nu_i^k) = (\lambda \phi_\delta + q)(\mathbf{x}), \qquad \mathbf{x} \in \Omega_k, \quad |i| = 1, \ldots, 2N_k, \\ \psi_k(\mathbf{x}, \nu_i^k) = 0, \qquad \mathbf{x} \in \Gamma_k \backslash \gamma, \quad \boldsymbol{\nu}_i^k \cdot \boldsymbol{n}(\mathbf{x}) < 0, \quad |i| = 1, \ldots, 2N_k, \\ \psi_k(\mathbf{x}, \nu) = \left(\mathcal{Q}^D \psi_{k'}\right)(\mathbf{x}, \nu), \quad \mathbf{x} \in \gamma, \quad \nu \in D, \quad D \in \mathcal{D}_k \\ \qquad \mathcal{D}_1 = \{NW, SW\}, \quad \mathcal{D}_2 = \{NE, SE\}, \end{array}\right. \tag{3.23}$$

where the approximated scalar flux is given by

$$\phi_\delta(\mathbf{x}) = \sum_{|i|=1}^{2N_k} \omega_i^k \psi_k(\mathbf{x}, \nu_i^k), \quad \mathbf{x} \in \Omega_k, \tag{3.24}$$

and where the coupling linear operators admit the following domains of definition for $\mathbf{x} \in \gamma$:

$$\begin{array}{lll} \mathcal{Q}^D & : & \mathbb{P}_{N_1-1}(D) \to \mathbb{P}_{N_2-1}(D), \text{ for } D = NE,\ SE \\ \mathcal{Q}^D & : & \mathbb{P}_{N_2-1}(D) \to \mathbb{P}_{N_1-1}(D), \text{ for } D = NW,\ SW. \end{array}$$

These coupling operators will be defined precisely in the next section. Note that we only have to compute the quantities $\psi_k(\mathbf{x}, \nu_i^k)$, $|i| = 1, \ldots, 2N_k$ at the points

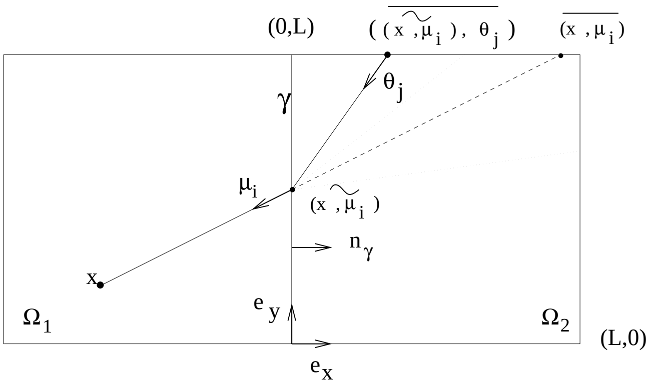

Fig. 1. Simplified Geometry

of the quadrature rule Q_k. The function $\psi_k(\mathbf{x},\boldsymbol{\nu})$, $\forall\boldsymbol{\nu}\in S$ is then reconstructed by polynomial interpolation. Similarly to the continuous problem, we recast equation (3.23) as an integral equation. We need the following characteristic functions, defined for each $\boldsymbol{\nu}\in S$ by

$$\chi_{\boldsymbol{\nu}}(\mathbf{x})=\begin{cases}1, & \text{if } \{\mathbf{x}-t\boldsymbol{\nu}, t>0\}\cap\gamma\neq\emptyset,\\ 0, & \text{otherwise.}\end{cases}\tag{3.25}$$

For $k=1,2$, we define the projections on the boundaries (see Fig.1)

$$\begin{array}{lll}(\overline{\mathbf{x},\boldsymbol{\nu}}) &= \mathbf{x}-d(\mathbf{x},\boldsymbol{\nu})\boldsymbol{\nu}, & (\mathbf{x},\boldsymbol{\nu})\in\Omega\times S,\\ (\overline{\mathbf{x},\boldsymbol{\nu}})_k &= \mathbf{x}-d_k(\mathbf{x},\boldsymbol{\nu})\boldsymbol{\nu}, & (\mathbf{x},\boldsymbol{\nu})\in\Omega_k\times S,\\ (\widetilde{\mathbf{x},\boldsymbol{\nu}}) &= \mathbf{x}-d_\gamma(\mathbf{x},\boldsymbol{\nu})\boldsymbol{\nu}, & (\mathbf{x},\boldsymbol{\nu})\in\Omega_k\times S, \text{ and } \chi_{\boldsymbol{\nu}}(\mathbf{x})=1.\end{array}\tag{3.26}$$

The distances d, d_k and d_γ are defined in section 2.1. We check that the solution of the non-homogeneous boundary value problem

$$\begin{array}{ll}\boldsymbol{\mu}\cdot\boldsymbol{\nabla}w+w=0 & \text{in }\Omega\\ w=h & \text{on }\Gamma^-_{\boldsymbol{\mu}},\end{array}$$

is given by $w(\mathbf{x})=e^{-d(\mathbf{x},\boldsymbol{\mu})}h(\overline{\mathbf{x},\boldsymbol{\mu}})$. According to (3.23) this yields the following integral formulation

$$\psi_k(\mathbf{x},\nu_i^k)=T^{\Omega_k}_{\boldsymbol{\nu}_i^k}(\lambda\phi_\delta+q)(\mathbf{x})\,+\,\chi_{\boldsymbol{\nu}_i^k}(\mathbf{x})e^{-d_\gamma(\mathbf{x},\boldsymbol{\nu}_i^k)}\psi_k\left((\widetilde{\mathbf{x},\boldsymbol{\nu}_i^k}),\nu_i^k\right),$$

for each $\mathbf{x}\in\Omega_k$ and $k=1,2$. It then remains to sum up this expression over all directions ν_i^k of the quadrature rule Q_k. We obtain the following integral problem: Find $\phi_\delta\in L^2(\Omega)$ such that

$$(I-\lambda T_\delta)\phi_\delta=T_\delta q,\tag{3.27}$$

where for $(k,k')=(1,2)$ or $(k,k')=(2,1)$, we have defined

$$\begin{aligned}
T_\delta g(\mathbf{x}) &= \sum_{|i|=1}^{2N_k} \omega_i^k\, T_{\boldsymbol{\nu}_i^k}\, g\,(\mathbf{x}) + \sum_{|i|=1}^{2N_k} \omega_i^k\, T_{\boldsymbol{\nu}_i^k}^{\Omega_{k'}\to\Omega_k}\, g\,(\mathbf{x}), \quad \mathbf{x}\in\Omega_k \\
T_{\boldsymbol{\nu}}^{\Omega_{k'}\to\Omega_k}\, g\,(\mathbf{x}) &= \chi_{\boldsymbol{\nu}}(\mathbf{x}) e^{-d_\gamma(\mathbf{x},\boldsymbol{\nu})}\, T_{\boldsymbol{\nu}}^{\Omega_{k'}\to\gamma}\, g(\widetilde{\mathbf{x},\boldsymbol{\nu}}), \quad \mathbf{x}\in\Omega_k \quad \boldsymbol{\nu}\in S \\
T_{\boldsymbol{\nu}}^{\Omega_{k'}\to\gamma} g(\boldsymbol{t}) &= \Big(\mathcal{Q}^D \mathcal{J}_D^{N_{k'}-1}\, T_{\boldsymbol{\nu}}\, g\,(\boldsymbol{t})\Big)\,(\boldsymbol{\nu}), \quad \nu\in D, \quad D\in\mathcal{D}_k, \quad \boldsymbol{t}\in\gamma.
\end{aligned} \tag{3.28}$$

Problems (3.23) and (3.27) are equivalent. Indeed, once (3.27) is solved, we obtain the solution to (3.23) by inverting free transport problems. Thus, it remains to prove that problem (3.27) admits a unique solution for a judicious choice of coupling operators and that it converges to the solution of (2.7). This study is directly related to the analysis of the sequence of operators T_δ, which we will compare to T and to the discretized operators

$$T_{N_k} = \sum_{|i|=1}^{2N_k} \omega_i^k T_{\boldsymbol{\nu}_i^k} \quad k=1,2. \tag{3.29}$$

3.2. *Coupling operators*

The proof of convergence given in the next section depends on the particular choice of the coupling operators $\mathcal{Q}^D$ we consider. However, the central idea in the construction of the coupling is to preserve the interface currents, or more precisely a polynomial approximation that has a similar form (see (3.38) below). Let $\pi_{w_D}^{N-1}$ be the $L^2_{w_D}$ projection on the set $I\!P_{N-1}(D)$ where $L^2_{w_D}$ is equipped with the scalar product $(f,g)_{w_D}=\int_D w_D\, f\, g$. This projection is defined by the following relations:

$$(\pi_{w_D}^{N-1} f, P)_{w_D} \;=\; (f,P)_{w_D}\,, \quad \forall P\in I\!P_{N-1}(D). \tag{3.30}$$

It is stable in $L^2_{w_D}$:

$$\int_D \left(\pi_{w_D}^{N-1} f\right)^2\, w_D \le \int_D f^2\, w_D. \tag{3.31}$$

For $D\in\mathcal{D}_k$, $(k,k')=(1,2)$ or $(k,k')=(2,1)$, we define the *coupling operators* by

$$\begin{aligned}
\mathcal{Q}^D \quad &: \quad I\!P_{N_k-1}(D)\to I\!P_{N_{k'}-1}(D) \\
\mathcal{Q}^{\mathcal{D}} \quad &= \quad \pi_{w_D}^{N_k-1}.
\end{aligned} \tag{3.32}$$

Recall that for $N_k > N_{k'}$, $\pi_{w_D}^{N_k-1}$ is the embedding operator from $I\!P_{N_{k'}-1}(D)$ into $I\!P_{N_k-1}(D)$. The advantages of the polynomial interpolation formulation appears clearly in this definition. The coupling operators act on the interpolated function and not on its values at discrete directions. According to the properties of the quadrature rule, the polynomial interpolation is an accurate approximation of the exact angular flux on γ.

Let us now state some useful properties of these operators. First, from the definition of the projections, we obtain that for each pair of functions $h, g \in L^\infty(D)$, we have

$$\int_D w_D \left(\mathcal{Q}^D \mathcal{J}_D^{N_k-1} h \right) \mathcal{J}_D^{N_{k'}-1} g = \int_D w_D \mathcal{J}_D^{N_k-1} h \mathcal{J}_D^{N_{k'}-1} g. \tag{3.33}$$

Proposition 3 *For $h \in L^\infty(D)$, we have the following stability condition*

$$\sum_{i \in I_{k'}^D} \omega_i^{k'} w_D(\nu_i^{k'}) \left[\left(\mathcal{Q}^D \mathcal{J}_D^{N_k-1} \right)(h) \right]^2 (\nu_i^{k'}) \le \sum_{j \in I_k^D} \omega_j^k w_D(\nu_j^k) h^2(\nu_j^k). \tag{3.34}$$

Proof. According to (2.15) and (3.31), and since w_D is a polynomial of degree 1 and $\mathcal{Q}^D f$ a polynomial of degree $N_{k'} - 1$, we have

$$\sum_{i \in I_{k'}^D} \omega_i^{k'} w_D(\nu_i^{k'}) \left[\left(\mathcal{Q}^D \mathcal{J}_D^{N_k-1} \right)(h) \right]^2 (\nu_i^{k'}) = \int_D w_D(\mu) \left[\left(\mathcal{Q}^D \mathcal{J}_D^{N_k-1} \right)(h) \right]^2 (\mu) d\mu$$
$$\le \int_D w_D(\mu) \left(\mathcal{J}_D^{N_k-1} h \right)^2 (\mu) d\mu = \sum_{j \in I_k^D} \omega_j^k w_D(\nu_j^k) h^2(\nu_j^k).$$

This concludes the proof. □

Proposition 4 *For each function $h \in L^\infty(D)$, the following error estimate is satisfied*

$$\int_D w_D \left[\left(\mathcal{Q}^D \mathcal{J}_D^{N_{k'}-1} - \mathcal{J}_D^{N_k-1} \right) h \right]^2 \le \int_D w_D \left[\left(\mathcal{J}_D^{N_{k'}-1} - \mathcal{J}_D^{N_k-1} \right) h \right]^2. \tag{3.35}$$

Proof. According to (3.33) and (3.31), we have

$$\int_D w_D \left(\mathcal{Q}^D \mathcal{J}_D^{N_{k'}-1} h - \mathcal{J}_D^{N_k-1} h \right)^2 \tag{3.36}$$
$$= \int_D w_D \left(\mathcal{Q}^D \mathcal{J}_D^{N_{k'}-1} h \right)^2 - 2 \int_D w_D \left(\mathcal{Q}^D \mathcal{J}_D^{N_{k'}-1} h \right) \left(\mathcal{J}_D^{N_k-1} h \right) + \int_D w_D \left(\mathcal{J}_D^{N_k-1} h \right)^2$$
$$\le \int_D w_D \left(\mathcal{J}_D^{N_{k'}-1} h \right)^2 - 2 \int_D w_D \left(\mathcal{J}_D^{N_{k'}-1} h \right) \left(\mathcal{J}_D^{N_k-1} h \right) + \int_D w_D \left(\mathcal{J}_D^{N_k-1} h \right)^2.$$

This concludes the proof of the proposition. □

We need the following stability properties:

Proposition 5 *Let $T_\nu^{\Omega_{k'} \to \Omega_k}$ be defined by (3.28). Then there exists a constant C such that for each function $g \in L^2(\Omega)$, for $k = 1, 2$ and for each quarter of circle D, we have*

$$\begin{aligned} &\int_{\Omega_k} d\mathbf{x} \sum_{i \in I_k^D} \omega_i^k \left(T_{\nu_i^k}^{\Omega_{k'} \to \Omega_k} g \right)^2 (\mathbf{x}) \le C \|g\|^2 , \\ &\int_\gamma dt \sum_{i \in I_k^D} \omega_i^k |\boldsymbol{\nu}_i^k \cdot \mathbf{n}_\gamma| \left(T_{\nu_i^k}^{\Omega_{k'} \to \Omega_k} g \right)^2 (t) \le C \|g\|^2 . \end{aligned} \tag{3.37}$$

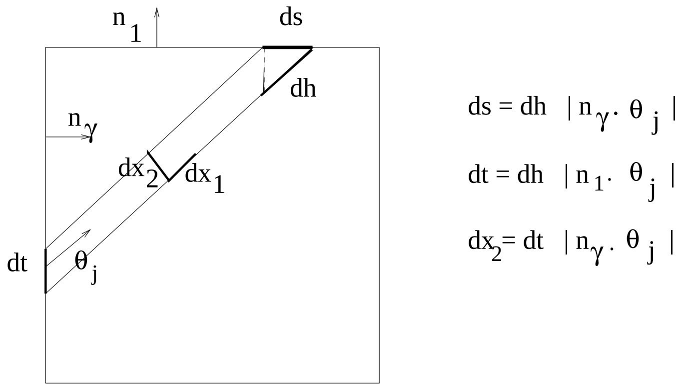

Fig. 2. Change of variables

Proof. Let $d\mathbf{x} = dx_1 dx_2$ be the measure of Ω_k given in the direct orthonormal basis such that $\boldsymbol{\nu}_i^k = (1,0)$ (see Fig.2). Then $dx_2 = |\boldsymbol{\nu}_i^k \cdot \boldsymbol{n}_\gamma| dt$, where dt is the measure of γ. Moreover, we check that

$$\int_{\Omega_k} d\mathbf{x}\,\chi_{\nu_i^k}(\mathbf{x}) \left(T^{\Omega_{k'}\to\Omega_k}_{\nu_i^k} g\right)^2(\mathbf{x}) = \int_{\Omega_k} d\mathbf{x}\chi_{\nu_i^k}(\mathbf{x}) e^{-2d_\gamma(\mathbf{x},\nu_i^k)} \left(\mathcal{Q}^D \mathcal{J}_D^{N_{k'}-1} T_\nu g(\widetilde{\mathbf{x},\nu_i^k})\right)^2(\nu_i^k)$$
$$= \int_\gamma dt |\boldsymbol{\nu}_i^k\cdot \mathbf{n}_\gamma| \left(\mathcal{Q}^D\mathcal{J}_D^{N_{k'}-1}T_\nu g(t)\right)^2(\nu_i^k)\int_0^{d((t,0),-\nu_i^k)} dx_1 e^{-2x_1} \le \int_\gamma dt|\boldsymbol{\nu}_i^k\cdot\boldsymbol{n}_\gamma|\left(T^{\Omega_{k'}\to\Omega_k}_{\nu_i^k} g\right)^2(t).$$

Thus it remains to prove the second inequality in (3.37) and the first one follows. Note that for $\nu \in D$,

$$|\boldsymbol{\nu}\cdot\boldsymbol{n}_\gamma| \le w_D(\nu) \le \frac{\pi}{2}|\boldsymbol{\nu}\cdot\boldsymbol{n}_\gamma|. \tag{3.38}$$

According to Proposition 3, (2.5), and (2.18) we have

$$\int_\gamma dt \sum_{i\in I_k^D} \omega_i^k |\boldsymbol{\nu}_i^k \cdot \mathbf{n}_\gamma| \left(T^{\Omega_{k'}\to\Omega_k}_{\nu_i^k} g\right)^2(t) \le \int_\gamma dt \sum_{i\in I_{k'}^D} \omega_i^{k'} w_D(\nu_i^k)\left(\mathcal{Q}^D\mathcal{J}_D^{N_{k'}-1}T_\nu g(t)\right)^2(\nu_i^k)$$
$$\le \int_\gamma dt \sum_{j\in I_{k'}^D} \omega_j^{k'} w_D(\nu_j^{k'})\left(T_{\nu_j^{k'}} g(t)\right)^2 \le \sum_{j\in I_{k'}^D}\omega_j^{k'}\frac{\pi}{2}\int_\gamma dt|\boldsymbol{\nu}_j^{k'}\cdot\boldsymbol{n}_\gamma|\left(T_{\nu_j^{k'}} g(t)\right)^2 \le C\|g\|^2.$$

This concludes the proof of the proposition. □

Finally, we state the

Corollary 6 *There exists a constant C independent of N_1 and N_2 such that*

$$\|T_{N_k}\|_{\mathcal{L}(L^2,L^2)} \le C, \quad k=1,2 \qquad \|T_\delta\|_{\mathcal{L}(L^2,L^2)} \le C. \tag{3.39}$$

Proof. This is an immediate consequence of (2.5), (2.18), and Proposition 5. □

3.3. *Convergence results*

Now we are in a position to state our main result.

Theorem 7 *Let $|\lambda| < \lambda_{max}$ and assume that $F(N) : \mathbb{N} \to \mathbb{N}$ satisfies $N < F(N)$ and $N^{-1/3}\log F(N) \to 0$ as $N \to \infty$. Assume further that the quadrature rules Q_N satisfy the properties on section 2.3. Then with the coupling operator defined by (3.32) and for sufficiently large N_{min}, $N_{min} \le M \le N_{max} < F(N_{min})$, equation (3.27) admits a unique solution given by*

$$\phi_\delta = (I - \lambda T_\delta)^{-1} T_\delta q.$$

Moreover, it satisfies the following stability and error estimate

$$\|\phi_\delta\| \le C\|q\|, \qquad \|\phi - \phi_\delta\| \le \frac{C_\eta}{N_{min}^{1-\eta}}, \tag{3.40}$$

provided $q \in H^1(\Omega)$ and $1 > \eta > 0$. In these expressions C and C_η are constants independent of q and N_k, $k = 1, 2$.

Remark 8 The constraint on the number of directions M in terms of N is not very restrictive. Indeed, $F(N) = \frac{\exp N}{N}$ satisfies the hypotheses of the theorem. Therefore M can be as large as CN^s for any constant C and s. Thus the coupling will be stable even for values of M that are much larger than N.

Remark 9 The error estimate given in the theorem relies on the H^1-regularity of ϕ. This condition readily stems from the properties of operator T stated in section 2. The problem of optimal regularity for solutions of transport equations is a difficult subject[23,6]. In bounded domains, the solution is not arbitrarily regular, even with smooth coefficients and boundaries. Assuming that $\phi \in H^1(\Omega)$, the error estimate given here, up to an arbitrarily small factor η, is of the same order as without coupling. In that sense the coupling is optimal and does not generate any unexpected singularities. More precisely, the stability of the coupling operator relies on inequality (2.17). This yields that the accuracy of the coupling operator will be of spectral type (i.e., the convergence of (2.17) is faster than any power of N) if ϕ is sufficiently regular. Therefore only the lack of regularity of the solution to the continuous problem is a burden to the spectral accuracy of the coupled solution.

4. Proof of the results

The proof of Theorem 7 relies on some lemmas that we state and prove now. The first one, due to Anselone[3], is stated without proof.

Lemma 10 *Let $T : L^2(\Omega) \to L^2(\Omega)$ be a bounded linear operator such that for some positive constant C*

$$\|(I - \lambda T)V\| \ge C\|V\| \quad \forall V \in L^2(\Omega),$$

and let $\{T_n\}_{n=1}^{\infty}$ be a uniformly bounded sequence of linear operators on $L^2(\Omega)$ such that for some positive integer m

$$\varepsilon_n \equiv \|(T - T_n)T_n^m\| \to 0 \quad \text{when } n \to \infty. \tag{4.41}$$

Then there exists a positive constant C_1 such that for n large enough

$$\|(I-\lambda T_n)V\| \geq C_1\|V\| \quad \forall V \in L^2(\Omega).$$

Lemma 11 *(Modification of Lemma 4.4[18].) Let $T_\mu g$ be given by (2.4) and $g \in H^1(\Omega_1)\cap H^1(\Omega_2)$. Then there exists a constant C such that*

$$\left\|\frac{\partial T_\mu g}{\partial \mu}\right\|_{L^1(L^2)} \leq C\,\left(\|g\|_{H^1(\Omega_1)}+\|g\|_{H^1(\Omega_2)}\right),$$

where $L^1(L^2)=L^1((-\pi,\pi);L^2(\Omega))$.

Proof. Let $\mathbf{x}\in\Omega_1$. We have

$$T_\mu g(\mathbf{x}) = \int_0^{d(\mathbf{x},\mu)} e^{-s} g(\mathbf{x}-s\boldsymbol{\mu})ds = \int_0^{d_1(\mathbf{x},\mu)} e^{-s} g(\mathbf{x}-s\boldsymbol{\mu})ds + \chi_\mu(\mathbf{x})\int_{d_1(\mathbf{x},\mu)}^{d(\mathbf{x},\mu)} e^{-s} g(\mathbf{x}-s\boldsymbol{\mu})ds.$$

For the points $\mu_0(\mathbf{x})$ satisfying $\left(\frac{\partial}{\partial\mu}\chi_\mu(\mathbf{x})\right)(\mu_0)\neq 0$ (in the space of distributions $\mathcal{D}'(\Omega)$), we also have $d(\mathbf{x},\mu_0)=d_1(\mathbf{x},\mu_0)$. This yields

$$\begin{aligned}\frac{\partial T_\mu g(\mathbf{x})}{\partial\mu}(\mathbf{x},\mu) &= e^{-d_1}g(\mathbf{x}^-,\mu)\frac{\partial d_1}{\partial\mu} + \int_0^{d_1} e^{-s}s\frac{\partial g}{\partial\boldsymbol{\mu}'}(\mathbf{x}-s\boldsymbol{\mu})ds\\ &+\chi_\mu e^{-d}g(\overline{\mathbf{x},\mu})\frac{\partial d}{\partial\mu} - \chi_\mu e^{-d_1}g(\mathbf{x}^+,\mu)\frac{\partial d_1}{\partial\mu}\\ &+\chi_\mu\int_{d_1}^{d} e^{-s}s\frac{\partial g}{\partial\boldsymbol{\mu}'}(\mathbf{x}-s\boldsymbol{\mu})ds,\end{aligned}$$

where $(\mathbf{x}^\pm,\mu)=\lim\limits_{\varepsilon\to0^+}(\mathbf{x}-(d_1(\mathbf{x},\mu)\pm\varepsilon)\boldsymbol{\mu})$, and $\boldsymbol{\mu}'=(\sin\mu,-\cos\mu)$. We recall the notation

$$\frac{\partial}{\partial\boldsymbol{\mu}} \equiv \boldsymbol{\mu}\cdot\boldsymbol{\nabla}.$$

The proof of Lemma 4.4[18] then applies to all terms above, either on Ω_1 or on Ω_2. This concludes the proof. □

Lemma 12 *Let $\boldsymbol{t}=(0,t_x)\in\gamma$, where $t_x\in(0,L)$. Then*

$$\begin{aligned}\|T_\nu g(\boldsymbol{t})\|_{H^1(D)} &\leq \frac{C}{\sqrt{t_x(L-t_x)}}\|g\|_{H^1(\Omega_k)},\\ \|T_\nu g(\boldsymbol{t})\|_{L^2(D)} &\leq C\|g\|_{H^1(\Omega_k)},\end{aligned}$$

where $(D,k)\in\{(NE,1),\,(SE,1),\,(NW,2),\,(SW,2)\}$.

Proof. The four cases $(D,k)\in\{(NE,1),\,(SE,1),\,(NW,2),\,(SW,2)\}$ are treated similarly. We consider $D=NE$. Let $\mu\in NE$ and $\boldsymbol{t}=(0,t_x)\in\gamma$. According to the Cauchy Schwartz inequality, we have

$$|T_\mu g(\boldsymbol{t})| \leq C\left(\int_0^{d(\boldsymbol{t},\mu)} g^2(\boldsymbol{t}-s\boldsymbol{\mu})ds\right)^{\frac12}.$$

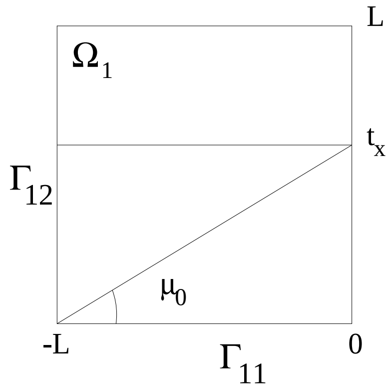

Fig. 3. Geometry of Ω_1.

Using successively the change of polar to Cartesian coordinates, the Hölder inequality (with $\frac{2}{3}+\frac{1}{3}=1$) and the Sobolev embedding properties[1] (H^1 is continuously embedded in L^6 in dimension 2), we prove that for $\boldsymbol{t}\in\gamma$, we have

$$\begin{aligned}\int_{NE} d\boldsymbol{\mu}\,|T_\mu g(\boldsymbol{t})|^2 &= \int_0^{\pi/2} d\mu \int_0^{d(\boldsymbol{t},\mu)} g^2(\boldsymbol{t}-s\boldsymbol{\mu})ds \le \int_{\Omega_1} \frac{d\mathbf{y}}{|\mathbf{y}-\boldsymbol{t}|} g^2(\mathbf{y}) \\ &\le \left(\int_{\Omega_1} \frac{d\mathbf{y}}{|\mathbf{y}-\boldsymbol{t}|^{3/2}}\right)^{2/3} \left(\int_{\Omega_1} g^6(\mathbf{y})d\mathbf{y}\right)^{1/3} \le C\|g\|^2_{H^1(\Omega_1)}.\end{aligned}$$

This yields

$$\|T_\mu g(\boldsymbol{t})\|_{L^2(D)} \le C\|g\|_{H^1(\Omega_1)}.$$

Differentiation with respect to the angular variable yields

$$\frac{\partial T_\mu g(\boldsymbol{t})}{\partial \mu} = e^{-d(\boldsymbol{t},\mu)} g(\overline{\boldsymbol{t},\mu}) \frac{\partial d}{\partial \mu} + \int_0^{d(\boldsymbol{t},\mu)} s e^{-s} \frac{\partial}{\partial \boldsymbol{\mu}'} g(\boldsymbol{t}-s\boldsymbol{\mu})ds, \tag{4.42}$$

where $\boldsymbol{\mu}' = (\sin\mu, -\cos\mu)$. Using again the Hölder inequality and the change of variables $sdsd\mu = d\mathbf{y}$, we have

$$\left\|\int_0^{d(\boldsymbol{t},\mu)} s e^{-s} \frac{\partial}{\partial \boldsymbol{\mu}'} g(\boldsymbol{t}-s\boldsymbol{\mu})ds\right\|_{L^2(D)} \le C\|g\|_{H^1(\Omega_1)}.$$

Let μ_0 be the angle defined by $0 \le \mu_0 \le \pi/4$ and $\tan\mu_0 = \frac{t_x}{L}$ (see Fig.3). For $\frac{\pi}{2} > \mu > \mu_0$ we have $d(\boldsymbol{t},\mu) = \frac{t_x}{\sin\mu}$ and then $\frac{\partial d}{\partial \mu} = -\frac{t_x \cos\mu}{(\sin\mu)^2}$. For $0 < \mu < \mu_0$ we have $d(\boldsymbol{t},\mu) = \frac{L}{\cos\mu}$ and then $\frac{\partial d}{\partial \mu} = \frac{L\sin\mu}{(\cos\mu)^2}$.

Let Γ_{11} be the bottom side of Ω_1 equipped with the measure $d\mathbf{s}$, and Γ_{12} the left side of Ω_1 equipped with the measure $d\mathbf{v}$. We check the relations: $d\mu = \frac{(\sin\mu)^2}{t_x} d\mathbf{s}$

on Γ_{11} and $d\mu = \frac{(\cos\mu)^2}{L} d\mathbf{v}$ on Γ_{12}. Since $d(\boldsymbol{t},\mu) = \frac{t_x}{\sin\mu} < C$, we deduce that

$$\int_0^{\pi/2} g^2(\overline{\boldsymbol{t},\mu}) \left(\frac{\partial d}{\partial \mu}\right)^2 d\mu = \left(\int_0^{\mu_0} d\mu + \int_{\mu_0}^{\pi/2} d\mu\right) \left(g^2(\overline{\boldsymbol{t},\mu}) \left(\frac{\partial d}{\partial \mu}\right)^2\right) \quad \leq$$
$$\int_{\Gamma_{12}} g^2(\mathbf{v}) L \frac{\sin^2\mu}{\cos^2\mu} d\mathbf{v} + \int_{\Gamma_{11}} \frac{t_x \cos^2\mu}{\sin^2\mu} g^2(\mathbf{s}) d\mathbf{s} \leq C \left(\int_{\Gamma_{12}} g^2(\mathbf{v}) d\mathbf{v} + \int_{\Gamma_{11}} \frac{1}{t_x} g^2(\mathbf{s}) d\mathbf{s}\right).$$

The first term of the right hand side of (4.42) is then bounded in $L^2(D)$ according to the classical trace inequality $\|u\|_{L^2(\partial X)} \leq C\|u\|_{H^1(X)}$ (see e.g. Adams[1]). This concludes the proof of the lemma. $\square$

Lemma 13 *There exist two operators A_δ and B_δ and a constant C such that*

$$\begin{aligned} T_\delta^2 &= A_\delta + B_\delta\,, \\ \|A_\delta g\|_{1,\Omega_k} &\leq C(\log N_{max})^{1/2}\varepsilon^{-2}\|g\|, \quad \forall g \in L^2(\Omega) \quad k=1,2 \\ \|B_\delta\|_{\mathcal{L}(L^2)} &\leq C\sqrt{\sigma(\varepsilon, N_1, N_2)}\,. \end{aligned}$$

Proof. Assume that $k=1$ and let $\mathbf{x} \in \Omega_1$. For simplicity, we denote by $\mu_i = \nu_i^1$ and $\theta_j = \nu_j^2$. Indexes i, j, k now refer to angular directions. We define

$$\begin{aligned} T_\delta^2 g(\mathbf{x}) &= \sum_i\sum_k \omega_i^1\omega_k^1 \left(T_{\mu_i}^{\Omega_1} T_{\mu_k}^{\Omega_1}\right) g(\mathbf{x}) + \sum_i\sum_k \omega_i^1\omega_k^1 \left(T_{\mu_i}^{\Omega_1} T_{\mu_i}^{\Omega_2\to\Omega_1}\right) g(\mathbf{x}) \\ &+ \sum_i\sum_j \omega_i^1\omega_j^2 \left(T_{\mu_i}^{\Omega_2\to\Omega_1} T_{\theta_j}^{\Omega_2}\right) g(\mathbf{x}) + \sum_i\sum_j \omega_i^1\omega_j^2 \left(T_{\mu_i}^{\Omega_2\to\Omega_1} T_{\theta_j}^{\Omega_1\to\Omega_2}\right) g(\mathbf{x}) \\ &= A_1 g(\mathbf{x}) + A_2 g(\mathbf{x}) + A_3 g(\mathbf{x}) + A_4 g(\mathbf{x}) \qquad (4.43) \\ &= \sum_i\sum_k \omega_i^1\omega_k^1 A_{1,ik} g(\mathbf{x}) + \sum_i\sum_k \omega_i^1\omega_k^1 A_{2,ik} g(\mathbf{x}) \\ &+ \sum_i\sum_j \omega_i^1\omega_j^2 A_{3,ij} g(\mathbf{x}) + \sum_i\sum_j \omega_i^1\omega_j^2 A_{4,ij} g(\mathbf{x}), \end{aligned}$$

with obvious notation. We define for $l=1,2$

$$\widetilde{A_l} g = \sum_{(i,k)\in I'_{\varepsilon,1}} \omega_i^1\omega_k^1 A_{l,ik} g,$$

and for $l=3,4$

$$\widetilde{A_l} g = \sum_{(i,j)\in J'_{\varepsilon,1,2}} \omega_i^1\omega_j^2 A_{l,ij} g.$$

In the sequel we denote by $I'_\varepsilon = I'_{\varepsilon,1}$ and $J'_\varepsilon = J'_{\varepsilon,1,2}$ for simplicity. Let $B_l = A_l - \widetilde{A_l}$. It is sufficient to prove the inequalities given in the lemma for the pairs $(\widetilde{A_l}, B_l)$, $l = 1, \ldots, 4$. The case $l=1$ is treated in the Remark p.957 and p.958[18]:

$$\|\widetilde{A_1} g\|_{1,\Omega} \leq C\varepsilon^{-\frac{3}{2}} N_1^{-1}\|g\|, \quad \|B_1\|_{\mathcal{L}(L^2)} \leq C\,\sigma(\varepsilon, N_1, N_2).$$

• Estimation of the terms indexed by $l = 2$. We denote by Y the quantities A_2, $\widetilde{A_2}$ and B_2 successively and by $D(Y)$ the set of summation. According to the Cauchy-Schwartz inequality, estimate (2.5), and Proposition 5 we have

$$\|Yg(\mathbf{x})\|^2_{L^2(\Omega_1)} \leq C\Big(\sum_{(i,k)\in D(Y)} \omega_i^1\omega_k^1\Big)\Big(\sum_{(i,k)}\omega_i^1\omega_k^1\int_{\Omega_1}\Big(T^{\Omega_1}_{\mu_i}T^{\Omega_2\to\Omega_1}_{\mu_k}g\Big)^2(\mathbf{x})d\mathbf{x}\Big) \tag{4.44}$$

$$\leq C\Big(\sum_{(i,k)\in D(Y)} \omega_i^1\omega_k^1\Big)\Big(\sum_{|k|=1}^{2N_1}\omega_k^1\int_{\Omega_1}\Big(T^{\Omega_2\to\Omega_1}_{\mu_k}g\Big)^2(\mathbf{x})d\mathbf{x}\Big) \leq C\Big(\sum_{(i,k)\in D(Y)} \omega_i^1\omega_k^1\Big)\|g\|^2.$$

Then, recalling (2.20), we have

$$\begin{aligned}\|\widetilde{A_2}g\|_{0,\Omega_1} &\leq C\|g\|,\\ \|B_2g\|_{0,\Omega_1} &\leq C\sqrt{\sigma(\varepsilon,N_1,N_2)}\|g\|.\end{aligned}$$

We check that

$$|\boldsymbol{\nabla} g|^2 \leq \frac{1}{\xi^2(\boldsymbol{\mu},\boldsymbol{\nu})}\left(\left|\frac{\partial g}{\partial\boldsymbol{\mu}}\right|^2 + \left|\frac{\partial g}{\partial\boldsymbol{\nu}}\right|^2\right).$$

Thus it remains to estimate the following quantity

$$\begin{aligned}|\widetilde{A_2}g|^2_{1,\Omega_1} &= \int_{\Omega_1}\Big|\sum_{(i,k)\in I'_\varepsilon}\omega_i^1\omega_k^1\boldsymbol{\nabla}\left(T^{\Omega_1}_{\mu_i}T^{\Omega_2\to\Omega_1}_{\mu_k}g\right)(\mathbf{x})\Big|^2 d\mathbf{x}\\ &\leq C\sum_{(i,k)\in I'_\varepsilon}\omega_i^1\omega_k^1\int_{\Omega_1}\left|\boldsymbol{\nabla}\left(T^{\Omega_1}_{\mu_i}T^{\Omega_2\to\Omega_1}_{\mu_k}g\right)(\mathbf{x})\right|^2 d\mathbf{x}\\ &\leq C\sum_{(i,k)\in I'_\varepsilon}\omega_i^1\omega_k^1\frac{1}{\varepsilon^2}\left\{\int_{\Omega_1}\left(\frac{\partial}{\partial\boldsymbol{\mu}_i}\left(T^{\Omega_1}_{\mu_i}T^{\Omega_2\to\Omega_1}_{\mu_k}g\right)(\mathbf{x})\right)^2 d\mathbf{x}+\right.\\ &\qquad\left.\int_{\Omega_1}\left(\frac{\partial}{\partial\boldsymbol{\mu}_k}\left(T^{\Omega_1}_{\mu_i}T^{\Omega_2\to\Omega_1}_{\mu_k}g\right)(\mathbf{x})\right)^2 d\mathbf{x}\right\},\end{aligned} \tag{4.45}$$

We deduce from the stability inequality (2.5) that

$$\int_{\Omega_1}\left(\frac{\partial}{\partial\boldsymbol{\mu}_i}\left(T^{\Omega_1}_{\mu_i}T^{\Omega_2\to\Omega_1}_{\mu_k}g\right)\right)^2 d\mathbf{x} \leq C\int_{\Omega_1}\left(T^{\Omega_2\to\Omega_1}_{\mu_k}g\right)^2 d\mathbf{x},$$

and the estimate of the first term in (4.45) is performed as for (4.44). On the other hand we have

$$\begin{aligned}\frac{\partial}{\partial\boldsymbol{\mu}_k}\left(T^{\Omega_1}_{\mu_i}T^{\Omega_2\to\Omega_1}_{\mu_k}g\right)(\mathbf{x}) &= \frac{\partial d_1(\mathbf{x},\mu_i)}{\partial\boldsymbol{\mu}_k}T^{\Omega_2\to\Omega_1}_{\mu_k}g(\overline{\mathbf{x},\mu_i})e^{-d_1(\mathbf{x},\mu_i)}\\ &\quad\underbrace{+\int_0^{d_1(\mathbf{x},\mu_i)}ds\,e^{-s}\frac{\partial}{\partial\boldsymbol{\mu}_k}(T^{\Omega_2\to\Omega_1}_{\mu_k}g)(\mathbf{x}-s\boldsymbol{\mu}_i),}_{-T^{\Omega_1}_{\mu_i}T^{\Omega_2\to\Omega_1}_{\mu_k}g(\mathbf{x}),}\end{aligned} \tag{4.46}$$

since, as can be seen from the definition (3.28), we have

$$\frac{\partial}{\partial\boldsymbol{\mu}_k}(T^{\Omega_2\to\Omega_1}_{\mu_k}g) = -T^{\Omega_2\to\Omega_1}_{\mu_k}g, \tag{4.47}$$

as $\frac{\partial}{\partial \boldsymbol{\mu}_k}\chi_{\mu_k} = 0$. The estimate of the second term of the right-hand side of (4.46) is already known. Let us now focus on the first term.

Let $\boldsymbol{n}$ be the outward unit normal to Ω_1 and $\boldsymbol{t}$ be such that $\boldsymbol{t}\cdot\boldsymbol{n} = 0$. Then according to

$$\frac{\partial}{\partial \boldsymbol{\mu}_k} = \boldsymbol{n}\cdot\boldsymbol{\mu}_k \frac{\partial}{\partial \boldsymbol{n}} + \boldsymbol{t}\cdot\boldsymbol{\mu}_k \frac{\partial}{\partial \boldsymbol{t}},$$

we verify that

$$\frac{\partial d_1}{\partial \boldsymbol{\mu}_k}(\mathbf{x},\mu_i) = \frac{\boldsymbol{n}\cdot\boldsymbol{\mu}_k}{\boldsymbol{n}\cdot\boldsymbol{\mu}_i}. \tag{4.48}$$

From $d\mathbf{x} = |\boldsymbol{\mu}\cdot\boldsymbol{n}|d\boldsymbol{t}$ (see Fig.2) we obtain as in the proof of Proposition 5 the L^2 estimate of the first term of the right-hand side of (4.46):

$$\int_{\Omega_1}\left(\frac{\boldsymbol{n}\cdot\boldsymbol{\mu}_k}{\boldsymbol{n}\cdot\boldsymbol{\mu}_i}\right)^2 \left(T^{\Omega_2\to\Omega_1}_{\mu_k} g\right)^2(\overline{\mathbf{x}_1,\mu_i})d\mathbf{x} \le \int_{\Gamma_1}\left(\frac{\boldsymbol{n}\cdot\boldsymbol{\mu}_k}{\boldsymbol{n}\cdot\boldsymbol{\mu}_i}\right)^2 \left(T^{\Omega_2\to\Omega_1}_{\mu_k} g\right)^2(\boldsymbol{t})|\boldsymbol{n}\cdot\boldsymbol{\mu}_i|d\boldsymbol{t}$$
$$\le \frac{1}{\varepsilon}\int_{\Gamma_1}|\boldsymbol{n}\cdot\boldsymbol{\mu}_k|\left(T^{\Omega_2\to\Omega_1}_{\mu_k} g\right)^2(\boldsymbol{t})d\boldsymbol{t} \le \frac{C}{\varepsilon}\int_{\gamma}|\boldsymbol{n}_\gamma\cdot\boldsymbol{\mu}_k|\left(T^{\Omega_2\to\gamma}_{\mu_k} g\right)^2(\mathbf{s})d\mathbf{s},$$

according to the definition (3.28) and to the change of integration measure from any side of Γ_1 to γ endowed with the measure $d\mathbf{s}$: $|\boldsymbol{n}\cdot\boldsymbol{\mu}_k|d\boldsymbol{t} = |\boldsymbol{n}_\gamma\cdot\boldsymbol{\mu}_k|d\mathbf{s}$. From Proposition 5 we deduce that the second term of (4.45) is bounded by $C\varepsilon^{-3}$ (after summation over I'_ε) and then $\|\widetilde{A_2}\|_{1,\Omega_1} \le C\varepsilon^{-\frac{3}{2}}\|g\|$.

• Estimation of the terms indexed by $l = 3, 4$. The same proof as in (4.44), replacing $T^{\Omega_1}_{\mu_i}$ by $T^{\Omega_2\to\Omega_1}_{\mu_i}$, allows us to deduce from (2.5) and Proposition 5 that

$$\|\widetilde{A_l}g\|_{0,\Omega_1} \le C\|g\|, \qquad \text{and} \qquad \|B_l g\|_{0,\Omega_1} \le C\sqrt{\sigma(\varepsilon,N_1,N_2)}\|g\|,$$

for $l = 3, 4$. So it remains to estimate

$$|\widetilde{A}_l g|^2_{1,\Omega_1} = \int_{\Omega_1}\Big|\sum_{(i,j)\in J'_\varepsilon}\omega^1_i\omega^2_j \boldsymbol{\nabla}\left(T^{\Omega_2\to\Omega_1}_{\mu_i} Z_{l,j} g\right)\Big|^2 d\mathbf{x} \qquad \le \tag{4.49}$$
$$C\sum_{(i,j)\in J'_\varepsilon}\omega^1_i\omega^2_j \frac{1}{\varepsilon^2}\Big\{\int_{\Omega_1}\left(\frac{\partial}{\partial\boldsymbol{\mu}_i}\left(T^{\Omega_2\to\Omega_1}_{\mu_i} Z_{l,j} g\right)\right)^2 d\mathbf{x} + \int_{\Omega_1}\left(\frac{\partial}{\partial\boldsymbol{\theta}_j}\left(T^{\Omega_2\to\Omega_1}_{\mu_i} Z_{l,j} g\right)\right)^2 d\mathbf{x}\Big\},$$

where $Z_{3,j} = T^{\Omega_2}_{\theta_j}$, $Z_{4,j} = T^{\Omega_1\to\Omega_2}_{\theta_j}$. According to (4.47) we have $\frac{\partial}{\partial\boldsymbol{\mu}_i}A_{l,ij} = -A_{l,ij}$ for $l = 3, 4$. Thus the estimate of the first term in (4.49) has already been derived.

For $\mu_i \in D$ we denote by $D(\mu_i) = D$. To simplify, we have to read $D = D(\mu_i)$ in $(\mathcal{Q}^D f)(\mu_i)\cdot\frac{\partial}{\partial\boldsymbol{\nu}}$, which is a spatial differentiation operator, and in $\left(\mathcal{Q}^D\mathcal{I}^{N_2-1}_D\right)$, which is an angular interpolation operator. Therefore we deduce that

$$\frac{\partial}{\partial\boldsymbol{\nu}}\left(\mathcal{Q}^D\mathcal{I}^{N_2-1}_D\right) = \left(\mathcal{Q}^D\mathcal{I}^{N_2-1}_D\right)\frac{\partial}{\partial\boldsymbol{\nu}}.$$

In the following equation, $\mathcal{Q}^D\mathcal{I}_D^{N_2-1}$ transforms a polynomial of θ on $D(\mu_i)$ into a polynomial of μ. We have:

$$\begin{aligned}\frac{\partial}{\partial\boldsymbol{\theta}_j}(T_{\mu_i}^{\Omega_2\to\Omega_1}g(\mathbf{x})) &= \chi_{\mu_i}(\mathbf{x})\left(\frac{\partial}{\partial\boldsymbol{\theta}_j}e^{-d_\gamma(\mathbf{x},\mu_i)}\right)\left(T_{\mu_i}^{\Omega_2\to\gamma}Z_{l,j}g\right)(\widetilde{\mathbf{x},\mu_i})\\ &+\chi_{\mu_i}(\mathbf{x})e^{-d_\gamma(\mathbf{x},\mu_i)}\left(\left(\mathcal{Q}^D\mathcal{I}_D^{N-1}\right)\left(\frac{\partial}{\partial\boldsymbol{\theta}_j}\left[T_\theta^{\Omega_2}Z_{l,j}g(\widetilde{\mathbf{x},\mu_i})\right]\right)\right)(\mu_i),\end{aligned}\tag{4.50}$$

since $< \frac{\partial}{\partial\boldsymbol{\theta}_j}\chi_{\mu_i}(\mathbf{x}), (\mathcal{Q}^D\mathcal{J}_D^{N_2-1}h(\widetilde{\mathbf{x},\mu_i}))(\mu_i) >_{(\mathcal{M}_B(\Omega),C^0(\Omega))}= 0$, where $\mathcal{M}_B(\Omega)$ is the set of bounded measures on Ω, for each continuous function h. This relation is necessary to differentiate $A_{l,ij}$. It relies on the distribution of the quadrature rule by quarters of circle. It means that no neutrons come from the domain Ω_2 through the point $(0,L)$ and enter the domain Ω_1, as requested from the boundary conditions of the continuous problem. Actually, $\frac{\partial}{\partial\boldsymbol{\theta}_j}\chi_{\mu_i}$ is a Dirac measure supported on the segment issued from point $(0,L)$ and oriented by $\boldsymbol{\mu}_i$. On this segment $(\mathcal{Q}^D\mathcal{J}_D^{N_2-1}h(\widetilde{\mathbf{x},\mu_i}))(\mu_i)$ vanishes according to the boundary condition, as for the continuous problem. This would no longer hold true if the coupling operators were defined as polynomial projection or interpolation on half circles.

We need the following results:

$$\begin{aligned}\frac{\partial}{\partial\boldsymbol{\theta}_j}\left[T_\theta^{\Omega_2}Z_{l,j}g(\widetilde{\mathbf{x},\mu_i})\right] &= \frac{\partial}{\partial\boldsymbol{\theta}_j}\left[\int_0^{d_2((\widetilde{\mathbf{x},\mu_i}),\theta)}e^{-s}(Z_{l,j}g)((\widetilde{\mathbf{x},\mu_i})-s\boldsymbol{\theta})ds\right]\\ &= \frac{\partial}{\partial\boldsymbol{\theta}_j}\left[d_2((\widetilde{\mathbf{x},\mu_i}),\theta)\right]e^{-d_2((\widetilde{\mathbf{x},\mu_i}),\theta)}(Z_{l,j}g)(\overline{(\widetilde{\mathbf{x},\mu_i}),\theta})_2\\ &+ \underbrace{\int_0^{d_2((\widetilde{\mathbf{x},\mu_i}),\theta)}e^{-s}\frac{\partial}{\partial\boldsymbol{\theta}_j}\left[(Z_{l,j}g)((\widetilde{\mathbf{x},\mu_i})-s\boldsymbol{\theta})\right]ds}_{\frac{\partial(\widetilde{\mathbf{x},\mu_i})}{\partial\boldsymbol{\theta}_j}T_\theta^{\Omega_2}\left[\frac{\partial}{\partial\boldsymbol{\theta}_j}Z_{l,j}g\right](\widetilde{\mathbf{x},\mu_i})},\end{aligned}\tag{4.51}$$

where we have used the notation (3.26), and

$$\begin{aligned}\frac{\partial}{\partial\boldsymbol{\theta}_j}\left[d_2((\widetilde{\mathbf{x},\mu_i}),\theta)\right] &= \frac{|\boldsymbol{n}_2\cdot\boldsymbol{\theta}_j|}{|\boldsymbol{n}_2\cdot\boldsymbol{\theta}|}\frac{\partial(\widetilde{\mathbf{x},\mu_i})}{\partial\boldsymbol{\theta}_j},\\ \frac{\partial}{\partial\boldsymbol{\theta}_j}\left[e^{-d_\gamma(\mathbf{x},\mu_i)}\right] &= e^{-d_\gamma(\mathbf{x},\mu_i)}\frac{|\boldsymbol{n}_\gamma\cdot\boldsymbol{\theta}_j|}{|\boldsymbol{n}_\gamma\cdot\boldsymbol{\mu}_i|},\\ \frac{\partial(\widetilde{\mathbf{x},\mu_i})}{\partial\boldsymbol{\theta}_j} &= (\boldsymbol{t}_\gamma\cdot\boldsymbol{\theta}_j)-\frac{(\boldsymbol{t}_\gamma\cdot\boldsymbol{\mu}_i)}{(\boldsymbol{n}_\gamma\cdot\boldsymbol{\mu}_i)}(\boldsymbol{n}_\gamma\cdot\boldsymbol{\theta}_j),\end{aligned}\tag{4.52}$$

where $(\boldsymbol{n}_\gamma,\boldsymbol{t}_\gamma)$ is a direct orthonormal basis of $I\!R^2$. Squaring (4.50) and integrating the first term over Ω_1 yields (see Fig.2):

$$\begin{aligned}&\int_{\Omega_1}\chi_{\mu_i}(\mathbf{x})\left(\frac{\partial}{\partial\boldsymbol{\theta}_j}\left[e^{-d_\gamma(\mathbf{x},\mu_i)}\right]\left(T_{\mu_i}^{\Omega_1\to\gamma}Z_{l,j}g\right)(\widetilde{\mathbf{x},\mu_i})\right)^2d\mathbf{x}\\ \le\;& C\int_\gamma|\boldsymbol{n}_\gamma\cdot\boldsymbol{\mu}_i|\left(\frac{|\boldsymbol{n}_\gamma\cdot\boldsymbol{\theta}_j|}{|\boldsymbol{n}_\gamma\cdot\boldsymbol{\mu}_i|}\right)^2\left(T_{\mu_i}^{\Omega_1\to\gamma}Z_{l,j}g\right)^2(t)d\boldsymbol{t}\\ \le\;& C\frac{1}{\varepsilon^2}\int_\gamma|\boldsymbol{n}_\gamma\cdot\boldsymbol{\mu}_i|\left(T_{\mu_i}^{\Omega_1\to\gamma}Z_{l,j}g\right)^2(t)d\boldsymbol{t}.\end{aligned}$$

The sum over i, j of the latter expression can be estimated as previously (see Proposition 5). The second term of (4.50) splits into two contributions (see (4.51)). The first one is

$$
\begin{aligned}
S_1 &= \sum_{(i,j)\in J'_\varepsilon} \omega_i^1\omega_j^2 \int_{\Omega_1} \Big(\chi_{\mu_i}(\mathbf{x})(\mathcal{Q}^D \mathcal{J}_D^{N-1})\Big[\frac{|\boldsymbol{n}_2\cdot\boldsymbol{\theta}_j|}{|\boldsymbol{n}_2\cdot\boldsymbol{\theta}|}\frac{\partial(\widetilde{\mathbf{x},\mu_i})}{\partial\boldsymbol{\theta}_j}e^{-d_2}(Z_{l,j}g)\big(\overline{(\widetilde{\mathbf{x},\mu_i}),\theta}\big)\Big](\mu_i)\Big)^2 d\mathbf{x} \\
&= \sum_{(i,j)\in J'_\varepsilon} \omega_i^1\omega_j^2 \int_{\Omega_1} \Big(\frac{\partial(\widetilde{\mathbf{x},\mu_i})}{\partial\boldsymbol{\theta}_j}\Big)^2 \chi_{\mu_i}(\mathbf{x}) \Big(\big(\mathcal{Q}^D \mathcal{J}_D^{N-1}\big)\Big[\frac{|\boldsymbol{n}_2\cdot\boldsymbol{\theta}_j|}{|\boldsymbol{n}_2\cdot\boldsymbol{\theta}|}e^{-d_2}\big(Z_{l,j}g\big)\big(\overline{(\widetilde{\mathbf{x},\mu_i}),\theta}\big)\Big](\mu_i)\Big)^2 d\mathbf{x} \\
&\overset{(a)}{\le} \sum_j \omega_j^2 \frac{1}{\varepsilon^2}\int_\gamma \sum_i \omega_i^1 |\boldsymbol{\mu}_i\cdot\boldsymbol{n}_\gamma| \Big(\big(\mathcal{Q}^D \mathcal{J}_D^{N-1}\big)\Big[\frac{|\boldsymbol{n}_2\cdot\boldsymbol{\theta}_j|}{|\boldsymbol{n}_2\cdot\boldsymbol{\theta}|}e^{-d_2}\big(Z_{l,j}g\big)\big(\overline{t,\theta}\big)\Big]\Big)^2 d\boldsymbol{t} \\
&\overset{(b)}{\le} \sum_j \omega_j^2 \frac{C}{\varepsilon^2}\int_\gamma \sum_k \omega_k^2 |\boldsymbol{\theta}_k\cdot\boldsymbol{n}_\gamma| \Big(\frac{|\boldsymbol{n}_2\cdot\boldsymbol{\theta}_j|}{|\boldsymbol{n}_2\cdot\boldsymbol{\theta}_k|}\Big)^2 \big(Z_{l,j}g(\overline{t,\theta})\big)^2 d\boldsymbol{t},
\end{aligned}
$$

where $^{(a)}$ stands for: according to (4.52) and $^{(b)}$ stands for: according to Proposition 3. The integration over γ is transformed into an integration over the boundary where the points $(\overline{t,\theta})$ are defined, by using the change of variables: $d\mathbf{s} = d\boldsymbol{t}\frac{|\boldsymbol{\theta}_k\cdot\boldsymbol{n}_\gamma|}{|\boldsymbol{\theta}_k\cdot\boldsymbol{n}_2|}$. Thus we have

$$
S_1 \le \frac{C}{\varepsilon^2}\Bigg(\sum_j \omega_j^2 \int_{\Gamma_2} |\boldsymbol{n}_2\cdot\boldsymbol{\theta}_j|\,(Z_{l,j}g)^2(\mathbf{s})d\mathbf{s}\Bigg)\Bigg(\sum_k \omega_k^2 \frac{1}{|\boldsymbol{\theta}_k\cdot\boldsymbol{n}_2|}\Bigg).
$$

The first term of this product is clearly bounded by $\|g\|$ according to (2.5) and Proposition 5. On the other hand

$$
\sum_k \omega_k^2 \frac{1}{|\boldsymbol{\theta}_k\cdot\boldsymbol{n}_2|} \le \frac{\pi}{2}\sum_k \omega_k^2 \frac{1}{\theta_k} \le C(\log N_2),
$$

according to (2.18). So it remains to estimate the second term of (4.51). We proceed as follows

$$
\begin{aligned}
&\sum_{(i,j)\in J'_\varepsilon} \omega_i^1\omega_j^2 \int_{\Omega_1} \chi_{\mu_i}(\mathbf{x}) \Big[\big(\mathcal{Q}^D \mathcal{J}_D^{N-1}\big)\Big(\frac{\partial(\widetilde{\mathbf{x},\mu_i})}{\partial\boldsymbol{\theta}_j} T_\theta^{\Omega_2}\frac{\partial}{\partial\boldsymbol{\theta}_j}Z_{l,j}g(\widetilde{\mathbf{x},\mu_i})\Big)(\mu_i)\Big]^2 d\mathbf{x} \\
&= \sum_{(i,j)\in J'_\varepsilon} \omega_i^1\omega_j^2 \int_{\Omega_1} \chi_{\mu_i}(\mathbf{x}) \Big(\frac{\partial(\widetilde{\mathbf{x},\mu_i})}{\partial\boldsymbol{\theta}_j}\Big)^2 \Big[\big(\mathcal{Q}^D \mathcal{J}_D^{N-1}\big)\Big(T_\theta^{\Omega_2}\frac{\partial}{\partial\boldsymbol{\theta}_j}Z_{l,j}g(\widetilde{\mathbf{x},\mu_i})\Big)(\mu_i)\Big]^2 d\mathbf{x} \\
&\overset{(a)}{\le} \sum_j \omega_j^2 \frac{1}{\varepsilon^2}\int_\gamma \sum_i \omega_i^1 |\boldsymbol{\mu}_i\cdot\boldsymbol{n}_\gamma| \Big[\big(\mathcal{Q}^D \mathcal{J}_D^{N-1}\big)\Big(T_\theta^{\Omega_2}\frac{\partial}{\partial\boldsymbol{\theta}_j}Z_{l,j}g(t)\Big)(\mu_i)\Big]^2 d\boldsymbol{t} \\
&\overset{(b)}{\le} \sum_j \omega_j^2 \frac{1}{\varepsilon^2}\int_\gamma \sum_k \omega_k^2 |\boldsymbol{\theta}_k\cdot\boldsymbol{n}_\gamma| \Big(T_{\theta_k}^{\Omega_2}\frac{\partial}{\partial\boldsymbol{\theta}_j}Z_{l,j}g(t)\Big)^2 d\boldsymbol{t} \\
&\overset{(c)}{\le} \frac{1}{\varepsilon^2}\int_{\Omega_2} \sum_j \omega_j^2 \Big(\frac{\partial}{\partial\boldsymbol{\theta}_j}Z_{l,j}g(\mathbf{x})\Big)^2 d\mathbf{x} \le \frac{1}{\varepsilon^2}\|g\|,
\end{aligned}
$$

where $^{(a)}$: according to (2.19) and (4.52), $^{(b)}$: according to Proposition 3 and (3.38), and $^{(c)}$: according to (2.5) if $l = 3$ and to Proposition 5 if $l = 4$.

• Let us define $A_\delta = \sum_{i=1}^4 \widetilde{A_i}$ and $B_\delta = T_\delta^2 - A_\delta$. We deduce from the previous estimates that for $g \in L^2(\Omega)$,

$$\|A_\delta g\|_{1,\Omega_1} \le C(\log N)^{1/2}\varepsilon^{-2}\|g\|, \quad \text{and} \quad \|B_\delta g\|_{0,\Omega_1} \le C(\sigma(\varepsilon, N_1, N_2))^{1/2}\|g\|.$$

Swapping N_1 and N_2, we obtain the same results on Ω_2. The coupling operators are different but satisfy the same stability estimates. The less regular term, corresponding to S_1 in the latter development, is bounded by $C(\log N_{max})^{1/2}\varepsilon^{-2}$. This concludes the proof of the lemma. □

Lemma 14 *For each* $0 < \delta < 1$ *there exists a constant* C_δ *such that*

$$\|(T - T_\delta)g\| \le C_\delta N^{-1+\delta}(\|g\|_{1,\Omega_1} + \|g\|_{1,\Omega_2}), \quad \forall g \in H^1(\Omega_1) \cap H^1(\Omega_2).$$

Proof. Let T_{N_k}, $k = 1, 2$ be defined by (3.29). We have:

$$\begin{aligned} \|(T - T_\delta)g\| &\le C\left(\|(T - T_\delta)g\|_{0,\Omega_1} + \|(T - T_\delta)g\|_{0,\Omega_2}\right) \\ &\le C\left(\|(T - T_{N_1})g\|_{0,\Omega_1} + \|(T_{N_1} - T_\delta)g\|_{0,\Omega_1}\right) \\ &\quad + C\left(\|(T - T_{N_2})g\|_{0,\Omega_2} + \|(T_{N_2} - T_\delta)g\|_{0,\Omega_2}\right). \end{aligned} \tag{4.53}$$

Following p.960[18], we have

$$\|(T - T_{N_k})g\| \le C N_k^{-1}\left\|\frac{\partial T_\mu g}{\partial \mu}\right\|_{L^1(L^2)},$$

for $k = 1, 2$. By Lemma 11, we have

$$\|(T - T_{N_k})g\| \le C N_k^{-1}(\|g\|_{1,\Omega_1} + \|g\|_{1,\Omega_2}), \tag{4.54}$$

for $k = 1, 2$. According to (3.28) and (3.29) we deduce that for $\mathbf{x} \in \Omega_1$,

$$\begin{aligned} (T_\delta - T_{N_1})g(\mathbf{x}) =& \sum_{i \in I_1^{NW}} \omega_i^1 e^{-d_\gamma(\mathbf{x},\mu_i)}\chi_{\mu_i}(\mathbf{x})\left[\left(\mathcal{Q}^{NW}\mathcal{J}_{NW}^{N_2-1} - \mathcal{J}_{NW}^{N_1-1}\right)T_\mu g(\widetilde{x,\mu_i})\right](\mu_i) \\ &+ \sum_{i \in I_1^{SW}} \omega_i^1 e^{-d_\gamma(\mathbf{x},\mu_i)}\chi_{\mu_i}(\mathbf{x})\left[\left(\mathcal{Q}^{SW}\mathcal{J}_{SW}^{N_2-1} - \mathcal{J}_{SW}^{N_1-1}\right)T_\mu g(\widetilde{x,\mu_i})\right](\mu_i), \end{aligned}$$

and for $\mathbf{x} \in \Omega_2$,

$$\begin{aligned} (T_\delta - T_{N_2})g(\mathbf{x}) =& \sum_{j \in I_2^{NE}} \omega_j^2 e^{-d_\gamma(\mathbf{x},\theta_j)}\chi_{\theta_j}(\mathbf{x})\left[\left(\mathcal{Q}^{NE}\mathcal{J}_{NE}^{N_1-1} - \mathcal{J}_{NE}^{N_2-1}\right)T_\theta g(\widetilde{x,\theta_j})\right](\theta_j) \\ &+ \sum_{j \in I_2^{SE}} \omega_j^2 e^{-d_\gamma(\mathbf{x},\theta_j)}\chi_{\theta_j}(\mathbf{x})\left[\left(\mathcal{Q}^{SE}\mathcal{J}_{SE}^{N_1-1} - \mathcal{J}_{SE}^{N_2-1}\right)T_\theta g(\widetilde{x,\theta_j})\right](\theta_j). \end{aligned}$$

So, according to the Cauchy inequality $(\sum_i \omega_i f_i)^2 \le (\sum_i \omega_i)(\sum_i \omega_i f_i^2)$, to (2.15) and to the proof of Proposition 5, we have

$$\begin{aligned}
&\int_{\Omega_k} [(T_\delta - T_{N_k})g(\mathbf{x})]^2 \\
\le\ & C \sum_{D\in\mathcal{D}_k} \int_{\Omega_k} \sum_{i\in I_k^D} \omega_i^k \chi_{\nu_i^k}(\mathbf{x}) e^{-2d_\gamma(\mathbf{x},\nu_i^k)} \Big[(\mathcal{Q}^D \mathcal{J}_D^{N_{k'}-1} - \mathcal{J}_D^{N_k-1}) T_\nu g(\widetilde{x,\nu_i^k})\Big]^2 d\mathbf{x} \\
\le\ & C \sum_{D\in\mathcal{D}_k} \int_\gamma \sum_{i\in I_k^D} |\boldsymbol{\nu}_i^k \cdot \boldsymbol{n}_\gamma|\, \omega_i^k \Big[(\mathcal{Q}^D \mathcal{J}_D^{N_{k'}-1} - \mathcal{J}_D^{N_k-1}) T_\nu g(\boldsymbol{t})\Big]^2 d\boldsymbol{t} \\
\le\ & C \sum_{D\in\mathcal{D}_k} \int_\gamma \int_D w_D(\nu) \Big[(\mathcal{Q}^D \mathcal{J}_D^{N_{k'}-1} - \mathcal{J}_D^{N_k-1}) T_\nu g(\boldsymbol{t})\Big]^2 d\nu d\boldsymbol{t},
\end{aligned}$$

where $(k,k') = (1,2)$ or $(2,1)$. From Proposition 4 for a given $\boldsymbol{t}$, we show that

$$\begin{aligned}
&\int_D w_D(\nu) \Big[(\mathcal{Q}^D \mathcal{J}_D^{N_{k'}-1} - \mathcal{J}_D^{N_k-1}) T_\nu g(\boldsymbol{t})\Big]^2 d\nu \\
\le\ & \int_D w_D(\nu) \Big[(\mathcal{J}_D^{N_{k'}-1} - \mathcal{J}_D^{N_k-1}) T_\nu g(\boldsymbol{t})\Big]^2 d\nu \\
\le\ & \int_D w_D(\nu) \Big[(\mathcal{J}_D^{N_{k'}-1} - Id) T_\nu g(\boldsymbol{t})\Big]^2 d\nu + \int_D w_D(\nu) \Big[(Id - \mathcal{J}_D^{N_k-1}) T_\nu g(\boldsymbol{t})\Big]^2 d\nu.
\end{aligned}$$

We deduce from (2.17) that

$$\left\{\int_D w_D(\nu) \Big[(Id - \mathcal{J}_D^{N_k-1}) T_\nu g(\boldsymbol{t})\Big]^2 d\nu\right\}^{1/2} \le C N_k^{-s} \|T_\nu g(\boldsymbol{t})\|_{H^s(D)},$$

for $1/2 < s$. Interpolating between both inequalities given in Lemma 12 (see Bergh and Löfström[12]) yields

$$\|T_\nu g(\boldsymbol{t})\|_{H^s(D)} \le C\,[t_x(L-t_x)]^{-s/2} \|g\|_{H^1(\Omega_k)},$$

the square of which is integrable over γ for $s < 1$. Then we deduce that

$$\int_{\Omega_k} ((T_\delta - T_{N_k})g)^2 \le C N_k^{-s} \|g\|_{H^1(\Omega_{k'})}, \quad \forall s < 1, \tag{4.55}$$

where $(k,k') = (1,2)$ or $(k,k') = (2,1)$. From inequalities (4.54) and (4.55), we obtain the estimate of the terms in (4.53) in which T_δ appears, and the proof is complete. □

Lemma 15 *Let $|\lambda| < \lambda_{max}$ and assume that $F(N) : \mathbb{N} \to \mathbb{N}$ satisfies $N < F(N)$ and $N^{-1/3} \log F(N) \to 0$ as $N \to +\infty$. Assume that the quadrature rules Q_N satisfy the properties on section 2. The coupling operators are those of section 3. Then, (4.41) is satisfied for $m = 2$ and there exists a constant C such that for sufficiently large N_{min} and N_{max} satisfying $N_{min} \le N_{max} \le F(N_{min})$, we have*

$$\|(I - \lambda T_\delta)V\| \ge C\|V\| \quad \forall V \in L^2(\Omega).$$

Proof. By Lemmas 13, 14 and Corollary 6, we deduce that

$$\begin{aligned} &\|(T-T_\delta)T_\delta^2 g\| = \|(T-T_\delta)(A_\delta+B_\delta)g\| \\ \leq\ & C_\delta((\log N_{max})^{1/2}\varepsilon^{-2}N_{min}^{-1+\delta} + \sigma(\varepsilon, N_1, N_2))\|g\|. \end{aligned}$$

Choosing for instance $\varepsilon = N_{min}^{-1/3}$, $\delta = 1/6$, and F such that $\log F(N) = \alpha(N)N^{1/3}$ where $\alpha(N) \to 0$ when $N \to \infty$, we obtain from (2.21) that

$$\|(T-T_\delta)T_\delta^2\| \to 0 \quad \text{when} \quad N_{min} \to \infty,\ N_{min} < N_{max} < F(N_{min}).$$

The desired results is then obtained from Lemma 10. □
Let us conclude with the proof of Theorem 7.
Proof. [Theorem 7] We consider the following argument[24]. Since T_δ is not compact, Lemma 15 does not allow us to prove the well-posedness of problem (3.27). Instead, we use Lemma 13 and the splitting $\lambda^2 T_\delta^2 = A + B$, where A is compact and B satisfies $\|B\| \to 0$ when $N_1, N_2 \to \infty$. Consider now the following problem: Find $V \in L^2$ such that

$$(I - \lambda^2 T_\delta^2)V \ = T_\delta q. \tag{4.56}$$

From (2.9), we have $\|(I \pm \lambda T)V\| \geq C\|V\|$, for $|\lambda| < \lambda_{max}$. So we deduce from Lemma 15 that $\|(I \pm \lambda T_\delta)V\| \geq C\|V\|$, provided that N_1 and N_2 are sufficiently large. So, still for N_1 and N_2 sufficiently large, we deduce from $\|B\| \to 0$ that

$$\|(I-A)V\| = \|(I+\lambda T_\delta)(I-\lambda T_\delta)+B)V\| \geq C\|V\|.$$

Then, $I - A : L^2 \to L^2$ is one-to-one according to the Fredholm alternative since A is compact. Recasting (4.56) as $[I-(I-A)^{-1}B]V = (I-A)^{-1}T_\delta f$, we deduce that it admits a unique solution V since $\|(I-A)^{-1}B\| \to 0$ when $N_1, N_2 \to \infty$. Thus,

$$\phi_\delta = (I + \lambda T_\delta)V$$

is a solution to (3.27); consequently, $(I - \lambda T_\delta)$ is onto. By Lemma 15, it is also into and therefore invertible. Consider now the error estimate. From (2.7) and (3.27), we have

$$(I - \lambda T_\delta)(\phi - \phi_\delta) \ = (T - T_\delta)(\lambda\phi + q). \tag{4.57}$$

We have proved that $(I - \lambda T_\delta)^{-1}$ is bounded from $L^2(\Omega)$ to $L^2(\Omega)$. According to Lemma 14 we deduce that $\|\phi - \phi_\delta\| \leq C_\delta N_{min}^{\eta-1}$, provided q is regular enough ($q \in H^1(\Omega)$) and $1 > \eta > 0$. □

5. Conclusions

We have shown in this paper how to couple two angular discretizations of a model transport equation posed on adjacent squares. The main result is that more angular directions are required close to the directions of spatial singularities ($\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ for a square). This ensures that the physical current, or a linearization of it (see (3.38)), is preserved at the interface. The same quadrature rules can be employed for any geometry where the interfaces between different domains are oriented along $\boldsymbol{e}_x$ or $\boldsymbol{e}_y$[7,10]. The procedure can also be easily extended to more complex geometries. In the case of hexagonal geometries for instance, the angular domain should be split into six subdomains of angle $\pi/3$ that are discretized using a Gaussian quadrature rule. The spatial discretization is crucial in practice. The refinement of angular discretizations will be efficient if combined with a refinement of the spatial discretization[7].

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1. R. A. Adams. *Sobolev Spaces.* Academic Press, NY, San Francisco, London, 1975.
2. G. Allaire and G. Bal. Homogenization of the criticality spectral equation in neutron transport. *M2AN Math. Model. Numer. Anal.*, 33:721–746, 1999.
3. P. M. Anselone. *Collectively compact operator approximation theory.* Prentice-Hall, Englewood Cliffs, NJ, 1971.
4. M. Asadzadeh. Analysis of a fully discrete scheme for neutron transport in two-dimensional geometry. *SIAM J. Math. Anal.*, 23:543–561, 1986.
5. M. Asadzadeh. L_p and eigenvalue error estimates for the discrete ordinates method for two-dimensional neutron transport. *SIAM J. Numer. Anal.*, 26:66–87, 1989.
6. M. Asadzadeh, P. Kumlin, and S. Larsson. The discrete ordinates method for the neutron transport equation in an infinite cylindrical domain. *Math. Models Meth. Appl. Sci.*, 2:317–338, 1992.
7. G. Bal. *Couplage d'équations et homogénéisation en transport neutronique. Thèse de Doctorat* de l'Université Paris 6 [in French], 1997.
8. G. Bal. Boundary layer analysis in the homogenization of neutron transport equations in a cubic domain. *Asymptotic Analysis*, 20(3-4):213–239, 1999.
9. G. Bal. First-order Corrector for the Homogenization of the Criticality Eigenvalue Problem in the Even Parity Formulation of the Neutron Transport. *SIAM J. Math. Anal.*, 30:1208–1240, 1999.
10. G. Bal and X. Warin. Discrete Ordinates Methods in xy-Geometry with spatially varying angular discretization. *Nuclear Science and Engineering*, 127(2):169–181, 1997.
11. A. Bensoussan, J. L. Lions, and G. Papanicolaou. Boundary Layers and Homogenization of Transport Processes. *RIMS, Kyoto Univ.*, 1979.
12. J. Bergh and L. Löfström. *Interpolation spaces.* Springer, New York, 1976.
13. C. Bernardi and Y.Maday. *Approximations spectrales de problèmes aux limites elliptiques.* Springer-Verlag, Paris, 1992.
14. C. Bernardi and Y.Maday. *Basic results on spectral methods.* Laboratoire d'analyse numérique R94022, Paris VI, 1992.
15. R. Dautray and J. L. Lions. *Mathematical analysis and numerical methods for Science and Technology. Vol.6.* Springer Verlag, Berlin, 1993.
16. R. A. de Vore and L. R. Scott. Error bounds for Gaussian quadrature and weighted-L^1

polynomial approximation. *SIAM J. Numer. Anal.*, 21:400–412, 1984.

17. J. J. Duderstadt and W. R. Martin. *Transport Theory.* Wiley-Interscience, 1979.
18. C. Johnson and J. Pitkäranta. Convergence of a fully discrete scheme for two-dimensional neutron transport. *SIAM J. Numer. Anal.*, 20:951–966, 1983.
19. E. W. Larsen and J. B. Keller. Asymptotic solution of neutron transport problems for small mean free paths. *J. Math. Phys.*, 15:75–81, 1974.
20. E. E. Lewis and W. F. Miller Jr. *Computational Methods of Neutron Transport.* John Wiley and sons, New York, 1984.
21. S. G. Mikhlin. *Multidimensional Singular Integrals and Integral Equations.* Pergamon Press, Oxford, 1965.
22. P. Nelson. Convergence of the Discrete-Ordinates Method for anisotropically Scattering Multiplying Particles in a Subcritical Slab. *SIAM J. Numer. Anal.*, 10:175–181, 1973.
23. J. Pitkäranta. Estimates for the derivatives of solutions to weakly singular Fredholm integral equations. *SIAM J. Math. Anal.*, 11:952–968, 1980.
24. J. Pitkäranta and L. R. Scott. Error estimates for the combined spatial and angular approximations of the transport equation for slab geometry. *SIAM J. Numer. Anal.*, 20:922–950, 1983.
25. G. Szegö. *Orthogonal polynomials.* American mathematical society colloquium publications Volume XXIII, 1967.
26. H. D. Victory. Convergence properties of discrete-ordinates solutions for neutron transport in three-dimensional media. *SIAM J. Numer. Anal.*, 17:71–83, 1980.
27. V. S. Vladimirov. *Mathematical problems in the one-velocity theory of particle transport.* Atomic energy of Canada Ltd, Chalk River, Ont. Report AECL 1661, 1963.