Diffusion Approximation of Radiative Transfer Problems with Interfaces

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Abstract

We derive the diffusion approximation of transport equations with discontinuities at interfaces. The transport equations model the energy density of acoustic waves. The waves are reflected and transmitted at the interface between different media, which leads to discontinuities of the energy density across the interface. The diffusion approximation, which is valid inside each region is not correct at the vicinity of the interface. However using interface layer analysis, we prove that the transport solution can be approximated by a diffusion term plus an interface layer which decays exponentially fast. We systematically derive the correct form of the interface conditions for this diffusion term.

1 Introduction

The propagation of the energy density of waves in heterogeneous media with weak random fluctuations can be modeled in the high frequency regime by a radiative transfer equation. This model has been first proposed phenomenologically [12], and was successfully applied to describe the propagation of light through a turbulent atmosphere. Transition from waves to transport has been later derived and analyzed from first principles, both for Schrödinger [14, 16, 23, 31] and classical waves [4, 10, 27, 35]. We consider in this paper only scalar waves, where polarization effects or conversion of energy between different modes is either absent or may be neglected for practical purposes, as in sound propagation.

The radiative transport equation for the evolution of the average phase space energy density $a(t, \mathbf{x}, \mathbf{k})$ is given at time $t \geq 0$, position $\mathbf{x} \in I\!R^3$ and wave vector $\mathbf{k} \in I\!R^3$ by

$$\frac{\partial a}{\partial t} + v\hat{\mathbf{k}} \cdot a = \int \sigma(\mathbf{x}, \mathbf{k}, \mathbf{k}')\delta(v|\mathbf{k}| - v|\mathbf{k}'|)(a(t, \mathbf{x}, \mathbf{k}') - a(t, \mathbf{x}, \mathbf{k}))d\mathbf{k}'. \tag{1.1}$$

Here $\hat{\mathbf{k}}$ is the unit vector in the direction $\mathbf{k}$, and v is the background sound speed. We assume for simplicity that v is constant, but its variations may be also taken into account. The differential scattering cross section $\sigma(\mathbf{x}, \mathbf{k}, \mathbf{k}')$ is explicitly given in terms of the power spectrum of the random fluctuations of the material properties [27] and is usually symmetric in $\mathbf{k}$ and $\mathbf{k}'$. Moreover, it typically depends only on the angle between $\mathbf{k}$ and $\mathbf{k}'$:

$$\sigma(\mathbf{x}, \mathbf{k}, \mathbf{k}') = \sigma(\mathbf{x}, \mathbf{k} \cdot \mathbf{k}'). \tag{1.2}$$

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The acoustic energy flux is given in phase space by

$$\mathcal{F}(t,\mathbf{x}) = v \int \hat{\mathbf{k}}\, a(t,\mathbf{x},\mathbf{k})\, d\mathbf{k}. \tag{1.3}$$

When the transport mean free path $l = v/\Sigma$, measuring the distance between two successive collisions, is small, it is well known that the energy distribution tends to be isotropic $a(t,\mathbf{x},\mathbf{k}) \approx a_0(t,\mathbf{x},|\mathbf{k}|)$. The function a_0 is then solution to a diffusion equation given by

$$\frac{\partial a_0}{\partial t} = \nabla_{\mathbf{x}} \cdot [D(\mathbf{x},|\mathbf{k}|)\nabla_{\mathbf{x}} a_0], \tag{1.4}$$

where D is a positive definite diffusion tensor [13], which is known explicitly. The approximation of linear transport by a diffusion equation has been studied at length in the past for various physical applications. First derived in the context of the transport of neutral particles in homogeneous media [11, 13, 22], it finds applications in many areas like homogenization of neutron transport in heterogeneous domains [1, 8, 9, 20, 21, 29], medical imaging [2, 25], or propagation of high frequency classical waves [27, 30, 32]. It is very popular since its numerical resolution is much faster than that of the transport equation. Therefore it allows to address problems of practical interest which would not be accessible with the transport theory, even in spite of the tremendous advances of modern computers.

Equation (1.4) is usually derived in the whole space $\mathbb{R}^3$. In the case of bounded domains or domains with interfaces between media having different properties, the ansatz $a(t,\mathbf{x},\mathbf{k}) \approx a_0(t,\mathbf{x},|\mathbf{k}|)$ may not be satisfied near the discontinuity, and specific boundary or interface conditions are required for (1.4). In the case of bounded domains, the derivation of boundary conditions for (1.4) has also been thoroughly analyzed in the past and is fairly well understood [6, 9, 13, 22]. It is shown that the transport solution is asymptotically equal to the sum of an isotropic part, which is a solution of a diffusion equation with Robin boundary conditions, and a boundary layer term, which is highly anisotropic and is exponentially decaying away from the boundary. Due to the exponential decay of the layer term, the postulate of asymptotically isotropic energy density in the interior of the domain is not modified by the presence of boundaries. However the derivation of correct boundary conditions is quite involved and necessitates an accurate description of the transport density at the vicinity of the boundary.

In the case of an interface between two different media with jump discontinuities of the energy density, we can expect that two boundary layers exist on both sides of the interface, and that interface conditions shlould be derived for an averaged diffusion density, which would be a good approximation of the transport density away from the interface. However, even assuming the existence of these boundary layers, we will see that they are not independent from one another. No theory is available at present for constructing such an interface layer. The goal of this paper is to address this issue.

The interface conditions for the radiative transport equation (1.1) arise naturally from the behavior of waves reflected and transmitted from flat or random rough interfaces separating two different media. The transition from waves to transport equations for time harmonic waves in bounded domains has been considered mathematically in several recent works [17, 24, 28] and has been extensively studied in the physical literature [3, 7, 15, 18, 26, 33, 34]. We consider here only the case of time harmonic waves, which result in steady state transport and diffusion equations, for simplicity. Time dependent problems may be also considered but we do not pursue that issue here, concentrating instead on the difficulties related to the interface problem. The energy densities $a_{1,2}(\mathbf{x},\mathbf{k})$ on the two sides of the interface are time independent in the time-harmonic case and are supported on the sets $\mathcal{S}_{1,2} = \{(\mathbf{x},\mathbf{k}) \text{ s.t. } v_{1,2}|\mathbf{k}| = \omega\}$, respectively, where ω is the time frequency. They satisfy the radiative transport equations (1.1) in the corresponding domains.

The interface conditions for the transport equation are as follows. Let the interface be described by $x_n = 0$ and the media 1 and 2 be above and below it, respectively. Given a horizontal wave vector $\mathbf{k}' \in I\!R^{n-1}$, let k_n^i, $i = 1,2$ be the normal component of the vector $\mathbf{k} \in \mathcal{S}_i$ that has horizontal component $\mathbf{k}'$ and is pointing inside medium i. Then the restriction of $a_i(\mathbf{x}, \mathbf{k})$ for $i = 1,2$ at the boundary has the form

$$a_i(\mathbf{x}', 0, \mathbf{k}) = \nu_{in}^i \delta(k_n - k_n^i) + \nu_{out}^i \delta(k_n + k_n^i), \quad i = 1,2. \tag{1.5}$$

The incoming energy densities ν_{in}^i are related to the outgoing densities ν_{out}^i by the following interface conditions

$$\nu_{in}^i(\mathbf{x}', \mathbf{k}') = \sum_{j=1}^{2} \int \tau_{ij}(\mathbf{x}', \mathbf{k}', \mathbf{p}') \nu_{out}^j(\mathbf{x}', \mathbf{p}') d\mathbf{p}'. \tag{1.6}$$

The cross sections τ_{ij} are positive bounded measures. Their form has been obtained for various regimes in the references cited above and we do not specify them here. Our analysis applies to a fairly general case, assuming only the conservation of energy flux at the interface. The singular part of τ_{ij} corresponds to the specular reflection at the boundary, and the regular part is due to the diffuse scattering of waves by a rough boundary. The measures τ_{ij} are related by the requirement that energy flux (1.3) be conserved across the interface. It turns out that the energy density is not necessarily continuous across the interface. Moreover the angular dependent form of the reflection and transmission operators may be incompatible with an asymptotically isotropic solution at the interface for the transport equation in the diffusion.

The above mentioned boundary layer analysis alone is not sufficient to treat the interface problem. The first main contribution of this paper is to show that there are indeed interface layers on both parts of the interface, which enable to define interface conditions for a diffusion approximation. The requirement that these interface layers decay exponentially away from the boundary provides the jump condition at the interface for the solution of the diffusion equation (1.4):

$$a_0(0^-) = \alpha a_0(0^+). \tag{1.7}$$

Here $a_0(0^+)$ and $a_0(0^-)$ refer to the limits of a_0 from above and below the interface, respectively. The existence of the boundary layer in the next order in the parameter $\varepsilon = l/\Sigma$ provides the jump condition for the normal derivatives on the two sides of the interfacein (1.7). It turns out to be given by the continuity of the diffusion flux across the interface so that

$$D(0^+)\frac{\partial a_0}{\partial n}(0^+) = D(0^-)\frac{\partial a}{\partial n}(0^-).$$

The jump condition (1.7) for the energy density itself is, in general, different from the one obtained by averaging the anisotropic boundary conditions over the space of directions. The second contribution of this work is to give an explicit way to compute the constant α (see our Lemma 6.2), which involves the resolution of an eigenvalue problem posed at the interface.

This paper is organized as follows. In section 2, we recall the transport equation in planar geometry. In the general 3D setting, interface conditions, as well as boundary conditions, localize in the diffusive regime. Therefore to simplify the analysis we only consider here geometries satisfying the planar invariance. We present our main results in section 3. It is shown that the diffusion approximation is still valid outside a boundary layer in the limit of small mean free path and prescribe the jump conditions at the interface. In section 4, we give some results on the transport equation in a half space. The theory presented in section 3 relies on the existence of a solution to an eigenvalue problem in infinite medium. The positive eigenfunction describes how the boundary layers are related to one another on both sides of the interface. The construction of this interface layer is presented in section 5. Then section 6 gives a proof of existence in L^1 space of a solution to the transport equation with interface conditions, and of the results on the asymptotic expansion claimed in section 3. Finally, we give in section 7 some examples of interface conditions for the transport equation and their translation into interface conditions for the diffusion approximation.

2 The Plane Symmetry Transport Equation

We assume here that the geometry and the physical data satisfy a planar symmetry, i.e. are invariant by translation in directions perpendicular to x_n. Since we are interested here in the behavior of the energy density close to a slowly varying interface in the diffusion regime, the interface can be asymptotically considered as flat and the underlying medium with constant physical data on both sides of the interface. Therefore, even in the general multi-dimensional setting, the interface conditions localize and give rise to a mono-dimensional analysis (see [9, 13] for an analog result in linear transport theory with boundaries).

To simplify the analysis and avoid technical difficulties that are independent of the interface conditions investigated here, we also assume that the planar symmetry is satisfied and the source term is isotropic. Then equation (1.1) in a homogeneous medium with a monochromatic source reduces to

$$v\hat{k}_n \frac{\partial a}{\partial x_n} = \int \sigma(x_n, \mathbf{k}\cdot\mathbf{k}')[a(x_n,\mathbf{k}') - a(x_n,\mathbf{k})]\delta(v|\mathbf{k}| - v|\mathbf{k}'|)d\mathbf{k}' + S(x_n)\delta(v|\mathbf{k}| - \omega). \tag{2.1}$$

Using the same symbol for the energy, we make the changes of variables

$$a(x_n,\mathbf{k}) = v^2 a(x_n,\mu)\delta(v|\mathbf{k}| - \omega) \quad \text{and} \quad \mu = \hat{k}_n \in [-1,1]$$

We denote $x = x_n$ for simplicity. Then the function $a(x,\mu)$ satisfies the one-dimensional transport equation

$$\mu \frac{\partial a}{\partial x} = \int_{-1}^{1} \sigma(x,\mu,\mu')[a(x,\mu') - a(x,\mu)]d\mu' + \tilde{S}(x). \tag{2.2}$$

We keep the same notation for the scattering cross section after the change of variables:

$$\sigma(x,\mu,\mu') = \frac{\omega^2}{v^4}\int_0^{2\pi} \sigma(x_n, \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos\phi + \mu\mu')d\phi \tag{2.3}$$

and the source $\tilde{S}(x)$ is

$$\tilde{S}(x) = \frac{S(x_n)}{v^3}. \tag{2.4}$$

The normal component of the flux (1.3) is given then in terms of $a(x,\mu)$ by

$$\mathcal{F}(x) = 2\pi\omega^2 \int_{-1}^{1} \mu a(x,\mu)d\mu. \tag{2.5}$$

The interface conditions (1.6) may be expressed in terms of the solutions $a_{1,2}(x,\mu)$ on the two sides of the interface $x = 0$ as follows. The functions ν_{ij} are isotropic in $\mathbf{k}'$, $\nu_{ij}(\mathbf{k}') = \nu_{ij}(|\mathbf{k}'|)$, and are related to a by

$$\begin{aligned} a_1(0,\mu) &= \frac{\mu}{v_1}\nu_{in}^1(\frac{\omega}{v_1}\sqrt{1-\mu^2}),\ \mu > 0, \qquad & a_1(0,\mu) &= \frac{\mu}{v_1}\nu_{out}^1(\frac{\omega}{v_1}\sqrt{1-\mu^2}),\ \mu < 0 \\ a_2(0,\mu) &= \frac{\mu}{v_2}\nu_{out}^2(\frac{\omega}{v_2}\sqrt{1-\mu^2}),\ \mu > 0, \qquad & a_2(0,\mu) &= \frac{\mu}{v_2}\nu_{in}^2(\frac{\omega}{v_2}\sqrt{1-\mu^2}),\ \mu < 0. \end{aligned} \tag{2.6}$$

The general interface conditions for (2.2) are formulated as follows. Let us introduce the sets $\Gamma_-^{1,2}$ of wave vectors pointing inside domains 1 and 2, respectively, defined by $\Gamma_-^1 = [0,1]$, $\Gamma_-^2 = [-1,0]$, and the sets $\Gamma_+^{1,2}$ of wave vectors pointing outside those domains $\Gamma_+^1 = [-1,0]$, $\Gamma_+^2 = [0,1]$. The reflection and transmission operators $R^{ij} : L^1_\mu(\Gamma_+^j) \to L^1_\mu(\Gamma_-^i)$, $i,j = 1,2$ map the functions defined on Γ_+^j onto the functions on Γ_-^i. The operators R^{ii} correspond to reflection, while the operators R^{ij}, $i \neq j$ describe the transmission across the interface. The space $L^1_\mu(X)$ is defined by $\|f\|_{L^1_\mu(X)} = \int_X |\mu| F(\mu) d\mu$. Then the interface conditions for the transport equations (2.2) in media 1 and 2 are

$$a_i(0,\mu)|_{\Gamma_-^i} = \sum_{j=1}^{2} R^{ij} \left(a_j|_{\Gamma_+^j} \right)(\mu). \tag{2.7}$$

The energy flux (2.5) is conserved across the boundary $x = 0$. Therefore the following constraints are imposed on the reflection and transmission operators,

$$\int_{\Gamma_+^i} \mu h(\mu) d\mu + \int_{\Gamma_-^i} \mu [R^{ii}(h)](\mu) d\mu = \int_{\Gamma_-^j} \mu [R^{ji}(h)](\mu) d\mu, \quad (i,j) = (1,2),(2,1), \tag{2.8}$$

for all functions $h(\mu) \in L^1_\mu(\Gamma_+^i)$. Another natural restriction is the positiveness of the operators R^{ij}. We summarize our assumptions on the operators R^{ij} in the following:

Hypothesis 2.1 *The operators R^{ij} satisfy the following properties:*

(H1) They are positive in the sense that

$$R^{ij}(h)(\mu) \geq 0 \text{ for a.e. } \mu \in \Gamma_-^i \text{ for all } h \in L^1_\mu(\Gamma_+^j) \text{ such that } h \geq 0 \text{ a.e. on } \Gamma_+^j.$$

Moreover, there exist constants $c_j > 0$ so that $R^{jj}(1)(\mu) > c_j$ for a.e. $\mu \in \Gamma_-^j$, $j = 1,2$.

(H2) They satisfy the flux continuity (2.8).

(H3) The reflection operators are not totally reflecting, that is there exist positive constants η_j and subsets $I_j \in \Gamma_+^j$ of positive measure such that for all non negative functions h of support $supp(h) \subset I_j$ we have

$$\int_{\Gamma_j^-} |\mu| R^{jj}(h) d\mu \;\leq\; (1-\eta_j) \int_{I_j} |\mu| h(\mu) d\mu. \tag{2.9}$$

Remark 2.2 *The lower bounds on $R^{11}(1)$, $R^{22}(1)$ in (H1) may be replaced by similar lower bounds for $R^{12}(1)$, $R^{21}(1)$. All our results below are readily extended to this new condition. One of these technical conditions is sufficient to guarantee the existence of a non-zero matching constant α given in (3.6).*

3 Diffusion approximation

The diffusion regime takes place when the scattering mean free path, measuring the average distance between two successive collisions, becomes small. We denote it by ε. The scattering cross section is inverse proportional to the mean free path and is therefore of order ε^{-1}. In order to have a solution bounded independently of ε, the source term has to be of order ε. Contrary to the Cauchy problem (1.1), which admits a solution for all time, the time harmonic case requires the existence of an equilibrium. Therefore the creation of energy should be compensated by some loss. Then we add some absorption Σ_a to the system in order to obtain finite energy solutions to our equation. This absorption term also has to be of order ε in order for the diffusive regime to appear. The scaling presented here is classical in diffusion theory [9, 13, 20, 21, 22].

After rescaling, problem (2.2)-(2.7) takes the following form

$$\begin{aligned} \frac{1}{\varepsilon}\mu\frac{\partial}{\partial x}a^i_\varepsilon + \frac{1}{\varepsilon^2}Q_i(a^i_\varepsilon) + \Sigma_a a^i_\varepsilon = S(x) \qquad & \text{in } X^i \times V \\ a^i_\varepsilon = \sum_{j=1}^{2} R^{ij}(a^j_\varepsilon) \qquad & \text{for } x = 0 \text{ and } \mu \in \Gamma^i_-. \end{aligned} \tag{3.1}$$

Here $V = [-1, 1]$, $X^1 = I\!R^+$ and $X^2 = I\!R^-$. The scattering operator Q_i is defined by

$$Q_i(h)(x, \mu) = \int_{-1}^{1} \sigma^i(\mu, \mu')(h(x, \mu) - h(x, \mu'))d\mu'. \tag{3.2}$$

We assume that $\sigma^i(\mu, \mu')$ are independent of x inside each half-space. Spatial variations of the differential scattering cross-section may be also taken into account but we do not pursue this issue here. Typically, σ is constant on both domains X^i, $i = 1, 2$. More generally, we assume that σ is a bounded function on $V \times V$, bounded from above and below:

$$0 < \sigma_1 \le \sigma^i(\mu, \mu') \le \sigma_2 < \infty, \qquad \text{for } i = 1, 2. \tag{3.3}$$

We denote by Σ the total scattering cross section

$$\Sigma^i = \int_{-1}^{1} \sigma^i(\mu, \mu')d\mu. \tag{3.4}$$

It is independent of μ' by assumptions (1.2) and (2.3).

The asymptotic behavior of the solution a_ε of (3.1) as ε tends to 0 is described by the following theorem, which is the main result of this paper.

Theorem 3.1 *Let us assume that the cross-sections $\sigma^i(\mu, \mu')$ satisfy (3.3) and the total scattering cross-sections Σ^i are independent of μ, and that the source term S is smooth and compactly supported. We assume in addition that the reflection-transmission operators satisfy Hypothesis 2.1. Then for every $\varepsilon > 0$ there exists a unique solution a_ε of (3.1) in $L^1(I\!R \times V)$.*

The family $a^i_\varepsilon(x, \mu)$ converges to a function $u^i(x) \in H^1(X^i)$, that is independent of μ, strongly in $L^1(X^i \times V)$ as $\varepsilon \to 0$ and there exists a constant C independent of ε so that

$$\sum_{i=1}^{2} \|a^i_\varepsilon - u^i\|_{L^1(X^i \times V)} \le C\varepsilon. \tag{3.5}$$

Moreover, there exists a positive constant α such that u^i is the solution of the diffusion equation

$$\begin{aligned} -\frac{\partial}{\partial x}\left[D^i\frac{\partial u^i}{\partial x}\right] + \Sigma_a u^i &= S \quad \text{in } X^i \\ u^2(0^-) &= \alpha\ u^1(0^+) \\ D^2\frac{\partial u^2}{\partial x}(0^-) &= D^1\frac{\partial u^1}{\partial x}(0^+). \end{aligned} \tag{3.6}$$

The diffusion coefficient D^i in (3.6) is obtained as follows. We define $\theta^i(\mu)$ as the zero mean solution of

$$\theta^i(\mu) - \int_{-1}^{1} \frac{\sigma^i(\mu,\mu')}{\Sigma^i}\theta^i(\mu')d\mu' \;=\; -\mu. \tag{3.7}$$

Then D^i is given by

$$D^i = \frac{1}{\Sigma^i}\int_{-1}^{1}\mu\theta^i(\mu)d\mu. \tag{3.8}$$

This coefficient is positive, and D^1 and D^2 may be different. The definition of the constant α, which requires additional notation related to boundary and interface problems, is given explicitly in Lemma 6.2 of section 6. The positiveness of α and $D^{1,2}$ ensures the existence of a unique solution to equation (3.6).

The proof of this theorem is given in the three following sections. The next two sections contain some necessary preliminary results. First we provide some auxiliary results on the half space problem in Section 4. These results allow us to derive a solution to the interface eigenvalue problem in Section 5. Its construction is given in Theorem 5.1. Then in Section 6 we give the proof of Theorem 3.1. First we prove the existence result for equation (3.1) in Theorem 6.1. The rest of the proof is based on an asymptotic expansion for a^i_ε given in Section 6. The formal convergence to diffusion in the interior is classical [9, 13]. Close to the interface however, the two-scale asymptotic expansion is not sufficient and an interface layer constructed in Section 5 is required to account for the anisotropic boundary conditions.

4 Boundary Layer Analysis

The problem posed in infinite space can be recast as a problem only on the interface by defining the response operators associated with each half space. They are the solution operators for the Milne problem in each half space and map the incoming current in a half space onto the outgoing density of the solution of the transport equation at the boundary. This problem is well known in the literature (see e.g. [6, 9, 13, 29]). We present here the results we use in the next section.

Consider the half space problem

$$\begin{aligned} \mu\frac{\partial a}{\partial x} + \int_{-1}^{1}\sigma(\mu,\mu')(a(x,\mu)-a(x,\mu'))d\mu' \;&=\; 0 \qquad \text{in } \mathbb{R}^+\times[-1,1] \\ a(0,\mu) \;&=\; g(\mu) \qquad \text{in } [0,1]. \end{aligned} \tag{4.1}$$

Up to spatial rescaling we can assume that $\Sigma = \displaystyle\int_{-1}^{1}\sigma(\mu',\mu)d\mu' = 1$. We have the following result [6].

Lemma 4.1 *Let $g\in L^\infty[0,1]$. Then equation (4.1) admits a unique bounded solution in $L^\infty(\mathbb{R}^+\times[-1,1])$. Moreover there exists a linear form $G: L^\infty[0,1]\to\mathbb{R}$ and two constants C and $\beta>0$ independent of g such that*

$$|\mu\frac{\partial a(x,\mu)}{\partial x}| + |a(x,\mu)-G(g)| \le Ce^{-\beta x}\|g\|_{L^\infty[0,1]}.$$

We define the response operator as follows. Let a be the solution to (4.1). Then

$$\mathcal{R}^1 \;:\; \begin{array}{rcl} L^\infty(0,1) & \to & L^\infty(-1,0) \\ a_{|\Gamma_-}=g & \mapsto & a_{|\Gamma_+}. \end{array} \tag{4.2}$$

This operator has some properties that we present now. First we have conservation of flux. Indeed no absorption is present in the half space and the flux at infinity tends to zero (since the solution converges to a constant). Therefore we have

$$\int_{-1}^{0} |\mu| \mathcal{R}^1(a_{|\Gamma_-})\, d\mu = \int_0^1 \mu a(0,\mu)\, d\mu. \tag{4.3}$$

This equality is easily deduced from Lemma 4.1 by integrating equation (4.1) over $[0, L] \times [-1, 1]$ and letting L go to infinity. Then $\mathcal{R}^1$ can be extended to an operator $\mathcal{R}^1 : L^1_\mu(\Gamma_-) \to L^1_\mu(\Gamma_+)$ by a density argument.

The linear form G may be written as [6, 13]

$$G(g) \;=\; -D^{-1} \int_{-1}^{1} \mu\theta(\mu) u(0,\mu) d\mu, \tag{4.4}$$

where D and θ are given by (3.8) and (3.7) respectively, and $u(x, \mu)$ is the solution of (4.1). The diffusion coefficient D is positive and θ is bounded in $L^\infty(V)$ (see Section 6). Therefore, using (4.3) we deduce that G can be extended to a linear form in $L^1_\mu[0, 1]$ by density.

Moreover, due to multiple scattering inside the domain, the energy density is positive for every exiting direction. More precisely we have the Doeblin condition

Lemma 4.2 *There exists a positive constant $\delta_0 > 0$ such that for all positive function $h \in L^1_\mu[0, 1]$ we have*

$$\mathcal{R}^1(h) \geq \delta_0 \int_0^1 \mu h(\mu) d\mu \qquad a.e. \; in \; [-1, 0]. \tag{4.5}$$

Proof. We use the integral formulation of the transport equation (4.1). It may be rewritten as

$$\begin{aligned} a(x,\mu) &= \int_0^x ds \frac{1}{\mu} e^{-s/\mu} \int_{-1}^1 d\mu' \sigma(\mu,\mu') a(x-s,\mu') + e^{-x/\mu} g(0,\mu) && \text{for } \mu > 0, \\ &= \int_0^\infty ds \frac{1}{|\mu|} e^{-s/|\mu|} \int_{-1}^1 d\mu' \sigma(\mu,\mu') a(x+s,\mu') && \text{for } \mu < 0. \end{aligned} \tag{4.6}$$

We define the zero scattering term

$$a_0(x,\mu) = e^{-x/\mu} g(0,\mu) \qquad \text{for } \mu > 0, \; x > 0$$

and $a_0(x, \mu) = 0$ elsewhere. The single scattering distribution is defined by

$$\begin{aligned} a_1(x,\mu) &= \int_0^x ds \frac{1}{\mu} e^{-s/\mu} \int_{-1}^1 d\mu' \sigma(\mu,\mu') a_0(x-s,\mu') + e^{-x/\mu} g(0,\mu) && \text{for } \mu > 0, \\ &= \int_0^\infty ds \frac{1}{|\mu|} e^{-s/|\mu|} \int_{-1}^1 d\mu' \sigma(\mu,\mu') a_0(x+s,\mu') && \text{for } \mu < 0. \end{aligned}$$

Then we have that $a \geq a_1 \geq a_0$ a.e. in $I\!R^+ \times [-1, 1]$ since $g(0, \mu) \geq 0$. Moreover, we obtain for $\mu < 0$

$$\begin{aligned} a_1(0,\mu) &= \int_0^\infty ds \frac{1}{|\mu|} e^{-s/|\mu|} \int_0^1 d\mu' \sigma(\mu,\mu') e^{-s/\mu'} g(0,\mu') \\ &= \int_0^1 d\mu' \sigma(\mu,\mu') \frac{\mu'}{|\mu| + \mu'} g(0,\mu') \geq \frac{\sigma_1}{2} ||g||_{L^1_\mu[0,1]} \end{aligned}$$

with σ_1 given in (3.3). This proves the Doeblin condition (4.5) for the response operator. □

An important corollary of this result is that partial reflection can be allowed at the boundary of the domain. Consider the equation

$$\begin{aligned} &\mu\frac{\partial a}{\partial x} + \int_{-1}^{1} \sigma(\mu,\mu')(a(x,\mu) - a(x,\mu'))d\mu' = 0 \qquad \text{in } I\!R^+ \times [-1,1] \\ &a(0,\mu) = R^{11}(a) + g(\mu) \qquad \text{in } (0,1). \end{aligned} \tag{4.7}$$

Then we have

Corollary 4.3 *Assume that R^{11} satisfies Hypothesis 2.1. Then the operator $(I - R^{11}\mathcal{R}^1)^{-1}$ exists and is bounded in $L^1_\mu[0,1]$ and problem (4.7) admits a unique solution defined by its value at the boundary*

$$a_{\Gamma_-} \;=\; (I - R^{11}\mathcal{R}^1)^{-1}\, g.$$

Proof. Consider a non negative function $h \in L^1_\mu(0,1)$ and let $I_1 \in [-1,0]$ be the set on which R^{11} is not totally reflecting (2.9) with the constant η_1. Then we have using (2.9), (2.8) and (4.3)

$$\begin{aligned} \int_0^1 \mu R^{11}\mathcal{R}^1 h d\mu &= \int_0^1 \mu R^{11}(\chi_{I_1}\mathcal{R}^1(h))d\mu + \int_0^1 \mu R^{11}((1-\chi_{I_1})\mathcal{R}^1(h))d\mu \\ &\le (1-\eta_1)\int_{-1}^0 |\mu|\chi_{I_1}\mathcal{R}^1(h)d\mu + \int_{-1}^0 |\mu|(1-\chi_{I_1})\mathcal{R}^1(h)d\mu \\ &\le \int_{-1}^0 |\mu|\mathcal{R}^1(h)d\mu - \eta_1\int_{-1}^0 |\mu|\chi_{I_1}\mathcal{R}^1(h)d\mu \\ &\le (1-\eta_1\delta_0|I_1|_\mu d\mu)\int_0^1 \mu h(\mu)d\mu \end{aligned}$$

with

$$|I|_\mu = \int_I |\mu| d\mu. \tag{4.8}$$

Let $\theta = \eta_1\delta_0|I_1|_\mu$ then we have that for every non negative function $h \in L^1_\mu[0,1]$

$$\|R^{11}\mathcal{R}^1 h\|_{L^1_\mu[0,1]} \le (1-\theta)\|h\|_{L^1_\mu[0,1]}.$$

Consider now any given $h \in L^1_\mu[0,1]$. We decompose it as $h = h^+ - h^-$ where $h^+ = \sup(h,0)$ and $h^- = -\inf(0,h)$ are non negative functions. Since $R^{11}\mathcal{R}^1$ is a positive operator we easily deduce that

$$\begin{aligned} (R^{11}\mathcal{R}^1 h)^+ &= (R^{11}\mathcal{R}^1 h^+ - R^{11}\mathcal{R}^1 h^-)^+ \;\le\; R^{11}\mathcal{R}^1(h^+) \\ (R^{11}\mathcal{R}^1 h)^- &= (R^{11}\mathcal{R}^1 h^+ - R^{11}\mathcal{R}^1 h^-)^- \le\; R^{11}\mathcal{R}^1(h^-). \end{aligned}$$

Hence

$$\begin{aligned} \int_0^1 \mu|R^{11}\mathcal{R}^1(h)|d\mu &= \int_0^1 \mu(R^{11}\mathcal{R}^1 h)^+ + (R^{11}\mathcal{R}^1 h)^- d\mu \\ &\le \int_0^1 \mu R^{11}\mathcal{R}(h^+ + h^-)d\mu = \int_0^1 \mu R^{11}\mathcal{R}(|h|)d\mu \le (1-\theta)\int_0^1 \mu|h|(\mu)d\mu. \end{aligned}$$

Therefore $\|R^{11}\mathcal{R}^1\|_{L^1_\mu(0,1)} \le (1-\theta)$, from which we easily conclude the proof of the result. □

5 Interface Layer Analysis

This section is devoted to the analysis of the interface problem. It reads as follows

$$\begin{aligned} \mu\frac{\partial}{\partial x}a^i + \int_{-1}^{1}\sigma^i(\mu,\mu')(a^i(x,\mu)-a^i(x,\mu'))d\mu' &= 0 \qquad \text{in } X^i\times V \\ a^i(0,\mu) &= \sum_{j=1}^{2}R^{ij}(a^j(0,\mu)) \qquad \text{for } \mu\in\Gamma^i_-. \end{aligned} \tag{5.1}$$

Recall that $X^1 = I\!R^+$, $X^2 = I\!R^-$ and $V = [-1,1]$. Let us write the solution of the transport equation (5.1) defined on both half spaces more synthetically. Let $\mathcal{R}^1$ be defined as in (4.2). The response operator $\mathcal{R}^2$ is defined similarly on Γ_2^-. Define the new set of variables involving the trace of the solution to (5.1) at the interface:

$$a^j_\pm(\mu) = a^j(0,\mu) \quad \text{for } \mu\in\Gamma^j_\pm. \tag{5.2}$$

Accordingly problem (5.1) can be recast as

$$\begin{aligned} a^i_- &= \sum_{j=1}^{2}R^{ij}(a^j_+) \\ a^i_+ &= \mathcal{R}^i(a^i_-). \end{aligned} \tag{5.3}$$

We may rewrite (5.3) using the result of Corollary 4.3 as an equation for a^1_- only. Easy algebra yields

$$a^1_- = (I-R^{11}\mathcal{R}^1)^{-1}R^{12}\mathcal{R}^2(I-R^{22}\mathcal{R}^2)^{-1}R^{21}\mathcal{R}^1a^1_-. \tag{5.4}$$

Define the operator $\mathcal{B}$ by

$$\mathcal{B} = (I-R^{11}\mathcal{R}^1)^{-1}R^{12}\mathcal{R}^2(I-R^{22}\mathcal{R}^2)^{-1}R^{21}\mathcal{R}^1. \tag{5.5}$$

We have the following result:

Theorem 5.1 *The operator $\mathcal{B}$ defined by (5.5) has the following properties.*

1. *It is bounded in $L^1_\mu[0,1]$. Moreover there exists a positive constant δ such that for all positive functions $h\in L^1_\mu(0,1)$ we have*

   $$\mathcal{B}h(\mu) \geq \delta\phi_0(\mu)\int_0^1\mu' h(\mu')d\mu' \qquad \text{a.e. in } [0,1] \tag{5.6}$$

   with the function $\phi_0(\mu) = R^{12}(1)(\mu)$. According to Hypothesis 2.1, $\|\phi_0\|_{L^1_\mu[0,1]} > 0$.

2. *There exists a unique normalized solution a^1_- in $L^1_\mu[0,1]$ of problem (5.4). Moreover this solution is positive and is bounded away from zero.*

3. *The adjoint operator $\mathcal{B}^*$ is also bounded in $L^\infty[0,1]$ and there exists a unique (up to normalization) function $\pi\in L^\infty[0,1]$ satisfying*

   $$\mathcal{B}^*\pi = \pi \tag{5.7}$$

   Moreover for $h\in L^1_\mu[0,1]$ the equation

   $$a = \mathcal{B}a \ + \ h \tag{5.8}$$

admits a unique solution in $L^1_\mu[0,1]$ if and only if

$$\langle \pi, h\rangle = \int_0^1 \mu\pi(\mu)h(\mu)d\mu = 0. \tag{5.9}$$

Proof. Step 1. The operator $\mathcal{B}$ is clearly bounded in $L^1_\mu[0,1]$. The estimate (5.6) is shown as follows. Note that for any positive function $h(\mu) \geq h_0 > 0$ we have

$$R^{12}(h)(\mu) \geq h_0\phi_0(\mu). \tag{5.10}$$

Then given a positive function $h \in L^1_\mu[0,1]$ we have by (5.10) and the Doeblin property (4.5)

$$\mathcal{B}h(\mu) \geq R^{12}\mathcal{R}^2R^{21}\mathcal{R}^1h(\mu) \geq \phi_0(\mu)\delta_0\int_{-1}^0 |\mu|R^{21}\mathcal{R}^1h(\mu)d\mu. \tag{5.11}$$

Then, using the flux conservation (2.8) and partial transmission property (2.9) we have

$$\begin{aligned}\mathcal{B}h(\mu) \geq &\phi_0(\mu)\delta_0\int_{-1}^0 |\mu|R^{21}(\chi_{I_1}\mathcal{R}^1(h))(\mu)d\mu \geq \eta_1\delta_0\phi_0(\mu)\int_{I_1}|\mu|\mathcal{R}^1h(\mu)d\mu\\ &\geq \eta_1\delta_0^2\phi_0(\mu)|I_1|_\mu||h||_{L^1_\mu[0,1]},\end{aligned}$$

which shows that (5.6) holds with $\delta = \delta_0^2\eta_1|I_1|_\mu$ where $|I_1|_\mu$ is given by (4.8).
Step 2. In order to prove the second part of the theorem we introduce the new operator

$$\mathcal{T} = (I - R^{11}\mathcal{R}^1)\mathcal{B}(I - R^{11}\mathcal{R}^1)^{-1} = R^{12}\mathcal{R}^2(I - R^{22}\mathcal{R}^2)^{-1}R^{21}\mathcal{R}^1(I - R^{11}\mathcal{R}^1)^{-1}.$$

The operators $\mathcal{B}$ and $\mathcal{T}$ have the same eigenvalues with the same multiplicity, so it is sufficient to show that $\mathcal{T}$ has a unique positive eigenfunction that corresponds to the eigenvalue $\lambda = 1$. The proof of this fact relies on the conservation of energy flux at the boundary. Indeed we have

$$\int_0^1 \mu R^{21}\mathcal{R}^1(I - R^{11}\mathcal{R}^1)^{-1}(h)d\mu \;=\; \int_0^1 \mu h(\mu)d\mu \quad \text{for all } h \in L^1_\mu(0,1) \tag{5.12}$$

and a similar expression for $R^{12}\mathcal{R}^2(I-R^{22}\mathcal{R}^2)^{-1}$. This follows from the conservations of energy flux (4.3) and (2.8). Then we have

$$\int_0^1 \mu\mathcal{T}(h)d\mu = \int_0^1 \mu h(\mu)d\mu \quad \text{for all } h(\mu) \in L^1_\mu[0,1]. \tag{5.13}$$

Next we observe that $\mathcal{T}$ satisfies the generalized Doeblin condition (5.6):

$$\mathcal{T}(h) \geq R^{12}\mathcal{R}^2R^{21}\mathcal{R}^1(I - R^{11}\mathcal{R}^1)^{-1}h \geq \delta_0\phi_0(\mu)\int_0^1 \mu h(\mu)d\mu \quad \text{a.e. in } (0,1) \tag{5.14}$$

for all $h \in L^1_\mu(0,1)$ such that $h \geq 0$. This follows from (5.10), (4.5) and (5.12).

Let us construct a function b such that $\mathcal{T}b = b$. We start with $b_0 = 1$ and define by iteration $b_{n+1} = \mathcal{T}b_n$. All functions b_n are positive and have the same flux

$$\int_0^1 \mu b_n(\mu)d\mu \;=\; \frac{1}{2}.$$

Therefore they are bounded in $L^1_\mu[0,1]$ and a subsequence converges weakly to b in $L^1_\mu[0,1]$. Since $\mathcal{T}$ is continuous in $L^1_\mu[0,1]$ (for the strong topology hence for the weak one) we readily deduce that

$$\mathcal{T}b \;=\; b \tag{5.15}$$

and $\|b\|_{L^1_\mu[0,1]} = \frac{1}{2}$. The corresponding eigenfunction a^1_- of the operator $\mathcal{B}$ is uniformly positive:

$$a^1_- = (I - R^{11}\mathcal{R}^1)^{-1} b \geq R^{11}\mathcal{R}^1 b \geq \frac{\delta_0 c_1}{2} \tag{5.16}$$

with c_1 defined in (H1). The uniqueness of the eigenvector b is shown as follows. Let us decompose the operator $\mathcal{T}$ as

$$\begin{aligned} \mathcal{T} &= T_1 + T_2 \\ T_1(h) &= \delta_0 \phi_0(\mu) \int_0^1 \mu h(\mu) d\mu \qquad \text{and} \qquad T_2 = \mathcal{T} - T_1. \end{aligned}$$

The operator T_1 is compact and T_2 is a positive operator as seen from (5.14). Moreover, we have

$$\begin{aligned} \int_0^1 \mu |T_2 h|(\mu) d\mu &= \int_0^1 \mu((T_2 h)^+ + (T_2 h)^-) d\mu \leq \int_0^1 \mu(T_2(h^+) + T_2(h^-)) = \int_0^1 \mu T_2(|h|)(\mu) d\mu \\ &\leq \int_0^1 \mu \mathcal{T}(|h|)(\mu) d\mu - \delta_0 \int_0^1 \mu \phi_0(\mu) d\mu \int_0^1 \mu' |h(\mu')| d\mu' \\ &\leq \left(1 - \delta_0 \|\phi_0\|_{L^1_\mu[0,1]}\right) \|h\|_{L^1_\mu[0,1]}. \end{aligned}$$

This proves that $\|T_2\|_{L^1_\mu[0,1]} \leq 1 - \delta_0 \|\phi_0\|_{L^1_\mu[0,1]}$ and so is its spectral radius. Therefore $(I - T_2)$ is invertible and has continuous inverse in $L^1_\mu[0,1]$. Then we may rewrite (5.15) as

$$b = (I - T_2)^{-1} T_1 b = Zb. \tag{5.17}$$

The operator T_1 has a one-dimensional image and the solution b of (5.15) or equivalently (5.17) is necessarily proportional to $(I - T_2)^{-1}\phi_0$. This proves that the eigenvalue $\lambda = 1$ for $\mathcal{T}$, hence for $\mathcal{B}$, is simple.

Step 3. Equation (5.8) admits a unique solution a if and only if equation

$$a' = \mathcal{T} a' + (I - R^{11}\mathcal{R}^1) h \tag{5.18}$$

admits a unique solution $a' = (I - R^{11}\mathcal{R}^1) a$. Equation (5.18) is equivalent to

$$b = (I - T_2)^{-1} T_1 b + (I - T_2)^{-1} (I - R^{11}\mathcal{R}^1) h. \tag{5.19}$$

Here the operator $(I - T_2)^{-1} T_1$ is compact. Let $\pi_{\mathcal{T}} = (I - R^{11}\mathcal{R}^1)^{-1*}\pi$ be the eigenfunction of $\mathcal{T}^*$ corresponding to the eigenvalue $\lambda = 1$ with π given by (5.7). The operator $Z^* = T_1^*(I - T_2^*)^{-1}$ has a unique eigenfunction associated with $\lambda = 1$ given by $\pi_1 = T_1^* \pi_{\mathcal{T}}$. Then the usual Fredholm alternative for compact operators implies that (5.19) admits a solution if and only if

$$\langle \pi_1, (I - T_2)^{-1} (I - R^{11}\mathcal{R}^1) h \rangle = 0. \tag{5.20}$$

This however is equivalent to

$$\langle (I - T_2^*)^{-1} T_1^* \pi_{\mathcal{T}}, (I - R^{11}\mathcal{R}^1) h \rangle = \langle \pi_{\mathcal{T}}, (I - R^{11}\mathcal{R}^1) h \rangle = \langle \pi, f \rangle = 0.$$

This is condition (5.9) stated in the theorem. Moreover we know that this solution is unique up to the addition of Ca^1_-. $\square$

Remark 5.2 *Our proof also provided an explicit expression for the eigenfunction a^1_- of $\mathcal{B}$. The image of the operator T_1 consists of functions proportional to ϕ_0, so a^1_- is given up to renormalization by*

$$a^1_- = (I - R^{11}\mathcal{R}^1)^{-1} b = (I - R^{11}\mathcal{R}^1)^{-1} (I - T_2)^{-1} \phi_0. \tag{5.21}$$

The eigenfunction for the incoming flux on the other side of the interface is given by

$$a^2_- = (I - R^{22}\mathcal{R}^2)^{-1} R^{21}\mathcal{R}^1 a^1_-. \tag{5.22}$$

6 Proof of Convergence

We begin this section with the proof of existence of a unique solution to the transport equation (3.1). Because of the specific nature of the interface conditions, this problem has not been treated in the literature to the best of our knowledge. We consider the more general source problem

$$\begin{aligned}\frac{1}{\varepsilon}\mu\frac{\partial a_\varepsilon^i}{\partial x}+\frac{1}{\varepsilon^2}Q(a_\varepsilon^i)+\Sigma_a a_\varepsilon^i = S(x,\mu) \qquad & \text{in } X_i\times V,\quad i=1,2\\ a_\varepsilon^i \;=\; \sum_{j=1}^{2}R^{ij}(a_\varepsilon^j)+f^i \qquad & \text{for } x=0 \text{ and } \mu\in\Gamma_-^i\end{aligned} \tag{6.1}$$

with $X^1=I\!R^+$, $X^2=I\!R^-$ and $V=[-1,1]$ as in (3.1). Then we have the following result.

Theorem 6.1 *Let the source $S\in L^1(I\!R\times V)$ and the incoming fluxes $f^j\in L^1_\mu(\Gamma_-^j)$. Then the problem (6.1) admits a unique solution in $L^1(I\!R\times[-1,1])$ satisfying the following estimate*

$$\|a_\varepsilon\|_{L^1(I\!R\times V)}\le C\left(\|S\|_{L^1(I\!R\times V)}+\frac{1}{\varepsilon}\sum_{j=1}^{2}\|f^j\|_{L^1_\mu(\Gamma_-^j)}\right). \tag{6.2}$$

Here the constant C is independent of the source terms and of ε.

Proof. We can decompose each source term into positive and negative part and assume without loss of generality that all the source terms are non negative using the linearity of the transport equation. Starting with $a_{\varepsilon,0}=0$, we define iteratively the sequence $a_{\varepsilon,k}$, $k=0,1,\dots$ as the solution to

$$\begin{aligned}\frac{1}{\varepsilon}\mu\frac{\partial}{\partial x}a_{\varepsilon,k+1}^i+(\frac{1}{\varepsilon^2}\Sigma+\Sigma_a)a_{\varepsilon,k+1}^i &= \frac{1}{\varepsilon^2}\int\sigma(\mu,\mu')a_{\varepsilon,k}^i(x,\mu')d\mu'+S(x,\mu) \text{ in } X_i\times V,\quad i=1,2\\ a_{\varepsilon,k+1}^i &= \sum_{j=1}^{2}R^{ij}(a_{\varepsilon,k+1}^j)+f^i \qquad \text{for } x=0 \text{ and } \mu\in\Gamma_-^i.\end{aligned} \tag{6.3}$$

Assuming no flux coming from infinity, it is easily seen that each of these problems admits a unique solution obtained by the method of characteristics (see [13] for instance). Denoting by $\delta a_{\varepsilon,k}^i = a_{\varepsilon,k+1}^i-a_{\varepsilon,k}^i$, we have by induction that $\delta a_{\varepsilon,k}^i$ is non negative. Moreover an integration of (6.3) over $I\!R\times V$ yields

$$\int_{I\!R\times V}(\frac{1}{\varepsilon^2}\Sigma+\Sigma_a)\delta a_{\varepsilon,k+1}^i \;=\; \int_{I\!R\times V}\frac{1}{\varepsilon^2}\Sigma\;\delta a_{\varepsilon,k}^i$$

according to (3.4) and (2.8). Therefore

$$\|\Sigma\delta a_{\varepsilon,k+1}^i\|_{L^1(I\!R\times V)}\le\left(1+\varepsilon^2\frac{\min\{\Sigma_a^{1,2}\}}{\max\{\Sigma^{1,2}\}}\right)^{-1}\|\Sigma\delta a_{\varepsilon,k}^i\|_{L^1(I\!R\times V)}.$$

Then $\delta a_{\varepsilon,k+1}^i$ converges to 0 exponentially fast and therefore $a_{\varepsilon,k+1}^i$ converges in $L^1(I\!R\times V)$ to some function a_ε^i. Using the integral formulation of transport, one verifies that a_ε^i is actually a solution of (6.1) [5]. Moreover by integration of (6.1) over $I\!R\times V$, we deduce the inequality (6.2). This implies the uniqueness of the solution to (6.1). Notice that the rate of convergence of the iterative process presented here becomes slower when $\varepsilon\to 0$. □

The proof of Theorem 3.1 is split in three parts. First we perform a two-scale asymptotic expansion of the solution in powers of ε. This allows us to define a volume equation for the limit. Second we look at the behavior of the transport equation at the interface $x=0$. This enables us to derive matching boundary conditions for the diffusion equation on both parts of the interface. Finally we prove that the volume equation with interface conditions is actually a good approximation of the solution of the transport equation.

Step 1: Derivation of the volume equations. This first part is classical and is given, for instance, in [9, 13]. We recall it here briefly for completeness. Consider the following Ansatz

$$a_\varepsilon^i \;=\; a_0^i + \varepsilon a_1^i + \varepsilon^2 a_2^i + \zeta_\varepsilon^i. \tag{6.4}$$

We insert (6.4) into (3.1) and equate the terms of the same order in ε. Then we have

$$\begin{aligned} Q_i(a_0^i) &= 0 \\ Q_i(a_1^i) &= -\mu \frac{\partial a_0^i}{\partial x} \\ Q_i(a_2^i) &= -\mu \frac{\partial a_1^i}{\partial x} - \Sigma_a^i a_0^i + S^i(x) \end{aligned} \tag{6.5}$$

with the operator Q_i given by (3.2). These equations may be solved using the results of Lemma 1 in Chapter 21, §5 of [13]. The first equation implies that a_0^i is independent of μ. Moreover there exists a solution $\theta^i(\mu) \in L^\infty(V)$ of

$$\theta^i(\mu) - \int_{-1}^{1} \frac{\sigma^i(\mu,\mu')}{\Sigma^i} \theta^i(\mu') d\mu' = -\mu.$$

The solution for the second equation in (6.5) is then given for each $x \neq 0$ by

$$a_1^i = -\frac{\theta^i(\mu)}{\Sigma^i} \frac{\partial}{\partial x} a_0^i + a_{10}^i(x) \tag{6.6}$$

with the function a_{10}^i not being determined yet. We integrate the second equation in (6.5) over $\mu \in [-1,1)]$ and obtain the solvability condition

$$-\frac{\partial}{\partial x} D^i \frac{\partial}{\partial x} a_0^i + \Sigma_a a_0^i = S \tag{6.7}$$

where the diffusion coefficient is defined by

$$D^i = \frac{1}{\Sigma^i} \int_{-1}^{1} \mu \theta^i(\mu) d\mu.$$

We obtain from the same lemma in [13] that D^i is positive.

Step 2: The interface conditions. Now we derive the interface conditions for the diffusion equations (6.7) on the two sides of the interface $x = 0$. If we insert directly the Ansatz (6.4) into the interface conditions then the term of order zero yields

$$a_0^i|_{\Gamma_-^i} \;=\; \sum_{j=1}^{2} R^{ij}(a_0^j|_{\Gamma_+^j}).$$

Since the terms a_0^i do not depend on μ, this equality is not necessarily satisfied for all directions $\mu \in [-1,1]$. The reason is that the solution of a transport equation can be non-isotropic at the vicinity of an interface due to the anisotropic behavior of the operators R^{ij}. However we show below that it is possible to introduce an interface layer which matches the diffusion approximations on both sides of the interface and decays exponentially away from it. Therefore the resulting diffusion approximation is not accurate at the interface, where the error is of order $O(1)$, but is globally accurate since the error is of order $O(\varepsilon)$ in $L^1(I\!R \times V)$. We introduce accordingly the new asymptotic expansion

$$a_\varepsilon^i \;=\; a_0^i + b_0^i(\frac{\cdot}{\varepsilon}, .) + \varepsilon a_1^i + \varepsilon b_1^i(\frac{\cdot}{\varepsilon}, .) + \varepsilon^2 a_2^i + \zeta_\varepsilon^i, \tag{6.8}$$

where the terms b_k^i play the role of interface layers. We perform a change of variables $x \to \dfrac{x}{\varepsilon}$ in (3.1) and formally let ε go to zero. The two first equations in the asymptotic expansion read

$$\begin{aligned}
\mu\frac{\partial}{\partial x}b_0^i(x,\mu) \;&+\; \int_{-1}^{1}\sigma^i(\mu,\mu')(b_0^i(x,\mu) - b_0^i(x,\mu')) = 0\\
\mu\frac{\partial}{\partial x}b_1^i(x,\mu) \;&+\; \int_{-1}^{1}\sigma^i(\mu,\mu')(b_1^i(x,\mu) - b_1^i(x,\mu')) = 0\\
[a_0^i + b_0^i]_{\Gamma_-^i} \;&=\; \sum_{j=1}^{2} R^{ij}[(a_0^j + b_0^j)_{\Gamma_+^j}]\\
\left[-\frac{\theta^i}{\Sigma^i}\frac{\partial}{\partial x}a_0^i + b_1^i\right]_{\Gamma_-^i} \;&=\; \sum_{j=1}^{2} R^{ij}\left[(-\frac{\theta^j}{\Sigma^j}\frac{\partial}{\partial x}a_0^j + b_1^j)_{\Gamma_+^j}\right].
\end{aligned} \tag{6.9}$$

We may rewrite the boundary conditions (6.9) using the response operators $\mathcal{R}^j$, corresponding to media with scattering cross-sections $\sigma^i(\mu,\mu')$, as

$$[a_0^i + b_0^i]|_{\Gamma_-^i} = \sum_{j=1}^{2} R^{ij}\mathcal{R}^j[(a_0^j + b_0^j)_{\Gamma_-^j}] \tag{6.10}$$

and

$$\left[-\frac{\theta^i}{\Sigma^i}\frac{\partial}{\partial x}a_0^i + b_1^i\right]_{\Gamma_-^i} = \sum_{j=1}^{2} R^{ij}\left[\left(-\frac{\theta^i}{\Sigma^j}\frac{\partial}{\partial x}a_0^j\right)_{\Gamma_+^j}\right] + R^{ij}\mathcal{R}^j[b_1^j|_{\Gamma_-^j}]. \tag{6.11}$$

We used here the fact that $\mathcal{R}^i(1) = 1$, which is readily deduced from (4.1) and (4.2). Let us still denote by b_k^j the restriction of b_k^j on Γ_-^i. We may eliminate b_0^2 to obtain an equation for b_0^1 on Γ_-^1:

$$a_0^1 + b_0^1 \;=\; (I - R^{11}\mathcal{R}^1)^{-1}R^{12}\mathcal{R}^2(I - R^{22}\mathcal{R}^2)^{-1}R^{21}\mathcal{R}^1(a_0^1 + b_0^1) = \mathcal{B}(a_0^1 + b_0^1). \tag{6.12}$$

We see that $(a_0^1 + b_0^1)$ is a solution of a problem posed at the interface that involves the response operators of transport equations on both sides of the interface. This interface layer problem was analyzed in Section 5. We deduce from Theorem 5.1 that there exists a constant C so that

$$a_0^1 + b_0^1 \;=\; Ca_-^1,$$

where a_-^1 is the normalized positive solution of the interface problem (5.4) studied in section 5. On the other hand we have

$$[a_0^2 + b_0^2]_{\Gamma_-^2} = (I - R^{22}\mathcal{R}^2)^{-1}R^{21}\mathcal{R}^1(a_0^1 + b_0^1) = C(I - R^{22}\mathcal{R}^2)^{-1}R^{21}\mathcal{R}^1(a_-^1) = Ca_-^2$$

with a_-^2 given by (5.22). The construction of the constant α that appears in the matching conditions in (3.6) in Theorem 3.1 is given explicitly in the following

Lemma 6.2 *Let the constants $G_{1,2}$ be defined by $G_1 = G^1(a_-^1)$ and $G_2 = G^2(a_-^2)$, where G^i is the linear form defined for each domain i in Lemma 4.1 of Section 4. The constants G_1 and G_2 are positive and the matching constant α is defined as*

$$\alpha = \frac{G_2}{G_1}. \tag{6.13}$$

Proof. We want the interface layer to vanish at infinity. Hence

$$a_0^1 \;=\; CG_1 \qquad \text{and} \qquad a_0^2 \;=\; CG_2.$$

Note that (5.21), Lemma 4.2 and (H1) imply that (recall that $\phi_0(\mu) = R^{12}(1)(\mu)$)

$$a_-^1 \geq (I + R^{11}\mathcal{R}^1)\phi_0 \geq c_1\delta_0\|\phi_0\|_{L^1_\mu[0,1]}$$

and similarly

$$a_-^2 \geq c_2\delta_0\|R^{21}(1)\|_{L^1_\mu[-1,0]}.$$

Thus G_1 and G_2 are positive and we obtain the interface condition

$$a_0^2 = \frac{G_2}{G_1}a_0^1 = \alpha a_0^1. \qquad \square \tag{6.14}$$

This is the first interface condition linking a_0^1 and a_0^2. The second interface condition is derived for the normal derivatives of a_0^1 and a_0^2 using the first order interface conditions in (6.11). It is convenient to rewrite them first as follows. Let us define the auxiliary operators

$$M^1 = (I - R^{11}\mathcal{R}^1)^{-1}R^{12}\mathcal{R}^2, \qquad M^2 = (I - R^{22}\mathcal{R}^2)^{-1}R^{21}\mathcal{R}^1,$$

and the functions

$$\tilde{f}^i = (I - R^{ii}\mathcal{R}^i)^{-1}\left(\frac{\theta^i}{\Sigma^i}\frac{\partial}{\partial x}a_0^i + \sum_{j=1}^{2} R^{ij}(-\frac{\theta^i}{\Sigma^j})\frac{\partial}{\partial x}a_0^j\right), \quad i = 1,2,$$

$$f^1 = \tilde{f}^1 + M^1\tilde{f}^2, \qquad f^2 = \tilde{f}^2 + M^2\tilde{f}^1.$$

Then equations (6.11) for the boundary layers b_1^i may be rewritten as

$$b_1^1 = M^1M^2b_1^1 + f^1, \qquad b_1^2 = M^2M^1b_1^2 + f^2. \tag{6.15}$$

Recall that according to Theorem 5.1, the Fredholm alternative is applicable to the operators $\mathcal{B} = M^1M^2$ and $\mathcal{B}' = M^2M^1$. The corresponding left eigenfunctions $\pi_{\mathcal{B}}$ and $\pi_{\mathcal{B}'}$ are related by $\pi_{\mathcal{B}} = M^{2*}\pi_{\mathcal{B}'}$. Therefore we have

$$\langle\pi_{\mathcal{B}}, f^1\rangle = \langle\pi_{\mathcal{B}'}, M^2f^1\rangle = \langle\pi_{\mathcal{B}'}, f^12 - \tilde{f}^2 + \mathcal{B}'\tilde{f}^2\rangle = \langle\pi_{\mathcal{B}'}, f^2\rangle,$$

and the compatibility conditions for the two equations (6.15) coincide. Then we have

$$\int_0^1 \mu\pi_{\mathcal{B}}(\mu)f^1(\mu)d\mu = \langle\pi, f^1\rangle = 0$$

as the only solvability condition for both equations in (6.15). Then there exist two constants G_3 and G_4 such that

$$G_3\frac{\partial a_0^1}{\partial x} = G_4\frac{\partial a_0^2}{\partial x}. \tag{6.16}$$

These constants are given explicitly by

$$G_3 = \langle\pi_{\mathcal{B}}, (I - R^{11}\mathcal{R}^1)^{-1}\left[\theta^1|_{\Gamma_-} + R^{11}(\theta^1|_{\Gamma_+}) + R^{12}\mathcal{R}^2(I - R^{22}\mathcal{R}^2)^{-1}R^{21}(\theta^1|_{\Gamma_+})\right]\rangle \tag{6.17}$$

$$G_4 = \langle\pi_{\mathcal{B}}, (I - R^{11}\mathcal{R}^1)^{-1}R^{12}\left[-\theta^2|_{\Gamma_+} + \mathcal{R}^2(I - R^{22}\mathcal{R}^2)^{-1}\left[\theta^2|_{\Gamma_-} - R^{22}(\theta^2|_{\Gamma_+})\right]\right]\rangle. \tag{6.18}$$

The ratio G_3/G_4 may be computed explicitly. We multiply both equations in (6.11) by μ, integrate the first equation over Γ^1_-, the second over Γ^2_- and subtract them. We obtain

$$\begin{aligned}
&\left(-\int_0^1 \mu\theta^1(\mu)d\mu + \int_0^1 \mu R^{11}(\theta^1)(\mu)d\mu - \int_{-1}^0 \mu R^{21}(\theta^1)(\mu)d\mu\right)\frac{1}{\Sigma^1}\frac{\partial a^1}{\partial x} \\
&+\int_0^1 \mu b_1^1(\mu)d\mu - \int_0^1 \mu R^{11}(b_1^1)(\mu)d\mu + \int_{-1}^0 \mu R^{21}(b_1^1)(\mu)d\mu \\
&= \left(-\int_0^1 \mu R^{12}(\theta^2)(\mu)d\mu - \int_{-1}^0 \mu\theta^2(\mu)d\mu + \int_{-1}^0 \mu R^{22}(\theta^2)(\mu)d\mu\right)\frac{1}{\Sigma^2}\frac{\partial a^2}{\partial x} \\
&+\int_{-1}^0 \mu b_1^2(\mu)d\mu - \int_{-1}^0 \mu R^{22}(b_1^2)(\mu)d\mu + \int_0^1 \mu R^{12}(b_1^2)(\mu)d\mu.
\end{aligned} \tag{6.19}$$

Recall that (4.3) implies that

$$\int_{-1}^1 \mu b_1^1(\mu)d\mu = \int_{-1}^1 \mu b_1^2(\mu)d\mu = 0.$$

We use this and the flux conservation (2.8) to get from (6.19)

$$\left[\int_{-1}^1 \mu\theta^1 d\mu\right]\frac{1}{\Sigma^1}\frac{\partial a_0^1}{\partial x} = \left[\int_{-1}^1 \mu\theta^2 d\mu\right]\frac{1}{\Sigma^2}\frac{\partial a_0^2}{\partial x}.$$

Therefore we have

$$G_3/G_4 = D^1/D^2,$$

the ratio of the diffusion constants from both sides of the interface. This implies that the conservation of the diffusion flux

$$D^1\frac{\partial a_0^1}{\partial x} = D^2\frac{\partial a_0^2}{\partial x}$$

is the correct interface relation. Notice however that the boundary term b_1 does not vanish in general, even though its computation is not necessary here in order to derive the correct interface conditions. Since b_1^1 is defined up to a multiple of a^1_-, and b_1^2 up to a multiple of a^2_-, we choose them so that b_1^j decay exponentially fast away from the interface.

Step 3: Proof of convergence. In order to prove convergence, we show that the remainder term ζ^ε in the Ansatz (6.8) is of order ε. The term a_0 is uniquely defined as a solution to (3.6). Then the terms a_1, a_2 and the interface layers b_0 and b_1 are obtained by solving the equations given in step 2. Here a_1 and a_2 are defined up to the addition of constants that we assume to be zero. Accordingly the equation for ζ^ε is

$$\begin{aligned}
&\frac{1}{\varepsilon}\mu\frac{\partial\zeta^i_\varepsilon}{\partial x} + \frac{1}{\varepsilon^2}Q^i(\zeta^i_\varepsilon) + \Sigma_a\zeta^i_\varepsilon = g^i_\varepsilon \qquad \text{in } X^i\times[-1,1], \quad i=1,2 \\
&\zeta^\varepsilon|_{\Gamma^i_-} = \sum_{j=1}^2 R^{ij}(\zeta^\varepsilon|_{\Gamma^j_+}) - \varepsilon^2(a_2^i|_{\Gamma^i_-} - \sum_{j=1}^2 R^{ij}(a_2^i|_{\Gamma^j_+})) \qquad \text{for } x=0 \text{ and } \mu\in\Gamma^i_-
\end{aligned} \tag{6.20}$$

with the source term g^i_ε given by

$$g^i_\varepsilon(x,\mu) = -\varepsilon^2\mu\frac{\partial a_2^i(x,\mu)}{\partial x} - \Sigma_a\left[\varepsilon a_1^i(x,\mu) + \varepsilon^2 a_2^i(x,\mu) + b_0^i(\frac{x}{\varepsilon},\mu) + \varepsilon b_1^i(\frac{x}{\varepsilon},\mu)\right].$$

Since the source term S is assumed to be smooth, all functions a_k^i are regular and then $\|g^i\|_{L^1(X^i\times V)} \le C\varepsilon$. Therefore Theorem 6.1 implies that the remainder ζ_i^ε is of order $O(\varepsilon)$ in $L^1(I\!R\times V)$. Recall that $a^i_\varepsilon - a^i_0$ is given by (6.8), and the proof of Theorem 3.1 is complete.

7 Examples of interface conditions

Flat Interface. The most natural example of interface conditions arises from the modeling of transmission and reflection of waves incident on a flat interface. In that case, equation (1.6) simplifies to

$$\begin{aligned} \nu^1_{in}(\mathbf{x}',\mathbf{k}') &= \mathcal{F}^{11}(\mathbf{k}')\nu^1_{out}(\mathbf{k}') + \mathcal{F}^{12}(\mathbf{k}')\nu^2_{out}(\mathbf{x}',\mathbf{k}') \\ \nu^2_{in}(\mathbf{x}',\mathbf{k}') &= \mathcal{F}^{22}(\mathbf{k}')\nu^2_{out}(\mathbf{k}') + \mathcal{F}^{21}(\mathbf{k}')\nu^2_{out}(\mathbf{x}',\mathbf{k}'). \end{aligned} \tag{7.1}$$

Here $\mathcal{F}^{ij}$ are the squares of the usual Fresnel coefficients (see [19] for instance). It is convenient to express them as functions of μ and not $\mathbf{k}'$. Let us assume that $v_1 > v_2$, and let the cosines of the angles of reflection and transmission, μ_1 and μ_2, be related by Snell's law

$$v_2\sqrt{1-\mu_1^2} = v_1\sqrt{1-\mu_2^2}, \qquad \mu_{1,2} > 0. \tag{7.2}$$

Then the reflection-transmission coefficients are given by

$$\mathcal{F}^{11}(\mu_1) = \frac{|\zeta_2\mu_1 - \zeta_1\mu_2|^2}{|\zeta_2\mu_1 + \zeta_1\mu_2|^2}, \qquad \mathcal{F}^{21}(\mu_1) = \frac{4\rho_2\zeta_1^2\mu_1^2}{\rho_1(\zeta_1\mu_2+\zeta_2\mu_1)^2} \tag{7.3}$$

and

$$\mathcal{F}^{22}(\mu_2) = \mathcal{F}^{11}(\mu_1), \qquad \mathcal{F}^{12}(\mu_2) = \frac{4\rho_1\zeta_2^2\mu_2^2}{\rho_2(\zeta_1\mu_2+\zeta_2\mu_1)^2} \qquad \text{for } \mu_2 > \mu_2^c = \sqrt{1-\frac{v_2^2}{v_1^2}},$$

and

$$\mathcal{F}^{22}(\mu_2) = 1, \qquad \mathcal{F}^{12}(\mu_2) = 0 \qquad \text{for } \mu_2 \le \mu_2^c.$$

Here $\zeta_i = \rho_i v_i$ is the impedance of medium i and ρ_i is its density. The conservation of flux (2.8) is expressed by the relation

$$v_1\mu_1\mathcal{F}^{11}(\mu_1) + v_2\mu_2\mathcal{F}^{21}(\mu_1) = v_2\mu_2. \tag{7.4}$$

In terms of the transport solution a, the interface conditions are

$$\begin{aligned} a_1(0,\mu_1) &= \mathcal{F}^{11}(\mu_1)a_1(0,-\mu_1) + \frac{\mu_1 v_2}{\mu_2 v_1}\mathcal{F}^{12}(\mu_2)a_2(0,\mu_2), \quad \mu_1 > 0, \\ a_2(0,-\mu_2) &= \mathcal{F}^{22}(\mu_2)a_2(0,\mu_2) + \frac{\mu_2 v_1}{\mu_1 v_2}\mathcal{F}^{21}(\mu_1)a_1(0,-\mu_1), \quad \mu_2 > 0. \end{aligned} \tag{7.5}$$

In the first equality of (7.5), the term μ_2 means $\mu_2(\mu_1)$ whereas in the second equality μ_1 replaces $\mu_1(\mu_2)$.

This practically important case of interface conditions turns out to be fairly simple since the zero order interface layer term b_0 in (6.8) vanishes. Indeed, one may use the explicit form of the reflection and transmission coefficients given above to show that $a_2 = 1$ and $a_1 = v_2^2/v_1^2$ is a solution to (7.5). Therefore no boundary layer is needed to match isotropic solutions of the two sides of the interface. Then $\alpha = v_1^2/v_2^2$ in (3.1) independently of the scattering operators in media 1 and 2.

More generally, whenever the leading order boundary conditions in (6.9) may be satisfied with $b_0^{1,2} = 0$, or otherwise the traces $a^i_\pm$ of the solutions on the boundary are approximately isotropic, we have

$$\begin{aligned} a^1\int_0^1 \mu d\mu &= a^1\int_0^1 \mu R^{11}(1)(\mu)d\mu + a^2\int_0^1 \mu R^{12}(1)(\mu)d\mu \\ a^2\int_{-1}^0 \mu d\mu &= a^2\int_{-1}^0 \mu R^{22}(1)(\mu)d\mu + a^1\int_{-1}^0 \mu R^{21}(1)(\mu)d\mu. \end{aligned}$$

Then the flux conservation (2.8) implies that

$$\alpha = \frac{\|1 - R^{11}(1)\|_{L^1_\mu[0,1]}}{\|1 - R^{22}(1)\|_{L^1_\mu[-1,0]}} \tag{7.6}$$

independent of the scattering operators inside the two media.

Diffuse Transmission. We consider now a more complicated case where the constant α can not be obtained without solving the interface problem. We assume that the reflection operators are defined as in the previous example, and that the transmission operators are isotropically diffusive. Therefore we define

$$R^{11}[\theta](\mu) = \mathcal{F}^{11}(\mu)\theta(-\mu), \qquad \mu > 0$$

and

$$R^{21}(\theta)(\mu) = 2\int_0^1 \mu' R_0^{21}(\theta)(\mu')d\mu' = 2\int_0^1 \mu_2 \frac{\mu_2 v_1}{\mu_1 v_2}\mathcal{F}^{21}(\mu_1)\theta(\mu_1)d\mu_2,$$

where $\mu_1 = \mu_1(\mu_2)$, and R_0^{12} refers to the transmission operator in (7.5). The operators R^{22} and R^{21} are defined similarly. Then Hypothesis 2.1 is satisfied and Theorem 3.1 applies. We assume that both media are homogeneous and that the scattering cross section is isotropic $\sigma^i(\mu, \mu') = 1/2$ and $\Sigma^i = 1$. The response operators $\mathcal{R}^i$ can be computed explicitly using the Chandrasekhar's function H [12, 13].

We consider medium 1 for which the density $\rho_1 = 0.5$ and the velocity $v_1 = 1.2$, and medium 2 with $\rho_2 = 1$ and $v_2 = 1$. The eigenfunctions $a_-^{1,2}$ (see Remark 5.2) of the interface problem are given in Figure 7.1. Notice that a_-^1 is normalized in $L^1_\mu[0,1]$ whereas $a_-^2 = (I - R^{22}\mathcal{R}^2)^{-1}R^{21}\mathcal{R}^1 a_-^1$ is not.

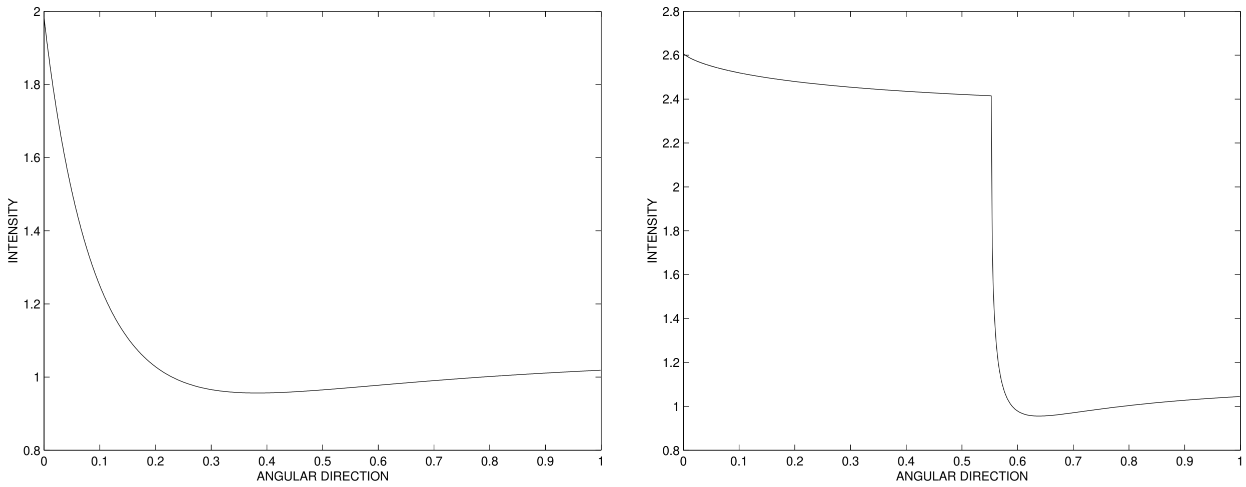

Figure 7.1: Left: Function $g(\mu) = a_-^1(\mu)$ and Right: Function $h(\mu) = a_-^2(-\mu)$

The computations have been performed using a Matlab code on a PC with Linux environment.

The ratio α of Theorem 3.1 and Lemma 6.2 is given by

$$\alpha \;=\; 1.3539. \tag{7.7}$$

If we assume that the diffusion approximation is valid up to the interface, which amounts to considering that the boundary layer terms b_0^i vanish, we obtain that $\alpha = v_1^2/v_2^2 = 1.44$, given by (7.6) since the reflection operators R^{11}, R^{22} are the same as for the flat interface problem.

The estimate (7.6) can be used in practice as an approximation for α. The gain obtained by solving the equation for a^i_- however, is in this case

$$\frac{1.44 - 1.3539}{1.3539} \sim 6.36\%,$$

which may not be negligible for specific applications. Notice also that the error made at the interface by calculating an approximate α propagates in the solution of the diffusion equation at any distance of order 1 from the interface. This simple example, although of no real physical interest, gives an estimate of the error that can be expected when the asymptotically correct interface condition is replaced by an approximation of diffusion.

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