FIRST-ORDER CORRECTOR FOR THE HOMOGENIZATION OF THE CRITICALITY EIGENVALUE PROBLEM IN THE EVEN PARITY FORMULATION OF THE NEUTRON TRANSPORT.

GUILLAUME BAL [†]

Abstract. We consider the homogenization of the criticality eigenvalue problem for the even parity flux of neutron transport in a domain with isotropic and periodically oscillating coefficients. We prove that the neutron density is factored in the product of two terms. The first one describes local behavior of the density at the cell level. It is a solution of a heterogeneous transport problem with periodic boundary conditions. The second term gives global behavior on the whole domain. It satisfies a homogeneous diffusion equation posed on the whole domain with Dirichlet boundary conditions. We also give the asymptotic analysis of the corresponding eigenvalues. This expansion gives rise to errors of the order of the cell size. It does not account for neutron leakage at the boundary of the core and yields unacceptable errors in practice. We derive a more accurate expansion of the eigen-elements in the case of a symmetric and cubic domain. The analysis of a boundary layer allows us to derive modified boundary conditions for the diffusion eigenvalue problem. The resulting approximation for the leading transport eigenvalue is proven to be accurate to one order higher than previously. Numerical experiments confirm the accuracy of the reconstructed eigenvectors in realistic settings.

Key words. neutron transport, eigenvalue problem, homogenization, even parity flux formulation, half space problem.

AMS subject classifications. 35B27

1. Introduction.

Transport equations are solved in industrial applications in order to determine the power distribution of neutrons in nuclear reactors. In the case of a stable reactor, only a steady-state solution is required. Hence the time variable can be eliminated. An eigenvalue problem, called the criticality problem for neutron transport, is solved to figure out whether a steady state exist. The unknowns are the multiplicative factor k_{eff}, which expresses the balance between the production of neutrons by fission and its absorption and leakage at the boundary of the core, and the neutron density $\phi(x, \mu)$, defined in phase space at position x and velocity μ. They are the largest eigenvalue and the associated positive eigenvector of the following equation

$$\mu \cdot \nabla\phi(x,\mu) + \Sigma(x,\mu)\phi(x,\mu) = \int_V \Sigma_s(x,\mu',\mu)\phi(x,\mu')d\mu' + \frac{1}{k_{eff}}\int_V \sigma_f(x,\mu',\mu)\phi(x,\mu')d\mu' \tag{1}$$

posed with appropriate boundary conditions in an open bounded domain $\Omega \in \mathbb{R}^d$, for a velocity space V. With the usual notation, $\mu\cdot\nabla = \sum_{i=1}^d \mu_i \frac{\partial}{\partial x_i}$, where $\mu = (\mu_1, \ldots, \mu_d)$ and $d \in \mathbb{N}^*$ is the spatial dimension. It turns out that the eigenvector $\phi(x, v)$ is the only positive normalized eigenvector of (1), hence the only one of physical interest. We have the following interpretation for the multiplicative factor k_{eff}. If $k_{eff} = 1$, the core is stable and fission exactly compensates absorption and leakage. When $k_{eff} > 1$, fission is too important and the reactor is super-critical. If $k_{eff} < 1$, fission must be increased, or the chain reaction dies out.

In equation (1), Σ, Σ_s, and σ_f are the total cross section, the scattering cross section, and the fission cross section respectively. They characterize the nuclear reactors, which are highly heterogeneous. Therefore, numerical simulations with the transport equation are very demanding.

[†]Électricité de France DER/IMA/MMN, 92141 Clamart, France and Laboratoire d'Analyse Numérique, Université Paris-VI, 75252 Paris Cedex 5, France. Current address: Stanford University, Department of Mathematics, Stanford CA 94305 (bal@math.Stanford.EDU).

A first approximation consists in assuming that the core is periodic. It allows us to homogenize the transport equation (1), which yields a homogeneous second-order elliptic eigenvalue problem. We consider here a physical domain Ω^ε composed of roughly $\frac{1}{\varepsilon^d}$ identical cells. The cross sections Σ, Σ_s and σ_f are supposed to be isotropic, i.e. only depend on the spatial position x, and to be $Y-$periodic in the domain Ω^ε, where Y is the unit cell. For simplicity, the velocity space V will be here the unit sphere $V = S^{d-1} = \{\mu \in I\!R^d,\, |\mu| = 1\}$.

The homogenization of transport equations has been studied at length in the past, physically [12, 15, 21] and mathematically [14, 26, 27, 31, 33]. We have recently revisited the criticality eigenvalue problem in [4, 5, 8]. To the best of our knowledge, no theory is available for the homogenization of eigenvalue problems with periodic coefficients taking account for neutron leakage at the boundary of the core. In the same spirit, let us mention the works of [6, 7, 29] concerning the homogenization of heterogeneous diffusion eigenvalue problems. Notice that various result on the homogenization of transport equations have been obtained in different contexts [1, 22, 23].

The theory of transport is usually done using equation (1). However, it is also interesting to analyze the so-called *even parity flux formulation*. The *even parity flux* is defined by

$$\psi^+(x,\mu) = \frac{1}{2}(\phi(x,\mu) + \phi(x,-\mu)).$$

When no direction is privileged with respect to its opposite one, such as in the diffusion limit, this symmetrized flux can be of interest. We deduce from (1) and Appendix A the following equation

$$(2)\quad -\mu\cdot\nabla\frac{1}{\Sigma(x)}\mu\cdot\nabla\psi^+(x,\mu) + \Sigma(x)\psi^+(x,\mu) \;=\; (\Sigma_s(x) + \frac{1}{k^\varepsilon_{eff}}\sigma_f(x))\int_V \psi^+(x,\mu')d\mu' \quad \text{in } \Omega^\varepsilon\times V.$$

The derivation of this symmetrized transport equation requires that the cross sections be isotropic. By definition of the even parity flux, we have $\psi^+(x,-\mu) = \psi^+(x,\mu)$. Consequently we only need to consider solutions of (2) satisfying this symmetry condition. It is well known that the homogenization of transport equations yields homogeneous second order elliptic equations. Thus, the second order differential operator in the even parity formulation is similar to that of diffusion. More precisely, we will see that the variational formulation associated with the transport equation (2) is well suited to the derivation of the variational formulation of diffusion, and simplifies the analysis of the leakage at the boundary of the core and of the associated boundary layer problem.

Let us now define our main framework. It is convenient to recast the sequence of problems (2), parameterized by ε, on a fixed domain. By change of variables $x \to \varepsilon x$, the spatial domain becomes Ω. Since the core is periodic, Ω is composed of roughly ε^{-d} identical cells of the kind $\varepsilon Y = (0,\varepsilon)^d$ where $Y = (0,1)^d$ is the unit cell. The period of the heterogeneities in Ω is thus given by $\varepsilon > 0$. No further geometrical assumption is needed when no leakage is accounted for. However, in order to derive a more accurate approximation for the leading eigenvalue, neutron leakage at the boundary cannot be ignored. We have to make more precise the local geometry at the boundary of the core. We assume here that Ω is the unit cube and that ε^{-1} is an integer. The main characteristic is that the boundary of Ω always coincides with the boundary of a unit cell.

Let us define

$$(3)\qquad \Sigma^\varepsilon(x) = \Sigma(\frac{x}{\varepsilon}),\ \Sigma^\varepsilon_s(x) = \Sigma_s(\frac{x}{\varepsilon}),\ \sigma^\varepsilon_f(x) = \sigma_f(\frac{x}{\varepsilon}),$$

where Σ, Σ_s and σ_f are $Y-$ periodic measurable functions, bounded from above and below by positive constants and such that the absorption cross section $\Sigma - \Sigma_s$ be also bounded from below by a positive constant. The latter condition ensures that absorption is positive everywhere, hence fission is allowed. For simplicity, we introduce $\lambda_\varepsilon = \frac{1}{k_{eff}}$. We consider here an absorbing boundary condition. It means that no particle enters the core. The solution to (1) satisfies

$$\phi = 0 \text{ on } \Gamma_- = \{(x,\mu) \in \partial\Omega \times V \text{ s.t. } \mu \cdot n(x) < 0\},$$

where $n(x)$ denotes the outward unit normal to $\partial\Omega$ at $x \in \partial\Omega$. In practice, this boundary conditions is a first approximation only. The addition of a reflector around the core yields a correction of order ε. Its analysis is much more involved and is not considered here.

Let us define $\varphi^{\varepsilon+}(x,\mu) = \psi^+(\frac{x}{\varepsilon},\mu)$. These assumptions enable us to recast (2) as:

$$\begin{aligned}
&-\varepsilon^2\mu\cdot\nabla\frac{1}{\Sigma^\varepsilon(x)}\mu\cdot\nabla\varphi^{\varepsilon+}(x,\mu) + \Sigma^\varepsilon(x)\varphi^{\varepsilon+}(x,\mu) = (\Sigma_s^\varepsilon(x) + \lambda^\varepsilon\sigma_f^\varepsilon(x))\int_V \varphi^{\varepsilon+}(x,\mu')d\mu' \quad \text{in } \Omega\times V \\
&\varphi^{\varepsilon+}(x,\mu) - \frac{\varepsilon}{\Sigma^\varepsilon(x)}\mu\cdot\nabla\varphi^{\varepsilon+}(x,\mu) = 0 \quad \text{on } \Gamma_-.
\end{aligned} \tag{4}$$

The boundary conditions have been obtained following rules recalled in Appendix A.

An outline of this paper is as follows. In section 2, we recall existing results on problem (4) at ε given. We also address the transport problem with periodic boundary conditions, which characterizes the neutron density at the small scale. In section 3, we present our main results on the asymptotic behavior of the eigenvalues and eigenvectors of (4) as $\varepsilon \to 0$. First an analysis is given without accounting for the leakage at the boundary of the core. Second, a first-order correction of the previous results is given for the largest eigenvalue k_{eff}. This correction characterizes the amount of neutron leakage at the boundary of a nuclear reactor. We give in sections 4 to 6 a detailed proof of these results. In section 4, we give some a priori estimates and existence results with ε fixed. Our results are based on the analysis of a source problem introduced in section 3. The asymptotic expansion of this problem is given in section 5. The analysis of a genuine multi-dimensional boundary layer problem, or Milne problem, that is used in section 5, is given in section 6. At last, we present a numerical experiment in section 7.

2. The criticality eigenvalue problem in bounded and periodic domain. In this section, we state some results of existence and regularity for the eigenvalues and eigenvectors of the even parity transport. They are very close to known results on eigenvalue problems in transport theory (see [20, Chapter 21]). We recall them here for the sake of completeness. The proofs are sketched only and we refer to the thesis [8] for the details and to the equivalence between the first-order formulation and the even parity flux formulation given in Appendix A. Our main hypothesis on the physical data are given now.

HYPOTHESIS 2.1.

(H1) *Ω is a convex open bounded subset of $\mathbb{R}^d$.*
(H2) *$V = S^{d-1} = \{v \in \mathbb{R}^d$ s.t. $|v| = 1\}$.*
(H3) *The cross sections $\Sigma(x)$, $\Sigma_s(x)$ and $\sigma_f(x)$ are positive functions in $L^\infty(\Omega)$. Moreover they are $Y-$ periodic, where $Y = (0,1)^d$ is the periodicity cell.*
(H4) *There exists a constant $\eta > 0$ such that $\Sigma(x) - \Sigma_s(x) \geq \eta$ and $\sigma_f(x) \geq \eta$.*

Let us introduce the Hilbert space

$$W^2(\Omega\times V) = \{u \in L^2(\Omega\times V),\ \mu\cdot\nabla u \in L^2(\Omega\times V)\}. \tag{5}$$

We deduce from [8, Theorem II.2.1.1] and Appendix A the following result.

THEOREM 2.2. *Assume that Hypothesis 2.1 is satisfied. Then problem (4) admits a countable number of real eigenvalues and of associated eigenvectors, which are elements of $W^2(\Omega \times V)$. Moreover there exists a simple, positive and real eigenvalue of smallest modulus, such that its associated eigenvector be the unique normalized positive eigenvector of (4).*

The solutions of (4) can be seen as the eigenvalues and eigenvectors of a positive compact operator. The reality of these eigenvalues, which holds true in the simplified setting of isotropic cross sections, is given in [34] for instance. The first part of the theorem is then proven. The second part relies on the Krein Rutman theory of positive operators, which asserts that the spectral radius of this compact operator is an eigenvalue and that the corresponding eigenvector is positive. Following a proof given in [20, chapter 21], one proves that this eigenvalue is simple and that there exists a unique positive normalized eigenvector.

As we shall see in the next section, the asymptotic limit as $\varepsilon \to 0$ of the solutions to (4) involves the small scale behavior of the neutron density. It is obtained by considering the solutions of the following criticality eigenvalue problem in periodic domain:

Find the smallest eigenvalue λ_∞ and the associated positive eigenvector ψ^+_∞ of

$$\begin{aligned} -\mu\cdot\nabla_y \frac{1}{\Sigma}\mu\cdot\nabla_y \psi^+_\infty + \Sigma\psi^+_\infty &= (\Sigma_s + \lambda_\infty \sigma_f) \int_V \psi^+_\infty(y,\mu')d\mu' \quad \text{in } Y \times V \\ y \mapsto \psi^+_\infty(y,\mu) \;\; &\text{is } Y-\text{periodic}. \end{aligned} \tag{6}$$

Still following the equivalence between the first-order formulation and the even parity flux formulation given in Appendix A, we deduce from [8, Theorem II.2.2.4] and [8, Theorem II.2.2.7] the

THEOREM 2.3. *Assume that Hypothesis 2.1 is satisfied. Then problem (6) admits a simple, positive and real eigenvalue of smallest modulus, such that its associated eigenvector be the unique normalized positive eigenvector of (6). Moreover assume that the cross sections Σ, Σ_s and σ_f belong to $C^m(Y)$ for $m \in \mathbb{N}^*$. Then we have that $\psi^+_\infty \in H^m(Y \times V)$.*

REMARK 2.4. *The regularity result stated in the previous theorem is a characteristic property of the criticality problem with periodic boundary conditions. It is well known that the solutions to (4) are not arbitrarily regular, even with smooth physical data (see [30]). For example, we have a non vanishing outgoing density at the boundary of the core, whereas the incoming density is zero by hypothesis. Let us also mention the following regularity result [5]. Assume that the cross sections are bounded from above and below by positive constants. Then ψ^+_∞ and $\frac{1}{\psi^+_\infty}$ belong to $L^\infty(Y \times V)$.*

Consider the associated source problem:

$$\begin{aligned} -\mu\cdot\nabla_y \frac{1}{\Sigma}\mu\cdot\nabla_y \psi^+ + \Sigma\psi^+ &= (\Sigma_s + \lambda_\infty \sigma_f) \int_V \psi^+(y,\mu')d\mu' + S \quad \text{in } Y \times V \\ y \mapsto \psi^+(y,\mu) \;\; &\text{is } Y-\text{periodic}, \end{aligned} \tag{7}$$

where S is a given function in $L^2(Y \times V)$ satisfying $S(y,\mu) = S(y,-\mu)$. Then we deduce from [8, Theorem II.2.2.5] and [8, Theorem II.2.2.8] the following result.

THEOREM 2.5. *Let be $(\lambda_\infty, \psi^+_\infty)$ the solution to (6). Then problem (7) admits a solution if and only if the source term S satisfies the following compatibility condition:*

$$\int_Y \int_V S(y,\mu)\psi^+_\infty(y,\mu)\, d\mu\, dy = 0. \tag{8}$$

Furthermore, if a solution exists, it is unique up to the addition of a multiple of ψ_∞^+. *Assume moreover that the cross sections* Σ, Σ_s *and* σ_f *are of class* $C^m(Y)$ *and the source term* S *belongs to* $H^m(Y \times V)$ *for* $m \in \mathbb{N}^*$. *Then we have that* $\psi^+ \in H^m(Y \times V)$.

3. Main results on the Asymptotic analysis. In this section, we present our main results on the asymptotic analysis of the criticality eigenvalue problem (4). Let us introduce the function $\psi_\varepsilon^+(x,\mu) = \psi_\infty^+(\frac{x}{\varepsilon},\mu)$, where ψ_∞^+ is the positive normalized eigenvector of (6) extended by Y-periodicity on $\mathbb{R}^d \times V$. Clearly ψ_ε^+ is εY-periodic and satisfies equation (4) on $\Omega \times V$. The difference between ψ_ε^+ and $\varphi^{\varepsilon+}$ is the definition of their boundary conditions on $\partial\Omega$. Therefore we can expect some similarities in the behavior of both solutions away from the boundary. This is confirmed more precisely in Theorem 3.2 below. It is based on the asymptotic expansion of solutions to source problems, which requires sufficient regularity for the physical parameters. We do not dwell on optimal regularity results here and assume the physical parameters to be smooth.

HYPOTHESIS 3.1.

(H5) *In addition to Hypothesis 2.1, we assume the cross sections* Σ, Σ_s *and* σ_f *to be of class* $C^\infty(\Omega)$.

(H6) *The domain* Ω *is either cubic, or has a boundary* $\partial\Omega$ *of class* C^∞.

THEOREM 3.2. *Assume that Hypothesis 3.1 is satisfied. Let be* $0 < \lambda_1^\varepsilon < \lambda_2^\varepsilon \le \lambda_3^\varepsilon \le \ldots \le \infty$ *the eigenvalues of (4) ranked in increasing order, and* $\varphi_l^{\varepsilon+}$ *the normalized eigenvector associated with* λ_l^ε. *Then, up to a subsequence, we have*

$$\varphi_l^{\varepsilon+}(x,\mu) = u_l(x)\psi_\infty^+(\frac{x}{\varepsilon},\mu) + O(\varepsilon) \quad \text{and} \quad \lambda_l^\varepsilon = \lambda_\infty + \varepsilon^2\nu_l + O(\varepsilon^3), \tag{9}$$

where λ_∞ *is the first eigenvalue of (6),* ψ_∞^+ *its associated normalized eigenvector and where* (ν_l, u_l) *are the* l^{th} *eigenvalue (in the sense that* $0 < \nu_1 < \nu_2 \le \nu_3 \le \ldots \le \infty$*) and corresponding normalized eigenvector of the homogenized diffusion problem*

$$\begin{aligned} -\nabla D \nabla u &= \nu \overline{\sigma_f} u \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{10}$$

In the convergence of the eigenvectors, $O(\varepsilon)$ *is to be understood in the sense of the* $L^2(\Omega \times V)-$*norm. The positive definite homogeneous tensor* $D = (D_{ij})_{1\le i,j\le d}$ *and the homogeneous fission cross section* $\overline{\sigma_f}$ *in (10) are defined by*

$$\begin{aligned} D_{ij} &= \int_V \int_Y \frac{(\psi_\infty^+(y,\mu))^2}{\Sigma(y)} \left(\mu_i + \mu\cdot\nabla\theta^i(y,\mu)\right)\mu_j d\mu\, dy \\ \overline{\sigma_f} &= \int_Y \sigma_f(y) \left(\int_V \psi_\infty^+(y,\mu) d\mu\right)^2 dy, \end{aligned} \tag{11}$$

where $(\mu_i)_{1\le i\le d}$ *are the coordinates of* μ *in* $\mathbb{R}^d$ *and* $(\theta^i)_{1\le i\le d}$ *are the zero mean solutions of*

$$\begin{aligned} &-\mu\cdot\nabla_y \frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla_y\theta^i + Q\theta^i = \mu_i\left(\mu\cdot\nabla_y\frac{(\psi_\infty^+)^2}{\Sigma}\right) \\ &y \mapsto \theta^i(y,\mu) \ \text{ is } Y-\text{ periodic} \end{aligned} \tag{12}$$

with the local scattering operator being defined by

$$\begin{aligned} Qw &= \Sigma^\infty\left(\psi_\infty^+ w \int_V \psi_\infty^+(x,\mu')d\mu' - \psi_\infty^+ \int_V w(x,\mu')\psi_\infty^+(x,\mu')d\mu'\right) \\ \Sigma^\infty &= \Sigma_s + \lambda_\infty\sigma_f. \end{aligned} \tag{13}$$

Due to the even parity formulation, the homogeneous coefficients given in this theorem are non-standard. However they are similar to those obtained in the homogenization of heterogeneous diffusion problems [29], and it can be shown [8] that they correspond to those derived in the setting of first-order transport [5, 26, 27] . Notice however that they slightly differ from the coefficients derived physically in [12].

REMARK 3.3. *Hypothesis* (H6) *is not optimal. We do not need it to prove the convergence of the eigenvalues and eigenvectors, i.e. to replace the rates of convergence* $O(\varepsilon)$ *and* $O(\varepsilon^3)$ *in (9) by* $o(1)$ *and* $o(\varepsilon^2)$ *respectively. Hypothesis* (H5) *can also be considerably weakened. We obtained in [5] the convergence of the eigenelements for cross sections that are only uniformly bounded from above and below by positive constants.*

In practical nuclear reactor computations, the number of assemblies, or equivalently ε^{-1}, is not very large and the expansion given in the previous theorem is not sufficiently accurate. The aim of the next theorem is to give a third order corrector for the smallest eigenvalue λ_1^ε, the only one of physical interest. Numerical simulations also show a better accuracy for the associated positive eigenvector [9], even if a theoretical proof is given only for source problems.

First order correctors have already been addressed in homogeneous transport theory [33] as well as in heterogeneous diffusion theory [3, 32]. The results obtained in these work show that the first-order corrector for the eigenvalues can strongly depend on the geometry. In transport theory, the first-order corrector is driven by the neutron leakage at the boundary of the core. Since the neutron mean free path is comparable to the size of the unit cell, the boundary $\partial\Omega$ has a direct influence on the leakage. We need to define our geometry more precisely in order to obtain an asymptotic neutron leakage. We study here the case of a cubic domain with symmetric cells:

HYPOTHESIS 3.4.

(H7) Ω *is the unit cube* $(0,1)^d$. *It is composed of* N^d *identical cells, where* $N = \frac{1}{\varepsilon} \in \mathbb{N}$.

(H8) *Hypotheses* (H5) *and* (H6) *are satisfied.*

(H9) *The periodicity cell* Y *is symmetric in the following sense. The cross sections* Σ, Σ_s *and* σ_f *are symmetric with respect to the (hyper)planes parallel to the sides of* Y *and splitting* Y *in two identical parts. Furthermore these cross sections are invariant by the rotations that preserve* Y *(rotations of angle* $\pi/2$*).*

Then we have the following first-order corrector for the smallest eigenvalue.

THEOREM 3.5. *Assume that Hypothesis 3.4 is satisfied. Then there exists a constant extrapolation length* L *such that the first eigenvalue* ω_1^ε *of the elliptic problem*

$$\begin{aligned} -\nabla D \nabla \Phi^\varepsilon &= \omega^\varepsilon \overline{\sigma_f} \Phi^\varepsilon \qquad \text{in } \Omega \\ \Phi^\varepsilon + \varepsilon L \frac{\partial \Phi^\varepsilon}{\partial n} &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{14}$$

satisfy

$$\lambda_1^\varepsilon = \lambda_\infty + \varepsilon^2 \omega_1^\varepsilon + O(\varepsilon^{7/2}). \tag{15}$$

REMARK 3.6. *To our knowledge, this extrapolation length can unfortunately not be computed explicitly. In the case of homogeneous cells, the value of* L *can be derived from Chandrasekhar's* H *function [18]. An approximate value is* $L_0 = 0.7104$ *(see e.g. [20, Chapter 21]). In general,* L *is* *defined as the limit when the first coordinate $x_1 \to \infty$ of the solution to a conservative transport problem posed on the half space $x_1 > 0$ with some suitable boundary conditions. The analysis of this problem is part of our section 6. Since the neutron leakage is positive, we believe that L is always positive, although we do not have a mathematical proof for it. In case $L < 0$, which is physically unrealistic, (14) is well posed for ε small enough only.*

REMARK 3.7. *Up to some slight modifications in the proof, the theorem can be extended to the case of cells satisfying all symmetries stated in* (H9) *but the invariance by rotation. Then L is constant on each side of $\partial\Omega$, but not necessarily on the whole $\partial\Omega$.*

We now give a proof for Theorems 3.2 and 3.5. They will rely on some results interesting in themselves for related source problems. We present them in the remaining part of this section and postpone the proofs to the following sections. The proof of Theorem 3.2 relies on the analysis of an equivalent eigenvalue problem for the factored function u_ε defined by

$$u_\varepsilon = \frac{\varphi^{\varepsilon+}}{\psi_\varepsilon^+}, \tag{16}$$

where $\varphi^{\varepsilon+}$ is a solution of (4). Since ψ_∞^+ is positive and regular by virtue of Theorem 2.3, $\varphi^{\varepsilon+} \mapsto u_\varepsilon$ is uniquely defined by (16). The derivation of a transport equation for u_ε uses the following identity

$$\mu\cdot\nabla\frac{1}{\Sigma}\mu\cdot\nabla(u\psi) = u\mu\cdot\nabla\frac{1}{\Sigma}\mu\cdot\nabla\psi + \frac{1}{\psi}\mu\cdot\nabla\frac{\psi^2}{\Sigma}\mu\cdot\nabla u. \tag{17}$$

An analogous relation was first used in the homogenization of heterogeneous diffusion eigenvalue problems [29]. Plugging (16) into (4) and using (6) and (17), we obtain

$$\begin{aligned} -\varepsilon^2\mu\cdot\nabla\frac{(\psi_\varepsilon^+)^2}{\Sigma^\varepsilon}\mu\cdot\nabla u_\varepsilon + \Sigma_s^\varepsilon\left(\psi_\varepsilon^+ u_\varepsilon\int_V \psi_\varepsilon^+(x,\mu')d\mu' - \psi_\varepsilon^+\int_V u_\varepsilon(x,\mu')\psi_\varepsilon^+(x,\mu')d\mu'\right) \\ = \lambda^\varepsilon\sigma_f^\varepsilon\psi_\varepsilon^+\int_V u_\varepsilon(x,\mu')\psi_\varepsilon^+(x,\mu')d\mu' - \lambda_\infty\sigma_f^\varepsilon u_\varepsilon\psi_\varepsilon^+\int_V \psi_\varepsilon^+(x,\mu')d\mu' \quad \text{in } \Omega\times V. \end{aligned} \tag{18}$$

Let us now derive the boundary conditions satisfied by u_ε. We deduce from (4) and (16) that

$$u_\varepsilon(\psi_\varepsilon^+ - \frac{\varepsilon}{\Sigma^\varepsilon}\mu\cdot\nabla\psi_\varepsilon^+) - \frac{\varepsilon}{\Sigma^\varepsilon}\psi_\varepsilon^+\mu\cdot\nabla u_\varepsilon = 0 \qquad \text{on } \Gamma_-.$$

Introduce $\psi_\varepsilon = \psi_\varepsilon^+ - \frac{\varepsilon}{\Sigma^\varepsilon}\mu\cdot\nabla\psi_\varepsilon^+$. From the equivalence presented in Appendix A, ψ_ε is the solution of the first-order criticality eigenvalue problem

$$\frac{1}{\varepsilon}\mu\cdot\nabla\psi_\varepsilon + \frac{1}{\varepsilon^2}\left(\Sigma^\varepsilon\psi_\varepsilon - \Sigma_\infty^\varepsilon\psi_\varepsilon\right) = 0, \tag{19}$$

where $\Sigma_\varepsilon^\infty = \Sigma_s^\varepsilon + \lambda_\infty\sigma_f^\varepsilon$. It is positive (see [5] and [8, Theorem II.2.1.1]). Thus we obtain the boundary conditions for u_ε

$$u_\varepsilon - \frac{\varepsilon\psi_\varepsilon^+}{\psi_\varepsilon\Sigma^\varepsilon}\mu\cdot\nabla u_\varepsilon = 0 \qquad \text{on } \Gamma_-\,. \tag{20}$$

This enables us to recast equation (18) as

$$\begin{aligned} A_\varepsilon u_\varepsilon &= \nu^\varepsilon F_\varepsilon u_\varepsilon \quad \text{in } \Omega\times V \\ u_\varepsilon - \frac{\varepsilon\psi_\varepsilon^+}{\psi_\varepsilon\Sigma^\varepsilon}\mu\cdot\nabla u_\varepsilon &= 0 \quad \text{on } \Gamma_-, \end{aligned} \tag{21}$$

where we have defined

$$A_\varepsilon u = -\mu\cdot\nabla\frac{(\psi_\varepsilon^+)^2}{\Sigma^\varepsilon}\mu\cdot\nabla u + \frac{1}{\varepsilon^2}Q_\varepsilon u \tag{22}$$

$$\nu^\varepsilon = \frac{\lambda^\varepsilon - \lambda_\infty}{\varepsilon^2} \tag{23}$$

$$Q_\varepsilon u = \Sigma_\varepsilon^\infty\left(\psi_\varepsilon^+ u\int_V \psi_\varepsilon^+(x,\mu')d\mu' - \psi_\varepsilon^+\int_V u(x,\mu')\psi_\varepsilon^+(x,\mu')d\mu'\right) \tag{24}$$

$$F_\varepsilon u = \psi_\varepsilon^+\sigma_f^\varepsilon\int_V \psi_\varepsilon^+(x,\mu')\,u(x,\mu')d\mu'. \tag{25}$$

Following the results given in section 2, ψ_ε^+ is smooth and uniformly positive. Thus problems (4) and (21) are equivalent in $W^2(\Omega\times V)$. We write (21) as

$$\begin{aligned}\frac{1}{\nu^\varepsilon}u_\varepsilon &= S_\varepsilon u_\varepsilon\\ S_\varepsilon &= A_\varepsilon^{-1}F_\varepsilon.\end{aligned} \tag{26}$$

We will see in Theorem 3.9 that $S_\varepsilon \in \mathcal{L}(L^2(\Omega\times V))$. The first step in the asymptotic analysis of problem (26), or equivalently (4), consists in studying the following source problem

$$\begin{aligned}&A_\varepsilon w_\varepsilon = F_\varepsilon q &&\text{in } \Omega\times V\\ &w_\varepsilon - \frac{\varepsilon\psi_\varepsilon^+}{\psi_\varepsilon\Sigma^\varepsilon}\mu\cdot\nabla w_\varepsilon = 0 &&\text{on } \Gamma_-\end{aligned} \tag{27}$$

where $q(x,\mu)\in L^2(\Omega\times V)$ is a given source term satisfying $q(x,\mu)=q(x,-\mu)$. Let us recall that we are interested only in solutions of the form $w_\varepsilon(x,-\mu)=w_\varepsilon(x,\mu)$. We first state an a priori estimate for the solutions of source problems.

LEMMA 3.8. *Let be $q(x,\mu)\in L^2(\Omega\times V)$ and $g\in L^2(\Gamma_-,d\xi)$. Assume that w_ε is a solution to*

$$\begin{aligned}&A_\varepsilon w_\varepsilon = F_\varepsilon q &&\textit{in } \Omega\times V\\ &w_\varepsilon - \frac{\varepsilon\psi_\varepsilon^+}{\psi_\varepsilon\Sigma^\varepsilon}\mu\cdot\nabla w_\varepsilon = g &&\textit{on } \Gamma_-.\end{aligned} \tag{28}$$

Then we have

$$\|\mu\cdot\nabla w_\varepsilon\| + \|w_\varepsilon\| + \frac{1}{\sqrt{\varepsilon}}\|w_\varepsilon\|_{L^2(\partial\Omega\times V,d\xi)} + \frac{1}{\varepsilon}\|w_\varepsilon - \int_V w_\varepsilon\| \le C\|q\| + \frac{C}{\sqrt{\varepsilon}}\|g\|_{L^2(\Gamma_-,d\xi)}, \tag{29}$$

where $\|\cdot\|$ is the $L^2(\Omega\times V)$ norm and where the measure $d\xi$ on $\partial\Omega\times V$ is given by $d\xi = |\mu\cdot n(x)|d\mu d\sigma$, with $d\sigma$ the surface measure on $\partial\Omega$.

The proof of this lemma is given in section 4. This energy estimate is a key result in our analysis. It enhances the interest of the variational formulation associated with the even parity flux formulation. The asymptotic behavior of the source problem is stated in the following result.

THEOREM 3.9. *Assume that Hypothesis 3.1 is satisfied. Let be $q\in L^2(\Omega\times V)$ a given source term. Then problem (27) admits a unique solution w_ε, which converges strongly to $w\in H_0^1(\Omega)$ in $L^2(\Omega\times V)$ as $\varepsilon\to 0$, where w is the solution of the diffusion problem*

$$\begin{aligned}-\nabla D\nabla w &= \overline{q} &&\textit{in } \Omega\\ w &= 0 &&\textit{on } \partial\Omega.\end{aligned} \tag{30}$$

Here we have

$$\overline{q}(x) = \int_Y \sigma_f(y) \left\{ \int_V \psi_\infty^+(y,\mu')d\mu' \int_V \psi_\infty^+(y,\mu) q(x,\mu) d\mu \right\} dy.$$

Moreover assume that there exists $M \in I\!N$ such that $q(x,\mu) = \sum_{m=1}^M q_m(x) h_m(\mu)$ where $q_m \in C^{2,\alpha}(\Omega)$ and $h_m \in L^2(V)$, $1 \le m \le M$. Then we have the error estimate in $L^2(\Omega \times V)$:

$$w_\varepsilon - w = O(\varepsilon).$$

The proof of the well posedness of (27) is given in section 4. The proof of the asymptotic behavior stated in this theorem is postponed to section 5. We deduce from this theorem the point-wise convergence of S_ε to the homogenized operator $S \in \mathcal{L}(L^2(\Omega \times V))$ defined by

$$q \mapsto Sq = w$$

where w is the solution to (30). The *compact convergence* of the sequence S_ε (in the sense given in Appendix B) is asserted by the following lemma

LEMMA 3.10. *Let x_ε be a sequence of elements in the unit ball of $L^2(\Omega \times V)$. Then $S_\varepsilon x_\varepsilon$ is relatively compact.*

Proof. We deduce from the a priori estimate (29) of Lemma 3.8 that

$$\|\mu\cdot\nabla(S_\varepsilon x_\varepsilon)\|^2 + \|S_\varepsilon x_\varepsilon\| + \|S_\varepsilon x_\varepsilon\|^2_{L^2(\partial\Omega\times V, d\xi)} \quad \le \quad C.$$

Then the averaging lemma given in [25] yields that $\int_V S_\varepsilon x_\varepsilon \, d\mu$ is relatively compact in $L^2(\Omega)$. Using (29) once again, we have

$$\|S_\varepsilon x_\varepsilon - \int_V S_\varepsilon x_\varepsilon \, d\mu\| \quad \le \quad C\varepsilon,$$

which asserts that $S_\varepsilon x_\varepsilon$ is also relatively compact. This completes the proof of this lemma. ◻
We deduce from Theorem 3.9 and Lemma 3.10 that the sequence of operators S_ε converges compactly to S. Theorem 3.2 is then a straight consequence of Theorem B.1. Since the eigenvectors of the diffusion eigenvalue problem are regular, the error estimates (9) given in Theorem 3.2 are easily derived from Theorem B.2.

In order to prove Theorem 3.5, we need to characterize first-order correctors for the source problem (27). The source term in the criticality eigenvalue problem, coming from the fission term, is regular and isotropic. Therefore we only consider source terms $q = q(x) \in C^{3,\alpha}(\Omega)$ with $\alpha > 0$.

THEOREM 3.11. *Let be w_ε the solution to (28) with $g = 0$ and $(\theta^i)_{1\le i\le d}$ defined by (12). Assume that Hypothesis 3.4 is satisfied and that $q = q(x) \in C^{3,\alpha}(\Omega)$. Then there exists a constant L, independent of q, such we have the following result.*

Let w_0 and w_{10} be the solutions of

$$\left\{ \begin{array}{rcll} -\nabla D \nabla w_0 & = & q & \text{in } \Omega \\ w_0 & = & 0 & \text{on } \partial\Omega, \end{array} \right. \tag{31}$$

$$\left\{ \begin{array}{rcll} -\nabla D \nabla w_{10} & = & 0 & \text{in } \Omega \\ w_{10} + L\dfrac{\partial w_0}{\partial n} & = & 0 & \text{on } \partial\Omega. \end{array} \right. \tag{32}$$

Then, denoting by $\theta_\varepsilon^i(x,\mu)=\theta^i(\frac{x}{\varepsilon},\mu)$, we have

$$\left\| w_\varepsilon - \left[w_0 + \varepsilon \left(\theta_\varepsilon^i \frac{\partial w_0}{\partial x_i} + w_{10} \right) \right] \right\|_{L^2(\Omega\times V)} = O(\varepsilon^{3/2}). \tag{33}$$

This theorem is proven in section 5. Let us focus on the proof of Theorem 3.5. We obtained in Theorem 3.2 that $\lambda^\varepsilon=\lambda_\infty+\varepsilon^2\nu^\varepsilon$, where the l^{th} eigenvalue ν_l^ε converges to ν_l as $\varepsilon\to 0$. Now we are interested in the limit of the corrector for the smallest eigenvalue $\xi_1^\varepsilon=\frac{\nu_1^\varepsilon-\nu_1}{\varepsilon}$. Denote by s^ε the solution of

$$A_\varepsilon s^\varepsilon = \nu_1 F_\varepsilon u_1.$$

This problem admits a unique solution, as seen in Theorem 3.9. Multiplying this equation by u_ε, solution of $A_\varepsilon u_\varepsilon=\nu_1^\varepsilon F_\varepsilon u_\varepsilon$, and integrating over $\Omega\times V$, we obtain, since A_ε is self-adjoint, that

$$\begin{aligned} \nu_1^\varepsilon(s^\varepsilon,F_\varepsilon u_\varepsilon) &= \nu_1(u_1,F_\varepsilon u_\varepsilon)\\ (\nu_1^\varepsilon-\nu_1)(u_1,F_\varepsilon u_\varepsilon) &= \nu_1^\varepsilon(u_1-s^\varepsilon,F_\varepsilon u_\varepsilon)\\ (\frac{\nu_1^\varepsilon-\nu_1}{\varepsilon})(u_1,F_\varepsilon u_\varepsilon) &= \nu_1^\varepsilon(\frac{u_1-s^\varepsilon}{\varepsilon},F_\varepsilon u_\varepsilon). \end{aligned} \tag{34}$$

Since u_1 is sufficiently smooth, we deduce from Theorem 3.11 the following expansion in $L^2(\Omega\times V)$:

$$s^\varepsilon = u_1 + \varepsilon\theta_\varepsilon^i \frac{\partial u_1}{\partial x_i} + \varepsilon w + O(\varepsilon^{3/2}), \tag{35}$$

where w is the solution to (32) with w_0 replaced by u_1. Then

$$(\frac{s^\varepsilon-u_1}{\varepsilon},F_\varepsilon u_1) \;=\; (w,F_\varepsilon u_1)+(F_\varepsilon\theta_\varepsilon^i\frac{\partial u_1}{\partial x_i},u_1)+O(\varepsilon).$$

The first term on the right-hand side clearly converges to $\overline{\sigma}_f(u_1,w)$. The second term is given by

$$\int_\Omega \left[\int_V \theta^i(\frac{x}{\varepsilon},\mu)\sigma_f(\frac{x}{\varepsilon})\psi_\infty^+(\frac{x}{\varepsilon},\mu)d\mu \int_V \psi_\infty^+(\frac{x}{\varepsilon},\mu')d\mu'\right]\frac{\partial u_1}{\partial x_i}(x)u_1(x)dx.$$

From the symmetry properties of Y given in Hypothesis 3.4, we easily deduce (see also Lemma 5.1 in section 5) that

$$\int_Y\int_V \theta^i(y,\mu)\sigma_f(y)\psi_\infty^+(y,\mu)d\mu\int_V\psi_\infty^+(y,\mu')d\mu'dy=0.$$

Since u_1 is regular, a Taylor expansion of u_1 yields that $(F_\varepsilon\theta_\varepsilon^i\frac{\partial u_1}{\partial x_i},u_1)=O(\varepsilon)$. Therefore, since $u_\varepsilon=u_1+O(\varepsilon)$, the right hand-side of (34) is equal to $-\nu_1\overline{\sigma}_f(u_1,w)+O(\varepsilon^{1/2})$. The left hand side of (34) is equal to $\xi_1^\varepsilon\overline{\sigma}_f+O(\varepsilon)$. The first-order corrector for the eigenvalue takes the form:

$$\xi_1^\varepsilon \;=\; -\nu_1(u_1,w)+O(\varepsilon^{1/2}).$$

At last we deduce from [20, Proposition 3, §5, Chapter 21], or from elementary computations on the eigenvalues of a homogeneous diffusion problem in a cube, that the smallest eigenvalue of

$$\begin{aligned} -\nabla D\nabla\Phi^\varepsilon &= \omega^\varepsilon\overline{\sigma_f}\Phi^\varepsilon \qquad \text{in } \Omega\\ \Phi^\varepsilon+\varepsilon L\frac{\partial\Phi^\varepsilon}{\partial n} &= 0, \qquad \text{on } \partial\Omega, \end{aligned} \tag{36}$$

satisfies, u_1 and Φ^ε being normalized in $L^2(\Omega)$,

$$\omega_1^\varepsilon=\nu_1-\varepsilon\nu_1(u_1,w)+O(\varepsilon^{3/2}). \tag{37}$$

This concludes the proof of Theorem 3.5.

4. A priori estimates and analysis of the source problem. This section presents some preliminary results on the source problem (27), which will prove useful in the study of its asymptotic behavior as $\varepsilon \to 0$. We give a proof for Lemma 3.8 and for the well posedness of problem (27) as stated in Theorem 3.9. We will use the following notation

$$< u >= \int_V u(\mu) d\mu.$$

One of the interesting properties of the even parity flux is that is allows us to use a variational formulation. Introduce the bilinear form $a_\varepsilon(u,v)$

$$\begin{aligned} a_\varepsilon(u,v) &= \int_\Omega \int_V \frac{(\psi_\varepsilon^+)^2}{\Sigma^\varepsilon} \mu\cdot\nabla u\, \mu\cdot\nabla v\, d\mu\, dx + \frac{1}{\varepsilon}\int_{\partial\Omega}\int_V |\mu\cdot n|\, h_\varepsilon\, uv\, d\mu\, d\sigma \\ &+ \frac{1}{\varepsilon^2}\int_\Omega\int_V (Q_\varepsilon u)\, v\, d\mu\; dx, \end{aligned} \tag{38}$$

where the function h_ε is defined on $\partial\Omega \times V$ by $h_\varepsilon = \psi_\varepsilon \psi_\varepsilon^+$ on Γ_- and $h_\varepsilon(x,\mu) = h_\varepsilon(x,-\mu)$. According to the results of section 2, h_ε is uniformly positive. An integration by parts shows that (28) is equivalent to finding $w_\varepsilon \in \mathcal{V}$ such that

$$a_\varepsilon(w_\varepsilon, v) = (F_\varepsilon q, v) + \frac{1}{\varepsilon}\int_{\partial\Omega}\int_V |\mu\cdot n| h_\varepsilon\, g\, v\, d\mu\, d\sigma \qquad \forall v \in \mathcal{V}, \tag{39}$$

where $\mathcal{V}$ is the following Hilbert space

$$\mathcal{V} = \{v \in L^2(\Omega\times V) \text{ s.t. } \mu\cdot\nabla v \in L^2(\Omega\times V),\, v \in L^2(\partial\Omega\times V, d\xi),\, v(x,\mu) = v(x,-\mu)\}. \tag{40}$$

The bilinear form a_ε is bi-continuous in $\mathcal{V}$. We deduce that a_ε is symmetric from the identity $\int_V (Q_\varepsilon u) v\, d\mu = \int_V u(Q_\varepsilon v)\, d\mu$. Choosing now $v = w_\varepsilon$ in (39), we obtain that

$$\begin{aligned} & \|\mu\cdot\nabla w_\varepsilon\|^2 + \frac{1}{\varepsilon}\|w_\varepsilon\|^2_{L^2(\partial\Omega\times V,d\xi)} + \frac{1}{\varepsilon^2}(Q_\varepsilon w_\varepsilon, w_\varepsilon) \\ \le\ & C a_\varepsilon(w_\varepsilon, w_\varepsilon) \le C\|q\|\,\|w_\varepsilon\| + \frac{C}{\varepsilon}\|g\|_{L^2(\partial\Omega\times V,d\xi)}\|w_\varepsilon\|_{L^2(\partial\Omega\times V,d\xi)}. \end{aligned} \tag{41}$$

A Poincaré-like inequality in transport theory (see [5, 8]) yields that

$$\|w_\varepsilon\| \le C\,(\|\mu\cdot\nabla w_\varepsilon\| + \|w_\varepsilon\|_{L^2(\partial\Omega\times V,d\xi)}), \tag{42}$$

where C is a constant independent of w_ε. Therefore the coercivity of a_ε is easily deduced from the positiveness of the collision operator Q_ε that we prove now.

LEMMA 4.1. *Let $f \in L^\infty(V)$ be a positive function. Then the operator $\mathcal{Q}$ defined by*

$$\mathcal{Q}u(\mu) = f(\mu)u(\mu) < f > - f(\mu) < f u >$$

satisfies the following property

$$(\mathcal{Q}u, u) \ge (\inf_V f)^2 \|u - < u >\|^2_{L^2(V)}$$

where $(\cdot,\cdot)$ is the usual scalar product of $L^2(V)$.

Proof. Some computations yield

$$\begin{aligned}(Qu,u) &= \int_V\int_V f(\mu)f(\mu')u(\mu)(u(\mu)-u(\mu'))d\mu d\mu' \\ &= \frac{1}{2}\int_V\int_V f(\mu)f(\mu')(u(\mu)-u(\mu'))^2 d\mu d\mu' \\ &\geq (\inf_V f)^2\frac{1}{2}\int_V\int_V (u(\mu)-u(\mu'))^2 d\mu d\mu' \;=\; (\inf_V f)^2\int_V (u(\mu)-<u>)^2\,d\mu.\end{aligned}$$

This concludes the proof of the lemma. □

From this lemma and the Poincaré inequality (42), we obtain that

$$\begin{aligned}&\|\mu\cdot\nabla w_\varepsilon\| + \|w_\varepsilon\| + \frac{1}{\sqrt{\varepsilon}}\|w_\varepsilon\|_{L^2(\partial\Omega\times V,d\xi)} + \frac{1}{\varepsilon}\|w_\varepsilon - \int_V w_\varepsilon\| \\ \leq\;\; & C\sqrt{a_\varepsilon(w_\varepsilon,w_\varepsilon)} \leq C(\|q\| + \frac{1}{\sqrt{\varepsilon}}\|g\|_{L^2(\partial\Omega\times V,d\xi)}).\end{aligned} \tag{43}$$

This concludes the proof of Lemma 3.8. Moreover the existence and uniqueness of a solution to (28) is a straight consequence of the Lax-Milgram theory. This also asserts the well posedness and boundedness of S_ε in $\mathcal{L}(L^2(\Omega\times V))$, and the first part of Theorem 3.9 is complete. We also need an analogous result for non homogeneous boundary conditions. We have the maximum principle

PROPOSITION 4.2. *Let be $g\in L^\infty(\Gamma_-)$. There exists a unique solution to (28) with $q=0$ in $L^\infty(\Omega\times V)$. Furthermore we have that $\|w_\varepsilon\|_{L^\infty(\Omega\times V)}\leq\|g\|_{L^\infty(\Gamma_-)}$.*

Proof. We use the maximum principle stated for first-order transport equations in [20, Chapter 21], and the equivalence presented in appendix A. Let be $\alpha>0$ and the sequence of problems

$$\begin{aligned}-\mu\cdot\nabla\frac{(\psi_\varepsilon^+)^2}{\Sigma^\varepsilon}\mu\cdot\nabla w_\alpha^\varepsilon + \frac{1}{\varepsilon^2}Q_\varepsilon w_\alpha^\varepsilon + \alpha w_\alpha^\varepsilon &= 0 \quad \text{in } \Omega\times V \\ w_\alpha^\varepsilon - \frac{\varepsilon\psi_\varepsilon^+}{\psi_\varepsilon\Sigma^\varepsilon}\mu\cdot\nabla w_\alpha^\varepsilon &= g \quad \text{on } \Gamma_-.\end{aligned}$$

For every function f, we denote by $\check{f}(\mu)=f(-\mu)$. We deduce from the results recalled in Appendix A that

$$w_\alpha^\varepsilon = \frac{\psi_\varepsilon\psi_\alpha^\varepsilon + \check{\psi}_\varepsilon\check{\psi}_\alpha^\varepsilon}{\psi_\varepsilon+\check{\psi}_\varepsilon}.$$

Here, ψ_ε is the positive solution of (19) and ψ_α^ε is the solution to

$$\begin{aligned}\frac{1}{\varepsilon}\mu\cdot\nabla\psi_\alpha^\varepsilon + \frac{\Sigma_\varepsilon^\infty}{\varepsilon^2}\left(\frac{1}{\psi_\varepsilon}\psi_\alpha^\varepsilon\int_V\psi_\varepsilon - \frac{1}{\psi_\varepsilon}\int_V\psi_\alpha^\varepsilon\psi_\varepsilon\right) + \alpha\psi_\alpha^\varepsilon &= 0 \quad \text{in } \Omega\times V \\ \psi_\alpha^\varepsilon &= g \quad \text{on } \Gamma_-.\end{aligned}$$

We deduce from [20, Proposition 7, Chapter 21, section 2] that $\|\psi_\alpha^\varepsilon\|_{L^\infty}\leq\|g\|_{L^\infty}$ independently of α. Therefore, $\|w_\alpha^\varepsilon\|_{L^\infty}\leq\|g\|_{L^\infty}$ independently of α. So there exists a subsequence of w_α^ε which converges to w_ε in $L^\infty(\Omega\times V)$ weak$*$ as $\alpha\to 0$. Using standard techniques (see Bardos [10] and Appendix A), we verify that w_ε is a solution of (28) and satisfies $\|w_\varepsilon\|_{L^\infty(\Omega\times V)}\leq\|g\|_{L^\infty(\Gamma_-)}$. We know from Lemma 3.8 that

$$\|w_\varepsilon\| \leq \frac{C}{\sqrt{\varepsilon}}\|g\|_{L^2(\Gamma_-,d\xi)}.$$

Therefore, the solution to (28) in $L^\infty(\Omega \times V)$ is unique. □

Before addressing the asymptotic convergence of the solutions to (27), we need one more result on source problems in infinite media. Let us introduce the factored flux φ^+ defined by $\psi^+ = \psi^+_\infty \varphi^+$. Then we easily obtain the following corollary for Theorem 2.5.

COROLLARY 4.3. *Assume that the hypotheses of Theorem 2.5 are satisfied. Then problem*

$$
(44) \qquad \begin{aligned}
-\mu \cdot \nabla_y \frac{(\psi^+_\infty)^2}{\Sigma} \mu \cdot \nabla_y \varphi^+ + Q\varphi^+ \; &= \; S \qquad \textit{in } Y \times V \\
y \mapsto \varphi^+(y,\mu) \;\; \textit{is } \; & Y - \textit{periodic},
\end{aligned}
$$

admits a unique zero mean solution if and only if

$$
\int_Y \int_V S(y,\mu) dy \, d\mu \; = \; 0.
$$

The same regularity results as in Theorem 2.5 hold true.

Since ψ^+_∞ is periodic, we deduce from this corollary that (12) admits a unique zero mean solution. Then the diffusion coefficients are uniquely defined by (11) and do not depend on the mean over $Y \times V$ of the functions θ^i. The following lemma asserts the positive definiteness of the diffusion tensor D and consequently the well posedness of the diffusion problem (10).

LEMMA 4.4. *The homogeneous tensor D defined by (11) is positive definite.*

Proof. Let us multiply equation (12) by a Y-periodic smooth test function. After integration by parts, we obtain

$$
\begin{aligned}
&\int_Y \int_V \frac{(\psi^+_\infty)^2}{\Sigma} \mu \cdot \nabla_y \theta^i \mu \cdot \nabla_y v \, d\mu dy + (Q\theta^i, v) - \int_{\partial Y} \int_V \frac{(\psi^+_\infty)^2}{\Sigma} \mu \cdot \nabla_y \theta^i (\mu \cdot n) v \, d\mu d\sigma \\
= \; & - \int_Y \int_V \mu_i \frac{(\psi^+_\infty)^2}{\Sigma} \mu \cdot \nabla_y v \, d\mu dy + \int_{\partial Y} \int_V \mu_i \frac{(\psi^+_\infty)^2}{\Sigma} v(\mu \cdot n) \, d\mu d\sigma.
\end{aligned}
$$

The boundary terms in this equation cancel out since $\frac{(\psi^+_\infty)^2}{\Sigma} \mu \cdot \nabla_y \theta^i v$ and $\mu_i \frac{(\psi^+_\infty)^2}{\Sigma} v$ are Y-periodic whereas the sign of $(\mu \cdot n)$ is reversed from one side of Y to the opposite one. Thus we have

$$
\int_Y \int_V \frac{(\psi^+_\infty)^2}{\Sigma} (\mu \cdot \nabla_y \theta^i + \mu_i) \mu \cdot \nabla_y v \, d\mu dy + (Q\theta^i, v) = 0
$$

for any periodic test function v, and in particular for the functions θ^j. We recall that the coefficients D_{ij} are defined by (11). For each vector $\xi \in \mathbb{R}^n$, of components ξ_i, we deduce that (with the convention of summation over the repeated indexes)

$$
\begin{aligned}
D_{ij}\xi_i\xi_j \;\; &= \int_Y \int_V \frac{(\psi^+_\infty)^2}{\Sigma} \mu_j \xi_j (\mu \cdot \nabla_y \theta^i + \mu_i) \xi_i \, d\mu dy \\
&= \int_Y \int_V \frac{(\psi^+_\infty)^2}{\Sigma} (\mu \cdot \nabla_y \theta^j + \mu_j) \xi_j (\mu \cdot \nabla_y \theta^i + \mu_i) \xi_i \, d\mu dy \\
&\quad - \int_Y \int_V \frac{(\psi^+_\infty)^2}{\Sigma} \mu \cdot \nabla_y \theta^j \xi_j (\mu \cdot \nabla_y \theta^i + \mu_i) \xi_i \, d\mu dy \\
&= \int_Y \int_V \frac{(\psi^+_\infty)^2}{\Sigma} \left\{ \left(\sum_{i=1}^n (\mu \cdot \nabla_y \theta^i + \mu_i) \xi_i \right)^2 + \left(Q(\sum_{i=1}^n \xi_i \theta^i), \sum_{i=1}^n \xi_i \theta^i \right) \right\} d\mu dy.
\end{aligned}
$$

We deduce from Lemma 4.1 that this expression is non negative. Therefore, D is positive. Assume now that it is not definite. Then there exists a vector $\xi \neq 0$ such that $(\mu\cdot\nabla_y\theta^i + \mu_i)\xi_i = 0$ a.e. $(y,\mu) \in Y\times V$. Integrate this equality over Y for $\mu \in V$ given. We deduce from the $Y-$ periodicity of θ^i that $\int_Y \mu_i\xi_i = \mu_i\xi_i = 0$ a.e. $\mu\in V$. This yields $\xi = 0$ and a contradiction. Thus the tensor D is positive definite. ◻

5. Analysis of the source problem. This section is devoted to the proof of Theorems 3.9 and 3.11. It is based on the classical method of two-scale asymptotic expansions. Notice that the two-scale convergence introduced in [2] represents a natural framework to prove the convergence stated in Theorem 3.9 [5]. It allows for very mild regularity assumptions on the physical data; however it does not enable us to obtain the rates of convergence stated in section 3.

5.1. Asymptotic convergence of the source problem. The point-wise convergence of the operator S_ε is stated in Theorem 3.9. We now give a proof of this result.

Proof. It is based on a two-scale analysis of the neutron density. Let us define the following ansatz

$$w_\varepsilon(x,\mu) = w_0(x,\frac{x}{\varepsilon},\mu) + \varepsilon w_1(x,\frac{x}{\varepsilon},\mu) + \varepsilon^2 w_2(x,\frac{x}{\varepsilon},\mu) + \zeta^\varepsilon(x,\mu), \tag{45}$$

where the functions $y\mapsto w_k(x,y,\mu)$ are Y-periodic for $0\le k\le 2$. We derive the equations that $w_k(x,y,\mu)$ must verify to justify this asymptotic expansion. This provides us with an explicit choice for $w_k(x,y,\mu)$. Next, we prove that ζ^ε defined by (45) is of order $O(\varepsilon)$ in $L^2(\Omega\times V)$. This is done first for a source term $q(x,\mu) = \tilde{q}(x)h(\mu)$, and then extended to the case $q\in L^2(\Omega\times V))$ by density.

i) Remark that the differentiation operator is now given by $\mu\cdot\nabla = \mu\cdot\nabla_x + \frac{1}{\varepsilon}\mu\cdot\nabla_y$. Inserting (45) into (27) and neglecting ζ_ε, we obtain

$$\begin{aligned} &- (\mu\cdot\nabla_x + \frac{1}{\varepsilon}\mu\cdot\nabla_y)\frac{(\psi_\infty^+)^2}{\Sigma}(\mu\cdot\nabla_x + \frac{1}{\varepsilon}\mu\cdot\nabla_y)(w_0+\varepsilon w_1+\varepsilon^2 w_2)\\ &+ \frac{1}{\varepsilon^2}Q(w_0+\varepsilon w_1+\varepsilon^2 w_2) = \sigma_f\psi_\infty^+\int_V \psi_\infty^+(x,\mu')q(x,\mu')d\mu'. \end{aligned} \tag{46}$$

The term of order -2 in the expansion in powers of ε yields

$$-\mu\cdot\nabla_y\frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla_y w_0 + Qw_0 = 0. \tag{47}$$

Following the same development as in section 3 we obtain that $w_0\psi_\infty^+$ is a solution of (6) for any given x. Then by virtue of Theorem 2.3, we have $w_0 = w_0(x)$. From the boundary conditions for w_ε we deduce that w_0 must vanish on $\partial\Omega$.

Taking into account the form of w_0, we obtain for the terms of order ε^{-1} that

$$-\mu\cdot\nabla_y\frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla_y w_1 + Qw_1 = \mu\cdot\nabla_y\frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla_x w_0. \tag{48}$$

This equation posed on $Y\times V$ for every given $x\in\Omega$ and we deduce from Corollary 4.3 that $w_1(y,\mu) = \theta^i(y,\mu)\frac{\partial w_0}{\partial x_i}(x) + w_{10}(x)$ where θ^i is defined by (12) and $w_{10}(x)$ is still undetermined. We choose here $w_{10}(x) = 0$.

Consider now the zeroth order equation. We have

$$\begin{aligned} &-\frac{(\psi_\infty^+)^2}{\Sigma}(\mu\cdot\nabla_x)^2 w_0 - (\mu\cdot\nabla_y\frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla_x + \mu\cdot\nabla_x\frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla_y)w_1\\ &-\mu\cdot\nabla_y\frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla_y w_2 + Qw_2 = \sigma_f\psi_\infty^+\int_V\psi_\infty^+(y,\mu')q(x,\mu')d\mu'. \end{aligned} \tag{49}$$

Following Corollary 4.3, this equation admits a solution when the source term is of zero mean. It implies that w_0 satisfies the following equation

$$\begin{aligned}&\int_V\int_Y \left[-\frac{(\psi_\infty^+)^2}{\Sigma}(\mu\cdot\nabla_x)^2 w_0 - \left(\mu\cdot\nabla_y\frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla_x + \mu\cdot\nabla_x\frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla_y\right)w_1\right]d\mu dy\\ = &\int_V\int_Y\left(\sigma_f(y)\psi_\infty^+(y,\mu)\int_V \psi_\infty^+(y,\mu')q(x,\mu')d\mu'\right)d\mu dy.\end{aligned}$$

Replacing w_1 by its expression in terms of w_0, we obtain that w_0 is a solution to (30). Since D is positive definite according to Lemma 4.1, we deduce that w_0 is uniquely defined in $H_0^1(\Omega)$. The expression of w_1 is also known. It remains to define w_2. Equation (49) can be recast as

$$-\mu\cdot\nabla_y\frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla_y w_2 + Qw_2 = \left(\sigma_f\psi_\infty^+\int_V\psi_\infty^+ q\; - \;\overline{q}\right) + (h_{ij}(y,\mu)-D_{ij})\frac{\partial^2 w_0}{\partial x_j\partial x_i}$$

owing to the expression of w_1 and (30), where

$$h_{ij} = \frac{(\psi_\infty^+)^2}{\Sigma}\mu_j(\mu_i+\mu\cdot\nabla_y\theta^i) + \mu_j\,\mu\cdot\nabla_y\frac{(\psi_\infty^+)^2}{\Sigma}\theta^i.$$

We define the functions w_{2a} and w_{2ij}, $1\le i,j\le n$, recalling that $q(x,\mu)=\tilde{q}(x)h(\mu)$, by

$$\begin{aligned}&-\mu\cdot\nabla_y\frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla_y w_{2a} + Qw_{2a} = \sigma_f\psi_\infty^+\int_V\psi_\infty^+ h\,d\mu' \;-\;\int_Y\int_V\left(\sigma_f\psi_\infty^+\int_V\psi_\infty^+ h\,d\mu'\right)d\mu dy\\ &-\mu\cdot\nabla_y\frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla_y w_{2ij} + Qw_{2ij} = h_{ij}-D_{ij}.\end{aligned}$$

We easily check that the source terms have zero mean. Consequently the functions w_{2a} and w_{2ij} are defined up to an additive constant and are smooth according to Corollary 4.3. By linearity of transport, we obtain that w_2 is equal to

$$w_2(x,y,\mu) = w_{2a}(y,\mu)\tilde{q}(x) + w_{2ij}(y,\mu)\frac{\partial^2 w_0}{\partial x_j\partial x_i}(x) + w_{20}(x),$$

where w_{20} is undetermined. We choose here $w_{20}=0$.

ii) It remains to derive an equation for ζ^ε and prove that this error term is small. Inserting (45) in (27), we obtain from the explicit expressions of w_0, w_1 and w_2 that

$$\begin{aligned}-\mu\cdot\nabla\frac{(\psi_\varepsilon^+)^2}{\Sigma^\varepsilon}\mu\cdot\nabla\zeta^\varepsilon + \frac{1}{\varepsilon^2}Q_\varepsilon\zeta^\varepsilon &= \varepsilon\zeta_1^\varepsilon + \varepsilon^2\zeta_2^\varepsilon \quad \text{in } \Omega\times V\\ \zeta^\varepsilon - \frac{\varepsilon\psi_\varepsilon^+}{\psi_\varepsilon\Sigma^\varepsilon}\mu\cdot\nabla\zeta^\varepsilon &= \varepsilon\zeta_3^\varepsilon + \varepsilon^2\zeta_4^\varepsilon + \varepsilon^3\zeta_5^\varepsilon, \qquad \text{on } \Gamma_-\end{aligned} \tag{50}$$

where $\zeta_i^\varepsilon(x,\mu) = \zeta_i(x,\frac{x}{\varepsilon},\mu)$ and

$$\begin{aligned}\zeta_1 &= -\frac{(\psi_\infty^+)^2}{\Sigma}(\mu\cdot\nabla_x)^2 w_1 - \mu\cdot\nabla_x\frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla_y w_2 - \mu\cdot\nabla_y\frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla_x w_2\\ \zeta_2 &= -\frac{(\psi_\infty^+)^2}{\Sigma}(\mu\cdot\nabla_x)^2 w_2, \qquad \zeta_3 = -w_1 + \frac{\psi_\infty^+}{\psi_\infty\Sigma}(\mu\cdot\nabla_x w_0 + \mu\cdot\nabla_y w_1)\\ \zeta_4 &= -w_2 + \frac{\psi_\infty^+}{\psi_\infty\Sigma}(\mu\cdot\nabla_x w_1 + \mu\cdot\nabla_y w_2), \qquad \zeta_5 = \frac{\psi_\infty^+}{\psi_\infty\Sigma}\mu\cdot\nabla_x w_2.\end{aligned} \tag{51}$$

Assume first that $\tilde{q}$ is of class $C^{2,\alpha}(\Omega)$. Since D is positive definite, we deduce from [24] that w_0 is of class $C^{4,\alpha}(\Omega)$. The terms w_1 and w_2 are the sum of products of functions only depending on x and of functions only depending on (y,μ). The part depending only on x in the expression of w_1: $\frac{\partial w_0}{\partial x_i}$, is of class $C^{3,\alpha}(\Omega)$ and the one in the expression of w_2: $\tilde{q}$ and $\frac{\partial^2 w_0}{\partial x_j x_i}$, is of class $C^{2,\alpha}(\Omega)$. The parts depending on (y,μ) in both terms: θ^i, w_{2a} and w_{2ij}, are regular according to Corollary 4.3. Therefore the terms ζ_i^ε are of class $C^{0,\alpha}(\Omega\times V)$. From the variational formulation of equation (50), see Lemma 3.8, and Proposition 4.2, we deduce that $\|\zeta^\varepsilon\|\le C\varepsilon$, where C is independent of ε. Therefore we have proven that $w_\varepsilon - w = O(\varepsilon)$, provided q is of the form $q(x,\mu)=\sum_{m=1}^M q_m(x)h_m(\mu)$ where all q_m are of class $C^{2,\alpha}(\Omega)$.

The proof of the convergence in the general case follows from two density arguments. Let us first assume that $\tilde{q}$ belongs to $L^2(\Omega)$. Consider a sequence $\tilde{q}_i$ of functions of class $C^{2,\alpha}(\Omega)$ converging to $\tilde{q}$ strongly in $L^2(\Omega)$. Then for any $\eta>0$ and i large enough we have $\|\tilde{q}_i-\tilde{q}\|\le C\eta$. Let w_i^ε be the solution of (27). From the linearity of the transport equation and from its variational formulation, we deduce that $\|w_i^\varepsilon - w_\varepsilon\|\le C\eta$ independently of η and ε. On the other hand we have proven that $w_i^\varepsilon\to w_i$, where w_i is defined as the solution of the diffusion solution with a source term equal to $\tilde{q}_i$ instead of $\tilde{q}$. We easily check that $\|w_i-w_0\|\le C\eta$, where w_0 is the solution of (30) with source term $\tilde{q}$, and deduce the convergence of w_ε to w_0 as $\varepsilon\to 0$.

The same argument is used for the general case $q\in L^2(\Omega\times V)$. Every function $q\in L^2(\Omega\times V)$ admits a spectral decomposition in the canonical Schauder basis of $L^2(\Omega\times V)$

$$q(x,\mu) = \sum_{m=1}^{\infty}\sum_{p=1}^{\infty}\alpha_{mp}\tilde{q}_m(x)h_p(\mu),$$

where the functions $\tilde{q}_m$ are vectors of an orthogonal basis in $L^2(\Omega)$ and h_p are vectors of an orthogonal basis in $L^2(V)$. We have that $\sum_{m=1}^\infty\sum_{p=1}^\infty \alpha_{mp}^2<\infty$. Thus the sequence

$$q_{MP} = \sum_{m=1}^{M}\sum_{p=1}^{P}\alpha_{mp}\tilde{q}_m(x)h_p(\mu)$$

converges strongly to q in $L^2(\Omega\times V)$. We deduce that $\|w_{MP}^\varepsilon - w_{MP}\|$ converges to 0 independently of ε and that w_{MP} converges strongly to w (using obvious notations) as $M,P\to\infty$. On the other hand for fixed M,P large enough, the linearity of the transport and diffusion equations yields the strong convergence in $L^2(\Omega\times V)$ of w_{MP}^ε to w_{MP} as $\varepsilon\to 0$. □

5.2. First-order corrector for the source problem. This subsection is devoted to the derivation of the first-order corrector of the source problem (27) as stated in Theorem 3.11.

First we introduce some notations. To simplify we assume here that the spatial dimension is $d=3$. Let O be the center of Y. We denote by $\mathcal{P}_p$ the plane (O,x_m,x_n), $m,n\in\{i,j,k\}$ spanned by the vectors e_m, e_n and such that $O\in\mathcal{P}_p$. The index $p\in\{i,j,k\}$ is such that $p\neq m$ and $p\neq n$.

We say that two pairs (y,μ) and (y',μ') of $Y\times V$ are symmetric with respect to $\mathcal{P}_p$ if they satisfy

$$d(y,\mathcal{P}_p)=d(y',\mathcal{P}_p),\quad (y'-y)\parallel e_p,\quad \mu'=\mu-2(\mu\cdot e_p)e_p.$$

For a pair (y,μ) and (y',μ') of symmetric points we say that a function ψ is symmetric with respect to $\mathcal{P}_p$ if $\psi(y',\mu')=\psi(y,\mu)$, and that it is skew-symmetric if $\psi(y',\mu')=-\psi(y,\mu)$. We now state the

LEMMA 5.1. *Assume that the cross sections Σ and Σ_s are symmetric in Y with respect to the planes $\mathcal{P}_p$, $p=i,j,k$. Let (y,μ) and (y',μ') be symmetric points with respect to $\mathcal{P}_p$ for $p\in\{i,j,k\}$.*

i) Let f be a source term satisfying $\check{f} = f$ and $f(y,\mu) = \pm f(y',\mu')$ a.e. $(y,\mu) \in Y \times V$. Then the solution of

$$-\mu\cdot\nabla_y \frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla_y\varphi + Q\varphi = f \qquad \text{in } Y \times V$$
$$y \mapsto \varphi(y,\mu) \ \text{ is } Y-\text{periodic},$$

satisfies the following relation of symmetry $\varphi(y,\mu) = \pm\varphi(y',\mu')$.

ii) Let f be a function satisfying that $f(y,\mu) = \pm f(y',\mu')$, a.e. $(y,\mu) \in Y \times V$. Then we have

$$\mu'\cdot\nabla_{y'} f(y',\mu') = \mu\cdot\nabla_y f(y,\mu).$$

Proof. i) Define $h(y',\mu') = \varphi(y,\mu)$. We check that $\mu'\cdot\nabla_{y'}h(y',\mu') = \mu\cdot\nabla_y\varphi(y,\mu)$. Since Y is symmetric, the solution ψ_∞^+ and the cross sections are also symmetric. Then

$$-\mu'\cdot\nabla_{y'}\frac{(\psi_\infty^+)^2}{\Sigma}(y',\mu')\mu'\cdot\nabla_{y'}h(y',\mu') = \mu\cdot\nabla_y\frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla_y\varphi.$$

and

$$(Qh)(y',\mu') = (Q\varphi)(y,\mu).$$

Therefore h is solution of the following equation

$$-\mu'\cdot\nabla_{y'}\frac{(\psi_\infty^+)^2}{\Sigma}(y',\mu')\mu'\cdot\nabla_{y'}h(y',\mu') + (Qh)(y',\mu') = \pm f(y',\mu').$$

We deduce from the uniqueness of the solution for this equation that $h(y',\mu') = \pm\varphi(y,\mu)$. It concludes the first part of the proof.

ii) Consider the plane of symmetry $\mathcal{P}_k$. We have $\mu\cdot\nabla_y f = \mu_i\frac{\partial f}{\partial y_i}$. If $i = k$, then $\frac{\partial f}{\partial y_i}(y',\mu') = -\frac{\partial f}{\partial y_i}(y,\mu)$ and $(\mu_i)' = \mu_i' = -\mu_i$. If $i \neq k$, then $\frac{\partial f}{\partial y_i}(y',\mu') = \frac{\partial f}{\partial y_i}(y,\mu)$ and $(\mu_i)' = \mu_i$. In any case we have $\left(\mu_i\frac{\partial f}{\partial y_i}\right)(y',\mu') = \left(\mu_i\frac{\partial f}{\partial y_i}\right)(y,\mu)$, and the proof is complete. □

We are now in a position to prove Theorem 3.11.

Proof. [Theorem 3.11] The first part of the proof is very similar to that of Theorem 3.9. We assume the following ansatz on w_ε

$$w_\varepsilon(x,\mu) = w_0(x,\frac{x}{\varepsilon},\mu) + \varepsilon w_1(x,\frac{x}{\varepsilon},\mu) + \varepsilon^2 w_2(x,\frac{x}{\varepsilon},\mu) + \varepsilon^3 w_3(x,\frac{x}{\varepsilon},\mu) + \zeta^\varepsilon(x,\mu), \tag{52}$$

where the functions $y \mapsto w_i(x,y,\mu)$ are Y-periodic for $0 \le i \le 3$. Our objective is to derive some conditions on the terms w_i such that ζ^ε be of order $O(\varepsilon^{3/2})$. The main difference with the proof of Theorem 3.9 is that the asymptotic expansion does not provide suitable boundary conditions for the functions w_i. This difficulty will be overcome by the analysis of a boundary layer problem, which allows us to find boundary conditions for the terms w_i such that the error ζ^ε be of order ε in the vicinity of the boundary but of order $\varepsilon^{3/2}$ globally in $L^2(\Omega\times V)$.

i) Plugging (52) into (27) and neglecting ζ^ε, we obtain

$$\begin{aligned}&-(\mu\cdot\nabla_x + \frac{1}{\varepsilon}\mu\cdot\nabla_y)\frac{(\psi_\infty^+)^2}{\Sigma}(\mu\cdot\nabla_x + \frac{1}{\varepsilon}\mu\cdot\nabla_y)(w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \varepsilon^3 w_3)\\ &+\frac{1}{\varepsilon^2}Q_\varepsilon(w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \varepsilon^3 w_3) = \sigma_f\psi_\infty^+\int_V \psi_\infty^+(y,\mu')q(x)d\mu'.\end{aligned} \tag{53}$$

From the terms of order ε^{-2}, we have $w_0 = w_0(x)$, and from the terms of order ε^{-1}, $w_1 = \theta^i \frac{\partial w_0}{\partial x_i} + w_{10}$. Here we cannot choose $w_{10} = 0$ since it is of order ε. The term of order ε^0 is given by

$$-\frac{(\psi_\infty^+)^2}{\Sigma}(\mu\cdot\nabla_x)^2 w_0 - (\mu\cdot\nabla_y \frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla_x + \mu\cdot\nabla_x \frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla_y) w_1$$
$$-\mu\cdot\nabla_y \frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla_y w_2 + Qw_2 = \sigma_f \psi_\infty^+ \int_V \psi_\infty^+(y,\mu')q(x)d\mu'.$$

This equation admits a solution when (31) is satisfied. We recast the equation for w_2 as follows

$$\begin{aligned} &-\mu\cdot\nabla_y \frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla_y w_2 + Qw_2 \\ = \ & (\sigma_f \psi_\infty^+ \int_V \psi_\infty^+ q - \overline{q}) + (h_{ij}(y,\mu) - D_{ij})\frac{\partial^2 w_0}{\partial x_j \partial x_i} + \mu\cdot\nabla_y \frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla_x w_{10}, \end{aligned}$$

where $h_{ij} = \frac{(\psi_\infty^+)^2}{\Sigma}\mu_j(\mu_i + \mu\cdot\nabla_y\theta^i) + \mu_j\mu\cdot\nabla_y \frac{(\psi_\infty^+)^2}{\Sigma}\theta^i$. Since $q = q(x)$, we have

$$\mu\cdot\nabla_y \frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla_y w_{2a} + Qw_{2a} = \sigma_f\psi_\infty^+ \int_V \psi_\infty^+ \, d\mu' \; - \int_Y\int_V \left(\sigma_f \psi_\infty^+ \int_V \psi_\infty^+ \, d\mu'\right) d\mu dy$$
$$-\mu\cdot\nabla_y \frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla_y w_{2ij} + Qw_{2ij} = h_{ij} - D_{ij}.$$

In these equations the source term has zero mean. Thus w_2 can be written as

$$w_2(x,y,\mu) = w_{2a}(y,\mu)q(x) + w_{2ij}(y,\mu)\frac{\partial^2 w_0}{\partial x_j \partial x_i}(x) + \theta^i \frac{\partial w_{10}}{\partial x_i} + w_{20}(x).$$

Because we are not interested in the terms of order ε^2 we can choose $w_{20} = 0$. Let us go one step further in the expansion in powers of ε. The term of order ε yields

$$-\frac{(\psi_\infty^+)^2}{\Sigma}(\mu\cdot\nabla_x)^2 w_1 - (\mu\cdot\nabla_y \frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla_x + \mu\cdot\nabla_x \frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla_y) w_2$$
$$-\mu\cdot\nabla_y \frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla_y w_3 + Qw_3 = 0.$$

This equation admits a solution if and only if the source term

$$\frac{(\psi_\infty^+)^2}{\Sigma}(\mu\cdot\nabla_x)^2 w_1 + (\mu\cdot\nabla_y \frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla_x + \mu\cdot\nabla_x \frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla_y) w_2$$

has zero mean. Replacing the functions w_i by their expressions in terms of w_0, w_{10} and q, and denoting for simplicity by $\partial_i = \frac{\partial}{\partial x_i}$, we rewrite this compatibility conditions as

$$\left(\int_V\int_Y \frac{(\psi_\infty^+)^2}{\Sigma}\left[(\mu_k\mu_j\theta^i) + \mu\cdot\nabla_y(\mu_k w_{2ij})\right] d\mu dy\right)\partial^3_{kji} w_0 + \left(\int_V\int_Y \frac{(\psi_\infty^+)^2}{\Sigma}\mu_i\mu\cdot\nabla_y w_{2a} d\mu dy\right)\partial_i q$$
$$+\left(\int_V\int_Y \frac{(\psi_\infty^+)^2}{\Sigma}\left[(\mu\cdot\nabla_x)^2 w_{10} + \mu\cdot\nabla_y\theta^i\mu\cdot\nabla_x\partial_i w_{10}\right] d\mu dy\right) = 0.$$

The last term of the left side is given by

$$\left(\int_V\int_Y \frac{(\psi_\infty^+)^2}{\Sigma}\left[(\mu\cdot\nabla_x)^2 w_{10} + \mu\cdot\nabla_y\theta^i\mu\cdot\nabla_x\partial_i w_{10}\right] d\mu dy\right) = \nabla D\nabla w_{10}.$$

Using the hypotheses of symmetry on Y, we prove now that the other terms in this expression vanish. We use results stated in Lemma 5.1. First, we easily obtain that

$$\int_V\int_Y \frac{(\psi_\infty^+)^2}{\Sigma}(\mu_k\mu_j\theta^i) d\mu dy = 0.$$

Indeed θ^i is skew-symmetric with respect to $\mathcal{P}_i$ and μ_j with respect to $\mathcal{P}_j$, whereas these functions are symmetric with respect to the other planes of symmetry. The product of three skew-symmetric functions being skew-symmetric at least with respect to one hyper plane, we deduce from the symmetries of $\frac{(\psi_\infty^+)^2}{\Sigma}$ that the integral vanishes. Consider now the term

$$\int_V \int_Y \frac{(\psi_\infty^+)^2}{\Sigma} \mu_k \mu \cdot \nabla_y (w_{2ij}) d\mu dy.$$

We easily check by symmetry that $D_{ij} = 0$ if $i \neq j$. So the source term in the equation for w_{2ij} equals h_{ij}. Since θ^i is skew-symmetric with respect to $\mathcal{P}_i$, $\left(\mu_i + \mu \cdot \nabla_y \theta^i + \mu \cdot \nabla \left(\frac{(\psi_\infty^+)^2}{\Sigma} \theta^i\right)\right)$ is also skew-symmetric. Then w_{2ij} and therefore $\mu \cdot \nabla_y w_{2ij}$ are skew-symmetric with respect to $\mathcal{P}_i$. Here again, for all values of k, the integral vanishes. The last term is

$$\int_V \int_Y \frac{(\psi_\infty^+)^2}{\Sigma} \mu_i \mu \cdot \nabla_y w_{2a} d\mu dy.$$

The source terms in the equation for w_{2a}, and then w_{2a} and $\mu \cdot \nabla_y w_{2a}$ are symmetric. Since μ_i is skew-symmetric with respect to $\mathcal{P}_i$, the corresponding integral vanishes. It follows that w_{10} satisfies (32) on $\Omega \times V$.

We do not give here the equation satisfied by w_3 explicitly. However, w_3, like w_2, is the sum of products of functions of (y, μ) and of functions of x. The terms depending on (y, μ) are regular by hypothesis and the terms depending on x depend on fifth order derivatives of w_0, first order derivatives of q and second order derivatives of w_{10}. By hypothesis for q, w_0 and w_{10} are sufficiently regular. This yields that w_3 is well defined and has continuous second order derivatives. Again w_3 is defined up to an additive function $w_{30}(x)$. We choose $w_{30} = 0$.

ii) It remains to define a constant L such that the term ζ^ε be small. Assume that L is given and insert the expressions of the functions w_i given by (52) into (27). We obtain

$$\begin{aligned} -\mu \cdot \nabla \frac{(\psi_\varepsilon^+)^2}{\Sigma^\varepsilon} \mu \cdot \nabla \zeta^\varepsilon + \frac{1}{\varepsilon^2} Q_\varepsilon \zeta^\varepsilon &= \varepsilon^2 \zeta_1^\varepsilon + \varepsilon^3 \zeta_2^\varepsilon \qquad \text{in } \Omega \times V \\ \zeta^\varepsilon - \frac{\varepsilon \psi_\varepsilon^+}{\psi_\varepsilon \Sigma^\varepsilon} \mu \cdot \nabla \zeta^\varepsilon &= \varepsilon \zeta_3^\varepsilon + \varepsilon^2 \zeta_4^\varepsilon + \varepsilon^3 \zeta_5^\varepsilon + \varepsilon^4 \zeta_6^\varepsilon, \qquad \text{on } \Gamma_- \end{aligned} \tag{54}$$

where $\zeta_i^\varepsilon(x, \mu) = \zeta_i(x, \frac{x}{\varepsilon}, \mu)$ and

$$\begin{aligned} \zeta_1 &= -\frac{(\psi_\infty^+)^2}{\Sigma} (\mu \cdot \nabla_x)^2 w_2 - \mu \cdot \nabla_x \frac{(\psi_\infty^+)^2}{\Sigma} \mu \cdot \nabla_y w_3 - \mu \cdot \nabla_y \frac{(\psi_\infty^+)^2}{\Sigma} \mu \cdot \nabla_x w_3 \\ \zeta_2 &= -\frac{(\psi_\infty^+)^2}{\Sigma} (\mu \cdot \nabla_x)^2 w_3, \qquad \zeta_3 = -w_1 + \frac{\psi_\infty^+}{\psi_\infty \Sigma} (\mu \cdot \nabla_x w_0 + \mu \cdot \nabla_y w_1) \\ \zeta_4 &= -w_2 + \frac{\psi_\infty^+}{\psi_\infty \Sigma} (\mu \cdot \nabla_x w_1 + \mu \cdot \nabla_y w_2) \\ \zeta_5 &= -w_3 + \frac{\psi_\infty^+}{\psi_\infty \Sigma} (\mu \cdot \nabla_x w_2 + \mu \cdot \nabla_y w_3), \qquad \zeta_6 = \frac{\psi_\infty^+}{\psi_\infty \Sigma} \mu \cdot \nabla_x w_3. \end{aligned} \tag{55}$$

Each term ζ_i gives rise to contributions of order ε^2 except ζ_3. We would like to derive some boundary conditions on w_{10}, which enable to cancel out this first order term. This cannot be done in general because ζ_3 depends on the variables x, y and μ, whereas w_{10} only depends on x. Replacing w_1 by its expression in terms of w_0 and w_{10}, we find that

$$\zeta_3(x,y,\mu) = \kappa_i(y,\mu)\frac{\partial w_0}{\partial x_i}(x) - w_{10}(x),$$

where $\kappa_i(y,\mu) = \left(\frac{\psi_\infty^+}{\psi_\infty\Sigma}(\mu_i + \mu\cdot\nabla_y\theta^i) - \theta^i\right)(y,\mu)$. Let us define the first order term b^ε as a solution of

$$\begin{aligned} A_\varepsilon b^\varepsilon = 0 \qquad & \text{in } \Omega\times V \\ b^\varepsilon - \frac{\varepsilon\psi_\varepsilon^+}{\psi_\varepsilon\Sigma^\varepsilon}\mu\cdot\nabla b^\varepsilon = \zeta_3^\varepsilon \qquad & \text{on } \Gamma_-. \end{aligned} \tag{56}$$

From the linearity of the transport equations, the function b^ε is the sum of $2d$ terms having vanishing boundary conditions on each side of Ω but one. For all $x\in I\!R^d$, we introduce the notation $x = (x_1, x')$, where x_1 is the first coordinate of x and x' the last $(d-1)^{th}$ ones. By symmetry we only consider the side $x_1 = 0$. Denote by b_1^ε the function satisfying the same equation as b^ε on $\Omega\times V$, with the same condition on the boundary where $x_1 = 0$ and vanishing boundary conditions on the other sides of Ω. For conciseness, we do not write the equation satisfied by b_1^ε. Since $w_0 = 0$ on $\partial\Omega$, we obtain for $x = (0, x')$ the relation:

$$\zeta_3^\varepsilon(x,\mu) \;=\; \kappa_1(\frac{x}{\varepsilon},\mu)\frac{\partial w_0}{\partial x_1}(x) - w_{10}(x).$$

We are interested in the asymptotic behavior of b_1^ε. Let ε go to 0 and consider a point $(0,x')$ on the side $x_1 = 0$. We perform a stretching around the point $(0,x')$ in order to obtain an equation for $b_1^\varepsilon(\varepsilon x,\mu)$. In the limit $\varepsilon = 0$ we obtain formally the following equation for $b_\infty(y,\mu)$, where x' is a parameter:

$$\begin{aligned} -\mu\cdot\nabla_y\frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla_y b_\infty + Qb_\infty &= 0 \qquad \text{in } I\!R^+\times I\!R^{d-1}\times V, \\ b_\infty - \frac{\psi_\infty^+}{\psi_\infty\Sigma}\mu\cdot\nabla_y b_\infty &= \kappa_1(y,\mu)\left[\frac{\partial w_0}{\partial x_1}(0,x')\right] - [w_{10}(0,x')] \qquad \text{on } \Gamma_-. \end{aligned} \tag{57}$$

This problem is similar to the Milne problem presented for instance in [14, 20, 33]. For some particular matching conditions between w_{10} and $\frac{\partial w_0}{\partial x_i}$ we obtain that b_∞ decays exponentially fast as $x_1\to+\infty$. This problem is studied in Lemma 6.2, given in section 6. We deduce from this lemma that (57) admits a unique solution in $W^2(I\!R^+\times I\!R^{d-1}\times V)$, which converges, as x_1 goes to infinity, to a constant term $G(\kappa_1\frac{\partial w_0}{\partial x_1} - w_{10}) = G(\kappa_1)\frac{\partial w_0}{\partial x_1} - w_{10}$, where $G(\kappa_1) = L$ is a constant. We obtain that the layer b_∞ vanishes when $x_1\to\infty$ provided $G(\kappa_1\frac{\partial w_0}{\partial x_1} - w_{10}) = 0$. In other words,

$$L\frac{\partial w_0}{\partial n} + w_{10} = 0. \tag{58}$$

This corresponds to the boundary conditions (32).

Let us now prove that b^ε is of order $O(\varepsilon^{1/2})$ in $L^2(\Omega\times V)$ when (58) is satisfied. We define

$$\gamma^0 = \{x\in\partial\Omega \text{ s.t. } x_1 = 0\}.$$

The variational formulation for b_1^ε is given for every test function $v\in\mathcal{V}$ by

$$\begin{aligned} &\int_\Omega\int_V \frac{(\psi_\varepsilon^+)^2}{\Sigma^\varepsilon}\mu\cdot\nabla b_1^\varepsilon\,\mu\cdot\nabla v\,d\mu dx + \frac{1}{\varepsilon^2}\int_\Omega\int_V (Q_\varepsilon b_1^\varepsilon)v\,d\mu dx + \frac{1}{\varepsilon}\int_{\partial\Omega}\int_V |\mu\cdot n|h_\varepsilon b_1^\varepsilon v\,d\mu d\sigma \\ =\;& \frac{1}{\varepsilon}\int_{\gamma^0}\int_V |\mu\cdot n|h_\varepsilon(\kappa_1(\frac{x}{\varepsilon},\mu) - L)\partial_1 w_0(x')v\,d\mu d\sigma. \end{aligned} \tag{59}$$

Let us introduce

$$d^\varepsilon(x_1, x', \mu) = -\partial_1 w_0(x') U(x_1) b(\frac{x}{\varepsilon}, \mu), \tag{60}$$

where $U(x_1) \in C^\infty(\mathbb{R}^+)$ is such that $U(0) = 1$ and $U(x_1) = 0$ for $x_1 \geq 1$, and where b is the solution to (66) with $g = \kappa_1 - L$, $S = 0$ and $T = 0$. According to Lemma 6.2, b is exponentially decaying to 0. We have that

$$\begin{aligned} A_\varepsilon d^\varepsilon &= \tilde{S} - \mu\cdot\nabla\tilde{T} \quad \text{in } X \\ d^\varepsilon - \frac{\varepsilon\psi_\varepsilon^+}{\psi_\varepsilon\Sigma^\varepsilon}\mu\cdot\nabla d^\varepsilon &= (L - \kappa_1(\frac{x}{\varepsilon}, \mu))\partial_1 w_0(x') \quad \text{on } \Gamma_-^0, \end{aligned} \tag{61}$$

with the notations of section 6. Here we have

$$\begin{aligned} \tilde{T} &= 2b(\frac{x}{\varepsilon}, \mu)\mu\cdot\nabla(-\partial_1 w_0(x')U(x_1))\frac{(\psi_\varepsilon^+)^2}{\Sigma^\varepsilon} \\ \tilde{S} &= (\mu\cdot\nabla)^2(-\partial_1 w_0(x')U(x_1))\frac{(\psi_\varepsilon^+)^2}{\Sigma^\varepsilon} b(\frac{x}{\varepsilon}, \mu). \end{aligned}$$

According to the Y'-periodicity of b (see section 6 for the notations), we obtain the following variational formulation for d^ε

$$\begin{aligned} &\int_X \frac{(\psi_\varepsilon^+)^2}{\Sigma^\varepsilon}\mu\cdot\nabla d^\varepsilon\, \mu\cdot\nabla v\, d\mu dx + \frac{1}{\varepsilon^2}\int_X (Q_\varepsilon d^\varepsilon) v\, d\mu dx + \frac{1}{\varepsilon}\int_{\Gamma^0} |\mu\cdot n| h_\varepsilon d^\varepsilon v\, d\mu d\sigma \\ =\ & \frac{1}{\varepsilon}\int_{\Gamma^0} [|\mu\cdot n| h_\varepsilon (\kappa_1(\frac{x}{\varepsilon}, \mu) - L)\partial_1 w_0(x') v - \varepsilon(\mu\cdot n)\tilde{T} v] d\mu d\sigma + \int_X (\tilde{S} v + \tilde{T}\mu\cdot\nabla v) d\mu dx \end{aligned} \tag{62}$$

Let us define $\delta^\varepsilon = d^\varepsilon - b_1^\varepsilon$. Since $w_0 = 0$ on $\partial\Omega$, we have $d^\varepsilon = 0$ on $\partial\Omega\backslash\gamma^0$. Therefore, subtracting (59) from (62), we obtain that

$$\begin{aligned} &\int_\Omega\int_V \frac{(\psi_\varepsilon^+)^2}{\Sigma^\varepsilon}\mu\cdot\nabla\delta^\varepsilon\, \mu\cdot\nabla v\, d\mu dx + \frac{1}{\varepsilon^2}\int_\Omega\int_V (Q_\varepsilon\delta^\varepsilon) v\, d\mu dx + \frac{1}{\varepsilon}\int_{\partial\Omega}\int_V |\mu\cdot n| h_\varepsilon\delta^\varepsilon v\, d\mu d\sigma \\ =\ & -\int_{\gamma^0}\int_V (\mu\cdot n)\tilde{T} v d\mu d\sigma + \int_\Omega\int_V (\tilde{S} v + \tilde{T}\mu\cdot\nabla v) d\mu dx. \end{aligned} \tag{63}$$

Because b decays exponentially fast, we deduce that

$$\|b(\frac{x}{\varepsilon}, \mu)\| \leq C\sqrt{\varepsilon}.$$

Therefore the source terms $\tilde{T}$ and $\tilde{S}$ satisfy the same bound. Choose $v = \delta^\varepsilon$ is this expression. We have

$$\|\mu\cdot\nabla\delta^\varepsilon\|^2 + \frac{1}{\varepsilon}\|\delta^\varepsilon\|^2_{L^2(\Gamma_-, d\xi)} \leq C\|\delta^\varepsilon\|_{L^2(\Gamma_-, d\xi)} + C\sqrt{\varepsilon}(\|\delta_\varepsilon\| + \|\mu\cdot\nabla\delta^\varepsilon\|).$$

Recalling the Poincaré inequality (42), we easily obtain that

$$\|\delta^\varepsilon\| + \|\mu\cdot\nabla\delta^\varepsilon\| + \varepsilon^{-1/2}\|\delta^\varepsilon\|_{L^2(\Gamma_-, d\xi)} \leq C\sqrt{\varepsilon}.$$

Since $b_1^\varepsilon = d^\varepsilon - \delta^\varepsilon$, we deduce that $\|b_1^\varepsilon\| \leq C\sqrt{\varepsilon}$. This concludes the proof of the theorem. ◻

6. The conservative multi-D Milne problem in a periodic half space. In this section, we state the lemma we used in the proof of Theorem 3.11. The study of problem (57) plays a crucial role in the construction of the first order corrector. For homogeneous problems in slab geometry, it is often referred to as the conservative Milne problem. It has been studied by many authors in the setting of the first-order integro-differential form of the transport equation. First introduced by astrophysicists [18], it has been analyzed in [14] using probabilistic techniques. More recently, it has been revisited in [11] using results of functional analysis. The tools introduced by these authors, such as the exponential decay analysis, are used here. The specific structure of the second order formulation in a genuine multi-D geometry necessitates to employ additional methods.

Although we do not need it in the proof of convergence, we also study the half space problem with exponentially decaying source term away from the boundary. Similar results were derived in diffusion theory for first order correctors of source problems [13].

Let us introduce some notations. For $x \in \mathbb{R}^d$, we have $x = (x_1, x')$. We denote by $Y' = (0,1)^{d-1}$ and for $A \in \mathbb{R}^+$, $\Gamma^A = \{A\} \times Y' \times V$. Let be $X_{AB} = (A,B) \times Y' \times V$ and $X = X_{0,+\infty}$. Its boundary is decomposed as $\partial X = \Gamma^{\#} \cup \Gamma^0$. The functional space $D(X)$ is defined by

$$D(X) = \{u \in W^2_{\#,loc}(X), \|\mu\cdot\nabla u\|_{L^2(X)} + \|u- <u> \|_{L^2(X)} + \|u\|_{L^2(\Gamma^0_-,d\xi)} < +\infty\}. \tag{64}$$

Here $W^2_{\#,loc}(X)$ is the space of functions of $W^2_{loc}(X)$ that are $Y'-$ periodic with respect x', and Γ^0_- is the set of incoming boundary conditions for X at side Γ^0.

LEMMA 6.1. *The space $D(X)$ defined in (64) is Hilbert.*

Proof. We have to prove that the semi-norm given in (64) is a norm for $D(X)$. Actually we prove that

$$\|u\|_{X_{0M}} \le CM(\|\mu\cdot\nabla u\|_{L^2(X)} + \|u- <u> \|_{L^2(X)} + \|u\|_{L^2(\Gamma^0_-,d\xi)}).$$

This is enough to obtain the local integrability. We first replace the domain of integration of the velocity by a smaller set of positive directions. Let be B a ball centered at μ such that $\mu_1 = 0$ of given radius. Let be $\theta \in V = S^{d-1}$ and B_θ the image of B by a rotation R_θ of angle θ. Let be $f \in L^2(S^{d-1})$. We have

$$\begin{aligned}\int_{B_\theta} f^2(\mu)d\mu &= \int_{B_\theta} [(f- <f>) - (R_{-\theta}f - <f>) + R_{-\theta}f]^2 d\mu\\ &\le 3\int_B f^2(\mu)d\mu + 6\int_V (f- <f>)^2 d\mu.\end{aligned}$$

Let be $B = \{\mu \in V \text{ s.t. } \mu_1 > 1/2\}$ for instance. A finite number of angles θ is sufficient so that the union of the corresponding B_θ cover V. Then we have that

$$\|u\|_{L^2(X_{0M})} \le C(\|u\|_{(0,M)\times Y'\times B} + \|u- <u> \|_{L^2(X)})$$

where C is a universal constant. Since u is bounded at the boundary Γ^0_- and $\mu\cdot\nabla u$ is bounded in $L^2(X)$, it is not difficult to obtain at bound for $\|u\|_{(0,M)\times Y'\times B}$. We have that

$$u(x,\mu) = \int_0^{d(x,\mu)} \mu\cdot\nabla u(x - s\mu,\mu)ds + u(\overline{x},\mu), \quad (x,\mu) \in (0,M)\times Y'\times B.$$

Here $d(x,\mu)$ is the distance between $x \in \mathbb{R}^+ \times \mathbb{R}^{d-1}$ and the surface $x_1 = 0$ in the direction $-\mu$ and $\overline{x}$ the point of the interface defined by $\overline{x} = x - d(x,\mu)\mu$. We deduce from the Cauchy-Schwartz inequality that

$$|u(x,\mu)|^2 \leq CM \int_0^{d(x,\mu)} (\mu\cdot\nabla u)^2(x-s\mu,\mu)ds + 2|u(\overline{x},\mu)|^2, \tag{65}$$

where C is a universal constant. For $\mu \in B$ given, we integrate and obtain

$$\int_0^M \int_{Y'} |u(x,\mu)|^2 dx \leq CM^2 \|\mu\cdot\nabla u\|^2_{L^2(X)} + CM\|u\|^2_{L^2(\Gamma^0_-,d\xi)}.$$

It remains to integrate this relation over B to complete the proof of the lemma. ☐

We can now state the main result of this section.

LEMMA 6.2. *Let be $g \in L^2(\Gamma^0_-,d\xi)$, $S \in L^2(X)$, $T \in L^2(X)$ sufficiently regular so that its trace on γ^0 be defined, and $\nu > 0$. We assume that $\check{S} = S$ and $\check{T} = -T$. We denote by $\mathcal{N} = \|g\|_{L^2(\Gamma^0_-,d\xi)} + \|S\|_{L^2(X)} + \|T\|_{L^2(X)} + \|T\|_{L^2(\gamma^0_-,d\xi)}$. There exists a unique solution in $D(X)$ to the following boundary layer problem*

$$\begin{aligned} -\mu\cdot\nabla\frac{(\psi^+_\infty)^2}{\Sigma}\mu\cdot\nabla b_\infty + Qb_\infty &= Se^{-\nu x_1} - \mu\cdot\nabla(Te^{-\nu x_1}) \quad \text{in } X \\ b_\infty - \frac{\psi^+_\infty}{\psi_\infty\Sigma}\mu\cdot\nabla b_\infty &= g \quad \text{on } \Gamma^0_-. \end{aligned} \tag{66}$$

The solution b_∞ satisfies $\check{b}_\infty = b_\infty$. Moreover there exist three linear forms G and H and J such that b_∞ decay exponentially as $x_1 \to \infty$ to a constant $L = G(g) + H(S) + J(T)$ satisfying $|L| \leq C\mathcal{N}$. More specifically, we have

$$\begin{aligned} &\|b_\infty - L\|_{L^2(X_{M,\infty})} + \|\mu\cdot\nabla b_\infty\|_{L^2(X_{M,\infty})} \\ &+ \|(\mu\cdot\nabla)^2 b_\infty\|_{L^2(X_{M,\infty})} + \|Qb_\infty\|_{L^2(X_{M,\infty})} \leq C\mathcal{N}e^{-\beta M/2}, \end{aligned} \tag{67}$$

where C and β are constants independent of b_∞ and M. Notice that $G(1) = 1$.

Proof. i) *Existence and uniqueness of the solution.* Let be $v \in D(X)$ a test function with compact support. Multiplying (66) by v and integrating by parts yields

$$\begin{aligned} &\int_X (\frac{(\psi^+_\infty)^2}{\Sigma}\mu\cdot\nabla b_\infty \mu\cdot\nabla v + Qb_\infty v) + \int_{\Gamma^0} (\psi^+_\infty\psi_\infty)|\mu\cdot n|(b_\infty - g)v \\ = &\int_X e^{-\nu x_1}Sv + e^{-\nu x_1}T\mu\cdot\nabla v - \int_{\Gamma^0}(\mu\cdot n)Tv. \end{aligned} \tag{68}$$

The measures of integration are $d\mu dx$ on X_{AB} and $d\mu dx'$ on Γ^A. We drop them for simplicity. By density, the same equation holds true for every $v \in D(X)$. With obvious notations, problem (66) is equivalent to

$$a(b_\infty, v) = \mathcal{F}(v) \quad \forall v \in D(X). \tag{69}$$

The form a is clearly bi-continuous and coercive in $D(X)$. We deduce from the proof of Lemma 6.1 that

$$|\int_X e^{-\nu x_1}T\mu\cdot\nabla v| + |\int_X e^{-\nu x_1}Sv| + |\int_{\Gamma^0}(\psi^+_\infty\psi_\infty)|\mu\cdot n|g\,v| + |\int_{\Gamma^0}|\mu\cdot n|T\,v| \leq C\mathcal{N}\|v\|_{D(X)},$$

where C is a constant independent of g, S and $v \in D(X)$. Therefore the form $\mathcal{F}$ is linear in $D(X)$. By Lax-Milgram lemma, there exists a unique solution to (66) in $D(X)$. From equation (66), we also deduce that $\mu\cdot\nabla_y b_\infty \in D(X)$

ii) *Convergence of* b_∞ *when* $x_1 \to \infty$. Let be $A < B$. Integrating (66) by parts yields

$$\begin{aligned}(70)\qquad &\int_{X_{AB}} \Big(\frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla b_\infty \mu\cdot\nabla v + Q b_\infty v\Big) - \int_{\Gamma^A\cup\Gamma^B}\Big(\frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla b_\infty - Te^{-\nu x_1}\Big)(\mu\cdot n)v \\ = &\int_{X_{AB}} e^{-\nu x_1}Sv + e^{-\nu x_1}T\mu\cdot\nabla v.\end{aligned}$$

Let us choose $v = 1$ in this variational formulation. Recalling that $\mu\cdot n = \mu_1$ on Γ^B and $\mu\cdot n = -\mu_1$ on Γ^A, we have for $B = \infty$

$$(71)\qquad \int_{\Gamma^A} \frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla b_\infty \mu_1 = \int_{X_{A,\infty}} e^{-\nu x_1}S - \int_{\Gamma^A}(\mu\cdot n)e^{-\nu A}T.$$

Choosing now $v = b_\infty$ yields

$$\begin{aligned}&\int_{\Gamma^A} \frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla b_\infty \mu_1 b_\infty = \int_{X_{A,\infty}}\Big(-\frac{(\psi_\infty^+)^2}{\Sigma}(\mu\cdot\nabla b_\infty)^2 - (Qb_\infty)b_\infty + e^{-x_1\nu}Sb_\infty + e^{-x_1\nu}T\mu\cdot\nabla b_\infty\Big) \\ &-\int_{\Gamma^A}(\mu\cdot n)e^{-\nu A}Tb_\infty \le \int_{X_{A,\infty}}(e^{-x_1\nu}Sb_\infty + e^{-x_1\nu}T\mu\cdot\nabla b_\infty) - \int_{\Gamma^A}(\mu\cdot n)e^{-\nu A}Tb_\infty.\end{aligned}$$

We deduce from this equality, Lemma 4.1 and a classical trace theorem [16, 17] that

$$(72)\qquad e^{A\nu/2}\int_{\Gamma^A} \frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla b_\infty \mu_1 b_\infty \le \|b_\infty\|_{D(X)}\mathcal{N}.$$

Let us now choose $v = b_\infty e^{\beta x_1}$ on X_{0B}, where $0 < \beta < \nu/2$ will be determined later on. We obtain that

$$\begin{aligned}&\int_{X_{0B}}\Big(\frac{(\psi_\infty^+)^2}{\Sigma}(\mu\cdot\nabla b_\infty)^2 e^{\beta x_1} + \frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla b_\infty\mu_1 b_\infty \beta e^{\beta x_1} + (Qb_\infty)b_\infty e^{\beta x_1}\Big) + \int_{\Gamma^0}(\psi_\infty^+\psi_\infty)|\mu\cdot n| b_\infty^2 \\ &= \int_{\Gamma^0}(\psi_\infty^+\psi_\infty)|\mu\cdot n| b_\infty g + \int_{\Gamma^B}\frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla b_\infty \mu_1 b_\infty e^{\beta B} + \int_{X_{0B}} e^{-(\nu-\beta)x_1}Sb_\infty \\ &+ \int_{X_{0B}} e^{-(\nu-\beta)x_1}T(\mu\cdot\nabla b_\infty + \mu_1\beta b_\infty) - \int_{\Gamma^0}T(\mu\cdot n)b_\infty - \int_{\Gamma^B}e^{-(\nu-\beta)B}T(\mu\cdot n)b_\infty.\end{aligned}$$

We deduce from (72) and Lemma 4.1 that there exist positive constants η and C independent of B such that

$$(73)\qquad \int_{X_{0B}}\Big[\frac{(\psi_\infty^+)^2}{\Sigma}(\mu\cdot\nabla b_\infty)^2 + \beta\frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla b_\infty \mu_1 b_\infty + \eta|b_\infty - \langle b_\infty\rangle|^2\Big]e^{\beta x_1} \le C\mathcal{N}^2.$$

The second term in the left-hand side in (73) has to be estimated in terms of the two other ones, which are positive. Let us introduce the following average operators, defined for every regular function u by

$$(74)\qquad m_{\Gamma^B}(u) = \int_{\Gamma^B} u\,d\mu dx', \qquad m_M(u) = \int_{X_{M,M+1}} u\,d\mu dx, \qquad m_{Y_M}(u) = \int_{Y_M} u\,dx,$$

where $Y_M = (M, M+1)\times Y'$ is the M^{th} cell. Remark that m_{Y_M} is only a spatial averaging operator and that $m_M(u) = \int_V m_{Y_M}(u)$. We now prove that

$$(75)\qquad \|u - m_M(u)\|_{L^2(X_{M,M+1})} \le C\left(\|\mu\cdot\nabla u\|_{L^2(X_{M,M+1})} + \|u - \langle u\rangle\|_{L^2(X_{M,M+1})}\right).$$

Let be $\mu \in V$ given. Since the cell Y_M is bounded, we easily check by integration of 1D problems along the direction μ and along its transverse directions that

$$\|u(\cdot,\mu) - (m_{Y_M}(u))(\mu)\|^2_{L^2(Y_M)} \le C\|\mu\cdot\nabla u\|^2_{L^2(Y_M)},$$

where C is a constant independent of μ. Integrating over V yields

$$\|u - m_Y(u)\|_{L^2(X_{M,M+1})} \leq C\|\mu\cdot\nabla u\|_{L^2(X_{M,M+1})}.$$

On the other hand we check that

$$\|m_{Y_M}(u) - m_M(u)\|^2_{L^2(X_{M,M+1})} = \int_{X_{M,M+1}} \left(\int_{Y_M} (u- < u >)\right)^2 \leq C\|u- < u > \|^2_{L^2(X_{M,M+1})}.$$

This proves (75). In addition, we obtain that

$$\|u - m_{\Gamma^{x_1}}(u)\|_{L^2(X_{M,M+1})} \leq \|u - m_M(u)\|_{L^2(X_{M,M+1})}.$$

Therefore we have

$$\|u - m_{\Gamma^{x_1}}(u)\|_{L^2(X_{M,M+1})} \leq C\left(\|\mu\cdot\nabla u\|_{L^2(X_{M,M+1})} + \|u- < u > \|_{L^2(X_{M,M+1})}\right).$$

This inequality holds for all $M \in I\!R^+$ and since $e^{\beta M} \leq e^{\beta x_1} \leq e^{\beta}e^{\beta M}$ for $x_1 \in (M, M+1)$, we obtain

$$\|(u - m_{\Gamma^{x_1}}(u))e^{\frac{\beta}{2}x_1}\|_{L^2(X)} \leq C(\|(\mu\cdot\nabla u)e^{\frac{\beta}{2}x_1}\|_{L^2(X)} + \|(u- < u >)e^{\frac{\beta}{2}x_1}\|_{L^2(X)}). \tag{76}$$

We deduce from (71) that

$$\begin{aligned} &|\int_X \frac{(\psi_\infty^+)^2}{\Sigma}\mu\cdot\nabla b_\infty \mu_1 m_{\Gamma^{x_1}}(b_\infty)e^{\beta x_1}| = |\int_X m_{\Gamma^{x_1}}(b_\infty)e^{\beta x_1}(\int_{X_{x_1\infty}} e^{-\nu y_1}S - \int_{\Gamma^{x_1}}(\mu\cdot n)e^{-\nu x_1}T) \\ \leq\ & C\int_X e^{-(\nu-\beta)x_1}|m_{\Gamma^{x_1}}(b_\infty)|(\|S\|_{L^2(X)} + \|T\|_{L^2(X)}) \leq C\mathcal{N}^2. \end{aligned} \tag{77}$$

Here and in the sequel, C denotes a constant independent of b_∞. Then according to (76) we can rewrite (73) and obtain for some constants $\eta > 0$ and $\theta > 0$,

$$\begin{aligned} &\int_X (\mu\cdot\nabla b_\infty)^2 e^{\beta x_1} - \theta\beta\int_X |(\mu\cdot\nabla b_\infty)(b_\infty - m_{\Gamma^{x_1}}(b_\infty))| e^{\beta x_1} \\ &+\eta\int_X (b_\infty - m_{\Gamma^{x_1}}(b_\infty))^2 e^{\beta x_1} \leq C\mathcal{N}^2 \end{aligned} \tag{78}$$

Choosing now $\beta > 0$ small enough, we deduce from the Cauchy-Schwartz inequality that

$$\int_X (\mu\cdot\nabla b_\infty)^2 e^{\beta x_1} + \int_X (b_\infty - m_{\Gamma^{x_1}}(b_\infty))^2 e^{\beta x_1} \leq C\mathcal{N}^2. \tag{79}$$

Owing to (73) and (77), we deduce from (79) and the Cauchy-Schwartz inequality that

$$\int_X |b_\infty - < b_\infty >|^2 e^{\beta x_1} \leq C\mathcal{N}^2.$$

Therefore, we obtain

$$\int_X (\mu\cdot\nabla b_\infty)^2 e^{\beta x_1} + \int_X |b_\infty - < b_\infty >|^2 e^{\beta x_1} \leq C\mathcal{N}^2. \tag{80}$$

It follows from (75) and (80) that

$$\|u - m_M(u)\|_{L^2(X_{M,M+1})} \leq Ce^{-\frac{\beta M}{2}}\mathcal{N},$$

and averaging over $x_1 \in (M, M+2)$, we have

$$\|u - \frac{m_M(u) + m_{M+1}(u)}{2}\|_{L^2(X_{M,M+1})} \le \|u - \frac{m_M(u) + m_{M+1}(u)}{2}\|_{L^2(X_{M,M+2})} \le Ce^{-\frac{\beta M}{2}}\mathcal{N}.$$

Then we readily show that

$$|m_{M+1}(u) - m_M(u)| \le Ce^{-\frac{\beta M}{2}}\mathcal{N}.$$

This proves that $M \mapsto m_M(u)$ converges exponentially fast to a constant L by linearity of the transport equation. Moreover, due to the exponential rate of convergence, we have that

$$|L| \le |G(g)| + |H(S)| + |J(T)| \le C\mathcal{N}.$$

Let us introduce

$$m(u) = m_M(u) \qquad \text{for } x_1 \in (M, M+1).$$

Then $m(b_\infty)$ is an element of $L^2(X)$ that converges to L and

$$\|[b_\infty - m(b_\infty)]e^{\frac{\beta x_1}{2}}\|_{L^2(X)} \le C\mathcal{N}.$$

Hence we can conclude that b_∞ converges to L in the following sense

$$\|b_\infty - L\|_{L^2(X_{M,\infty})} \le Ce^{-\frac{\beta M}{2}}\mathcal{N}.$$

The same results are derived for $\mu \cdot \nabla_y b$ and for Qb. This completes the proof of the lemma. □

7. Numerical application and conclusion. It goes beyond the scope of this paper to present an extensive numerical application of the theory we have described. We refer to [8, 9] for a more detailed analysis. However, we show the convergence of the eigenvalues for the homogenization of a core composed of $N \times N$ uranium assemblies, which are common in thermal reactors. In its two dimensional approximation, each assembly is made of 17×17 fuel pins or control rods. We do not describe them here and refer to [9] for the details. We present the convergence of the eigenvalue $k_{eff} = 1/\lambda_1^\varepsilon$ for different values of the number of assemblies $N = 1/\varepsilon$. In table 1 are gathered the exact multiplication factor k_{eff}, the associated diffusion approximation given by $(\lambda_\infty + \varepsilon^2\nu_1)^{-1}$, corresponding to an extrapolation length $L = 0$ and the diffusion approximation $(\lambda_\infty + \varepsilon^2\omega_1^\varepsilon)^{-1}$ that corresponds to an extrapolation length of approximately $L = 0.71$. The exact extrapolation length has not been computed exactly. Instead we took the value of the extrapolation length in homogeneous media. This can be physically justified by the fact that uranium assemblies are not very heterogeneous.

As expected from the theory, the convergence of the eigenvalues without accounting for the leakage is of order ε^3. A numerical estimate is given by

$$v \;=\; \frac{\ln\frac{e_{20}}{e_{10}}}{\ln\frac{20}{10}} \simeq 2.9,$$

which is in good agreement with the theoretical value. When we account for the neutron leakage, the rate of convergence is too fast to be estimated numerically. Indeed two dimensional computations are already highly demanding, especially for a number of assemblies equal to 20×20 and 40×40, which correspond to 115600 and 462400 rods respectively. Hence we obtained an accuracy of 10^{-5} for

TABLE 1
Reference and reconstructed k_{eff} for cores composed of $N \times N$ uranium assemblies.

N	Reference	k_{eff}, $L=0$	error (10^{-5})	k_{eff}, $L=0.71$	error (10^{-5})
5	0.46598	0.45542	1056	0.46543	55
10	0.54521	0.54334	187	0.54515	6
20	0.57114	0.57089	25	0.57114	0
40	0.57825	0.57821	4	0.57825	0
∞	0.58070				

the eigenvalues. Nevertheless the gain obtained by adding an extrapolation length is clear according to the numerical results presented in table 1. We do not give here the reconstructed fluxes for both methods. However in both cases the shape of the exact fluxes is well reproduced by the reconstructed fluxes [9]. In the case of a zero extrapolation length, the reconstructed flux is shifted downwards. This is explained by the Dirichlet boundary conditions, which do not account for the neutron leakage. The error made is of the order of 5% for a core composed of 10×10 assemblies (usual cores have approximately 150 assemblies). When Robin-like boundary conditions are imposed for the diffusion approximation, the reconstructed flux has the correct shape, up to an error of less than 1%. Fine resolution close to the boundary also allows to see the exponential decay of the boundary layer [9].

In this paper, we have presented the homogenization of the transport criticality eigenvalue problem and a numerical experiment for a one-velocity periodic neutron transport problem. The generalization to multigroup and anisotropic equations is straightforward and does not require new techniques [4, 5]. The results can also be readily extended [8] to cores that are periodic up to a perturbation of order ε^2, i.e. for instance, $\Sigma(\frac{x}{\varepsilon},\mu)+\varepsilon^2\Sigma'(x,\frac{x}{\varepsilon},\mu)$. However, the extension to genuinely non periodic cores is still open. Periodicity is needed in order to obtain separation of slow and fast variables. It would be interesting, theoretically as well as practically, to understand the homogenization of locally periodic domains. The analysis of boundary layers is replace by that of interface layers, for which no theory is available at present.

Acknowledgments The author wishes to thank G. Allaire and G. Papanicolaou for many discussions and comments during the preparation of this paper, and X. Warin for his contribution to the numerical analysis presented here. He also would like to acknowledge helpful comments and remarks from the anonymous referees.

Appendix A. Equivalence between the first-order and even parity flux formulations. A linear integro-differential Boltzmann equation is usually solved to model neutron transport. In the simplified setting of one-velocity and isotropic source problems, it is given as follows. Given the cross sections Σ and Σ_s and the source term q, we want to find the neutron angular flux $\psi(x,\mu)$ satisfying

$$\begin{aligned} \mu\cdot\nabla\psi(x,\mu)+\Sigma(x)\psi(x,\mu) &= \Sigma_s(x)\int_V \psi(x,\mu')d\mu' + q(x) \quad \text{in } \Omega\times V \\ \psi(x,\mu) &= 0 \quad \text{on } \Gamma_- . \end{aligned} \tag{81}$$

The assumption of one-velocity neutrons is not essential in order to derive the even parity flux formulation, and the generalization to multigroup problems is straightforward. However isotropy of the scattering operator can be hardly avoided, even if complicated generalizations exist for non isotropic media. This restricts the use of the even parity formulation, although isotropy is a correct approximation in nuclear reactor computations.

The even parity flux formulation consists in deriving a second-order integro-differential equation for the *even parity flux*, which is defined by

$$\psi^+(x,\mu) = \frac{\psi(x,\mu)+\psi(x,-\mu)}{2}. \tag{82}$$

The main features of this formulation are that the second-order differential operator allows for the use of a variational formulation, which is convenient in theory, as well as for numerical implementations. For instance, standard finite element methods can be used as in the resolution of elliptic problems [28, chapter6]. Let us also emphasize that the number of angular directions, and therefore the computational cost, has been divided by two.

The derivation of the even parity flux equations from the first-order integro-differential equation is done as follow. Let us first introduce the *odd parity flux*

$$\psi^-(x,\mu) \;=\; \frac{\psi(x,\mu)-\psi(x,-\mu)}{2}$$

and the *scalar flux*

$$\phi(x) \;=\; \int_V \psi(x,\mu')d\mu' \;=\; \int_V \psi^+(x,\mu')d\mu'.$$

Then equation (81) written at points (x,μ) and $(x,-\mu)$ yields

$$(\mu\cdot\nabla+\Sigma)(\psi^+ + \psi^-) = \Sigma_s\phi + q, \quad (-\mu\cdot\nabla+\Sigma)(\psi^+ - \psi^-) = \Sigma_s\phi + q.$$

Summing and subtracting these equalities, we obtain that

$$\mu\cdot\nabla\psi^- + \Sigma\psi^+ \;=\; \Sigma_s\phi + q, \qquad \mu\cdot\nabla\psi^+ + \Sigma\psi^- \;=\; 0. \tag{83}$$

Since the total cross section Σ is positive, we have

$$\psi^- = -\frac{1}{\Sigma}\mu\cdot\nabla\psi^+. \tag{84}$$

It remains to insert this expression in (83) and obtain

$$-\mu\cdot\nabla\frac{1}{\Sigma(x)}\mu\cdot\nabla\psi^+(x,\mu) + \Sigma(x)\psi^+(x,\mu) = \Sigma_s(x)\int_V \psi^+(x,\mu')d\mu' + q(x). \tag{85}$$

From the boundary conditions for ψ, we deduce that

$$\begin{aligned} \psi^+ + \psi^- &= 0 \quad \text{on} \quad \Gamma_- = \{(x,\mu)\in\partial\Omega\times V;\, \mu\cdot n(x) < 0\}, \\ \psi^+ - \psi^- &= 0 \quad \text{on} \quad \Gamma_+ = \{(x,\mu)\in\partial\Omega\times V;\, \mu\cdot n(x) > 0\}. \end{aligned}$$

Then, according to (84), we have

$$\psi^+ - \frac{1}{\Sigma}\mu\cdot\nabla\psi^+ = 0 \text{ on } \Gamma_-, \quad \psi^+ + \frac{1}{\Sigma}\mu\cdot\nabla\psi^+ = 0 \text{ on } \Gamma_+. \tag{86}$$

Remark that due to the symmetry of the even parity flux, these two boundary conditions are redundant, so we only need to mention one of them. Equations (85) and (86) are called the even parity flux formulation of the neutron transport. With the assumption of sub-criticity given by

$$\Sigma(x) - \Sigma_s(x) \geq \eta > 0,$$

problem (81) admits a unique solution in $W^2(\Omega \times V)$ (see e.g. [20, chap.21]). Then from the definition (82), we obtain a weak solution of (85)-(86) in $W^2(\Omega \times V)$. The even parity flux ψ^+ also belongs to $\mathcal{V}$, defined by (40). This follows from a trace theorem [16, 17], which ensures that $\psi|_{\Gamma_+} \in L^2(\Gamma_+, d\xi)$ when $\psi \in W^2(\Omega \times V)$ and $\psi = 0$ on Γ_-.

Assume now that $\psi^+ \in \mathcal{V}$ is a weak solution of (85)-(86) and define $\psi = \psi^+ - \frac{1}{\Sigma}\mu\cdot\nabla\psi^+$. From (85), we easily deduce that

$$\mu\cdot\nabla(\psi - \psi^+) + \Sigma\psi^+ = \Sigma_s \int_V \psi + q,$$

and then (81) since (86) clearly implies that $\psi = 0$ on Γ_-. From (81) we deduce that $\psi \in W^2(\Omega \times V)$ and both formulations are equivalent. The same equivalence holds true for eigenvalue problems in bounded or periodic domains.

Appendix B. Some results on operator convergence. The results presented here are derived from the monograph [19]. Let be X a Banach space and $\mathcal{L}(X)$ the set of bounded linear operators from X to X. Let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence of operators in $\mathcal{L}(X)$. Then T_n is said to converge *compactly* to T if

- for all $x \in X$, $T_n x \to Tx$ as $n \to \infty$
- for any bounded sequence $\{x_n\}_{n\in\mathbb{N}}$ with $\|x_n\| \leq 1$, the sequence $\{(T - T_n)x_n\}_{n\in\mathbb{N}}$ is relatively compact in X.

Then we have the following result.

THEOREM B.1. *Let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence of operators in $\mathcal{L}(X)$ converging compactly to T. Let $\sigma(T)$ and $\sigma(T_n)$ be the spectra of T and T_n, respectively. Let λ be an isolated eigenvalue of T of finite multiplicity and let Γ be a closed Jordan curve in the complex plane around λ and isolating λ such that the domain Δ enclosed by Γ contain no other point of the spectrum $\sigma(T)$ than λ. Then, $\sigma(T_n) \cap \Delta$ contains a number of eigenvalues equal to the multiplicity of λ provided n is large enough.*

Moreover, let be λ_n a sequence of eigenvalues of T_n converging to λ and u_n is a sequence of normalized associated eigenvectors. Then, up to a subsequence, u_n converges to a limit u in X which is an eigenvector of T associated with λ.

The proof of this theorem relies on Theorem 5.5 p.232, Proposition 5.6 p.234, Theorem 5.10 p.236, Theorem 5.20 p.244 and Proposition 5.28 p.24 of [19]. We also have the following error estimate result. Let be λ an eigenvalue of T of finite multiplicity, and Δ defined as above. Let be P the projection on the eigenvectors of T associated with λ and P_n the projection on the eigenvectors associated with eigenvalues of T_n included in Δ. We denote by λ_{in}, $1 \leq i \leq m$ the m eigenvalues counting their multiplicities of T_n in Δ for n large enough. We note $M = PX$ and $M_n = P_n X$. for P and Q two orthogonal projections on X we define $M = PX$ and $N = QX$, and the distance between M and N

by

$$\begin{aligned}\Theta(M,N) &= \max\left(\sup_{x\in M,\,\|x\|=1}\|(1-Q)x\|,\ \sup_{x\in N,\,\|x\|=1}\|(1-P)x\|\right)\\ &= \max\left(\sup_{x\in M,\,\|x\|=1}\text{dist }(x,N),\ \sup_{x\in N,\,\|x\|=1}\text{dist }(x,M)\right).\end{aligned}$$

Then we prove the following result.

THEOREM B.2. *With the hypotheses and notations of Theorem B.1, the following quantities are at least of order* $\|(T-T_n)u\|$*:*

$$\Theta(M,M_n),\qquad \lambda-\frac{1}{m}\sum_{i=1}^{m}\lambda_{in},\qquad \frac{1}{\lambda}-\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\lambda_{in}}\quad (\textit{if } \lambda\neq 0).$$

Moreover if λ *is simple, then* $\lambda-\lambda_n$ *and* $u-u_n$ *are at least of order* $\|(T-T_n)u\|$.

REFERENCES

[1] R. Alexandre and K. Hamdache, *Homogénéisation d'équations cinétiques en milieu perforé*, C. R. Acad. Sci. Paris, t.313, Série I, (1991), pp. 339–344.

[2] G. Allaire, *Homogenization and two-scale convergence*, SIAM J. Math. Anal., No. 9, (1992), pp. 1482–1518.

[3] G. Allaire and M. Amar, *Boundary Layer Tails in Periodic Homogenization*, Preprint, (1998).

[4] G. Allaire and G. Bal, *Homogénéisation d'une équation spectrale de transport neutronique*, C. R. Acad. Sci. Paris, t.325, Série I, (1997), pp. 1043–1048.

[5] ———, *Homogenization of the criticality spectral equation in neutron transport*, Accepted in Math. Model. Anal. Num., (1998).

[6] G. Allaire and Y. Capdeboscq, *Homogenization of a spectral problem in neutronic multigroup diffusion*, Submitted to M3AS, (1998).

[7] G. Allaire and F. Malige, *Analyse asymptotique spectrale d'un problème de diffusion neutronique*, C. R. Acad. Sci. Paris, t.324, Série I, (1997), pp. 939–944.

[8] G. Bal, *Couplage d'équations et homogénéisation en transport neutronique*, *Thèse de Doctorat* de l'Université Paris 6 [in French], 1997.

[9] G. Bal and X. Warin, *Numerical results in the homogenization of the criticality eigenvalue problem*, preprint, (1998).

[10] C. Bardos, *Problèmes aux limites pour les équations aux dérivées partielles du premier ordre*, Ann. Sci. Ecole Norm. Sup. ,4, (1969), pp. 185–233.

[11] C. Bardos, R. Santos, and R. Sentis, *Diffusion, Approximation and computation of the critical size*, Trans. Am. Math. Soc., (1984), pp. 617–649.

[12] P. Benoist, *Théorie du coefficient de diffusion des neutrons dans un réseau comportant des cavités [in French]* , Note CEA-R 2278, (1964).

[13] A. Bensoussan, J. L. Lions, and G. Papanicolaou, *Boundary Layer Analysis in Homogeneization of Diffusion Equations with Dirichlet Conditions in the Half Space*, Proc. of Intern. Symp. SDE, Kyoto Univ., (1978), pp. 21–40.

[14] ———, *Boundary Layers and Homogenization of Transport Processes*, RIMS, Kyoto Univ., 1979.

[15] J. Bussac and P. Reuss, *Traité de neutronique [in French]*, Hermann, Paris, 1978.

[16] M. Cessenat, *Théorèmes de trace L^p pour des espaces de fonctions de la neutronique*, C.R. Acad. Sci. Paris, 299, Sér I, (1984), pp. 831–834.

[17] ———, *Théorèmes de trace pour des espaces de fonctions de la neutronique*, C.R. Acad. Sci. Paris, 300, Sér I, (1985), pp. 89–92.

[18] S. Chandrasekhar, *Radiative Transfer*, Oxford University Press, Cambridge, 1950.

[19] F. Chatelin, *Spectral approximation of linear operators*, Comp sci app math, Academic Press, 1983.

[20] R. Dautray and J. L. Lions, *Analyse Mathématique et calcul numérique. Tome 3*, Masson, 1984.

[21] V. Deniz, *The theory of neutron leakage in reactor lattices*, in Handbook of nuclear reactor calculations, Y. Ronen ed., vol II, (1968), pp. 409–508.

[22] L. Dumas and F. Golse, *Homogenization of transport equations* , Submitted to C.P.A.M., (1997).

[23] E. Frenod and K. Hamdache, *Homogenization of a transport kinetic equation with oscillating potentials* , Proc. Roy. Soc. Edinburgh, 126A, (1996), pp. 1247–1275.

[24] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1977.

[25] F. Golse, P.-L. Lions, B. Perthame, and R. Sentis, *Regularity of the moments of the solution of a transport equation*, Journal of Functional Analysis 76, (1988), pp. 110–125.

[26] E. W. Larsen, *Neutron transport and diffusion in inhomogeneous media. I*, J. Math. Phys.,16, (1975), pp. 1421–1427.

[27] ———, *Neutron transport and diffusion in inhomogeneous media. II*, Nuclear Science and Engineering, 60, (1976), pp. 357–368.

[28] E. E. Lewis and W. F. Miller Jr., *Computational Methods of Neutron Transport*, John Wiley and sons, New York, 1984.

[29] F. Malige, *Etude mathématique et numérique de l'homogénéisation des assemblages combustibles d'un cœur de réacteur nucléaire [in French]*, *Thèse de Doctorat* de l'Ecole polytechnique, 1996.

[30] J. Pitkäranta, *Estimates for the derivatives of solutions to weakly singular Fredholm integral equations*, SIAM J. Math. Anal. ,11 (1980) pp.952-968.

[31] J. Planchard, *Méthodes mathématiques en neutronique [in French]*, Collection de la Direction des Etudes et Recherches d'EDF, Eyrolles, 1995.

[32] F. Santosa and M. Vogelius, *First-order corrections to the homogenized eigenvalues of a periodic composite medium*, SIAM J. Appl. Math., 53 (1993), pp. 1636–1668.

[33] R. Sentis, *Study of the corrector of the eigenvalue of a transport operator*, SIAM J. Math. Anal. 16, (1985), pp. 151–166.

[34] V. S. Vladimirov, *Mathematical problems in the one-velocity theory of particle transport*, Atomic energy of Canada Ltd, Chalk River, Ont. Report AECL 1661, 1963.