Diffusion Approximation of Radiative Transfer Equations in a Channel

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Abstract

We address the propagation of elastic waves generated by an earthquake in the earth crust modeled by a channel separated from the atmosphere and the mantel by two horizontal interfaces. Geophysical studies have shown the validity of radiative transfer in this frequency regime to describe the phase space energy density of seismic waves. For long times and large distances, radiative transfer in weakly absorbing media can be approximated by a diffusion equation. However, the thickness of the crust is of the order of the transport mean free path, the average distance it takes for waves with a wavenumber v to be scattered into another wavenumber v' by interaction with the inhomogeneous underlying medium. Hence there cannot be diffusion in the vertical direction. This paper shows that diffusion is still valid in the following sense. The radiative transfer solution factors asymptotically in the limit of vanishing mean free paths as the product of a two-dimensional diffusion term in the horizontal directions and a one-dimensional transport term in the vertical direction.

1 Introduction

Radiative transport equations were first used to describe the propagation of light through turbulent atmospheres [1]. The propagation of the energy density of other high frequency waves, like acoustic or elastic waves, in heterogeneous media with weak random fluctuations is also successfully modeled by radiative transfer equations. Their derivation has been analyzed from first principles, both for quantum [2, 3, 4, 5] and classical waves [6, 7, 8, 9, 10].

One of the main objectives of seismic studies is to describe the propagation of elastic (seismic) waves generated by an earthquake. Several works confirm the validity of radiative transfer in this context. Pioneered by Aki [11], it has been the object of growing interest in the past decade [12, 13, 14]. See also the recent monograph [15]. Radiative transfer equations for elasticity involve five coupled unknowns [9]. One describes the energy density of shear waves, three the polarization state of shear waves, and the last one the energy density of pressure waves. As a first approximation, the energy density of pressure waves and the polarization parameters of shear waves can be neglected, leading to a closed equation involving only the energy density of the shear waves, a scalar quantity [15]. The results of this paper can theoretically be extended to the full system of radiative transfer equations for elastic waves. However, we will focus only on the scalar radiative transfer equation.

It goes beyond the scope of this paper to present a comprehensive derivation of radiative transfer. We instead refer to [9]. Let us still recall how the radiative transfer equations are constructed. To do so, we consider a simpler model than elastic waves, namely acoustic waves. The acoustics equations for the velocity $\mathbf{u}$ and pressure p are

$$\begin{aligned} \rho\frac{\partial \mathbf{u}}{\partial t} + \nabla p &= 0 \\ \kappa\frac{\partial p}{\partial t} + \nabla\cdot\mathbf{u} &= 0, \end{aligned} \tag{1}$$

where ρ and κ are the density and compressibility of the underlying medium, respectively. We consider high frequency solutions of (1). This means that the typical wavelength λ of the initial data is much smaller than the overall propagation distance L, so that $\eta = \lambda L^{-1} \ll 1$. The spatial energy density for the solutions of (1) is given by

$$\mathcal{E}(t,x) = \frac{\rho\mathbf{u}^2}{2} + \frac{\kappa p^2}{2},$$

as the sum of the volume density of kinetic energy and the volume density of deformation energy (see for instance [16]). No closed equation can be obtained for the energy density $\mathcal{E}(t,x)$, even in the high frequency limit $\eta \to 0$. It is however possible to obtain a transport equation for a phase space energy density $a(t,x,v)$ such that $\mathcal{E}(t,x) = \int_{I\!R^3} a(t,x,v)dv$. Here, $a(t,x,v)$ is the energy density of waves at time t and position x and propagating with *wavenumber* v. In particle transport, this is the analog of the particle density at time t, position x, and velocity v. The radiative transfer equation for $a(t,x,v)$ arises when the density ρ and compressibility κ are random and varying on the scale of the wavelength. We assume to simplify that only the density ρ is fluctuating:

$$\rho(x) = \rho_0 + \sqrt{\eta}\rho_1(\frac{x}{\eta}) \quad \text{and} \quad \kappa(x) = \kappa_0. \tag{2}$$

Here, ρ_0 and κ_0 are constant functions and ρ_1 is a mean zero spatially homogeneous random function with *autocorrelation function* $R(y) =< \rho_1(x)\rho_1(x+y) >$. We denote by $\hat{R}$ the *power spectrum* of the fluctuations, defined as the Fourier transform of R. The speed of propagation of the waves in the absence of fluctuations is given by $\nu = (\rho_0\kappa_0)^{-1/2}$. Because ρ_1 varies at the same scale as the wavelength, propagating waves interact fully with the fluctuations of the underlying medium. The macroscopic effect of these interactions is that waves with wavenumber v can be scattered into waves with different wavenumber v', similarly to the scattering of particles (neutrons or photons for instance) in scattering media. This scattering is described by the following *radiative transfer equation* for the energy density $a(t,x,v)$,

$$\frac{\partial a}{\partial t} + \frac{\nu}{|v|} v \cdot \nabla a = \frac{\pi\nu^2}{2|v|^2} \int_{I\!R^3} \delta(|v| - |v'|)(v \cdot v')^2 \hat{R}(v - v')[a(v') - a(v)]dv'. \tag{3}$$

It is in complete analogy with the linear Boltzmann equation describing the scattering of particles.

In acoustics, (3) models the propagation of acoustic waves in turbulent water. If the domain of application turns out to have the form of a channel, the results of this paper will be applicable. In seismology, which is our main application here since the geometry of the earth crust can be approximated by a channel, the scattering kernel on the right-hand side of (3) is modified to account for the interactions of the propagating shear waves with the underlying medium. In elasticity, inhomogeneities are caused by variations of the density or the Lamé parameters. Each parameter will be described by a power spectrum similar to $\hat{R}$ and a corresponding scattering operator as in (3). These scattering operators are detailed in [9, 15]. Thus, upon neglecting pressure waves and polarization effects, the energy density of shear waves propagating in the earth crust can be modeled by a radiative transfer equation of the form of (3).

The radiative transfer equation (3) needs to be supplemented with boundary conditions. As a first approximation, earth is roughly composed of two parts: the crust, which goes from the surface of the earth up to depths of roughly 50 kms, and the mantel, which corresponds to everything below the crust. It is natural to assume that no elastic energy is transfered into the atmosphere. In addition, we assume that scattering only occurs in the crust. The mantel is indeed found to be relatively homogeneous at the scale of seismic wavelengths (the density ρ_1 defined above, for instance, almost vanishes). Consequently, energy radiated into the mantel propagates away and is never scattered back to the crust. The radiative transfer problem is therefore posed only in the earth crust. A model for the transfer of energy from the crust to the mantel has been obtained in [17] assuming a random rough interface of small amplitude. Denoting by $a_{\rm inc}$ the incident energy at the surface and by $a_{\rm ref}$ the reflected energy from the surface, we have the generic expression

$$a_{\rm ref} = \mathcal{R}(a_{\rm inc}), \tag{4}$$

where $\mathcal{R}$ is a reflection operator, which is either conservative (no loss of energy at the interface with the atmosphere) or dissipative (leakage of energy in the earth mantel).

The main motivation for this paper is the following. Estimates of the transport mean free path (mfp), the average length between two successive changes of wavenumber v of the energy density by interaction with the underlying medium, range between 30 and 100 kms [12]. In the horizontal directions, propagation of energy can reach several hundreds of kilometers. For large distances compared with the mfp, it is well known that radiative transfer in lossless media is approximated by a diffusion equation [18, 19, 20]. The radiative transfer regime and its diffusive approximation have recently been used by geophysicists to model the *tail* of seismograms [12], where most of the energy of waves that undergo multiple scattering is concentrated. Since the crust is at best not much thicker than one mfp, there is no diffusion in the vertical direction. However, since the domain in the horizontal directions is large compared with the mfp, we can expect that some averaging occurs in the horizontal direction leading to diffusion and that solving the full radiative transfer equation is not necessary to obtain an accurate description of energy density.

This is the result we show in this paper. For large horizontal distances and long times, we observe a separation of variables. Asymptotically, the transport solution factors as the product of two terms. The first one is a local radiative transfer solution, which gives the energy distribution in the vertical direction and in the wavenumber. The second one is a diffusion term, which characterizes the behavior of the elastic energy density in the horizontal directions and in time. We also obtain the exponential decay in time of the radiative transfer solution that is induced by the leakage of energy at the interface between the crust and the mantel.

Let us mention that a similar separation of variables, with transport in one direction and diffusion in the others has been obtained as a singular perturbation about the classic planar analysis in [21]. An alternative approach to treat the interface between the crust and the mantel in the diffusive regime was considered in [22].

The paper is organized as follows. Section 2 presents the regime of validity of the diffusion approximation. Section 3 gives a formal expansion of the transport solution in powers of the mean free path. We obtain the asymptotic form of the transport solution and show the factorization in horizontal and vertical variables. We first treat the simpler case of conservative operators in section 3.1 and then address the general case in section 3.2. The theoretical hypotheses and results are summarized in section 4. Section 5 is devoted to the analysis of a one-dimensional eigenvalue transport problem that describes the transport solution in the vertical direction and characterizes the possible leakage of energy at the boundary of the channel. Section 6 is devoted to a proof of the main result Theorem 4.2.

2 Scaling of the Diffusive Regime

We consider the following general radiative transfer problem

$$\begin{aligned} &\partial_t a(t,x,v) + v\cdot\nabla_x a(t,x,v) + \Sigma_t(x,v)a(t,x,v) \\ &\qquad = \int_K \sigma(x,v',v)a(t,x,v')dv' + q(t,x,v) \qquad \text{in } [0,T]\times\Omega\times K \\ &a(0,x,v) = a_{\rm in}(x,v) \qquad \text{in } \Omega\times K \\ &a(t,x,v) = \mathcal{R}(a(t,x,v)_{|\Gamma_+}) \text{ on } \Gamma_-. \end{aligned} \tag{5}$$

The domain Ω is the channel $\Omega = I\!R^2\times(z_{min},z_{max})$ and K is the space of wavenumbers. Since the scattering operator in (3) is elastic, i.e. the moduli of the energy density wavenumbers before and after scattering are identical ($|v'|=|v|$), the integration in (3) is actually over the sphere of radius $|v|$. After suitable rescaling, we therefore assume to simplify that $K = S^2$, the unit sphere. The sets on which the boundary conditions are specified are defined by

$$\begin{aligned}\Gamma_- &= \{(x,v) \in \partial\Omega \times K \text{ such that } v \cdot n(x) < 0\} \\ \Gamma_+ &= \{(x,v) \in \partial\Omega \times K \text{ such that } v \cdot n(x) > 0\},\end{aligned} \tag{6}$$

where $n(x)$ represents the outward normal to Ω at point $x \in \partial\Omega$. The reflection operator at the boundary $\mathcal{R}$ describes how incident energy at the boundary is reflected back inside the domain. The coefficient Σ_t is split as

$$\Sigma_t(x,v) = \Sigma_a(x,v) + \Sigma(x,v), \qquad \text{where} \quad \Sigma(x,v) = \int_K \sigma(x,v,v')dv'. \tag{7}$$

Here, σ is the differential scattering coefficient, Σ the scattering coefficient, and Σ_a the absorption coefficient. The scattering operator in (3) is conservative, i.e. $\Sigma_a = 0$. For technical reasons and to simplify some of the proofs, we assume here that absorption is positive. Vanishing absorption can be dealt with as in [23]. We also assume that σ is a positive bounded function. The source term q is not present in (3), as earthquakes are theoretically modeled in the radiative transfer regime by imposing an initial condition. Since it does not modify the mathematical analysis, we have added it here for completeness.

The diffusive regime is reached when the transport mean free path, denoted by ε, is small compared with the size of the domain. This regime is valid for long times and large scattering coefficients. It also requires that the absorption be small enough. We scale our domain to obtain a diffusion equation independent of ε. All these changes of variables are classical for diffusion approximations [19, 20, 24]. They are summarized below:

$$\begin{array}{rclcrcl}\sigma(x,v',v) & \mapsto & \varepsilon^{-1}\sigma(x,v',v), & \qquad & \Sigma_a(x,v) & \mapsto & \varepsilon\Sigma_a(x,v) \\ t & \mapsto & \varepsilon t, & & q(t,x,v) & \mapsto & \varepsilon q(t,x,v).\end{array} \tag{8}$$

We assume here that the distance between the two surfaces of the boundary $\partial\Omega$ is on the order of the mean free path. Therefore we have $z_{max} - z_{min} = O(\varepsilon)$. To simplify, we set $z_{min} = 0$ and $z_{max} = \varepsilon$. With these new variables, the radiative transfer equation is given by

$$\begin{aligned}&\partial_t a^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x a^\varepsilon + \Sigma_a a^\varepsilon + \frac{1}{\varepsilon^2}\Big(\Sigma a^\varepsilon - \int_K \sigma(x,v',v)a^\varepsilon(t,x,v')dv'\Big) \\ &\qquad = q(t,x,v) \qquad \text{in} \quad [0,T] \times \Omega^\varepsilon \times K \\ &a^\varepsilon(0,x,v) = a^\varepsilon_{\text{in}}(x,v) \qquad \text{in} \quad \Omega^\varepsilon \times K \\ &a^\varepsilon(t,x,v) = \mathcal{R}(a^\varepsilon(t,x,v)_{|\Gamma^\varepsilon_+}) \quad \text{on} \quad \Gamma^\varepsilon_-.\end{aligned} \tag{9}$$

Here Ω^ε is the domain $I\!R^2 \times (0,\varepsilon)$ and Γ^ε_- the corresponding set of incoming boundary conditions.

The asymptotic limit of this equation is well known when Ω^ε is replaced by $I\!R^3$ (see [19, 20, 24]). The transport solution $a^\varepsilon(t,x,v)$ converges then to a limit $a_0(t,x)$, which is a solution of a diffusion equation. When Ω^ε is a fixed convex subset $\Omega \in I\!R^3$, the presence of a boundary already renders the analysis more complicated [19, 20, 24].

Here, the domain Ω^ε depends on ε and it is difficult to define in which sense the solution a^ε can converge. To overcome this difficulty, we make a change of variables to have a problem posed in a domain independent of ε. We denote by $z = x_3$ the vertical component and still denote by x the two-dimensional variable (x_1, x_2). Similarly, we denote by $\bar{v}$ the vertical component of the wavenumber $\bar{v} = (v_1, v_2)$ and by v_3 the horizontal component. Upon performing the change of variables

$$z \mapsto \frac{z}{\varepsilon}, \tag{10}$$

the radiative transfer equation (9) is recast as:

$$\begin{aligned}
&\partial_t a^\varepsilon + \frac{1}{\varepsilon}\bar{v}\cdot\nabla_x a^\varepsilon + \frac{1}{\varepsilon^2}v_3\nabla_z a^\varepsilon + \frac{1}{\varepsilon^2}\Big(\Sigma a^\varepsilon - \int_K \sigma(x,z,v',v)a^\varepsilon(v')dv'\Big)\\
&\qquad\quad +\Sigma_a a^\varepsilon = q(t,x,v) \qquad \text{in} \quad [0,T]\times\Omega\times K\\
&a^\varepsilon(0,x,z,v) = \psi_x(z,v)u_{\rm in}(x) \qquad \text{in} \quad \Omega\times K\\
&a^\varepsilon(t,x,z,v) = \mathcal{R}(a^\varepsilon(t,x,z,v)_{|\Gamma_+}) \qquad \text{on} \quad \Gamma_-.
\end{aligned} \tag{11}$$

Here we have $\Omega = I\!R^2 \times (0,1)$ independent of ε. To avoid initial layers that are theoretically well understood [24], we assume that the initial condition $a_{\rm in}$ has the specific form $\psi_x(z,v)u_{\rm in}(x)$ in (11); see section 3, Theorem 4.2, and Remark 4.4 below. We assume that the medium varies slowly in the horizontal directions x_1 and x_2, but not necessarily in the direction x_3. (This is consistent with a description of the earth crust as a layered medium for instance.) Hence the coefficient Σ_a and σ depend on $x \in I\!R^2$ and $z \in (0,1)$. We need to prescribe boundary condition on both interfaces. Let us define

$$\Gamma^i_\pm = \Gamma_\pm \cap \{z = i\} \qquad \text{for} \quad i = 0,1. \tag{12}$$

The operator $\mathcal{R}$ is given by $\mathcal{R}^i$ on the boundaries $\{z = i\}$, $i = 0,1$, acting on a regular function $f(x,v)$ defined on Γ^i_- as

$$\mathcal{R}^i f(x,v) = \mathcal{R}^i_x f(v) = \int_{K^i} r^i_x(v',v)f(x,v')dv' \quad \text{on} \quad \Gamma^i_-, \quad i = 0,1, \tag{13}$$

where $K^0 = K \cap \{v_3 < 0\}$ and $K^1 = K \cap \{v_3 > 0\}$.

The asymptotic behavior of the solutions of (11) is closely related to the boundary operator $\mathcal{R}$. We first consider the case of a conservative operator, that is to say satisfying

$$\int_{K^j} |v_3| r_x^i(v', v) dv = |v_3'| \qquad \text{a.e.} \qquad x \in I\!R^2, \tag{14}$$

for $(i, j) = (0, 1)$ and $(i, j) = (1, 0)$. This implies that incident energy at the interface is sent back inside the domain. Indeed, we readily verify that (14) implies the conservation of the energy flux at each boundary,

$$\int_{\Gamma_-^i} |v_3| a_{\text{ref}}(t, x, v) dv = \int_{\Gamma_+^i} |v_3| a_{\text{inc}}(t, x, v) dv,$$

with the notation of (4). With this assumption, we will show in section 3.1 that the solution of (11) converges asymptotically to the product of a diffusive term for the variables (t, x) and a transport term for the variables (z, v).

Energy conservation is physically satisfied at the interface between the crust and the atmosphere. However, it is not satisfied at the interface with the mantel. There, a model with a dissipative boundary operator is necessary. Since energy is lost at the interface, the problem (11) does not admit a non trivial limit as $\varepsilon \to 0$, and actually converges to 0 exponentially in time. We have to find out a suitable change of variables that allows one to capture this leakage at the interface. We will derive this exponential decay in section 3.2 and obtain the asymptotic behavior of the transport solution after suitable rescaling in time.

3 Formal Asymptotic Expansion

3.1 Conservative Boundary Operators

Let us first assume that the boundary operator is conservative on both interfaces, i.e. (14) is satisfied. To understand the asymptotic behavior as $\varepsilon \to 0$ of the solutions to (11), we consider the following expansion,

$$a^\varepsilon(t, x, z, v) = a_0(t, x, z, v) + \varepsilon a_1(t, x, z, v) + \varepsilon^2 a_2(t, x, z, v) + o(\varepsilon^2). \tag{15}$$

Plugging this relation into (11) and equating like powers of ε yields three equations. The first one is

$$\begin{aligned} & v_3 \nabla_z a_0 + \Big(\Sigma a_0 - \int_K \sigma(x,z,v',v) a_0(v') dv'\Big) = 0 \\ & a_0(0,x,z,v) = \psi_x(z,v) u_{\rm in}(x) \qquad \text{in} \quad \Omega \times K \\ & a_0(t,x,z,v) = \mathcal{R}(a_0(t,x,z,v)_{|\Gamma_+}) \quad \text{on} \quad \Gamma_-. \end{aligned} \tag{16}$$

Let $x \in I\!R^2$ and $t \in (0,T)$ be fixed. Define $\sigma_x(z,v',v) = \sigma(x,z,v',v)$ and $\Sigma_x(z,v) = \Sigma(x,z,v)$. We consider the problem:

$$\begin{aligned} & v_3 \nabla_z \psi_x + \Big(\Sigma_x \psi_x - \int_K \sigma_x(z,v',v) \psi_x(z,v') dv'\Big) = 0 \quad \text{in} \quad (0,1) \times K \\ & \psi_x(z,v) = \mathcal{R}_x(\psi_x(z,v)_{|\Gamma_+}) \quad \text{on} \quad \Gamma_-. \end{aligned} \tag{17}$$

It is shown in section 5, Theorem 5.4, that this problem admits a unique normalized positive solution when the boundary operators are point-wise conservative, i.e. satisfy (14). Therefore, a_0 satisfies the same equation as ψ_x for fixed t and x, and we deduce the existence of a function $u(t,x)$ such that

$$a_0(t,x,z,v) = u(t,x) \psi_x(z,v). \tag{18}$$

This is the separation of scales mentioned in the introduction. It now remains to derive an equation for $u(t,x)$.

The second equation obtained from (15) is

$$\begin{aligned} & \bar{v} \cdot \nabla_x a_0 + v_3 \nabla_z a_1 + \Big(\Sigma a_1 - \int_K \sigma(x,z,v',v) a_1(v') dv'\Big) = 0 \\ & a_1(0,x,z,v) = 0 \qquad \text{in} \quad \Omega \times K \\ & a_1(t,x,z,v) = \mathcal{R}(a_1(t,x,z,v)_{|\Gamma_+}) \quad \text{on} \quad \Gamma_-. \end{aligned} \tag{19}$$

Again, x and t are parameters in the equation. We obtain that a_1 has the form

$$a_1(t,x,z,v) = \sum_{i=1}^{2} \theta_x^i(z,v) \frac{\partial u}{\partial x_i}(t,x), \tag{20}$$

where the correctors θ_x^i are solutions of the source problems for $i = 1, 2$:

$$\begin{aligned} & v_3 \nabla_z \theta_x^i + \Big(\Sigma_x \theta_x^i - \int_K \sigma_x(z,v',v) \theta_x^i(z,v') dv'\Big) = -v_i \psi_x(z,v) \quad \text{in} \quad (0,1) \times K \\ & \theta_x^i(z,v) = \mathcal{R}(\theta_x^i(z,v)_{|\Gamma_+}) \quad \text{on} \quad \Gamma_-. \end{aligned} \tag{21}$$

These problems have a solution if the following Fredholm alternative is satisfied,

$$\int_0^1 \int_K v_i \psi_x(z,v) dz dv = 0. \tag{22}$$

This compatibility condition is obtained in (53) in Theorem 5.4 with $\psi^* \equiv 1$; see Corollary 5.5. The condition (22) is a *crucial* hypothesis, and is not always satisfied. We can verify that it is satisfied in the case of rotational invariant differential scattering coefficient $\sigma(x, z, v, v') = \sigma(x, z, v \cdot v')$; see also [23]. It is therefore satisfied for scattering kernels of the form given in (3) provided $\hat{R}(v - v')$ is a function of $|v - v'|$ only, which is the case in practice [9]. There are cases however where this symmetry is violated; see for instance [25]. We *assume* that the condition (22) is satisfied so that (20) holds.

The third equation obtained from (15) is

$$\begin{aligned} &\partial_t a_0 + \bar{v} \cdot \nabla_x a_1 + v_3 \nabla_z a_2 + \Sigma_a a_0 + \Big(\Sigma a_2 - \int_K \sigma(x, z, v', v) a_2(v') dv'\Big) = q \\ &a_2(0, x, z, v) = 0 \qquad \text{in} \qquad \Omega \times K \\ &a_2(t, x, z, v) = \mathcal{R}(a_2(t, x, z, v)_{|\Gamma_+}) \qquad \text{on} \qquad \Gamma_-. \end{aligned} \tag{23}$$

As before, for fixed x and t, this source equation admits a solution only if the following compatibility condition holds,

$$\int_0^1 \int_K \Big(\partial_t a_0 + \bar{v} \cdot \nabla_x a_1 + \Sigma_a a_0 - q\Big) dv dz = 0. \tag{24}$$

Let us assume that $a_{\text{in}}(x, z, v) = \psi_x(z, v) u_{\text{in}}(x)$ to avoid to deal with initial layers, where u_{in} is a smooth function. We already know that a_0 has the form (18). Therefore, this compatibility condition is equivalent to

$$\begin{aligned} &\partial_t u - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\Big(D_{ij}(x) \frac{\partial}{\partial x_j} u\Big) + \overline{\Sigma_a}(x) u = \overline{q}(x), \\ &u(0, x) = u_{\text{in}}(x), \end{aligned} \tag{25}$$

where we have, assuming the normalization $\int_0^1 \int_K \psi_x(z, v) dz dv = 1$,

$$\begin{aligned} D_{ij}(x) &= -\int_0^1 \int_K \theta_x^i(z, v) v_j \, dz \, dv \\ \overline{\Sigma_a}(x) &= \int_0^1 \int_K \Sigma_a(x, z, v) \psi_x(z, v) dz \, dv \\ \overline{q}(x) &= \int_0^1 \int_K q(x, z, v) dz \, dv. \end{aligned} \tag{26}$$

We will show that $D_{ij}(x)$ is a positive definite tensor. Therefore (25) is well-posed. We have obtained the following asymptotic behavior of a^ε as $\varepsilon \to 0$,

$$a^\varepsilon(t, x, z, v) = u(t, x) \psi_x(y, v) + O(\varepsilon), \tag{27}$$

where $\psi_x(y, v)$ is the normalized positive solution of (17) and u is a solution of the diffusion equation (25).

3.2 Dissipative Boundary Operators

When one or both boundary operators $\mathcal{R}^i$ are dissipative, problem (17) admits no positive solution. Since there is leakage of energy at the boundary of the domain, we expect that the solution decays exponentially in time. For $\gamma \in I\!\!R$, we introduce the change of variables

$$a^\varepsilon \mapsto \exp\Big(\frac{\gamma t}{\varepsilon^2}\Big) a^\varepsilon, \qquad \text{and} \qquad q \mapsto \exp\Big(\frac{\gamma t}{\varepsilon^2}\Big) q. \tag{28}$$

Then (11) becomes

$$\begin{aligned}
&\partial_t a^\varepsilon + \frac{1}{\varepsilon}\bar{v}\cdot\nabla_x a^\varepsilon + \frac{1}{\varepsilon^2} v_3 \nabla_z a^\varepsilon + \frac{1}{\varepsilon^2}\Big(\Sigma a^\varepsilon - \int_K \sigma(x,z,v',v) a^\varepsilon(v') dv'\Big) \\
&\quad + \Sigma_a a^\varepsilon = q + \frac{\gamma}{\varepsilon^2} a^\varepsilon \qquad \text{in} \quad [0,T]\times\Omega\times K \\
&a^\varepsilon(0,x,z,v) = \psi_x(z,v) u_{\text{in}}(x) \qquad \text{in } \Omega\times K \\
&a^\varepsilon(t,x,z,v) = \mathcal{R}(a^\varepsilon(t,x,z,v)_{|\Gamma_+}) \quad \text{on} \quad \Gamma_-.
\end{aligned} \tag{29}$$

We will see that there exists a unique positive coefficient γ that allows us to justify the asymptotic expansion (15) for the solution $a^\varepsilon(t,x,z,v)$ of (29). The difference with section 3.1 is the definition of the eigenvalue problem. Instead of (17), we now have

$$\begin{aligned}
&v_3 \nabla_z \psi_x + \Big(\Sigma_x \psi_x - \int_K \sigma_x(z,v',v)\psi_x(z,v') dv'\Big) = \gamma \psi_x \quad \text{in} \quad (0,1)\times K \\
&\psi_x(z,v) = \mathcal{R}(\psi_x(z,v)_{|\Gamma_+}) \quad \text{on} \quad \Gamma_-.
\end{aligned} \tag{30}$$

We will show in section 5, Theorem 5.4, that for every dissipative operator $\mathcal{R}$, there exists a unique positive eigenvalue γ associated with the unique normalized positive eigenvector ψ_x of (30). Its adjoint equation also admits a positive eigenvalue ψ^*_x:

$$\begin{aligned}
&-v_3 \nabla_z \psi^*_x + \Big(\Sigma_x \psi^*_x - \int_K \sigma_x(z,v,v')\psi^*_x(z,v') dv'\Big) = \gamma \psi^*_x \quad \text{in} \quad (0,1)\times K \\
&\psi^*_x(z,v) = \mathcal{R}^*(\psi_x(z,v)_{|\Gamma_-}) \quad \text{on} \quad \Gamma_+.
\end{aligned} \tag{31}$$

Notice that $\psi^*_x \equiv 1$ in section 3.1 because of (7) and (14). In the dissipative case however, it is not possible to obtain an explicit expression for ψ^*_x in general. Using the expansion (15), we obtain that a_0 can be factored as in (18). The second equation now reads

$$\begin{aligned}
&\bar{v}\cdot\nabla_x a_0 + v_3\nabla_z a_1 + \Big(\Sigma a_1 - \int_K \sigma(x,z,v',v) a_1(v') dv'\Big) = \gamma a_1 \\
&a_1(0,x,z,v) = 0 \qquad \text{in } \Omega\times K \\
&a_1(t,x,z,v) = \mathcal{R}(a_1(t,x,z,v)_{|\Gamma_+}) \quad \text{on } \Gamma_-.
\end{aligned} \tag{32}$$

We deduce from the Fredholm alternative that this equation admits a solution if and only if we have for $i = 1, 2$,

$$\int_0^1 \int_K v_i \psi_x(z, v) \psi_x^*(z, v) dz dv = 0. \tag{33}$$

Like (22), this symmetry condition is crucial, and is not always satisfied. It is again satisfied in the case of rotational invariant differential scattering coefficient $\sigma(x, z, v, v') = \sigma(x, z, v \cdot v')$.

We assume that this symmetry condition is satisfied. Then a_1 still takes the form (20). Here, the correctors θ_x^i solve

$$\begin{aligned}
&v_3 \nabla_z \theta_x^i + \Big(\Sigma_x \theta_x^i - \int_K \sigma_x(z, v', v) \theta_x^i(z, v') dv' \Big) = \gamma \theta_x^i - v_i \psi_x(z, v) \quad \text{in} \quad (0,1) \times K \\
&\theta_x^i(z, v) = \mathcal{R}(\theta_x^i(z, v)_{|\Gamma_+}) \quad \text{on} \quad \Gamma_-.
\end{aligned} \tag{34}$$

Assume that $a_{\rm in}(x, z, v) = \psi_x(z, v) u_{\rm in}(x)$ where $u_{\rm in}$ is smooth. It remains to utilize the equation for a_2 and the corresponding compatibility condition to obtain the following modified equation for u,

$$\begin{aligned}
&\partial_t u - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} \Big(D_{ij}(x) \frac{\partial}{\partial x_j} u \Big) + \overline{\Sigma_a} u = \overline{q}, \\
&u(0, x) = u_{\rm in}(x).
\end{aligned} \tag{35}$$

where we have, with the normalization $\int_0^1 \int_K \psi_x(y, v) \psi_x^*(y, v) dy dv = 1$,

$$\begin{aligned}
D_{ij}(x) &= - \int_0^1 \int_K \theta_x^i(z, v) v_j \psi_x^*(z, v) dz\, dv \\
\overline{\Sigma_a} &= \int_0^1 \int_K \Sigma_a(z, v) \psi_x(z, v) \psi_x^*(z, v) dz\, dv \\
\overline{q} &= \int_0^1 \int_K q(x, z, v) \psi_x^*(z, v) dz\, dv.
\end{aligned} \tag{36}$$

Again, we will show that the diffusion tensor D_{ij} is positive definite; hence (35) is well-posed. The asymptotic expansion in the conservative case (27) is now replaced by

$$a^\varepsilon(t, x, z, v) = \exp\Big(\frac{-\gamma t}{\varepsilon^2} \Big) \Big(\psi_x(y, v) u(t, x) + O(\varepsilon) \Big), \tag{37}$$

where $(\gamma, \psi_x(y, v))$ is the first eigenvalue/eigenvector of (30) and $u(t, x)$ the solution of (36).

4 Main Results on the Diffusion Approximation

This section states more rigorously and comments on the formal results obtained in the preceding section. The technical hypotheses are the following.

Hypothesis 4.1 We assume that:

(H1) The spatial domain Ω is the channel $\mathbb{R}^2 \times (0,1)$. The space of wavenumbers K is the unit sphere S^2.

(H2) The coefficients Σ_a and σ are measurable bounded functions on their domain of definition. There are two constants α and σ_0 such that

$$\begin{aligned} 0 < \alpha \leq \Sigma_a(x,v) \qquad & \text{a.e. in } \Omega \times K, \\ 0 < \sigma_0 \leq \sigma(x,v',v) \qquad & \text{a.e. in } \Omega \times K \times K. \end{aligned} \tag{38}$$

(H3) The boundary operators $\mathcal{R}^i$ in (13) have measurable non-negative bounded kernels r_x^0 and r_x^1 for all $x \in \mathbb{R}^2$. They are said to be dissipative when

$$\int_{K^j} |v_3| r_x^i(v',v)dv \leq |v_3'| \qquad \text{a.e. } x \in \mathbb{R}^2 \text{ and } v' \in K. \tag{39}$$

They are said to be conservative when the above inequality is replaced by an equality. We assume that the map $x \mapsto \mathcal{R}_x$ is smooth. Moreover we assume to simplify that there exists an index $i = 0$ or $i = 1$ such that $\mathcal{R}_x^i$ is a compact operator for all x in the L^p sense. This is the case for instance if the kernel $r_x^i(v',v)$ is a continuous function.

The main result of this paper is the following

Theorem 4.2 *Let us assume that Hypothesis 4.1 is satisfied. For $x \in \mathbb{R}^2$, we denote by (γ, ψ_x) the unique solution of (30) such that ψ_x is positive and normalized. We assume that the symmetry condition (33) is satisfied and that γ is independent of x. For all $\varepsilon > 0$, there exists a unique solution a^ε in $C^1([0,T]; L^1(\Omega \times K))$ to the transport equation (11). We assume that the initial condition a_{in} has the form*

$$a_{\text{in}}(x,z,v) = \psi_x(z,v) u_{\text{in}}(x), \tag{40}$$

where $u_{\text{in}} \in C^\infty(\Omega)$ with compact support. Then we have

$$a^\varepsilon(t,x,z,v) = \exp\Big(\frac{-\gamma t}{\varepsilon^2}\Big)\Big(\psi_x(z,v)u(t,x) + O(\varepsilon)\Big), \tag{41}$$

where $u(t,x) \in C^1([0,T];H^1(\Omega))$ is the solution of the diffusion problem (35). The diffusion tensor D given by (36) is positive definite. Here $O(\varepsilon)$ is to be understood in the $C^0([0,T];L^1(\Omega \times K))$ sense.

The proof of this theorem is given in section 6. It requires an analysis of the local transport problems (17), (21), (30), and (34), which is done in section 5. Before dealing with its proof, let us comment on the results of this theorem.

Remark 4.3 A physical interpretation of the theorem is as follows. In the limit of vanishing mean free paths ε, the transport solution $a^\varepsilon(t,x,z,v)$ asymptotically factors as the product of three terms. The first two terms $\exp\frac{-\gamma t}{\varepsilon^2}$ and $\psi_x(z,v)$ describe the local (in the slow variable x) equilibrium reached in the limit $\varepsilon \to 0$. The first term $\exp\frac{-\gamma t}{\varepsilon^2}$ describes the exponential decay of energy density caused by the leakage at the crust-mantel interface. The rate of decay depends on the transport solution in the whole vertical cross section of the channel, given by the second term $\psi_x(z,v)$. The pair $(\gamma, \psi_x(z,v))$ are the eigenvalue/eigenvector of the local problem (30). If we assume that no energy leaks into the mantel, the problem simplifies to finding the solution of (17), with $\gamma = 0$. If we further assume that $\sigma(x,z,v',v) = \sigma(x,z,v' \cdot v)$, we easily check in this case that $\psi_x(z,v) \equiv 1$. The behavior of $a^\varepsilon(t,x,z,v) = u(t,x) + O(\varepsilon)$ becomes then trivial in the fast variables t/ε^2 and z and the wavenumber variable v. We recover the "classical" diffusive regime that prevails in the absence of boundaries or interfaces.

When leakage is present, $\gamma > 0$ and $\psi_x(z,v)$ cannot equal 1 uniformly since 1 does not satisfy the interface conditions, as is easily verified by inspection. The full problem (30) must then be solved to obtain the local equilibrium in the fast time variable t/ε^2, vertical direction z, and wavenumber v. When the symmetry condition (33) is satisfied (which is the case when $\sigma(x,z,v',v) = \sigma(x,z,v' \cdot v)$ for instance, therefore for scattering kernels of the form in (3) when $\hat{R}(v - v')$ is a function of $|v - v'|$ only), the ratio of the transport solution a^ε with $\exp\frac{-\gamma t}{\varepsilon^2}\psi_x(z,v)$ asymptotically converges to the solution $u(t,x)$ of the diffusion equation (35), which depends only on the slow time and space variables. We therefore have a separation of scales between the fast scale, which requires to be treated by transport, and the slow scale, for which the diffusion approximation is sufficiently accurate. This bears some resemblance with the separation of scales that appears in the density of neutrons in periodic nuclear reactors (see [18, 23] for instance), with the exception that no periodicity is required here and that boundary leakage is accounted for. When the symmetry condition (33) is not satisfied, the diffusion approximation is no longer valid. We then have to account for a dominating drift term similar to the one analyzed in [25].

The practical interest of this decomposition is that we have reduced the calculation of a function of six independent variables (one in time, three in space, and two for the wavenumber) to two separate calculations, both involving three independent variables; a transport problem with one spatial variable and two wavenumber variables, and a diffusion problem with one time variable and two spatial variables.

Remark 4.4 The special form of the initial condition $a_{\rm in}(x,z,v) = \psi_x(z,v)u_{\rm in}(x)$ is imposed to avoid the formation of *initial layers.* The theory of initial layers is well known in unbounded domains –we refer to [24, §3] for the mathematical details– and is not treated here. We can infer from the general theory of initial layers the following result for more general initial conditions. Assume that $a_{\rm in}(x,z,v)$ does not necessarily have the form (40). We then define

$$\overline{u_{\rm in}}(x) = \int_0^1 \int_K a_{\rm in}(x,z,v)\psi_x^*(z,v)dvdz. \tag{42}$$

Let us denote by $u(t,x) \in C^1([0,T];H^1(\Omega))$ the solution of the diffusion problem (35) with initial condition $u(0,x) = \overline{u_{\rm in}}(x)$. We then have that

$$a^\varepsilon(t,x,z,v) = \exp\Big(\frac{-\gamma t}{\varepsilon^2}\Big)\Big(\psi_x(z,v)(u(t,x) + u^{IL}(t,x)) + O(\varepsilon)\Big). \tag{43}$$

The term $u^{IL}(t,x)$ is not uniformly small, but decays exponentially in time. Therefore (41) is still accurate once the initial layer is small enough. Notice that since the initial layer $u^{IL}(t,x)$ decays exponentially in time, (41) still holds, but now $O(\varepsilon)$ has to be understood in the $L^1((0,T)\times\Omega\times K))$ sense. Therefore, a seismogram detecting the energy density over a period of time $(0,T)$, which is large compared with the mean free time –of order ε here– will capture an amount of energy very close to that predicted by the diffusion approximation (41).

5 Eigenvalue Problem with General Boundary Conditions in a Slab

This section analyzes problems (17), (21), (30), and (34), which have been used in section 3 to describe the asymptotic behavior of the energy distribution inside the channel in the variables t/ε^2, z, and v. Some of the compactness results given here can also be found in [26, 27] in a more general setting. We have included them here for completeness.

Let λ be a constant. Define $X = (0,1) \times K$. Consider the following source problem, for $q \in L^p(X)$,

$$\begin{aligned}
&v_3 \nabla_z \psi(z,v) + \Big(\Sigma\psi - \int_K \sigma(z,v',v)\psi(z,v')dv'\Big) - \lambda\psi = q(z,v) \qquad \text{in} \quad X \\
&\psi(0,v) = \int_{K^0} r^0(v',v)\psi(0,v')dv' \qquad v \in K^1 \\
&\psi(1,v) = \int_{K^1} r^1(v',v)\psi(1,v')dv' \qquad v \in K^0.
\end{aligned} \tag{44}$$

We define the optical length, for $y, z \in (0,1)$ and $v \in K$, as

$$\alpha_\lambda(y,z,v) = \int_{(y,z)} |v_3|^{-1}[\Sigma(s,v) - \lambda]ds, \tag{45}$$

and the operators

$$Tu(z,v) \;=\; v_3\nabla_z u(z,v) + \Sigma u(z,v), \qquad Fu(z,v) \;=\; \int_K \sigma(z,v',v)u(z,v')dv'. \tag{46}$$

All problems (17), (21), (30), and (34) can be treated by analyzing the spectral decomposition of the linear operator $q \mapsto \psi = (T - F - \lambda)^{-1}q$, where ψ is a solution (if any) to (44). However, the spectrum of $(T - F - \lambda)^{-1}$ is fairly complicated. For instance, we know that in the absence of reflection at the boundary, i.e. $r^0 = r^1 \equiv 0$, the continuous part of the spectrum of $(T - F - \lambda)^{-1}$ is not empty [20, 28]. Therefore, we analyze instead the spectrum of $(T - \lambda)^{-1}F$, which is simpler because $(T - \lambda)^{-1}F$ is compact and positive. This will be sufficient to solve the problems (17), (21), (30), and (34). Let us recall an important result on compact operators that leave invariant the cone of positive functions [20].

Theorem 5.1 [Krein Rutman] Let T be a compact operator in some Banach space $L^p(\Omega)$ leaving invariant the cone of positive functions, i.e. $T\psi \geq 0$ if $\psi \geq 0$. Then T admits a unique positive eigenvector. It is associated with a real positive eigenvalue $\lambda = \rho(T)$, where $\rho(T)$ is the spectral radius of T. Moreover λ is simple, and every other eigenvalue μ of T satisfies $|\mu| \leq \lambda$.

Let us denote by $L^p_{|v_3|}(K^i)$ for $1 \leq p < \infty$ the set of functions u such that

$$\int_{K^i} |v_3||u(v)|^p dv < \infty. \tag{47}$$

We denote by $W^p(\Omega \times K)$ the set of functions

$$W^p(\Omega \times K) = \{u \in L^p(\Omega \times K) \text{ such that } v \cdot \nabla u \in L^p(\Omega \times K)\}. \tag{48}$$

We begin with a result for the free transport problem

Lemma 5.2 *Let $q \in L^p(X)$, $1 \leq p < \infty$. Then the following problem admits a unique solution ψ in $W^p(X)$ for $\lambda < \lambda^*$, where λ^* is an explicitly computable positive constant,*

$$\begin{aligned} v_3 \nabla_z \psi + (\Sigma - \lambda)\psi &= q(z,v) \qquad in \qquad X \\ \psi(0,v) &= \int_{K^0} r^0(v',v)\psi(0,v')dv' \qquad v \in K^1 \\ \psi(1,v) &= \int_{K^1} r^1(v',v)\psi(1,v')dv' \qquad v \in K^0. \end{aligned} \tag{49}$$

Moreover we have $\psi(0,v) \in L^p_{|v_3|}(K^1)$ and $\psi(1,v) \in L^p_{|v_3|}(K^0)$.

Proof. Solving the free transport equation by the method of characteristics yields

$$\begin{aligned} \psi(x,v) &= \int_0^x v_3^{-1} e^{-\alpha_\lambda(s,x,v)} q(x-s,v)ds + e^{-\alpha_\lambda(0,x,v)}\psi(0,v) \qquad v \in K^1, \\ \psi(x,v) &= \int_x^1 |v_3|^{-1} e^{-\alpha_\lambda(x,s,v)} q(s-x,v)ds + e^{-\alpha_\lambda(x,1,v)}\psi(1,v) \quad v \in K^0. \end{aligned}$$

With the boundary conditions of (49), we obtain a system of four equations for the unknown densities at the boundary. Let us define

$$\psi^k_\pm(v) = \psi(k,v) \quad \text{for} \quad k = 0,1 \quad \text{and} \quad v \in K^1 \text{ for } +;\ v \in K^0 \text{ for } -.$$

Using the solution of the free transport equation and the boundary conditions in (49), we eliminate three densities at the interface and obtain the equation for ψ^1_+,

$$\psi^+_1 = \mathcal{L}_\lambda \psi^+_1 + \mathcal{Q}, \tag{50}$$

where we easily check using Hölder inequalities that $\mathcal{Q} \in L^p_{|v_3|}(K^1)$ since the boundary operators are bounded in $L^p_{|v_3|}(K^i)$ for $i = 0,1$, and where the operator $\mathcal{L}_\lambda$ is given by

$$\mathcal{L}_\lambda \psi = \exp\left[-\alpha_\lambda(0,1,v)\right]\mathcal{R}^0\Big(\exp\left[-\alpha_\lambda(0,1,v')\right]\mathcal{R}^1\psi\Big). \tag{51}$$

We check that $\mathcal{L}_\lambda$ is a positive operator for all λ, in the sense that $\psi \geq 0$ implies $\mathcal{L}_\lambda \psi \geq 0$. Moreover, since one of the response operators is compact by assumption, $\mathcal{L}_\lambda$ is compact in every L^p. It follows from the Krein Rutman theorem that $\mathcal{L}_\lambda$ admits a unique positive normalized eigenvector ψ_λ, associated with a positive real eigenvalue ν_λ, which is simple and equal to the spectral radius of $\mathcal{L}_\lambda$. The spectral radius is therefore independent of p. We have the more explicit expression for $\mathcal{L}_\lambda$,

$$\mathcal{L}_\lambda\psi = \exp\left[\frac{\lambda}{|v_3|}\right]\exp\left[-\alpha_0(0,1,v)\right]\mathcal{R}^0\Big(\exp\left[\frac{\lambda}{|v_3'|}\right]\exp\left[-\alpha_0(0,1,v')\right]\mathcal{R}^1\psi\Big).$$

Since the reflection operators are conservative or dissipative in L^1, we easily deduce that $\rho(\mathcal{L}_0) < 1$ since Σ is uniformly bounded from below by a positive constant. Moreover using

$$\mathcal{L}_\lambda\psi_\lambda = \nu_\lambda\psi_\lambda$$

for two close values λ and $\lambda+\delta\lambda$ and letting $\delta\lambda$ go to zero yields that $\lambda \mapsto (\nu_\lambda, \psi_\lambda)$ is continuously differentiable. We easily obtain that $\lambda \mapsto \nu_\lambda$ is a strictly increasing function. In addition, since $|v_3| \in [0,1]$, we have

$$\rho(\mathcal{L}_\lambda) \geq \exp\left[2\lambda\right]\rho(\mathcal{L}_0)$$

hence $\rho(\mathcal{L}_\lambda)$ is increasing and goes to infinity when λ goes to infinity. There is therefore a unique λ^* such that $\rho(\mathcal{L}_{\lambda^*}) = 1$. For all $\lambda < \lambda^*$, we readily obtain the existence of a unique solution $\psi_+^1 \in L^p_{|v_3|}(K^1)$ to the boundary problem, hence the existence of a unique solution $\psi(x,v) \in L^p(X)$ to the full transport problem. Moreover, we deduce that

$$\psi_+^1 = \sum_{k=0}^{\infty}(\mathcal{L}_\lambda)^k\mathcal{Q}$$

hence ψ_+^1 and $\psi(x,v)$ are non-negative functions if q is non-negative, since we readily verify that the mapping $q \mapsto \mathcal{Q}$ is non-negative. $\Box$

Theorem 5.3 *For every $\lambda < \lambda^*$, the operators $F(T-\lambda)^{-1}$ and $(T-\lambda)^{-1}F$ are compact in $L^p(X)$ for $1 < p < \infty$ and leave invariant the cone of positive functions, i.e. $q \geq 0$ implies $F(T-\lambda)^{-1}q \geq 0$ and $(T-\lambda)^{-1}Fq \geq 0$.*

Proof. The positiveness of $F(T-\lambda)^{-1}$ and $(T-\lambda)^{-1}F$ is clear since F and $(T-\lambda)^{-1}$ are obviously positive. The proof is based on the averaging lemma in L^p space [28, 29]. Let $\mathcal{Q}$ be the unit ball in $L^p(X)$. We define $\mathcal{T} = (T-\lambda)^{-1}\mathcal{Q}$. According to Lemma 5.2, $u \in \mathcal{T}$ satisfies that u and $v_3\nabla_z u$ are in $L^p(X)$ and that $|v_3|^{1/p}u$ is in $L^p(\Gamma_-)$. Therefore the averaging lemma applies and we obtain that $F\mathcal{Q} = F(T-\lambda)^{-1}\mathcal{U}$ is relatively compact, implying that $F(T-\lambda)^{-1}$ is itself compact in $L^p(X)$ for $1 < p < \infty$. The same proof holds for the operator $F^*(T^*-\lambda)^{-1}$, which is compact and bounded. Therefore its adjoint operator $(T-\lambda)^{-1}F$ is also compact. $\Box$

We are now in a position to solve the problems (17) and (30) and show that the Fredholm alternative holds to solve the problems (21) and (34).

Theorem 5.4 *There exists a unique non-negative value $0 \leq \lambda_0 < \lambda^*$ such that*

$$\rho((T-\lambda_0)^{-1}F) = 1.$$

Let us denote by ψ the solution of $(T-\lambda_0)\psi = F\psi$ and by ψ^* the solution of $(T^*-\lambda_0)\psi^* = F^*\psi^*$, which is unique up to renormalization. The eigenvectors ψ and ψ^* are uniformly bounded from below by positive constants.

Let $q \in L^p(X)$, for $1 < p < \infty$. Then the problem

$$\begin{aligned} &(T-\lambda_0)\phi = F\phi + q \quad \text{in} \quad X \\ &\phi = \mathcal{R}(\phi_{|\Gamma_+}) \quad \text{on} \quad \Gamma_- \end{aligned} \tag{52}$$

admits a unique solution up to the addition of a multiple of ψ if and only if q satisfies

$$\int_X q(z,v)\psi^*(z,v)dzdv = 0. \tag{53}$$

Proof. Since the boundary operators and the scattering operator are conservative or dissipative, we clearly have that $\rho(T^{-1}F) \leq 1$. From the definition of λ^*, and since Σ is uniformly bounded from below by a constant, we deduce after straightforward algebra that

$$(T-\lambda)^{-1}F(1) \geq C(\lambda) \quad \text{a.e. in } X,$$

and that $C(\lambda)$ goes to infinity when $\lambda \to \lambda^*$, since $(T-\lambda)^{-1}(1)$ goes uniformly to ∞. From this we deduce that $\rho((T-\lambda)^{-1}F)$ tends to infinity when $\lambda \to \infty$. Let us now prove that $\lambda \mapsto \rho((T-\lambda)^{-1}F)$ is strictly increasing. It is relatively standard to prove that this map is continuous and differentiable by looking at the difference of $\rho((T-\lambda)^{-1}F)$ at two nearby points λ and $\lambda + \delta\lambda$. Let us denote by ψ_λ the unique positive normalized eigenvector such that

$$\psi_\lambda = \nu_\lambda (T-\lambda)^{-1} F \psi_\lambda.$$

We know that this eigenvector is unique according to Theorems 5.3 and 5.1. Denoting by $\phi_\lambda = F\psi_\lambda$, we know that $F(T-\lambda)^{-1}$ admits the same eigenvalue ν_λ associated with a positive eigenvector ϕ_λ. Therefore its adjoint operator $(T^*-\lambda)^{-1}F^*$ admits the same eigenvalue with a positive eigenvector ψ^*. We renormalize ψ^* such that $<\psi, \psi^*>_{L^2,L^2} = 1$. Since Σ is uniformly positive, we check using the method of characteristics that there exists a constant δ such that $(T-\lambda)^{-1}F(q) \geq \delta \|q\|_{L^1(\Omega\times K)}$ for every positive function q. This implies that ψ is uniformly bounded from below by a positive constant. (See [23] for a similar procedure.)

Differentiating the equality $(T-\lambda)\psi_\lambda = \nu_\lambda F \psi_\lambda$ with respect to λ yields

$$\partial_\lambda \nu_\lambda (T-\lambda)\psi_\lambda - \nu_\lambda \psi_\lambda + (T-\lambda)\nu_\lambda \partial_\lambda \psi_\lambda = F\partial_\lambda \psi_\lambda,$$

or equivalently

$$\partial_\lambda \nu_\lambda \psi_\lambda - \nu_\lambda (T-\lambda)^{-1}\psi_\lambda + \nu_\lambda \partial_\lambda \psi_\lambda = (T-\lambda)^{-1} F\partial_\lambda \psi_\lambda.$$

Multiplying this equation by ψ_λ^* and integrating over X yields

$$\partial_\lambda \nu_\lambda = \nu_\lambda \Big((T-\lambda)^{-1}\psi_\lambda, \psi_\lambda^*\Big).$$

Therefore $\partial_\lambda \nu_\lambda > 0$. Since $\nu_0 < 1$ and $\nu_\lambda \to \infty$ when $\lambda \to \lambda^*$, there exists a unique λ_0 such that $\nu_{\lambda_0} = \rho((T-\lambda_0)^{-1}F) = 1$. Condition (53) is simply the classical Fredholm alternative for compact operators, since we know from Theorem 5.1 that the leading eigenvalue $\nu_{\lambda_0} = 1$ is simple. $\square$

In the conservative case of section 3.1, some simplifications occur. We easily check the following result:

Corollary 5.5 *In the conservative case treated in section 3.1, we have $\lambda_0 = 0$ and $\psi^* = 1$ up to renormalization.*

6 Proof of Convergence

Let us begin with the following result:

Lemma 6.1 *The diffusion tensor $D_{ij}(x)$ defined by (26) is positive definite.*

Proof. Define $\chi_x^i = \psi_x^{-1}\theta_x^i$, where θ_x^i is a solution of (21). We check that χ_x^i satisfies

$$\begin{aligned} &v_3 \nabla_z \chi_x^i + \tilde{Q}_x(\chi^i) = -v_i \quad \text{in} \quad (0,1)\times K \\ &\chi_x^i(z,v) = \psi_x^{-1}\mathcal{R}(\psi_x \chi^i(z,v)_{|\Gamma_+}) \quad \text{on} \quad \Gamma_-. \end{aligned} \tag{54}$$

Here, we have

$$\tilde{Q}_x(f)(v) = \psi_x^{-1}\int_K \sigma(x,z,v',v)\psi_x(z,v')(f(v)-f(v'))dv'. \tag{55}$$

Let $\xi \in \mathbb{R}^3$ and $\chi_\xi = \sum_{i=1}^2 \xi_i \chi^i$. By virtue of (30) and (31), we obtain that

$$\begin{aligned} D(x)\xi \cdot \xi &= \int_X (v_3 \nabla_z \chi_\xi + \tilde{Q}(\chi_\xi))\chi_\xi \psi\psi^* dz dv \\ &= \frac{1}{2}\int_X \int_K \sigma_x(z,v',v)\psi_x^*(z,v)\psi_x(z,v)|\chi_\xi(z,v) - \chi_\xi(z,v')|^2 dz dv dv'. \end{aligned}$$

A similar result was obtained in the context of homogenization [23, 30]. Since σ_x, ψ_x, and ψ_x^* are uniformly bounded from below by positive constants, we deduce that $D(x)$ is definite. Assume now that $D(x)\xi \cdot \xi = 0$. Then we obtain that χ_ξ does not depend on v. Hence $\tilde{Q}(\chi_\xi) = 0$. Therefore $v_3 \cdot \nabla_z \chi_\xi = \bar{v} \cdot \xi$. Since $v_3 \cdot \nabla_z \chi_\xi$ does not depend on $\bar{v}$, we have $\bar{v} \cdot \xi$ independent of $\bar{v}$ for $\bar{v}$ in the unit disk. This is only possible if $\xi = 0$ and therefore $D(x)$ is positive definite. Since $x \mapsto D(x)$ is smooth, we have that $D(x)$ is uniformly positive definite. □

We now prove Theorem 4.2. Let us first show that the steady state transport problem is well posed.

Lemma 6.2 *Let $q \in L^1(\Omega \times K)$. Then for every $\varepsilon > 0$, there exists a unique solution $a^\varepsilon \in W^1(\Omega \times K)$ such that*

$$\begin{aligned} &\frac{1}{\varepsilon}\bar{v} \cdot \nabla_x a^\varepsilon + \frac{1}{\varepsilon^2} v_3 \nabla_z a^\varepsilon + \frac{1}{\varepsilon^2}\Big(\Sigma a^\varepsilon - \int_K \sigma(x,z,v',v)a^\varepsilon(v')dv'\Big) \\ &\qquad + \Sigma_a a^\varepsilon = q(x,z,v) + \frac{\gamma}{\varepsilon^2} a^\varepsilon \qquad \text{in} \quad \Omega \times K \\ &a^\varepsilon(x,z,v) = \mathcal{R}(a^\varepsilon(x,z,v)_{|\Gamma_+}) \qquad \text{on} \quad \Gamma_-. \end{aligned} \tag{56}$$

Furthermore, we have

$$\|a^\varepsilon\|_{L^1(\Omega\times K)} \le \frac{C}{\alpha}\|q\|_{L^1(\Omega\times K)}, \tag{57}$$

where C is a constant independent of q and α. Moreover, the solution a^ε is non-negative when q is non-negative.

Proof. Assume that such a solution a^ε exists and define

$$u^\varepsilon = \psi_x^{-1} a^\varepsilon,$$

where ψ_x is the positive solution of (30). Then u^ε satisfies the following equation,

$$\begin{aligned} &\frac{1}{\varepsilon}\bar{v} \cdot \nabla_x u^\varepsilon + \frac{1}{\varepsilon^2} v_3 \nabla_z u^\varepsilon + \frac{1}{\varepsilon^2}\tilde{Q}(u^\varepsilon) + \Sigma_a u^\varepsilon = \psi_x^{-1} q(x,z,v) \qquad \text{in} \quad \Omega \times K \\ &u^\varepsilon(x,z,v) = \psi_x^{-1}\mathcal{R}(\psi_x u^\varepsilon(x,z,v)_{|\Gamma_+}) \qquad \text{on} \quad \Gamma_-. \end{aligned} \tag{58}$$

The operator $\tilde{Q}$ is defined in (55). We also define

$$\tilde{\sigma}(x,z,v',v) = \psi_x^{-1}(z,v)\sigma(x,z,v',v)\psi_x(z,v'), \qquad \tilde{\Sigma}(x,z,v) = \int_K \tilde{\sigma}(x,z,v',v)dv'.$$

Upon writing $q = q^+ - q^-$ where $q^+ = \sup(q, 0)$ and $q^- = \sup(-q, 0)$, we can assume that q is non-negative. Following the proof of Lemma 5.2 and starting with $u_0^\varepsilon = 0$, we define the sequence u_k^ε by induction as follows:

$$\begin{aligned}
&\frac{1}{\varepsilon}\bar{v}\cdot\nabla_x u_{k+1}^\varepsilon + \frac{1}{\varepsilon^2} v_3 \nabla_z u_{k+1}^\varepsilon + \Big(\Sigma_a + \frac{1}{\varepsilon^2}\tilde{\Sigma}\Big) u_{k+1}^\varepsilon \\
&\qquad = \frac{1}{\varepsilon^2}\int_K \tilde{\sigma}(x,z,v',v) u_k^\varepsilon + \psi_x^{-1} q(x,z,v) \quad \text{in} \quad \Omega\times K \\
&u_{k+1}^\varepsilon(x,z,v) = \psi_x^{-1}\mathcal{R}(\psi_x u_{k+1}^\varepsilon(x,z,v)_{|\Gamma_+}) \quad \text{on} \quad \Gamma_-.
\end{aligned} \tag{59}$$

Let us denote by

$$\Delta_k = u_{k+1}^\varepsilon - u_k^\varepsilon.$$

We obtain by induction that Δ_k is non-negative. Moreover, integrating (59) over $\Omega\times K$ and using the fact that the response operators are conservative or dissipative, we get that

$$\int_{\Omega\times K}\Big(\Sigma_a + \frac{1}{\varepsilon^2}\tilde{\Sigma}\Big)\Delta_{k+1} dx dv \le \int_{\Omega\times K}\frac{1}{\varepsilon^2}\tilde{\Sigma}\Delta_k dx dv.$$

Since Σ_a is uniformly bounded from below by a positive constant, Δ_k converges to 0 exponentially fast for every $\varepsilon > 0$. Therefore, the sequence u_k^ε converges to a function u^ε. Using the integral formulation of (59), we deduce that u^ε is a solution of (58). Integrating this equation over $\Omega\times K$ yields an estimate for u^ε similar to (57) and the uniqueness of the solution. The map $a^\varepsilon \mapsto u^\varepsilon$ being clearly invertible, we have that (56) admits a unique solution in $L^1(\Omega\times K)$. $\square$

Let us denote by A the operator such that (56) is equivalent to

$$Aa^\varepsilon = q. \tag{60}$$

The domain of definition of A is $W^1(\Omega\times K)$, which is a dense subset of $L^1(\Omega\times K)$. We deduce from Lemma 6.2 that $I + \lambda A$ is continuously invertible from $D(A)$ to $L^1(\Omega\times K)$ for all $\lambda > 0$. Moreover since $q \ge 0$ implies that $a^\varepsilon \ge 0$, we have that

$$\|(I+\lambda A)^{-1}\|_{\mathcal{L}(L^1(\Omega\times K))} \le 1.$$

This implies that A is m-accretive (or equivalently $-A$ is m-dissipative). We then apply the Hille-Yosida theorem [31, 32] and obtain that A is the infinitesimal generator of a continuous semigroup of contractions. This implies that (29), hence (11), admits a unique solution in $C^1([0,T]; L^1(\Omega\times K))$.

Let us now prove the convergence properties stated in Theorem 4.2. We define, for a^ε a solution of (29),

$$u^\varepsilon(t,x,z,v) = \psi_x^{-1}(z,v) a^\varepsilon(t,x,z,v). \tag{61}$$

Then we write

$$u^\varepsilon = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \xi^\varepsilon. \tag{62}$$

We have that $u_k = \psi_x^{-1} a_k$ where the a_k are defined in section 3.2. We deduce from (29) and from the equations for the terms a_0, a_1, and a_2 of section 3.2 that ξ^ε solves

$$\begin{aligned} &\frac{\partial \xi^\varepsilon}{\partial t} + \frac{1}{\varepsilon}\bar{v}\cdot\nabla_x \xi^\varepsilon + \frac{1}{\varepsilon^2} v_3 \nabla_z \xi^\varepsilon + \frac{1}{\varepsilon^2}\tilde{Q}(\xi^\varepsilon) + \Sigma_a \xi^\varepsilon \\ &\qquad = q^\varepsilon = -\frac{\partial}{\partial t}(\varepsilon u_1 + \varepsilon^2 u_2) - \varepsilon \bar{v}\cdot\nabla_x u_2 \qquad \text{in} \quad [0,T]\times\Omega\times K \\ &\xi^\varepsilon(0,x,z,v) = 0 \qquad \text{in} \quad \Omega\times K \\ &\xi^\varepsilon(t,x,z,v) = \psi_x^{-1}\mathcal{R}(\psi_x \xi^\varepsilon(t,x,z,v)_{|\Gamma_+}) \qquad \text{on} \quad \Gamma_-. \end{aligned} \tag{63}$$

Since D_{ij} is positive definite according to Lemma 6.1, u_0 is smooth; hence u_1, u_2 and $\bar{v}\cdot\nabla_x u_2$ are also smooth. Therefore the source term q^ε is bounded by $C\varepsilon$ in $C^0([0,T], L^1(\Omega\times K))$. According to (57) and the Hille-Yosida theorem [31], we have that ξ^ε is also bounded by $C\varepsilon$ in $C^0([0,T]; L^1(\Omega\times K))$. This concludes the proof of Theorem 4.2.

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