Boundary Layer Analysis in the Homogenization of Neutron Transport Equations in a Cubic Domain.

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Abstract

The homogenization of neutron transport source problems in a finite domain with periodic structure is considered. It is known that the solution of such problems can be factored asymptotically as the product of two terms. The first one gives the local behavior of the neutron transport and is a solution of a periodic transport equation. The second term contains the large scale fluctuations of the neutron density and is a solution of a homogeneous diffusion equation in finite domain. In this paper we present a detailed analysis of the first-order correction to the expansion that accounts for the neutron leakage at the boundary of the domain. We study a multi-dimensional boundary layer equation by considering half space problems with heterogeneous boundary conditions and exponentially decaying source terms. Modified boundary conditions are then obtained for the homogeneous diffusion equation.

keywords: homogenization, linear transport, boundary layer, half space problem.

1 Introduction

The homogenization of criticality eigenvalue problems in neutron transport [2, 4, 23, 26] reduces to the analysis of the source problems that we consider in this paper. They are

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given for the neutron density $\phi_\varepsilon(x,v)$ by the following sequence of equations

$$\begin{cases} \dfrac{1}{\varepsilon} v \cdot \nabla \phi_\varepsilon + \dfrac{1}{\varepsilon^2} \Sigma^\varepsilon \phi_\varepsilon = \dfrac{1}{\varepsilon^2} \displaystyle\int_V \sigma^\varepsilon(x,v',v)\phi_\varepsilon(x,v')dv' + \gamma\phi_\varepsilon \ + \ S^\varepsilon & \text{in } \Omega \times V \\ \phi_\varepsilon = 0 \qquad \text{on } \Gamma_- = \{(x,v) \in \partial\Omega \times V \mid v \cdot n(x) < 0\}. \end{cases} \tag{1.1}$$

Here the position space Ω is an open subset of $\mathbb{R}^n$ and the velocity space V a closed subset of $\mathbb{R}^n$. The cross sections Σ^ε and σ^ε are positive and $\varepsilon Y-$ periodic in all spatial variables, where we denote by Y the cell of periodicity. They are chosen so that problem (1.1) is critical in a suitable sense. Roughly speaking, it amounts to saying that bounded source terms S^ε will give bounded solutions ϕ^ε to (1.1) independently of ε. The absorption term $\gamma \equiv \gamma(x)$ is non positive. This term vanishes in the homogenization of eigenvalue problems [2, 4]. The parameter ε describes the size of the heterogeneities in Ω. Therefore Ω is composed of the order of ε^{-n} periodicity cells of the form $\varepsilon Y = (0,\varepsilon)^n$. Equation (1.1) is scaled such that the scattering mean free path of the neutrons is of the size of the heterogeneities, as is physically correct. Our objective is to describe the asymptotic behavior of ϕ_ε as ε goes to 0.

It is well known [2, 4, 9, 17, 18, 19, 23, 25] that ϕ_ε can be factored as the product of two terms

$$\phi_\varepsilon(x,v) \ = \ \psi(\frac{x}{\varepsilon},v) \ u_\varepsilon(x,v),$$

where $\psi(x,v)$ is the positive normalized solution of an eigenvalue problem posed in $Y \times V$ with periodic boundary conditions and

$$u_\varepsilon \ \to \ u \quad \text{ in } L^2(\Omega \times V) \ \text{ strongly }.$$

Here $u \in H_0^1(\Omega)$ is a solution of the following homogeneous diffusion equation

$$\begin{cases} -\nabla \cdot D\nabla u(x) \ = \ \gamma u(x) + S \quad \text{in } \Omega \\ \qquad\qquad u(x) \ = \ 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where D is a homogeneous positive definite matrix, which can be computed explicitly using the solution of cell problems.

The difference between u_ε and u is known to be bounded by $O(\varepsilon)$. One reason is that the neutron leakage at the boundary of the core has not been accounted for correctly. It turns out that the neutron distribution depends highly upon the velocity of the particles near the boundary, since by assumption the incoming density vanishes whereas the outgoing density does not. Therefore, the diffusion approximation, which describes only the average of the density over the velocity space, is not accurate near the boundary.

This shortcoming can be overcome for cubic domains by introducing a boundary layer, whose role is to match the heterogeneous neutron distribution at the boundary of the domain with a more homogeneous distribution in the interior. This asymptotically velocity independent distribution can be described by a diffusion approximation. Modified boundary conditions for the homogeneous diffusion equation are then derived.

The boundary layer problem has been mathematically extensively addressed in the past for 1D structures [7, 9, 10, 25], and more recently in higher dimensions with some restrictions on the cross sections [4]. Notice that 1D boundary layers are not sufficient here, since the size of the periodicity cell is on the same order as the neutron mean free path. Therefore, the neutron density is heterogeneous in the tangential directions at the boundary and the boundary layer is genuinely multi-dimensional. In this paper, we present a detailed analysis of the boundary layer problem with boundary and volume source terms and show how it can be used for obtaining more accurate diffusion approximations of the transport equation.

An outline of this paper is as follows. In Section 2 we recall briefly some results on the eigenvalue problem in periodic domain. We also extend them to the case of exponentially decaying solutions in an infinite domain. Section 3 is devoted to a presentation of the main results. We give the asymptotic expansion of the transport equation and the analysis of the half space problem with heterogeneous boundary conditions and exponentially decaying volume source terms. The proofs of these results are given in sections 4 to 6.

Notice that similar results on the analysis of correctors and half space problems for diffusion equations can be found in [8, 24]. Other homogenizations of transport equations are presented for different regimes in other works [1, 13, 14]. A numerical analysis of the homogenization procedure presented here can be found in [5].

2 Eigenvalue Problem in Periodic Medium

In this section we present some properties satisfied by the solutions of neutron transport problems in a periodic setting. We want to find the eigenvalue of smallest modulus λ, which is real and positive, and the associated positive eigenvector ψ of the following problem posed in $Y \times V$

$$\begin{cases} v \cdot \nabla_y \psi + \Sigma(y,v)\psi = \lambda \int_V \sigma(y,v',v)\psi(y,v')dv' \\ y \mapsto \psi(y,v) \; Y - \text{periodic.} \end{cases} \tag{2.1}$$

We also introduce an adjoint problem to (2.1). Denoting by $\sigma^*(y,v',v) = \sigma(y,v,v')$ the adjoint cross-section, it is defined on $Y \times V$ by

$$\begin{cases} -v \cdot \nabla_y \psi^* + \Sigma(y,v)\psi^* = \lambda \int_V \sigma^*(y,v',v)\psi^*(y,v')dv' \\ y \mapsto \psi^*(y,v) \; Y - \text{periodic.} \end{cases} \tag{2.2}$$

This adjoint problem has the same first eigenvalue λ. In these problems, Σ is the total cross section and σ the sum of the scattering and fission cross section. They characterize the interaction between the neutral particles and the passive underlying medium. When a non-trivial solution to (2.1) exists with $\lambda = 1$, fission and scattering exactly compensate total absorption. The core is then said to be critical.

Criticality of the periodic problem is not enough for the regime we are interested in. We also want to ensure that no intrinsic drift takes place. Denote the *drift flux* by

$$J = \int_Y \int_V v\, \psi(y,v)\, \psi^*(y,v) dy\, dv, \tag{2.3}$$

which is a constant vector in $\mathbb{R}^n$. We require the additional hypothesis on the coefficients of (2.1) that the drift flux be zero

$$J = 0. \tag{2.4}$$

This symmetry condition in phase space is usually satisfied in practice. For instance (2.4) is true when $V = -V$ and when the cross sections do not depend on the velocity variable. It is also true when $V = -V$, when the cross-sections are symmetric with respect to v, and when the cell Y has a cubic symmetry [2, 3]. This condition is almost always satisfied in the practice of nuclear reactor computations. Notice that the assumption of vanishing drift flux is not only technical. It was shown in [20] that J can be non zero, and that the behavior of the neutron density is then dramatically different.

Define now the following cross sections:

$$\Sigma^{\varepsilon}(x,v) \;=\; \Sigma(\frac{x}{\varepsilon},v), \qquad \sigma^{\varepsilon}(x,v) \;=\; \sigma(\frac{x}{\varepsilon},v). \tag{2.5}$$

The condition of criticity mentioned in the introduction is defined as follows.

Definition 2.1 *We say that the cell Y with cross sections Σ and σ is critical if*

1. *There exists a solution to (2.1) with $\lambda = 1$,*

2. *The symmetry condition (2.4) is satisfied.*

Existence of a solution for the transport problems in periodic setting can be proved under the following conditions

Hypothesis 2.2 *Assume the following*

(H1) The domain Ω is a convex bounded open subset of $I\!R^n$.

(H2) The velocity space V is a compact subset of $I\!R^n$ which does not contain 0. Furthermore V is assumed to be the closure of an open set, and its n-dimensional measure is normalized so that $|V| = 1$. All the results hold true if V is the union of a finite number of spheres. In that case the $(n-1)$-dimensional measure is normalized so that $|V| = 1$.

(H3) The cross-sections $\Sigma(y,v)$ and $\sigma(y,v',v)$ are measurable non negative bounded functions of their arguments and there exists a positive constant $C > 0$ such that, for a.e. (y,v,v'), we have

$$\sigma(y,v',v) \;\geq\; C \quad \text{and} \quad \Sigma(y,v) \;\geq\; C. \tag{2.6}$$

(H4) Denoting by $Y = (0,1)^n$ the unit cube, the cross-sections $\Sigma(y,v)$ and $\sigma(y,v',v)$ are Y-periodic functions with respect to the space variable y.

These assumptions are natural. (H1) means that the particles escaping the domain never come back. It is here only for convenience and the results can be extended to more general cases. (H2) implies that the particles that are not scattered must leave the domain in a finite time. It ensures that the underlying evolution problem eventually relaxes to a steady state problem. The conditions imposed on V basically ensure that V cannot be included in a hyper plane of $\mathbb{R}^n$ as is necessary in order to obtain a non degenerate diffusion equation. In (H3), we see that fission is necessary in order to have criticity. We need the positiveness of σ in order to prove the uniqueness of a positive solution to (2.1).

Denote by W^p the following the Banach spaces (see [12])

$$W^p(\Omega \times V) = \{u \in L^p(\Omega \times V) \text{ s.t. } v \cdot \nabla u \in L^p(\Omega \times V)\}, \tag{2.7}$$

for $1 \leq p \leq \infty$. A first existence result is as follows.

Theorem 2.3 *There exists an eigenvalue λ, common to both problems (2.1) and (2.2), which is real, positive, of smallest modulus, and with multiplicity one. Its associated eigenvectors ψ and ψ^* are positive elements of $W^2(Y \times V)$. Up to renormalization, ψ and ψ^* are the unique positive solutions to (2.1) and (2.2) respectively. Moreover assume that the cross sections Σ and σ are of class C^m of their domain of definition for $m \in \mathbb{N}^*$. Then ψ and ψ^* belong to $H^m(Y \times V)$.*

This theorem is essentially a variation of a result given in [12, chapter 21]. A more detailed proof can be found in [3, 26]. It relies on the velocity averaging lemma of [16] (or similar compactness result in [21]) and on the Krein Rutman theory of positive operators. Then direct computations using the integral form of the transport equation show the uniqueness of the positive eigenvector and the simplicity of the leading eigenvalue. Only these two parameters have a physical meaning since the eigenvector models a particle density.

The following proposition ensures that the eigenvectors ψ and ψ^* be bounded from above and below by positive constants. Its proof can be found in [2].

Proposition 2.4 *Let ψ and ψ^* be the positive solutions to (2.1) and (2.2) respectively. There exists two positive constants $C' \geq C > 0$ such that*

$$0 < C \leq \psi(y,v) \leq C' \quad \text{and} \quad 0 < C \leq \psi^*(y,v) \leq C' \quad \text{a.e. in } Y \times V.$$

More generally every eigenvector of (2.1) or (2.2) belonging to $L^2(Y \times V)$ is also an element of $L^\infty(Y \times V)$.

We assume now that $\lambda = 1$. From the classical Fredholm alternative for compact operators, we deduce the following result for source problems.

Proposition 2.5 *Let $S(y,v)$ be a source term in $L^2(Y \times V)$. There exists a solution $\varphi(y,v) \in W^2(Y \times V)$ of*

$$\begin{cases} v \cdot \nabla_y \varphi + \Sigma(y,v)\varphi = \int_V \sigma(y,v',v)\varphi(y,v')dv' \; + \; S(y,v) \\ y \mapsto \varphi(y,v) \;\; Y - periodic \end{cases} \tag{2.8}$$

if and only if S is orthogonal to the positive eigenvector ψ^* of (2.2), i.e. if S satisfies the compatibility condition

$$\int_Y \int_V S(y,v)\psi^*(y,v)dydv \; = \; 0.$$

Furthermore, if it exists, the solution φ is unique up to the addition of a multiple of ψ. Assume moreover that the cross sections Σ and σ are of class C^m of their domain of definition for $m \in \mathbb{N}^*$, and that S is an element of $H^m(Y \times V)$. Then φ belongs to $H^m(Y \times V)$.

We consider now exponential solutions of the transport problem posed in an infinite and periodic medium. Let $\nu \neq 0 \in \mathbb{R}^n$ be given. The criticality eigenvalue problem reads as follows. Find the smallest eigenvalue and the associated positive eigenvector of

$$\begin{cases} v \cdot \nabla_y \psi_\nu + \Sigma(y,v)\psi_\nu = \lambda_\nu \int_V \sigma(y,v',v)\psi_\nu(y,v')dv' \\ y \mapsto e^{-\nu \cdot y}\psi_\nu(y,v) \; Y - \text{periodic.} \end{cases} \tag{2.9}$$

By the change of variables $\phi_\nu(y,v) = e^{-\nu \cdot y}\psi_\nu(y,v)$, we obtain the following equation

$$\begin{cases} v \cdot \nabla_y \phi_\nu + \Sigma(y,v)\phi_\nu + v \cdot \nu \; \phi_\nu = \lambda_\nu \int_V \sigma(y,v',v)\phi_\nu(y,v')dv' \\ y \mapsto \phi_\nu(y,v) \; Y - \text{periodic.} \end{cases} \tag{2.10}$$

Owing to Theorem 2.3, there exists a smallest eigenvalue for (2.10), still denoted by λ_ν, as soon as $\Sigma + v \cdot \nu$ remains positive. It is associated with a unique normalized positive eigenvector ϕ_ν. We obtain a solution for (2.9) by reverse change of variables. We also denote by ϕ^*_ν the solution to (2.2) where Σ has been replaced by $\Sigma + v\cdot\nu$. It is associated with the same eigenvalue λ_ν.

Consider the mapping $\nu \mapsto (\lambda_\nu, \phi_\nu)$. We shall not dwell on the differentiability of the solutions of (2.10) with respect to ν. Straightforward computations show a solution of the transport equation is twice continuously differentiable for a source problem given by replacing the right-hand side of (2.10) by a source term. Using then results on the approximation of operators given in [11], we can extend the results to the eigenvalue problem (2.10). We shall prove in section 6 the following result

Lemma 2.6 *Assume that $\nu \mapsto (\lambda_\nu, \phi_\nu)$ is twice differentiable with a continuous second derivative for $\nu \in \Lambda = \{\nu \in \mathbb{R};\ \inf_{y\in Y} \Sigma(y,v) + v\cdot\nu > 0$ a.e. $v \in V\}$. Then we have*

- $J_\nu \equiv \int_{Y\times V} v\ \phi_\nu(y,v)\phi^*_\nu(y,v)\ dydv = (\nabla_\nu \lambda_\nu) \int_{Y\times V\times V} \sigma(y,v',v)\phi_\nu(y,v')\phi^*_\nu(y,v)\ dydvdv'$
- *$\nu \mapsto -\lambda_\nu$ is strictly convex in the vicinity of $\nu = 0$. The matrix $D^0_{ij} \equiv -\left.\dfrac{\partial^2\lambda_\nu}{\partial\nu_i\,\partial\nu_j}\right|_{\nu=0}$, given explicitly in section 6, is positive definite.*

By virtue of the assumption of criticity given in Definition 2.1, we have $J \equiv J_0 = 0$. Then we obtain that $1 \equiv \lambda_0 > \lambda_\nu$ for all $\nu \neq 0$. Owing to the simplicity of the eigenvalue λ_ν, we deduce the existence of a positive constant ν_1 such that for all ν satisfying $|\nu| < \nu_1$, we have

$$\lambda_\nu < 1 = \lambda_0 < |\lambda'_\nu|$$

where λ'_ν is any eigenvalue of (2.10) which is not λ_ν. By virtue of Proposition 2.4, the corresponding eigenvectors belong to L^∞. Then we deduce from the Fredholm alternative the following result.

Proposition 2.7 *Let S be a periodic function in $L^\infty(Y\times V)$ and let be $\nu \in \mathbb{R}^n$, such that its modulus $|\nu|$ be small enough. Then the following problem*

$$\begin{cases} v\cdot\nabla_y\varphi + \Sigma(y,v)\varphi = \displaystyle\int_V \sigma(y,v',v)\varphi(y,v')dv' \ + \ e^{-\nu\cdot y}\ S(y,v) \\ y \mapsto e^{\nu\cdot y}\varphi(y,v)\ Y-\text{periodic} \end{cases} \tag{2.11}$$

admits a unique solution in $W^\infty(Y\times V)$.

This proposition states the existence of exponentially decaying solutions in the direction ν for problems posed in $\mathbb{R}^n \times V$ with exponentially decaying source terms.

3 Main Results

We assume that the medium is critical according to Definition 2.1. Let us denote by ψ_ε the $\varepsilon Y-$ periodic function defined by

$$\psi_\varepsilon(x,v) = \psi(\frac{x}{\varepsilon}, v) \tag{3.1}$$

where ψ is the positive solution to (2.1), and by u_ε the factored density given by

$$u_\varepsilon(x,v) \;=\; \frac{\phi_\varepsilon(x,v)}{\psi_\varepsilon(x,v)}, \tag{3.2}$$

where ϕ_ε is a solution to (1.1). Notice that ψ_ε is bounded from above and below by positive constants. Therefore the map $\phi_\varepsilon \mapsto u_\varepsilon$ defined by (3.2) is continuous in $L^p(\Omega \times V)$ for $1 \le p \le \infty$, and has continuous inverse. Using this change of variables, we easily check that (1.1) is equivalent to

$$\left\{ \begin{array}{l} \dfrac{1}{\varepsilon} v \cdot \nabla u_\varepsilon + \dfrac{1}{\varepsilon^2} Q_\varepsilon(u_\varepsilon) \;=\; \gamma u_\varepsilon \;+\; \dfrac{S^\varepsilon}{\psi_\varepsilon} \quad \text{in } \Omega \times V \\[2ex] u_\varepsilon \;=\; 0 \quad \text{on } \Gamma_-, \end{array} \right. \tag{3.3}$$

where

$$Q_\varepsilon(u)(x,v) = \int_V \sigma^\varepsilon(x,v',v) \frac{\psi_\varepsilon(x,v')}{\psi_\varepsilon(x,v)} \left[u(x,v) - u(x,v')\right] dv'. \tag{3.4}$$

Our main assumptions are the following

Hypothesis 3.1 *In addition to Hypothesis 2.2 we have*

(H5) *Ω is the unit cube $(0,1)^n$. It is composed of N^n identical cells εY, where $N = \frac{1}{\varepsilon} \in \mathbb{N}$.*

(H6) *The cross sections Σ and σ are regular and they satisfy the cubic symmetry, in the sense that they are invariant by the rotations preserving the unit cell. The absorption term γ is also regular.*

(H7) *The source term S^ε satisfies*

$$S^\varepsilon(x,v) \;=\; \psi_\varepsilon(x,v)\; h(\frac{x}{\varepsilon}, v)\; q(x)$$

where $h(y,v)$ is a bounded regular function in $Y \times V$ which is periodic in y, and q is a regular bounded function in Ω.

Theorem 3.2 *Assume that Hypothesis 2.2 is satisfied. Then problem (1.1) admits a unique solution in $W^2(\Omega \times V)$ bounded in $L^2(\Omega \times V)$ independently of the source term S^ε. Assume moreover that Hypothesis 3.1 is verified. Then there exists a constant L such that we have the following result. Let u_0 and u_{10} be the solutions to*

$$\begin{cases} -\nabla \cdot D\nabla u_0 = \gamma u_0 + \bar{S} & in\ \Omega \\ u_0 = 0 & on\ \partial\Omega \end{cases} \tag{3.5}$$

$$\begin{cases} -\nabla \cdot D\nabla u_{10} = \gamma u_{10} & in\ \Omega \\ u_{10} + L\dfrac{\partial u_0}{\partial n} = 0 & on\ \partial\Omega. \end{cases} \tag{3.6}$$

Here the homogenized source term $\bar{S}$ is given by

$$\bar{S}(x) = q(x) \int_Y \int_V h(y,v)\ \psi^*(y,v)\psi(y,v)\ dy\, dv \tag{3.7}$$

while the homogenized diffusion tensor $D = (D_{ij})_{1 \le i,j \le n}$ in (3.5) is defined by

$$D_{ij} = -\int_Y \int_V v_j \psi(y,v)\psi^*(y,v)\theta^i(y,v) dy\, dv, \tag{3.8}$$

and the functions $(\theta^i)_{1 \le i \le n}$ are solutions of the cell problems

$$\begin{cases} v \cdot \nabla\theta^i + Q(\theta^i) = -v_i & in\ Y \times V \\ y \mapsto \theta^i(y,v)\ Y-periodic, \end{cases} \tag{3.9}$$

where the local scattering operator Q is defined by

$$Q(\theta) = \int_V \sigma(y,v',v)\ \frac{\psi(y,v')}{\psi(y,v)}\ (\theta(y,v) - \theta(y,v'))\ dv'.$$

Assume that u_0 is of class $C^{5,\alpha}(\Omega)$ for some $\alpha > 0$ and that u_{10} is of class $C^{4,\alpha}(\Omega)$. Then we have the second order estimate in $L^2(\Omega \times V)$

$$u_\varepsilon(x,v) = u_0(x) + \varepsilon\ [\theta^i(\frac{x}{\varepsilon},v)\ \partial_{x_i} u_0(x) + u_{10}(x)] + \varepsilon b_\varepsilon(x,v) + O(\varepsilon^2), \tag{3.10}$$

where $b_\varepsilon(x,v)$ is a boundary layer term decaying exponentially away from the boundary. It satisfies $b_\varepsilon = O(\varepsilon^{1/p})$ in $L^p(\Omega \times V)$ for $1 \le p \le \infty$.

Remark 3.3 *Define* u_0^ε *as the solution to*

$$\left\{\begin{array}{rcl} -\nabla\cdot D\nabla u_0^\varepsilon & = & \gamma u_0^\varepsilon + \bar{S} \quad \text{in } \Omega \\ u_0^\varepsilon + \varepsilon L \dfrac{\partial u_0^\varepsilon}{\partial n} & = & 0 \quad \text{on } \partial\Omega. \end{array}\right. \tag{3.11}$$

If L *is positive or* ε *is small enough, this solution is uniquely defined. Then we check [12] that we have*

$$u_\varepsilon(x,v) \;=\; u_0^\varepsilon(x) \;+\; \varepsilon\,[\theta^i(\frac{x}{\varepsilon},v)\;\partial_{x_i}u_0^\varepsilon(x)] + O(\varepsilon^{3/2})$$

in $L^2(\Omega\times V)$*. The modified boundary conditions given in (3.11) enable us to approximate the transport solution* u_ε *with a better accuracy than by solving only (3.5). The computation of the extrapolation length* $L = -G(\theta^1)$*, where the linear form* G *is given in Theorem 3.4 below, is complicated. In the case of a homogeneous cell, where* $\sigma\equiv 1$*, it can be computed analytically, using Chandrasekhar's function* H *[10]. An approximate value is* $L_0 = 0.7104$ *[12].*

By linearity of the transport equation, S^ε can be replaced by a finite sum of terms satisfying (H7). Notice that u_0 and u_{10} satisfy the regularity assumptions requested in the theorem if $q(x)$ is given as the sum of a product of one-dimensional sine functions vanishing at the boundary of Ω. This is the form of the source terms obtained in the asymptotic analysis of eigenvalue problems.

The proof of this theorem will rely on two results interesting in themselves, Theorem 3.4 and Theorem 3.5 below. Let us introduce some notations. For $x\in\mathbb{R}^n$ we define $x=(x_1,x')$ where x_1 is the first component of x and x' the $(n-1)^{th}$ last ones. We rewrite the scattering operator as

$$Q(u) \;=\; \int_V \tilde{\sigma}(x,v',v)(u(x,v)-u(x,v'))dv' \tag{3.12}$$

where according to Proposition 2.4, $\tilde{\sigma}(x,v',v)=\psi^{-1}(y,v)\psi(y,v')\sigma(y,v',v)$ is bounded from above and below by positive constants

$$0<\sigma_0\le\tilde{\sigma}(x,v',v)\le\sigma_1 \qquad \text{a.e. } \Omega\times V\times V. \tag{3.13}$$

We denote by $\tilde{\Sigma}$ the total absorption $\tilde{\Sigma}(x,v)=\int_V\tilde{\sigma}(x,v',v)dv'$.

Theorem 3.4 *Let $g \in L^\infty(\Gamma_-)$ where $\Gamma_- = \{(x,v) \in \mathbb{R}^n \times V \;\; s.t. \;\; x_1 = 0 \text{ and } v_1 > 0\}$. We assume that Hypothesis 2.2 is satisfied. Then there exists a unique solution in $W^\infty(\mathbb{R}^+ \times \mathbb{R}^{n-1} \times V)$, which is $Y-$periodic with respect to x', to the following equation*

$$\begin{aligned} v \cdot \nabla b + Q(b) &= 0 \text{ in } \mathbb{R}^+ \times \mathbb{R}^{n-1} \times V \\ b &= g \quad \text{on } \Gamma_-. \end{aligned} \tag{3.14}$$

Moreover there exists a linear form G such that the solution of (3.14) decays exponentially to $G(g)$ as $x_1 \to \infty$. More precisely there exist $\beta > 0$ and C independent of g such that

$$(|b - G(g)| + |v \cdot \nabla b|)(x,v) \le Ce^{-\beta x_1} \|g\|_{L^\infty(\Gamma_-)} \quad a.e. \; (x,v) \in \mathbb{R}^+ \times \mathbb{R}^{n-1} \times V. \tag{3.15}$$

We have $G(1) = 1$. Assume that Hypothesis 3.1 is satisfied. Then $G(\theta^i) = 0$ for $i \neq 1$. If moreover $V = -V$ and $\sigma(x,v,v') = \sigma(v' \cdot v)$, the extrapolation length $L = G(-\theta^1)$ is positive.

Theorem 3.5 *Let $S \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^{n-1} \times V)$ and $\nu > 0$. We assume that S is periodic in x' of period mY, where m is an integer. Assuming Hypothesis 2.2, there exists a unique solution in $W^\infty(\mathbb{R}^+ \times \mathbb{R}^{n-1} \times V)$ to the following equation*

$$\begin{aligned} v \cdot \nabla b \; + \; Q(b) &= e^{-\nu x_1} S \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^{n-1} \times V \\ b &= 0 \quad \text{on } \Gamma_-. \end{aligned} \tag{3.16}$$

Moreover there exists a linear form $H(S)$ and two constants $\beta_m > 0$ independent of S and C independent of m and S such that

$$|b - H(S)|(x,v) \; + \; |v \cdot \nabla b|(x,v) \le Ce^{-\beta_m x_1} \; \|S\|_{L^\infty(Y \times V)}. \tag{3.17}$$

4 Proof of the Convergence Result

This section is devoted to the proof of Theorem 3.2.

Step 1: Existence Result. A key result in the proof of existence of a solution to the transport equation (3.3) is the following a priori estimate.

Lemma 4.1 *Assume that* $\|S^\varepsilon\|_{L^2(\Omega\times V)} \le C_S$ *independently of* ε. *Let* u_ε *be a solution of (3.3). There exists a positive constant* C, *which does not depend on* ε *and* C_S, *such that*

$$\begin{aligned}\|u_\varepsilon\|_{L^2(\Omega\times V)} + \|v\cdot\nabla u_\varepsilon\|_{L^2(\Omega\times V)} + \frac{1}{\sqrt{\varepsilon}}\|u_\varepsilon\|_{L^2(\Gamma_+,|v\cdot n|)} \\ +\frac{1}{\varepsilon}\|u_\varepsilon - \int_V u_\varepsilon\|_{L^2(\Omega\times V)} \ \le\ CC_S.\end{aligned} \tag{4.1}$$

Here $L^2(\Gamma_+,|v\cdot n|)$ *is the space of functions* u *satisfying* $\int_{\Gamma_+}(v\cdot n)\,|u|^2 d\Gamma < \infty$, *where* $\Gamma_+ = \{(x,v)\in\partial\Omega\times V \mid v\cdot n(x) > 0\}$ *and* $d\Gamma = dvd\sigma$, $d\sigma$ *being the surface measure on* $\partial\Omega$.

Proof. A complete proof of this lemma is given in [2, 3]. For completeness and because we shall use similar methods in the sequel, we present here the main ideas. We assume that $\gamma = 0$ for simplicity. Addition of absorption is easily handled. Multiplying equation (3.3) by $u_\varepsilon\psi_\varepsilon\psi_\varepsilon^*$ and integrating over $\Omega\times V$ yields

$$I_1 + I_2 = \int_\Omega\int_V S^\varepsilon\psi_\varepsilon^* u_\varepsilon dx\,dv,$$

where I_1 and I_2 are defined below. Multiplying the infinite medium equation (2.1) by ψ^* and subtracting the adjoint equation (2.2) multiplied by ψ, we obtain

$$v\cdot\nabla(\psi\psi^*) \ =\ \psi^*\int_V \psi(y,v')\sigma(y,v',v)dv' - \psi\int_V\psi^*(y,v')\sigma^*(y,v',v)dv'. \tag{4.2}$$

Integrating by parts, we obtain owing to the boundary conditions that

$$\begin{aligned} I_1 &= \frac{1}{\varepsilon}\int_\Omega\int_V v\cdot\nabla u_\varepsilon(u_\varepsilon\psi_\varepsilon\psi_\varepsilon^*)dx\,dv \\ &= \frac{1}{2\varepsilon}\int_{\Gamma_+}|u_\varepsilon|^2\psi_\varepsilon\psi_\varepsilon^* v\cdot n\,dvd\sigma - \frac{1}{2\varepsilon^2}\int_\Omega\int_V|u_\varepsilon|^2\left(\psi_\varepsilon^*\int_V\psi_\varepsilon\sigma^\varepsilon - \psi_\varepsilon\int_V\psi_\varepsilon^*\sigma^{\varepsilon*}\right)dxdv.\end{aligned}$$

Adding the second contribution yields

$$\begin{aligned} I_1 + I_2 &= \frac{1}{2\varepsilon}\int_{\Gamma_+}|u_\varepsilon|^2\psi_\varepsilon\psi_\varepsilon^* v\cdot n\,dv\,d\sigma \\ &+ \frac{1}{2\varepsilon^2}\int_\Omega\int_V\int_V\sigma^\varepsilon(x,v',v)\psi_\varepsilon^*(x,v)\psi_\varepsilon(x,v')|u_\varepsilon(x,v)-u_\varepsilon(x,v')|^2 dv\,dv'\,dx.\end{aligned}$$

Therefore we deduce the a priori estimate

$$\frac{1}{\varepsilon}\|u_\varepsilon\|^2_{L^2(\Gamma_+,|v\cdot n|)} + \frac{1}{\varepsilon^2}\|u_\varepsilon - \int_V u_\varepsilon dv\|^2_{L^2(\Omega\times V)} \le C\|u_\varepsilon\|_{L^2(\Omega\times V)}C_S.$$

According to a Poincaré-like inequality given in [2, 3], we have

$$\|u_\varepsilon\|_{L^2(\Omega\times V)} \le C\left[\|u_\varepsilon\|_{L^2(\Gamma_+,|v\cdot n|)} + \|v\cdot\nabla u_\varepsilon\|_{L^2(\Omega\times V)}\right]. \tag{4.3}$$

Using these two last inequalities and the bound on $v\cdot\nabla u_\varepsilon$ obtained from equation (3.3), we deduce the lemma. □

This a priori estimate enables us to prove existence and uniqueness of a solution to (1.1) or equivalently to (3.3). The original proof is due to G. Allaire. Define $T = v\cdot\nabla + Q$ and

$$D(T) = \{u \in L^2(\Omega\times V) \text{ s.t. } v\cdot\nabla u \in L^2 \text{ and } u_{|\Gamma_-} = 0\}.$$

Then $W^2(\Omega\times V) \subset D(T) \subset L^2(\Omega\times V)$. Hence $D(T)$ is dense in $L^2(\Omega\times V)$. Let us prove that $Im(T)$ is closed. Let $u_n \to u$ in $L^2(\Omega\times V)$ strong and $Tu_n = f_n \to f$ in $L^2(\Omega\times V)$ strong. Then according to Lemma 4.1, $v\cdot\nabla u_n$ is bounded in $L^2(\Omega\times V)$ independently of n and then converges weakly to $v\cdot\nabla u$. Therefore $u\in D(T)$. Moreover $Tu_n \rightharpoonup Tu$ and $f = Tu$. Accordingly, $Im(T)$ is closed and $Im(T) = (Ker(T^*))^\perp$. We also deduce from Lemma 4.1 that $Ker(T) = \emptyset$. This proves the uniqueness of the solution. A similar proof also yields that $Ker(T^*) = \emptyset$, where $T^* = -v\cdot\nabla + Q^*$ and Q^* is the adjoint operator to Q. Therefore, $Im(T) = L^2(\Omega\times V)$. This proves the existence and uniqueness of a solution to (3.3).

An analog result is true for heterogeneous boundary conditions.

Proposition 4.2 *Let $g\in L^\infty(\Gamma_-)$. Then there exists a unique solution in $L^\infty(\Omega\times V)$ to*

$$\left\{\begin{array}{l} \frac{1}{\varepsilon}v\cdot\nabla u_\varepsilon + \frac{1}{\varepsilon^2}Q_\varepsilon(u_\varepsilon) \;=\; \gamma u_\varepsilon \quad \text{in } \Omega\times V \\ u_\varepsilon \;=\; g \quad \text{on } \Gamma_-. \end{array}\right. \tag{4.4}$$

Moreover, there exists a constant C independent of g such that

$$\|u_\varepsilon\|_{L^\infty(\Omega\times V)} \le \|g\|_{L^\infty(\Gamma_-)}, \qquad \|u_\varepsilon\|_{L^2(\Omega\times V)} \le \frac{C}{\sqrt{\varepsilon}}\|g\|_{L^2(\Gamma_-,|v\cdot n|)}.$$

Proof. Define the sequence of solutions to

$$\left\{\begin{array}{l} \frac{1}{\varepsilon}v\cdot\nabla u^\alpha_\varepsilon + \frac{1}{\varepsilon^2}Q_\varepsilon(u^\alpha_\varepsilon) + \alpha u^\alpha_\varepsilon \;=\; \gamma u^\alpha_\varepsilon \quad \text{in } \Omega\times V \\ u^\alpha_\varepsilon \;=\; g \quad \text{on } \Gamma_-, \end{array}\right.$$

for $\alpha > 0$. From the maximum principle given in [12, Chapter 21], we deduce the existence and uniqueness of a solution to this equation for all $\alpha > 0$. Moreover $\|u_\varepsilon^\alpha\|_{L^\infty(\Omega\times V)} \leq \|g\|_{L^\infty(\Omega\times V)}$. Thus there exists a subsequence of u_ε^α converging to u_ε in $L^\infty(\Omega \times V)$ weak*. Using techniques given by Bardos [6], we prove that u_ε is a solution of (4.4). Using the same variational principle as in the proof of Lemma 4.1, we deduce that the solutions of (4.4) satisfy $\|u_\varepsilon\|_{L^2(\Omega\times V)} \leq \frac{C}{\sqrt{\varepsilon}}\|g\|_{L^2(\Gamma_-,|v\cdot n|)}$. This implies the uniqueness of the solution. □

Step 2: Derivation of the volume equations. We derive now the equations that u_0 and u_{10} must satisfy in order to justify the asymptotic expansion for u_ε. It is based on the usual two-scale analysis introduced in [9, 17, 18, 19]. Let us consider the following Ansatz:

$$u_\varepsilon(x,v) = u_0(x,\frac{x}{\varepsilon},v) + \varepsilon u_1(x,\frac{x}{\varepsilon},v) + \varepsilon^2 u_2(x,\frac{x}{\varepsilon},v) + \varepsilon^3 u_3(x,\frac{x}{\varepsilon},v) + \zeta_\varepsilon, \tag{4.5}$$

where the functions $y \mapsto u_k(x,y,v)$ are $Y-$periodic for $k = 0,\ldots,3$. For these functions, the differentiation operator $v\cdot\nabla$ has the form $\varepsilon^{-1}v\cdot\nabla_y + v\cdot\nabla_x$. Plugging these expressions into (3.3) and equating like powers of ε, we find that

$$v\cdot\nabla_y u_0 + Q(u_0) \;=\; 0 \tag{4.6}$$

$$v\cdot\nabla_y u_1 + Q(u_1) \;=\; -v\cdot\nabla_x u_0 \tag{4.7}$$

$$v\cdot\nabla_y u_2 + Q(u_2) \;=\; -v\cdot\nabla_x u_1 + \gamma u_0 + q(x)h(y,v) \tag{4.8}$$

$$v\cdot\nabla_y u_3 + Q(u_3) \;=\; -v\cdot\nabla_x u_2 + \gamma u_1. \tag{4.9}$$

From (4.6), we check that ψu_0 is a solution to (2.1) with $\lambda = 1$. Hence it is proportional to ψ for any given $x \in \Omega$. Therefore $u_0(x,y,v) \equiv u_0(x)$.

Owing to (4.7), we have

$$u_1(x,y,v) = \theta^i(y,v)\partial_{x_i}u_0(x) + u_{10}(x) \tag{4.10}$$

with convention of summation over repeated indices. Indeed according to Proposition 2.5 and assumption (2.4), $\psi\theta^i$ is a solution to (2.8) with $S = \psi v_i$. The functions θ^i are defined up to the addition of a constant. To simplify θ^i is supposed to have zero mean. For x given, u_1 is also defined up to an additive constant that we call $u_{10}(x)$.

The compatibility condition for the existence of a solution to (4.8), for a given x, is expressed by

$$\int_{Y\times V} \psi\psi^*(-v\cdot\nabla_x u_1 + \gamma u_0 + q(x)h(y,v))dy\,dv \;=\; 0.$$

Replacing u_1 by its expression (4.10), u_{10} vanishes in the integral because of (2.4) and we obtain equation (3.5). Boundary conditions for u_0 are derived from those for u_ε. The diffusion matrix D is positive definite according to Lemma 4.3 below. Therefore u_0 is defined uniquely.

Next by solving equation (4.8), we obtain that

$$u_2(x,y,v) \;=\; w_{2ij}(y,v)\frac{\partial^2 u_0}{\partial x_i \partial x_j}(x) + \gamma u_0 + w_{2a}(y,v)q(x) + \theta^i(y,v)u_{10}(x),$$

where w_{2ij} and w_{2a} are the zero-mean solutions of

$$\begin{aligned} v\cdot\nabla_y w_{2ij} + Q w_{2ij} &= -(v_i\theta^j - \int_{Y\times V} v_i\theta^j\ \psi\psi^* dy\ dv) \\ v\cdot\nabla_y w_{2a} + Q w_{2a} &= q(x)h(y) - \bar{S}(x). \end{aligned}$$

Here we assume the normalization $\int_{Y\times V}\psi\psi^* dydv = 1$. These equations are well posed according to the Fredholm alternative given in Proposition 2.5. The compatibility condition for the existence of a solution to (4.9) is given by $\int_{Y\times V}(-v\cdot\nabla_y u_2)\psi\psi^* dy\ dv = 0$. According to (H6) we can show by simple symmetry arguments (see [3, 4] for similar results) that $\int_{Y\times V} w_{2ij}\psi\psi^* dy\ dv = 0$ and $\int_{Y\times V} w_{2a}\psi\psi^* dy\ dv = 0$. The compatibility condition simplifies then to

$$-\nabla\cdot D\nabla u_{10} \;=\; \gamma u_{10}$$

as expected from (3.6). The equation for u_3 can be solved similarly to that of u_2 and its solution is expressed as a sum of products of known functions of x and (y,v). By assumption each of these terms is regular, at least of class $C^{1,\alpha}$ of its domain of definition. Now from (4.5) and the expressions of the u_i, $i = 0,\ldots,3$, we obtain the following equation for ζ_ε

$$\left\{ \begin{aligned} &\frac{1}{\varepsilon}v\cdot\nabla\zeta_\varepsilon + \frac{1}{\varepsilon^2}Q_\varepsilon(\zeta_\varepsilon) \;=\; \gamma\zeta_\varepsilon \;-\; \varepsilon^2 v\cdot\nabla_x u_3 \quad \text{in } \Omega\times V \\ &\varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \zeta_\varepsilon \;=\; 0 \quad \text{on } \Gamma_-. \end{aligned} \right. \tag{4.11}$$

Since $v\cdot\nabla_x u_3$, u_1, u_2 and u_3 are bounded functions, we deduce from Lemma 4.1 and Proposition 4.2 that ζ_ε is of order ε in $L^2(\Omega\times V)$ and would be of order ε^2 if the term u_1 vanished. It turns out that u_1 can not be equated to zero at the boundary Γ_- because the functions θ^i depend on (y,v) whereas u_{10} depends only on x. Surprisingly enough, it is possible to derive boundary conditions for u_{10}, which enable us to obtain a solution ζ_ε to (4.11) with the following properties. ζ_ε is of order ε in $L^\infty(\Omega \times V)$, and decays exponentially away from the boundary. Therefore it is also of order $\varepsilon^{3/2}$ in $L^2(\Omega \times V)$. This result comes from the analysis of a boundary layer given in our Step 3 below. Before, we prove a lemma we used to obtain the well-posedness of the diffusion equations.

Lemma 4.3 *The diffusion tensor $D = (D_{ij})_{1 \le i,j \le N}$, defined by*

$$D_{ij} \;=\; -\int_Y \int_V v_j \psi(y,v)\psi^*(y,v)\theta^i(y,v)dv\,dy,$$

is positive definite.

Proof. The proof is given in [2]. We shall use similar results in the sequel and therefore we sketch it here. Let $\xi \in \mathbb{R}^n$ and $\theta_\xi = \sum_{i=1}^n \xi_i \theta^i$, where θ^i is a solution of (3.9). Multiplying by $\theta_\xi \psi\psi^*$ the cell equation satisfied by θ_ξ, a similar computation to that of Lemma 4.1 yields

$$\begin{aligned} D\xi \cdot \xi \;&=\; \int_Y \int_V (v \cdot \nabla_y \theta_\xi + Q(\theta_\xi))\theta_\xi \psi\psi^* dy\,dv \\ &=\; \frac{1}{2}\int_Y \int_{V'} \int_V \sigma_\infty \psi^*(v)\psi(v')|\theta_\xi(v) - \theta_\xi(v')|^2 dy\,dv\,dv'. \end{aligned}$$

This implies that $D\xi \cdot \xi \ge 0$ for all $\xi \in \mathbb{R}^n$. Assume now that $D\xi \cdot \xi = 0$. Then θ_ξ does not depend on v, hence $Q(\theta_\xi) = 0$. This yields

$$v \cdot \nabla_y \theta_\xi \;=\; v \cdot \xi \qquad \text{in } Y \times V.$$

Since V contains n independent directions and θ_ξ does not depend on v, we conclude that $\nabla_y \theta_\xi = -\xi$, which is only possible for a Y-periodic function if $\xi = 0$. This proves that there exists a positive constant C such that $D\xi \cdot \xi \ge C|\xi|^2$. □

Step 3: Derivation of the boundary conditions for u_{10}. To simplify we assume here that $\gamma = 0$ since it yields only a corrector term of order ε^2. Consider the error term of order ε:

$$\left\{ \begin{array}{l} \dfrac{1}{\varepsilon} v \cdot \nabla b_\varepsilon + \dfrac{1}{\varepsilon^2} Q_\varepsilon(b_\varepsilon) \;=\; 0 \quad \text{ in } \Omega \times V \\ b_\varepsilon \;=\; -(\theta^i \partial_{x_i} u_0 + u_{10}) \quad \text{ on } \Gamma_-. \end{array} \right. \tag{4.12}$$

By linearity it is the sum of $m(n)$ terms vanishing at each side of Ω but one, where $m(n)$ is the number of sides of Ω. We consider here only the side $x_1 = 0$, which we call γ^0, and denote by b^1_ε the corresponding term in the sum. We denote by $\gamma^0_- = \Gamma_- \cap (\gamma^0 \times V)$. Since $u_0 = 0$ on $\partial\Omega$, we obtain that

$$\begin{cases} \dfrac{1}{\varepsilon} v \cdot \nabla b^1_\varepsilon + \dfrac{1}{\varepsilon^2} Q_\varepsilon(b^1_\varepsilon) \;=\; 0 \quad \text{in } \Omega \times V \\ b^1_\varepsilon \;=\; -(\theta^1 \partial_{x_1} u_0 + u_{10}) \quad \text{on } \gamma^0_- \\ b^1_\varepsilon \;=\; 0 \quad \text{on } \Gamma_- \backslash \gamma^0_- . \end{cases}$$

Let ε go to zero and consider the change of variables $x \to \frac{x}{\varepsilon}$ around a given point $(0, x')$ of the boundary of Ω. Formally at the limit $\varepsilon \to 0$ we obtain the following equation for $b(y, v)$

$$\begin{aligned} v \cdot \nabla b \; + \; Q(b) \;&=\; 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^{n-1} \times V \\ b \;&=\; -(\theta^1(y, v) \partial_{x_1} u_0(0, x') + u_{10}(0, x')) \quad \text{on } \Gamma_-, \end{aligned} \tag{4.13}$$

where we also use the symbol Γ_- for the set of incoming boundary conditions for $\mathbb{R}^+ \times \mathbb{R}^{n-1} \times V$. This is the half space problem treated in Theorem 3.4. We deduce that (4.13) admits an exponentially decaying solution as $y_1 \to \infty$ if

$$G(\theta^1(y, v)) \partial_{x_1} u_0(0, x') + u_{10}(0, x') = 0.$$

This relation is equivalent to

$$L \frac{\partial u_0}{\partial n} + u_{10} \;=\; 0, \tag{4.14}$$

which corresponds to the boundary condition given in (3.6). Let us prove now that b^1_ε and b_ε are of order ε in $L^2(\Omega \times V)$. We denote by b_∞ the exponentially decaying solution to

$$\begin{aligned} v \cdot \nabla b_\infty \; + \; Q(b_\infty) \;&=\; 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^{n-1} \times V \\ b_\infty \;&=\; \theta^1(y, v) + L \quad \text{on } \Gamma_- \end{aligned} \tag{4.15}$$

and d in $\mathbb{R}^+ \times \mathbb{R}^{n-1} \times V$ by

$$d(x_1, x', v) \;=\; -\partial_{x_1} u_0(\varepsilon x') \; b(x, v), \tag{4.16}$$

where $\partial_{x_1} u_0(0, \varepsilon x')$ is extended by periodicity outside γ_0 on the whole hyperplane $x_1 = 0$. Then d is a solution of the following equation in $\mathbb{R}^+ \times \mathbb{R}^{n-1} \times V$

$$
\begin{aligned}
v \cdot \nabla d \; + \; Q(d) \; &= \; b \; v \cdot \nabla[\partial_{x_1} u_0(\varepsilon x')] = \varepsilon b \; (v' \cdot \nabla_{x'} \partial_{x_1} u_0)(\varepsilon x, v) \\
d \; &= \; -(\theta^1(y,v) + L)\partial_{x_1} u_0 \quad \text{ on } \Gamma_- .
\end{aligned}
\tag{4.17}
$$

Consider also the solution h of

$$
\begin{aligned}
v \cdot \nabla h \; + \; Q(h) \; &= \; 0 \quad \text{ in } \mathbb{R}^+ \times \mathbb{R}^{n-1} \times V \\
h \; &= \; -(\theta^1(y,v) + L)\partial_{x_1} u_0 \quad \text{ on } \Gamma_-
\end{aligned}
\tag{4.18}
$$

where again $\partial_{x_1} u_0$ is extended by periodicity outside γ_0. The difference $d - h$ satisfies the same equation (4.17) as d with a homogeneous boundary condition. Since b is exponentially decaying, the source term in the previous equation satisfies the hypotheses of Theorem 3.5. Therefore $d - h$ and its derivative $v \cdot \nabla(d - h)$ are bounded by $0(\varepsilon)$ in $L^\infty(\mathbb{R}^+ \times \mathbb{R}^{n-1} \times V)$. Since $u_0 = 0$ at the boundary, we have $\partial_{x_1} u_0 = 0$ on each side of $\partial\Omega$ which is not parallel to the hyperplane $x_1 = 0$. Therefore d either vanishes or is exponentially small on all sides of $\partial\Omega^\varepsilon$ except the side γ_0. Consequently $h = (h - d) + d$ is bounded by $O(\varepsilon)$ on each side of $\partial\Omega^\varepsilon$ except γ_0. Consider now $\delta_\varepsilon(x, v) = b_1^\varepsilon(x, v) - h(\varepsilon^{-1}x, v)$ on Ω. It satisfies the following equation

$$
\begin{cases}
\dfrac{1}{\varepsilon} v \cdot \nabla \delta_\varepsilon + \dfrac{1}{\varepsilon^2} Q_\varepsilon(\delta_\varepsilon) \; = \; 0 \quad \text{ in } \Omega \times V \\
\delta_\varepsilon \; = \; 0 \quad \text{ on } \gamma_-^0 \\
\delta_\varepsilon \; = \; -h(\varepsilon \cdot, \cdot) \quad \text{ on } \Gamma_- \backslash \gamma_-^0 .
\end{cases}
$$

Therefore δ_ε is bounded by $O(\varepsilon)$ in $L^\infty(\Omega \times V)$ as well as $\delta_\varepsilon + (h - d)(\varepsilon^{-1} \cdot, \cdot)$. By definition (4.16), $d(\varepsilon^{-1} \cdot, \cdot)$ is of order $O(\varepsilon^{1/p})$ in $L^p(\Omega \times V)$ for $1 \le p \le \infty$ due to the exponential decay of b. We deduce that $b_\varepsilon^1 = \delta_\varepsilon + (h - d)(\varepsilon^{-1} \cdot, \cdot) + d(\varepsilon^{-1} \cdot, \cdot)$ is of order $O(\varepsilon^{1/p})$ in $L^p(\Omega \times V)$. This concludes the proof of the theorem. $\square$

5 Proof of Theorem 3.4

Let us introduce some notations. We denote by $\Gamma_Y = (0,1)^{d-1}$ the trace of Y on the plane $x_1 = 0$. For $(A, B) \in \mathbb{R}^2$, let be $X_{AB} = (A, B) \times \Gamma_Y \times V$ and $\Gamma_A = \{A\} \times \Gamma_Y \times V$.

We define also $\Gamma_{AB} = \partial((A,B) \times \Gamma_Y) \times V$ and $X = (0,\infty) \times \Gamma_Y \times V$. The proof of Theorem 3.4 relies on the following lemmas.

Lemma 5.1 *Let u be a regular function defined on $Y \times V$. Then there exists a constant C independent of u such that*

$$\|u - \int_{Y\times V} u \; dydv\|_{L^2(Y\times V)} \leq C \left(\|v \cdot \nabla u\|_{L^2(Y\times V)} + \|u - \int_V u \; dv\|_{L^2(Y\times V)} \right). \tag{5.1}$$

Proof. Let be $v \in V$ given. We have

$$u(x+tv, v) - u(x,v) \;=\; \int_0^t (v \cdot \nabla u)(x+sv, v) ds.$$

Squaring and integrating this equation along the direction v and then over the transverse directions yields the inequality

$$\|u(\cdot, v) - \int_Y u(y,v)dy\|_{L^2(Y)} \leq C\|v \cdot \nabla u\|_{L^2(Y)},$$

where C is a constant which does not depend on u. We also check that

$$\| \int_Y u(y,v)dy - \int_{Y\times V} u(y,v)dydv\|_{L^2(Y\times V)} \leq C\|u - \int_V u\|_{L^2(Y\times V)}.$$

From these two inequalities, we conclude the proof of the lemma. □

Lemma 5.2 *Let us define the local flux*

$$J(y) \;=\; \int_V v\psi(y,v)\psi^*(y,v)dv.$$

Then we have div $J(y) = 0$. Assume that (2.4) is satisfied. Then for a.e. $0 \leq B \leq 1$, we have

$$\int_{\Gamma_B} J(B, y')dy' \;=\; 0.$$

Proof. Using (4.2), we deduce the conservation of flux div $J(y) = 0$. We integrate now the equality over Γ_B. Owing to the periodicity of ψ and ψ^* we have $\partial_B \int_{\Gamma_B} J(B,y')dy' = 0$, which implies that

$$\int_{\Gamma_B} J(B, y')dy' = Cte.$$

Integrating this equality over $(0,1)$ yields that $Cte = J$ which is 0 according to (2.4). This proves the result. □

Let us come back to the proof of Theorem 3.4.

Step 1: Existence of a solution to (3.14). We define the following sequence of problems for $\alpha > 0$

$$\begin{aligned} v\cdot\nabla b^\alpha \ + \ Q(b^\alpha) + \alpha b^\alpha &= 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^{n-1} \times V \\ b^\alpha &= g \quad \text{on } \Gamma_-. \end{aligned} \tag{5.2}$$

For $\alpha > 0$, we deduce from the maximum principle in [12, Chapter 21] that this problem admits a unique solution in $L^\infty(\mathbb{R}^+ \times \mathbb{R}^{n-1} \times V)$. As was done in the proof of Proposition 4.2, we obtain that $b^\alpha \rightharpoonup b$ in $L^\infty(\mathbb{R}^+ \times \mathbb{R}^{n-1} \times V)$ weak*, where b is solution to (3.14) which is $Y-$periodic with respect to x'.

Step 2: Uniqueness. The proof of uniqueness of a solution to (3.14) and its asymptotic behavior rely on a variational formulation of (3.14) that we give now. Assume that there exists a solution to the following problem

$$\begin{aligned} v\cdot\nabla b \ + \ Q(b) &= e^{-\nu x_1}S \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^{n-1} \times V \\ b &= g \quad \text{on } \Gamma_-. \end{aligned} \tag{5.3}$$

Let $\theta(x)$ be a given regular and bounded test function defined on $\mathbb{R}^+ \times \mathbb{R}^{n-1}$. We multiply (5.3) by $b\,\theta\,\psi\psi^*$ and integrate by parts over X_{0B}. One first term is

$$\int_{X_{0B}} v\cdot\nabla b\; b\,\theta\,\psi\psi^* dxdv = \frac{1}{2}\int_{\Gamma_{0B}} b^2\theta(v\cdot n)\,\psi\psi^* d\sigma dv - \frac{1}{2}\int_{X_{0B}} b^2(\theta v\cdot\nabla(\psi\psi^*) + \psi\psi^* v\cdot\nabla\theta)dxdv.$$

Recall that $v\cdot\nabla(\psi\psi^*)$ is given by (4.2). Therefore we have

$$\begin{aligned} &\frac{1}{2}\int_{\Gamma_{0B}} b^2\theta(v\cdot n)\,\psi\psi^* d\sigma dv + \frac{1}{2}\int_{X_{0B}}\int_V \theta\psi^*\sigma\psi(b(x,v') - b(x,v))^2 dxdvdv' \\ &-\frac{1}{2}\int_{X_{0B}} b^2\,\psi\psi^* v\cdot\nabla\theta\; dxdv \ = \ \int_{X_{0B}} e^{-\nu x_1} S\ \psi\psi^*\ b\ \theta\ dxdv. \end{aligned} \tag{5.4}$$

Multiplying equation (5.3) by $\psi\psi^*$ and integrating by parts yields

$$\int_{\Gamma_{0B}} b(v\cdot n)\,\psi\psi^* d\sigma dv \ = \ \int_{X_{0B}} e^{-\nu x_1} S\psi\psi^*\ dxdv. \tag{5.5}$$

Assume now that b is a bounded solution of (5.3) and that $S = 0$. Take $\theta = 1$ in (5.4). Since b is bounded, there exists an increasing sequence B_N going to infinity such that

$$\frac{1}{2}\int_{X_{0B_N}}\int_V \psi^*\sigma\psi\,(b(x,v') - b(x,v))^2\, dxdvdv'$$

be bounded independently of N. Noticing that $\psi^*\sigma\psi$ is bounded from below by a positive constant and according to equation (5.3) with $S=0$ we obtain by letting N go to infinity that

$$\|v\cdot\nabla b\|_{L^2(X)}+\|b-\int_V b(\cdot,v')dv'\|_{L^2(X)} \;\leq\; C. \tag{5.6}$$

Integrating now (5.4) over $(M,M+1)$ with $\theta=1$ yields

$$\begin{aligned}&\frac{1}{2}\int_{\Gamma_0} b^2(v\cdot n)\,\psi\psi^* d\sigma dv+\frac{1}{2}\int_{X_{M,M+1}} b^2(v\cdot n)\,\psi\psi^* dxdv\\ &+\frac{1}{2}\int_{X_{0,M+1}}\int_V \theta_M\psi^*\sigma\psi(b(x,v')-b(x,v))^2 dxdvdv' \;=\; 0,\end{aligned}$$

where $\theta_M(x)=1$ for $0\leq x_1\leq M$ and $\theta_M(x)=1+M-x_1$ for $M\leq x_1\leq M+1$. According to (5.6), we have

$$\|v\cdot\nabla b\|_{L^2(X_{M,M+1})}+\|b-\int_V b(\cdot,v')dv'\|_{L^2(X_{M,M+1})}\to 0 \quad \text{as} \quad M\to\infty.$$

We denote by $<b>$ the average of b over $X_{M,M+1}\times V$. By virtue of Lemma 5.1, we have

$$\|b-\;<b>\|_{L^2(X_{M,M+1})}\to 0 \quad \text{as} \quad M\to\infty.$$

Since b is bounded and $J=0$ according to (2.4), we obtain that

$$\frac{1}{2}\int_{X_{M,M+1}} b^2(v\cdot n)\,\psi\psi^* dxdv \;=\; \frac{1}{2}\int_{X_{M,M+1}}(b^2-<b>^2)(v\cdot n)\,\psi\psi^* dxdv\to 0 \quad \text{as} \quad M\to\infty.$$

Thus letting M go to infinity, we have

$$\frac{1}{2}\int_{\Gamma_+} b^2|v\cdot n|\,\psi\psi^* d\sigma dv+\frac{1}{2}\int_X\int_V \psi^*\sigma\psi(b(x,v')-b(x,v))^2 dxdvdv' \;=\; \frac{1}{2}\int_{\Gamma_-} g^2|v\cdot n|\,\psi\psi^* d\sigma dv. \tag{5.7}$$

Taking $g=0$ yields $b(x,v)\equiv b(x)$ and $b=0$ on Γ_+. From (3.14) we obtain that $v\cdot\nabla b=0$, hence $b=0$. This completes the proof of uniqueness.

Step 3: Exponential decay in L^2. In order to prove the exponential decay of the solution, we consider test functions θ of the form $\theta(x)=e^{\beta x_1}$. Upon plugging this expression into (5.4) with $S=0$ and $B=+\infty$, we have

$$\begin{aligned}&\int_{\Gamma_+} b^2(v\cdot n)\,\psi\psi^* d\sigma dv+\int_X e^{\beta x_1}\psi^*\sigma\psi(b(x,v')-b(x,v))^2 dxdv-\int_X \beta v_1 b^2 e^{\beta x_1}\psi\psi^* dxdv\\ =\;&\int_{\Gamma_-} g^2|v\cdot n|\,\psi\psi^* d\sigma dv.\end{aligned}$$

Because $\psi^*\sigma\psi$ is bounded from below by a positive constant and according to equation (5.3) with $S=0$, there exists a positive constant C such that

$$C(\|e^{\frac{\beta x_1}{2}} v\cdot\nabla b\|^2_{L^2(X)} + \|e^{+\frac{\beta x_1}{2}}(b-\int_V b(\cdot,v')dv')\|^2_{L^2(X)})$$
$$+\int_X \beta v_1 b^2 e^{\beta x_1}\psi\psi^* dxdv \le \int_{\Gamma_-} g^2|v\cdot n|\,\psi\psi^* d\sigma dv.$$

Let us define the following mean operators over a cross-section and a cell

$$m_{\Gamma_B}(u)=\int_{\Gamma_B} u(B,x',v)\,dx'\,dv \quad \text{ and } \quad m_M(u)=\int_{X_{M,M+1}} u(x,v)\,dx\,dv.$$

We deduce from Lemmas 5.1 and 5.2 that

$$C\|e^{\frac{\beta x_1}{2}}(b-m_M(b))\|^2_{L^2(X)} + \int_X \beta v_1(b^2-m^2_{\Gamma_{x_1}}(b))e^{\beta x_1}\psi\psi^* \le \int_{\Gamma_-} g^2|v\cdot n|\,\psi\psi^*.$$

Assume that $\|g\|_{L^\infty(\Gamma_-)}=1$. Since b is bounded by 1, we deduce from the Cauchy Schwartz inequality that there exists a constant C independent of g so that

$$\int_{X_{M,M+1}} \beta|v_1|(b^2-m^2_{\Gamma_{x_1}}(b))e^{\beta x_1}\psi\psi^* dxdv \le C\int_{X_{M,M+1}} \beta e^{\beta x_1}(b-m_{\Gamma_{x_1}}(b))^2 dxdv.$$

We also have that

$$\int_{X_{M,M+1}} \beta e^{\beta x_1}(b-m_{\Gamma_{x_1}}(b))^2 dxdv \le C\int_{X_{M,M+1}} \beta e^{\beta x_1}(b-m_M(b))^2 dxdv.$$

Hence for β positive and small enough, we obtain that

$$\int_X e^{\beta x_1}(b-m_M(b))^2 dxdv \le C\int_{\Gamma_-} g^2|v\cdot n|\,\psi\psi^* d\sigma dv.$$

The result also holds true if we average over two cells. Hence we have

$$\int_X e^{\beta x_1}(b-\frac{1}{2}(m_M(b)+m_{M+1}(b)))^2 \le C\int_{\Gamma_-} g^2|v\cdot n|\,\psi\psi^*.$$

We deduce from this inequality that $m_M(b)$ converges exponentially to a constant $G(g)$. Moreover we check that

$$\|b-G(g)\|_{L^2(X_{M,\infty})} \le Ce^{-\frac{\beta M}{2}}, \tag{5.8}$$

where C and β are independent of g for $\|g\|_{L^\infty(\Gamma_-)}=1$. From the linearity of the maps $g\mapsto b$ and $g\mapsto G(g)$, we deduce that for the same constants C and β independent of g, we have

$$\|b-G(g)\|_{L^2(X_{M,\infty})} \le Ce^{-\frac{\beta M}{2}}\|g\|_{L^\infty(\Gamma_-)}. \tag{5.9}$$

The proof that $g \mapsto G(g)$ is a linear form in $L^\infty(\Gamma_-)$ is deduced from equation (5.14) below.

Step 4: L^∞ estimate. To prove bounds in L^∞ spaces, we use the following result

Lemma 5.3 *Let $g \in L^\infty(\Gamma_-)$ be a positive function which is periodic in x', and let u be a solution of*

$$\begin{aligned} v \cdot \nabla u \ + \ Q(u) \ &= \ 0 \quad \text{in } I\!R^+ \times \ I\!R^{n-1} \times V \\ u \ &= \ g \quad \text{on } \Gamma_-, \end{aligned} \tag{5.10}$$

which is also periodic in x'. Assume that $\phi(x) = \int_V u(x,v)dv$ converges exponentially to 0 in the sense that

$$\|\phi\|_{L^p(Y_M)} \leq Ce^{-\frac{\beta}{p}M}\|g\|_{L^\infty(\Gamma_-)}, \tag{5.11}$$

where β is positive and small enough and where Y_M is the M^{th} cell $Y_M = (M, M+1) \times (0,1)^{n-1}$. Then the density exiting or entering Y_M is also decaying exponentially

$$\left(\int_{\Gamma_\pm(Y_M)} |v \cdot n||u|^p \, dv \, d\sigma\right)^{1/p} \leq Ce^{-\frac{\beta}{p}M}\|g\|_{L^\infty(\Gamma_-)}.$$

Proof. Remark that the particles coming out from one cell go into the next one. Therefore we prove the result only for the density coming out from $\Gamma_+(Y_M)$. The proof of this result uses the integral formulation of transport given for instance in [12, Chapter 21]. The solution u can be written as

$$\begin{aligned} u(x,v) \ &= \ \int_0^{t(x,v)} ds \ e^{-\alpha(x,x-sv)} \int_V \tilde{\sigma}(x-sv,v',v)u(x-sv,v')dv' \\ &+ \ e^{-\alpha(x,x-t(v,x)v)} \ g(x-t(v,x)v,v), \end{aligned} \tag{5.12}$$

where $t(x,v)$ is the exit travel time defined by $x - t(x,v)v \in \partial\Omega$ and $0 \leq t(x,v) \leq +\infty$, and where $\alpha(x, x-sv)$ is the optical length between x and $x - sv$ defined by

$$\alpha(x,x-sv) \ = \ \int_0^s \tilde{\Sigma}(x-tv,v)dt.$$

Since $\tilde{\sigma}$ is uniformly bounded by σ_1, we have

$$e^{-\alpha(x,x-tv)} \geq e^{-\sigma_1 t}.$$

Therefore the second term in the right hand side of (5.12) decays exponentially and has the correct form on $\Gamma_+(Y_M)$ if β is small enough. Without loss of generality we proceed as if this term vanished. Consider now the first term in the right hand side of (5.12). For simplicity, we deal with the side of Y_M where $x_1 = M+1$ and we call it Z_M. According to the Hölder inequality, we have

$$\begin{aligned} |u(x,v)|^p &\le C \left(\int_0^{t(x,v)} ds\ e^{-\sigma_1 s} |\phi(x - sv)| \right)^p \\ &\le C \int_0^{t(x,v)} ds\ e^{-\sigma_1 s} |\phi(x-sv)|^p. \end{aligned}$$

Indeed for $\frac{1}{p} + \frac{1}{p'} = 1$, we have

$$\int_0^t ds\ e^{-\sigma_1 s} f^{\frac{1}{p}} = \int_0^t ds\ e^{-\frac{\sigma_1 s}{p'}} (e^{-\sigma_1 s} f)^{\frac{1}{p}} \le (\int_0^t ds\ e^{-\sigma_1 s})^{\frac{1}{p'}} (\int_0^t ds\ e^{-\sigma_1 s} f)^{\frac{1}{p}},$$

and the results follows from the integration of the exponential term, which is independent of t. Denoting the spatial integration along the direction v by ds, we also have that

$$|v|\ dx = |v \cdot n|\ dx'\ ds$$

where n is defined by $x \cdot n = x_1$. Therefore, we obtain that

$$\begin{aligned} &\int_{(v\cdot n)>0} dv \int_{Z_M} dx' |v \cdot n| |u(M+1, x', v)|^p \\ &\le C \int_{(v\cdot n)>0} dv \int_{Z_M} dx' |v\cdot n| \int_0^{t((M+1,x'),v)} ds\ e^{-\sigma_1 s} |\phi((M+1,x') - sv)|^p \\ &\le C \int_{(v\cdot n)>0} dv \int_{S(v)} dy\ e^{-\frac{\sigma_1}{v_{min}} |y_1 - (M+1)|} |\phi(y)|^p, \end{aligned}$$

where $S(v)$ is the subset of a cylinder defined *a.e.* in V by

$$S(v) = \{ y \in \mathbb{R}^+ \times \mathbb{R}^{n-1} \text{ s.t. } y = (x', M+1) - sv \text{ where } (x', M+1) \in Z_M \text{ and } s \in \mathbb{R}^+ \}.$$

By periodicity of ϕ and (5.11), we deduce that

$$\begin{aligned} \int_{(v\cdot n)>0} dv \int_{Z_M} dx' |v\cdot n| |u(M+1,x',v)|^p &\le C \int_{(0,M+1)\times(0,1)^{n-1}} e^{\frac{-\sigma_1}{v_{min}} |y_1 - (M+1)|} |\phi(y)|^p\ dy \\ &\le C \sum_{N=0}^{M} e^{-\frac{\sigma_1}{v_{min}} |M+1-N|} e^{-\beta N} \sim C e^{-\beta M}. \end{aligned}$$

The other sides of Y_M are handled similarly. This concludes the proof of the lemma. □

Notice that the rate of convergence $e^{-\frac{\beta M}{p}}$ tends to 0 when p goes to infinity. Therefore we can not deduce from this estimate any decay when $p = \infty$. In order to obtain a result in L^∞ spaces, we use the averaging lemma given in [16]: if u and $v \cdot \nabla u$ belong to $L^p(\Omega \times V)$ and the trace at the boundary $u_{|\Gamma_-}$ belongs to $L^p(\Gamma_-, |v \cdot n|)$ for $2 \leq p < \infty$, then $\int_V u(x,v)dv$ belongs to $W^{s,p}(\Omega)$, the usual Sobolev space, with continuous embedding for $0 < s < \frac{1}{p}$. Then by the Sobolev embedding theorem, $\int_V u(x,v)dv$ belongs to $L^q(\Omega)$ where

$$p \leq q < \frac{np}{n-1}.$$

Let u be a solution to (5.10) satisfying the conditions of Lemma 5.3 with $p = 2$. Then according to this lemma, we have

$$\|\phi\|_{L^2(Y_M)} + \|u\|_{L^2(\Gamma_-(Y_M))} \leq Ce^{-\frac{\beta}{2}M} \|g\|_{L^\infty(\Gamma_-)}.$$

From the averaging lemma and the Sobolev embedding, we obtain that $\phi \in L^q(Y_M)$ where $q < \frac{2n}{n-1}$. Therefore from Lemma 5.3, we deduce that

$$\|\phi\|_{L^q(Y_M)} + \|u\|_{L^q(\Gamma_-(Y_M))} \leq Ce^{-\frac{\beta}{q}M} \|g\|_{L^\infty(\Gamma_-)}.$$

Owing to the classical results [12] on the solution of the transport equation (5.10), we deduce that u and $v \cdot \nabla u$ are also in $L^q(Y_M \times V)$ and are bounded by $Ce^{-\frac{\beta}{q}M}\|g\|_{L^\infty(\Gamma_-)}$. We check by induction that the result holds true for $q < \infty$. Among others, it holds true for some $n + \eta$ where $\eta > 0$.

Replacing g by $g - G(g)$ in (3.14), we check that b satisfies the hypotheses of Lemma 5.3 with $p = 2$. Let us denote by $\phi(x) = \int_V b(x,v)dv$. We know that

$$\|\phi\|_{L^{n+\eta}(Y_M)} + \|\phi\|_{L^2(Y_M)} \;\leq\; Ce^{-\frac{\beta}{n+\eta}M} |g\|_{L^\infty(\Gamma_-)}.$$

A point-wise estimate for b will follow from the regularization property of an integral operator. Define

$$Tu \;=\; v \cdot \nabla u + \tilde{\Sigma} u \quad \text{and} \quad Ku \;=\; \int_V \tilde{\sigma}(x,v,v')u(x,v')dv'.$$

Then the integral formulation for b reads

$$b \;=\; T^{-1}Kb + Bg,$$

where $Bg(x,v) \;=\; e^{-\alpha(x,x-t(x,v)v)} g(x - t(x,v)v, v)$. Therefore we have

$$b \;=\; (T^{-1}K)^2 b + T^{-1}KBg + Bg.$$

Unlike $T^{-1}K$, the operator $KT^{-1}K$ is an integral operator with useful regularization properties, which have already been used in several works [12, 21, 22]. The remaining terms $T^{-1}KBg$ and Bg, corresponding to single scattering and no scattering events respectively, are clearly decaying exponentially, as would do any term involving only a finite number of scattering events.

Up to the addition of a constant, we assume that $G(g) = 0$. Let us denote by $\psi_2 = (T^{-1}K)^2 b$. According to the change of variables $y = x - sv - s'v'$ and the change of measure $dy = sdsd(\frac{v}{|v|})$, we deduce from (3.13) that

$$
\begin{aligned}
|\psi_2(x,v)| &\leq C\int_0^{t(x,v)} ds\ e^{-\sigma_0 s}\int_V dv' \int_0^{t(x-sv,v')} ds'\ e^{-\sigma_0 s'}|\phi(x-sv-sv')| \\
&\leq C\int_0^{t(x,v)} ds\ e^{-\sigma_0 s}\int_{\mathbb{R}^+\times\ \mathbb{R}^{n-1}} \frac{e^{-\frac{\sigma_0}{v_{min}}|y-(x-sv)|}}{|y-(x-sv)|^{n-1}}\ |\phi(y)|\ dy.
\end{aligned}
$$

For $a.e.$ $x \in \mathbb{R}^+\times\ \mathbb{R}^{n-1}$, we define $Y(x)$ as the unit cell to which x belongs, and $W(x)$ as the set of 3^{n-1} cells centered on $Y(x)$. For $d \in Z^n$ let $Y_d = \Pi_{i=1}^n(d_i, d_{i+1})$ and x_d the center of the cell Y_d. Then we have

$$
\begin{aligned}
\varphi(x) &\stackrel{def}{=} \int_{\mathbb{R}^+\times\ \mathbb{R}^{n-1}} \frac{e^{-\frac{\sigma_0}{v_{min}}|y-x|}}{|y-x|^{n-1}}\ |\phi(y)|\ dy \\
&\leq \int_{|y-x|<1} \frac{1}{|y-x|^{n-1}}\ |\phi(y)|\ dy\ +\ \int_{|y-x|>1} e^{-\frac{\sigma_0}{v_{min}}|y-x|}\ |\phi(y)|\ dy \\
&\leq C\|\phi\|_{L^{n+\eta}(W(x))} + C\sum_{d\in Z^n} e^{-\frac{\sigma_0}{v_{min}}|x-x_d|}\|\phi\|_{L^2(Y_d)},
\end{aligned}
$$

since $\frac{1}{|y-x|^{n-1}}$ belongs to $L^p(W(x))$ for any $1 \leq p < \frac{n}{n-1}$. This proves that

$$
\varphi(x)\ \leq\ Ce^{-\frac{\beta}{n+\eta}x_1}\|g\|_{L^\infty(\Gamma_-)}.
$$

As a consequence, we have the following estimate

$$
\begin{aligned}
|b(x,v)| &\leq \int_0^{t(x,v)} e^{-\sigma_0 s}(1 + e^{-\frac{\beta}{n+\eta}(x-sv)_1})\|g\|_{L^\infty(\Gamma_-)}ds + e^{-\sigma_0 t(x,v)}\|g\|_{L^\infty(\Gamma_-)} \\
&\leq C\ e^{-\frac{\beta}{n+\eta}x_1}\|g\|_{L^\infty(\Gamma_-)},
\end{aligned}
$$

where β is small enough. This concludes our fourth step.

Step 5: Positiveness of $G(-\theta^1)$. We check by symmetry that $G(\theta^i) = G(-\theta^i) = -G(\theta^i) = 0$ for $i \neq 1$. The proof of positiveness of $G(\theta^1)$ for specific configurations, is adapted from [12]. Let us first define the adjoint function θ_1^* as the zero mean solution to

$$\begin{aligned} -v\cdot\nabla(\theta_1^*\psi\psi^*) + Q^*(\theta_1^*\psi\psi^*) &= v_1\psi\psi^* \\ y\mapsto \theta_1^*(y,v):\ Y-\text{ periodic}, \end{aligned} \tag{5.13}$$

where Q^* is the adjoint operator to Q defined by

$$Q^*(u) \ = \ (\int_V \tilde\sigma(x,v',v)dv')\ u(x,v) \ - \ \int_V \tilde\sigma(x,v,v')u(x,v')dv'.$$

Similarly to (3.9), equation (5.13) admits a unique zero mean solution. Moreover, since $J=0$, we deduce from Lemma 5.2 that

$$\partial_B \int_{\Gamma_B} v(\theta_1^*+x_1)\psi\psi^* d\sigma dv \ = \ 0.$$

Therefore, by integrations by parts, we have

$$\int_{\Gamma_B} v(\theta_1^*+x_1)\psi\psi^* d\sigma dv = \int_{Y\times V} v\theta_1^*\psi\psi^* dydv = \int_{Y\times V} v\theta_1\psi\psi^* dydv.$$

Notice that x_1 can be replaced by any smooth function independent of v. Because $-v\cdot\nabla(\psi\psi^*)+Q^*(\psi\psi^*)=0$ and $Q^*(x_1\theta)=x_1Q^*(\theta)$ for every smooth function θ, we deduce that

$$-v\cdot\nabla((\theta_1^*+x_1)\psi\psi^*) + Q^*((\theta_1^*+x_1)\psi\psi^*) \ = \ 0.$$

Define now u as the unique solution to

$$\begin{aligned} &v\cdot\nabla u + Q(u) \ = \ 0 \quad \text{ in } \mathbb{R}^+\times\ \mathbb{R}^{n-1}\times V \\ &u = g \quad \text{ on } \Gamma_-, \end{aligned}$$

where g is a given smooth function. By integrations by parts, we obtain

$$0 = \int_{X_{0B}} v\cdot\nabla u(\theta_1^*+x_1)\psi\psi^* dxdv + \int_{X_{0B}} Q(u)(\theta_1^*+x_1)\psi\psi^* dxdv = \int_{\Gamma_{0B}} (v\cdot n)u(\theta_1^*+x_1)\psi\psi^* d\sigma dv.$$

Since $u\to G(g)$ exponentially when $B\to\infty$ and $\int_{\Gamma_B} v\theta_1^*\psi\psi^*$ is independent of B, we deduce that

$$0 \ = \ \int_{\Gamma_0}(v\cdot n)u\theta_1^*\psi\psi^* d\sigma dv + G(g)D_{11}$$

where D_{11}, defined by (3.8), is positive. This implies that

$$G(g) \ = \ -\frac{\displaystyle\int_{\Gamma_0}(v\cdot n)u\theta_1^*\psi\psi^* d\sigma dv}{D_{11}}, \tag{5.14}$$

and therefore $g \mapsto G(g)$ is a linear form in $L^\infty(\Gamma_-)$. Notice that the constant $G(g)$ depends only on the trace of the transport solution u at the boundary Γ_0.

Let us assume that $\theta^i = \theta_i^*$. This is for instance true when $\sigma(x, v', v) = \sigma(v \cdot v')$. Consider $g = -\theta^1$. We want to prove that $\int_{\Gamma_0} (v \cdot n) u \theta_1 \psi\psi^* > 0$. We deduce from the variational formulation (5.4) with $\theta = 1$ and $S = 0$ that

$$\frac{1}{2}\int_{\Gamma_{AB}} u^2 (v \cdot n)\, \psi\psi^* d\sigma dv + \frac{1}{2}\int_{X_{AB}} \psi(x,v)\sigma(x,v',v)\psi^*(x,v')(u(x,v') - u(x,v))^2 \, dxdv \;=\; 0$$

for all real numbers $A < B$. Therefore the function $B \mapsto \int_{\Gamma_B} u^2 (v \cdot n)\, \psi\psi^* d\sigma dv$ is decaying and converges to 0, hence is non negative. This yields, since $u \not\equiv 0$,

$$\int_{\Gamma_0^+} (v \cdot n) u^2 \psi\psi^* d\sigma dv > \int_{\Gamma_0^-} |v \cdot n| u^2 \psi\psi^* d\sigma dv \;=\; \int_{\Gamma_0^-} |v \cdot n| (\theta^1)^2 \psi\psi^* d\sigma dv.$$

We deduce from the cubic symmetry of Y that

$$h \stackrel{def}{=} \int_{\Gamma_0^+} (v \cdot n)(\theta^1)^2 \psi\psi^* d\sigma dv \;=\; \int_{\Gamma_0^-} (-v \cdot n)(\theta^1)^2 \psi\psi^* d\sigma dv \;>\; 0.$$

These integrals can not vanish because D_{11} would also be zero. Therefore we have

$$\begin{aligned}
\int_{\Gamma_0} v_1 u \theta^1 \psi\psi^* d\sigma dv \;&=\; \int_{\Gamma_0^+} v_1 u \theta^1 \psi\psi^* d\sigma dv + \int_{\Gamma_0^-} v_1 u \theta^1 \psi\psi^* d\sigma dv \\
&\geq\; h - \int_{\Gamma_0^-} (-v_1) u \theta^1 \psi\psi^* d\sigma dv \\
&\geq\; h^{1/2} \left[h^{1/2} - \left(\int_{\Gamma_0^-} (-v_1) u^2 d\sigma dv \right)^{1/2} \right].
\end{aligned}$$

Consequently $G(-\theta_1)$ given by (5.14) is positive. This concludes our fifth step and the proof of the theorem. □

6 Proof of Theorem 3.5

We begin this section with the proof of Lemma 2.6.

Proof of Lemma 2.6. Let us still denote by $K\phi(x, v) = \int_V \tilde{\sigma}(x, v', v)\phi(x, v')dv'$ and $T\phi = v \cdot \nabla\phi + \tilde{\Sigma}\phi$. Let $\xi \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^n$ be two normalized vectors, and define for $X = \phi_\nu$ or $X = \lambda_\nu$

$$X_\xi = \sum_{i=1}^n \xi_i \partial_{\nu_i} X \quad \text{and} \quad X_{\xi,\zeta} = \sum_{i,j=1}^n \xi_i \zeta_j \partial^2_{\nu_i \nu_j} X.$$

Upon differentiating $T\phi_\nu + v \cdot \nu\phi_\nu = \lambda_\nu K\phi_\nu$ with respect to ν, we obtain that

$$T\phi_\xi + v \cdot \nu\phi_\xi + v \cdot \xi\phi_\nu = \lambda_\xi K\phi_\nu + \lambda_\nu K\phi_\xi. \tag{6.1}$$

The compatibility condition, or Fredholm alternative, yields

$$\int_{Y\times V} v \cdot \xi\phi_\nu\phi_\nu^* dydv \;=\; \lambda_\xi(K\phi_\nu, \phi_\nu^*)$$

where $(\cdot,\cdot)$ denotes the usual scalar product in $L^2(Y \times V)$. We deduce from (2.4) that $\nabla_\nu\lambda_\nu|_{\nu=0} = 0$. Differentiating one more time with respect to ν yields

$$T\phi_{\xi,\zeta} + v \cdot \nu\phi_{\xi,\zeta} + v \cdot (\zeta\phi_\xi + \xi\phi_\zeta) = \lambda_{\xi,\zeta}K\phi_\nu + K(\lambda_\zeta\phi_\xi + \lambda_\xi\phi_\zeta) + \lambda_\nu K\phi_{\xi,\zeta}.$$

Here the compatibility condition reads

$$\int_{Y\times V} v \cdot (\zeta\phi_\xi + \xi\phi_\zeta)\phi_\nu^* \; dydv \;=\; (K(\lambda_\zeta\phi_\xi + \lambda_\xi\phi_\zeta), \phi_\nu^*) + \lambda_{\xi,\zeta}(K\phi_\nu, \phi_\nu^*). \tag{6.2}$$

We normalize the eigenvectors such that $(K\phi_\nu, \phi_\nu^*) = 1$. Define by $-D^0_{ij} \equiv \frac{\partial^2\lambda_\nu}{\partial\nu_i\,\partial\nu_j}|_{\nu=0}$. Then D^0 is a positive definite matrix. The proof is similar to that of Lemma 4.3. Multiplying (6.1) by $\phi_\xi\phi_\nu^*(\phi_\nu)^{-1}|_{\nu=0}$ and integrating over $Y \times V$, a similar computation to that of Lemma 4.1 (and Lemma 4.3) proves that (6.2) becomes, since $\lambda_\xi|_{\nu=0} = 0$,

$$\begin{aligned} D^0\xi \cdot \xi \;&=\; 2\int_{Y\times V}(T\phi_\xi + v \cdot \nu\phi_\xi - \lambda_\nu K\phi_\xi)\phi_\xi\phi_\nu^*(\phi_\nu)^{-1}dy\,dv\Big|_{\nu=0} \\ &=\; \int_{Y\times V\times V}\tilde\sigma(x,v',v)\;\phi_\nu^*(y,v)\phi_\nu(y,v')|\frac{\phi_\xi}{\phi_\nu}(x,v) - \frac{\phi_\xi}{\phi_\nu}(x,v')|^2\;dy\,dv\,dv'\Big|_{\nu=0}. \end{aligned}$$

Therefore it is positive. Assume that $D^0\xi \cdot \xi = 0$. Then $\frac{\phi_\xi}{\phi_\nu}$ is independent of v. This yields that

$$v \cdot \nabla\frac{\phi_\xi}{\phi_\nu}\Big|_{\nu=0} = \xi \cdot v \quad \text{ in } Y \times V,$$

Integrating this relation over Y, we obtain that $v \cdot \xi = 0$ for every v. Therefore $\xi = 0$ and D^0 is positive definite. □

We are now in a position to prove Theorem 3.5.

Step 1: Existence of a solution to (3.16). Consider $S = 1$ in equation (2.11). We deduce from Proposition 2.7 the existence of a solution φ_1 to (2.11) in $L^\infty(Y \times V)$. The solution is extended to $\mathbb{R}^n \times V$ by periodicity of $e^{\nu\cdot y}\varphi_1(y,v)$. There exists a constant

C_0 such that $\varphi = C_0 + \varphi_1$ be uniformly bounded from below by a positive constant on $\mathbb{R}^+ \times \mathbb{R}^{n-1} \times V$. Moreover φ is a solution to

$$\begin{aligned} v \cdot \nabla\varphi \ + \ Q(\varphi) \ &= \ e^{-\nu x_1} \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^{n-1} \times V \\ \varphi \ &= \ C_0 + \varphi_1 \quad \text{on } \Gamma_-. \end{aligned}$$

Assume now that $S \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^{n-1} \times V)$ and is periodic with respect to x' of period mY for $m \in \mathbb{N}^*$. The proof of existence is adapted from [15]. Consider the sequence of solutions

$$\begin{aligned} v \cdot \nabla b_{k+1} \ + \ \Sigma b_{k+1} \ &= \ \int_V \sigma(x, v', v) b_k(x, v') dv' + e^{-\nu x_1} S \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^{n-1} \times V \\ b_{k+1} \ &= \ \varphi \quad \text{on } \Gamma_-, \end{aligned} \tag{6.3}$$

where $|S| \leq \varphi$ and $b_0 = -\varphi$. The integral formulation of this equation reads

$$b_{k+1} \ = \ T^{-1}Kb_k + T^{-1}S + B\varphi. \tag{6.4}$$

φ is a solution to (6.4) if $S = 1$ and $b_k = \varphi$. Since $|S(x, \mu)| \leq \varphi$ we deduce that $b_1 \geq -\varphi$ and by induction that $b_{k+1} - b_k \geq 0$. Moreover $b_k \leq \varphi$ and therefore

$$-\varphi = b_0 \leq b_1 \leq b_2 \leq \ldots \leq b_k \leq \cdots \leq \varphi.$$

Passing to the limit in the integral equation (6.4), we deduce from the previous inequalities the existence of a solution to (3.16) with heterogeneous boundary conditions if $|S| \leq \varphi$. We deduce from Theorem 3.4 and from the linearity of the transport equation the existence of a solution with homogeneous boundary conditions. Remark that the solution is then bounded independently of the number of cells m. By linearity again, we also obtain a solution for every bounded source term S.

Step 2: Uniqueness and exponential decay. These proofs are very close to those of the steps 2 to 4 of Theorem 3.4. In step 2, the term $\int_{\Gamma_-} g^2 |v \cdot n| \, \psi\psi^* d\sigma dv$ can be replaced in (5.7) by $\int_X |S| \psi\psi^* dxdv$ and the uniqueness result follows. The same analogy holds true for step 3 and step 4, where Bg is replaced by $T^{-1}S$ that also converges exponentially to 0 away from the boundary. Therefore the proof presented for boundary source terms can be extended without difficulties to the case of problems with volume source terms. This concludes the proof of the theorem. $\square$

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References

[1] R. Alexandre and K. Hamdache, *Homogénéisation d'équations cinétiques en milieu perforé*, C. R. Acad. Sci. Paris, t.313, Série I, (1991), pp. 339–344.

[2] G. Allaire and G. Bal, *Homogenization of the criticality spectral equation in neutron transport*, to appear in Math. Model. Anal. Num., (1999).

[3] G. Bal, *Couplage d'équations et homogénéisation en transport neutronique*, *Thèse de Doctorat* de l'Université Paris 6 [in French], 1997.

[4] ——, *First-order Corrector for the Homogenization of the Criticality Eigenvalue Problem in the Even Parity Formulation of the Neutron Transport*, to appear in SIAM J. Math. Anal., (1999).

[5] G. Bal and X. Warin, *Numerical results in the homogenization of the criticality eigenvalue problem*, preprint, (1998).

[6] C. Bardos, *Problèmes aux limites pour les équations aux dérivées partielles du premier ordre*, Ann. Sci. Ecole Norm. Sup. ,4, (1969), pp. 185–233.

[7] C. Bardos, R. Santos, and R. Sentis, *Diffusion, Approximation and computation of the critical size*, Trans. Am. Math. Soc., (1984), pp. 617–649.

[8] A. Bensoussan, J. L. Lions, and G. Papanicolaou, *Boundary Layer Analysis in Homogeneization of Diffusion Equations with Dirichlet Conditions in the Half Space*, Proc. of Intern. Symp. SDE, Kyoto Univ., (1978), pp. 21–40.

[9] ——, *Boundary Layers and Homogenization of Transport Processes*, RIMS, Kyoto Univ., (1979).

[10] S. Chandrasekhar, *Radiative Transfer*, Oxford University Press, Cambridge, 1950.

[11] F. Chatelin, *Spectral approximation of linear operators*, Comp sci app math, Academic Press, 1983.

[12] R. Dautray and J. L. Lions, *Mathematical analysis and numerical methods for Science and Technology. Vol.6*, Springer Verlag, Berlin, 1993.

[13] L. Dumas and F. Golse, *Homogenization of transport equations*, to appear in SIAM, (1998).

[14] E. Frenod and K. Hamdache, *Homogenization of a transport kinetic equation with oscillating potentials*, Proc. Roy. Soc. Edinburgh, 126A, (1996), pp. 1247–1275.

[15] F. Golse, S. Jin, and D. Levermore, *The convergence of Numerical Transfer Schemes in Diffusive Regimes I: Discrete-Ordinate Method*, submitted to SIAM J. Num Anal, (1997).

[16] F. Golse, P.-L. Lions, B. Perthame, and R. Sentis, *Regularity of the moments of the solution of a transport equation*, Journal of Functional Analysis 76, (1988), pp. 110–125.

[17] E. W. Larsen, *Neutron transport and diffusion in inhomogeneous media. I*, J. Math. Phys., 16 (1975), pp. 1421–1427.

[18] ——, *Neutron transport and diffusion in inhomogeneous media. II*, Nucl. Sci. Engrg., 60 (1976), pp. 357–368.

[19] E. W. Larsen and J. B. Keller, *Asymptotic solution of neutron transport problems for small mean free paths*, J. Math. Phys., 15 (1974), pp. 75–81.

[20] E. W. Larsen and M. Williams, *Neutron Drift in Heterogeneous Media*, Nucl. Sci. Engrg., 65 (1978), pp. 290–302.

[21] M. Mokhtar-Kharroubi, *Les équations de la neutronique*, Thèse de doctorat d'état, Paris XIII, 1987.

[22] J. Pitkäranta, *Estimates for the derivatives of solutions to weakly singular Fredholm integral equations*, SIAM J. Math. Anal. ,11 (1980) pp.952-968.

[23] J. Planchard, *Méthodes mathématiques en neutronique [in French]*, Collection de la Direction des Etudes et Recherches d'EDF, Eyrolles, 1995.

[24] F. Santosa and M. Vogelius, *First-order corrections to the homogenized eigenvalues of a periodic composite medium*, SIAM J. Appl. Math., 53 (1993), pp. 1636–1668.

[25] R. Sentis, *Study of the corrector of the eigenvalue of a transport operator*, SIAM J. Math. Anal. 16, (1985), pp. 151–166.

[26] V. S. Vladimirov, *Mathematical problems in the one-velocity theory of particle transport*, Atomic energy of Canada Ltd, Chalk River, Ont. Report AECL 1661, 1963.