Transport Theory for Acoustic Waves with Reflection and Transmission at Interfaces

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Abstract

We derive systematically transport theoretic boundary conditions for acoustic wave reflection and transmission from a rough interface when its fluctuations are small. We use the Wigner distribution in order to go from waves to energy transport in the high frequency limit and calculate, in addition, the effect of the random rough surface. We also calculate the form of the transport theoretic boundary conditions with the smoothing method so as to remove the grazing angle singularity they have in the Born approximation. The results are presented in a form that is convenient both for theoretical analysis and for numerical computations.

Contents

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1 Introduction

Wave propagation in weakly fluctuating random media over long distances compared to the wave length can be modeled by incoherent energy transport. This is the radiative transport regime as we recapitulate in the next section. Near boundaries and interfaces the waves undergo coherent or partially coherent reflection and transmission, depending on whether the interfaces are smooth or randomly rough. In this paper we derive systematically, starting from the acoustic equations, boundary conditions for the radiative transport equation, which is asymptotically valid in the interior. The asymptotic limit that leads to radiative transport corresponds to weak fluctuations and high frequency waves, with the correlation length of the inhomogeneities taken to be comparable to the wave length. We use systematically the Wigner distribution to analyze this limit, as we did in [23]. The main result of this work is the derivation of the transport boundary conditions (1.4) and (1.5) for reflection and transmission at a rough interface. The boundary conditions are intuitively clear as they represent coherent or specular reflection and transmission, and incoherent or diffuse reflection and transmission in the form of a linear input-output relation. If there is no dissipation, as we assume here, energy flux is conserved (1.6). We calculate the explicit form of the coherent and incoherent energy flux reflection and transmission coefficients or cross-sections, respectively, when the fluctuations are weak. The usual form of the energy flux reflection and transmission coefficients from a flat interface appear as the principal terms of the coherent reflection and transmission coefficients in our formulas. Special cases of our formulas in the Born approximation have been obtained before [5, 10, 29]. We note also the general framework for transport boundary conditions introduced in [15]. Our results are summarized at the end of section 1.2.

The terms in the reflection and transmission coefficients that are due to the small random fluctuations are computed by a Born expansion as is usual. These terms are singular at grazing angles so that their use in boundary conditions can lead to difficulties, in Monte Carlo simulations, for example. It is important, therefore, to regularize their behavior at grazing angles by using the smoothing method, as was done in [31]. We do this in Appendix B by suitably rearranging the Born expansion.

1.1 The Radiative Transport Equation

Radiative transport is a phenomenological theory that was introduced to describe the propagation of light intensity through the Earth's atmosphere. It has been applied successfully to many other problems of wave propagation in a complex medium. In its simplest form, let $a(t, \mathbf{x}, \mathbf{k})$ denote the angularly resolved energy density defined for all wave vectors $\mathbf{k}$, position $\mathbf{x}$ and time t. Because of interaction with the inhomogeneous medium through which it propagates, a wave with wave vector $\mathbf{k}$ may be scattered into any other direction $\hat{\mathbf{k}}'$, where $\hat{\mathbf{k}} = \dfrac{\mathbf{k}}{|\mathbf{k}|}$. The transport equation gives the energy balance

$$\begin{aligned}\frac{\partial a(t,\mathbf{x},\mathbf{k})}{\partial t} &+ \nabla_{\mathbf{k}}\omega(\mathbf{x},\mathbf{k})\cdot\nabla_{\mathbf{x}}a(t,\mathbf{x},\mathbf{k}) - \nabla_{\mathbf{x}}\omega(\mathbf{x},\mathbf{k})\cdot\nabla_{\mathbf{k}}a(t,\mathbf{x},\mathbf{k}) \\ &= \int_{R^n}\sigma(\mathbf{x},\mathbf{k},\mathbf{k}')a(t,\mathbf{x},\mathbf{k}')d\mathbf{k}' - \Sigma(\mathbf{x},\mathbf{k})a(t,\mathbf{x},\mathbf{k}).\end{aligned} \tag{1.1}$$

Here n is the dimension of space ($n = 2$ or 3), $\omega(\mathbf{x}, \mathbf{k})$ is the local frequency at position $\mathbf{x}$ of the wave with wave vector $\mathbf{k}$, the differential scattering cross-section $\sigma(\mathbf{x}, \mathbf{k}, \mathbf{k}')$ is the rate at which energy with wave vector $\mathbf{k}'$ is converted to wave energy with wave vector $\mathbf{k}$ at position $\mathbf{x}$, and

$$\int \sigma(\mathbf{x}, \mathbf{k}', \mathbf{k}) d\mathbf{k}' = \Sigma(\mathbf{x}, \mathbf{k}) \tag{1.2}$$

is the total scattering cross-section. The function $\sigma(\mathbf{x}, \mathbf{k}, \mathbf{k}')$ is nonnegative and usually symmetric in $\mathbf{k}$ and $\mathbf{k}'$. The left side of (1.1) is the total time derivative of $a(t, \mathbf{x}, \mathbf{k})$ at a point moving along a trajectory in phase space $(\mathbf{x}, \mathbf{k})$ and may be written as a Liouville equation

$$\frac{\partial a}{\partial t} = \{\omega, a\}, \tag{1.3}$$

where $\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial k_i} - \frac{\partial f}{\partial k_i} \frac{\partial g}{\partial x_i} \right)$ is the Poisson bracket. The right side of (1.1) represents the effects of scattering.

The transport equation (1.1) is conservative when (1.2) holds because then

$$\iint a(t, \mathbf{x}, \mathbf{k}) d\mathbf{x} d\mathbf{k} = \text{const}$$

independent of time. Absorption may be accounted for easily by letting the total scattering cross-section be the sum of two terms

$$\Sigma(\mathbf{x}, \mathbf{k}) = \Sigma_{sc}(\mathbf{x}, \mathbf{k}) + \Sigma_{ab}(\mathbf{x}, \mathbf{k})$$

where $\Sigma_{sc}(\mathbf{x}, \mathbf{k})$ is the total scattering cross-section given by (1.2) and $\Sigma_{ab}(\mathbf{x}, \mathbf{k})$ is the absorption rate.

The radiative transport equation (1.1) was derived from microscopic equations in the sixties and seventies by many authors (see [23] for references). A nice overview of these methods and results is presented in a recent review [22]. We have recently considered scattering of high frequency waves in a random medium [23] and established validity of the radiative transport theory for scalar and vector waves, including mode conversion and polarization in the following regime:

- Distances of propagation L are much larger than the wave length λ,
- The medium parameters vary on the scale comparable to the wave length,
- The mismatch between the inhomogeneities and the background medium is small,
- Absorption is small.

This regime arises in many physically important situations. In seismic wave propagation, teleseismic events can be modeled by radiative transport equations [1, 13].

1.2 Half Space Problems and Boundary Conditions

The radiative transport equation (1.1) is derived in the whole space with no boundaries and interfaces. It is important in electromagnetic, geophysical and other applications to consider problems in bounded domains with the correct boundary conditions. A mathematical study of reflection and transmission of energy at boundaries and interfaces in the high frequency regime is in [24] for time harmonic solutions of the acoustic equations, in the absence of volume scattering. The background medium was assumed to be varying on a scale large compared to the wave length, and only flat boundaries were considered. The boundary conditions derived there generalize the usual plane wave

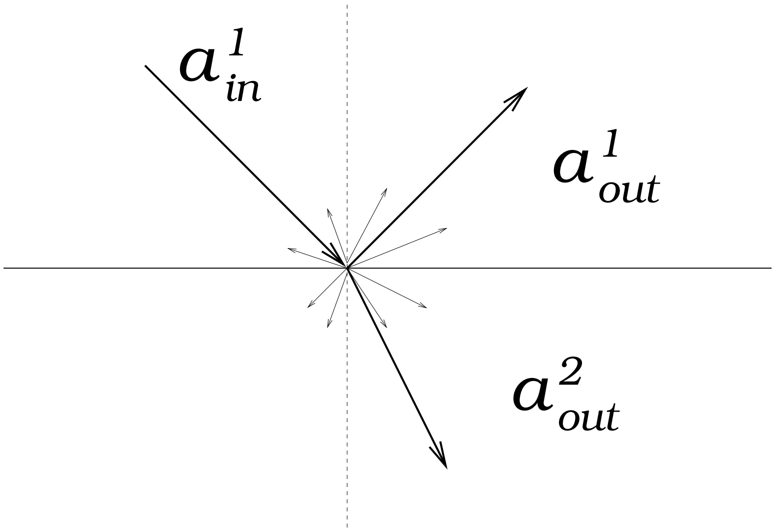

Figure 1.1: The incident and diffuse reflected waves.

energy reflection and transmission coefficients for an inhomogeneous medium. They were derived for the monochromatic,free transport equation

$$\nabla_{\mathbf{k}}\omega(\mathbf{x},\mathbf{k})\cdot\nabla_{\mathbf{x}}a(t,\mathbf{x},\mathbf{k})-\nabla_{\mathbf{x}}\omega(\mathbf{x},\mathbf{k})\cdot\nabla_{\mathbf{k}}a(t,\mathbf{x},\mathbf{k})=0$$

but may be applied also to the radiative transport equation (1.1) in a domain with curved boundaries and interfaces varying on a scale large compared to the wave length. These boundary conditions remain valid because of the separation of scales and weak scattering.

The problem of wave scattering from rough boundaries was extensively studied in the past [5, 10, 29]. We study here the reflection and transmission of the phase space energy density at such an interface in the high frequency limit. We show that scattering from a rough surface may be incorporated in the radiative transport theory under the following conditions:

- The wave length is small compared to the mean curvature of the interface
- The height of the rough surface is small or comparable to the wave length
- The correlation length of the surface fluctuations is comparable to the wave length.

The boundary conditions for the radiative transport equation (1.1) in this regime are as follows.

Let $a^1_{in}(\mathbf{x}',\mathbf{k}')$ and $a^2_{in}(\mathbf{x}',\mathbf{k}')$ be the energy densities of the waves coming to the interface from above and below, respectively, with $\mathbf{x}'$ being a horizontal coordinate on the rough surface, and $\mathbf{k}'$ the horizontal wave vector. Then the outgoing energy densities are given by

$$\begin{aligned}v^1\hat{k}^1_n a^1_{out}(\mathbf{x}',\mathbf{k}') &= v^1\hat{k}^1_n|R_{11}(\mathbf{k}')|^2a^1_{in}(\mathbf{x}',\mathbf{k}')+v^2\hat{k}^2_n|T_{12}(\mathbf{k}')|^2a^2_{in}(\mathbf{x}',\mathbf{k}')\\ &+\int d\mathbf{p}'\left(v^1\hat{k}^1_n(\mathbf{p}')\tau_{11}(\mathbf{x}',\mathbf{k}',\mathbf{p}')a^1_{in}(\mathbf{x}',\mathbf{p}')+v^2\hat{k}^2_n(\mathbf{p}')\tau_{12}(\mathbf{x}',\mathbf{k}',\mathbf{p}')a^2_{in}(\mathbf{x}',\mathbf{p}')\right)\end{aligned}\tag{1.4}$$

and

$$\begin{aligned}v^2\hat{k}^2_n a^2_{out}(\mathbf{x}',\mathbf{k}') &= v^2\hat{k}^2_n|R_{22}(\mathbf{k}')|^2a^2_{in}(\mathbf{x}',\mathbf{k}')+v^1\hat{k}^1_n|T_{21}(\mathbf{k}')|^2a^1_{in}(\mathbf{x}',\mathbf{k}')\\ &+\int d\mathbf{p}'\left(v^2\hat{k}^2_n(\mathbf{p}')\tau_{22}(\mathbf{x}',\mathbf{k}',\mathbf{p}')a^2_{in}(\mathbf{x}',\mathbf{p}')+v^1\hat{k}^1_n(\mathbf{p}')\tau_{21}(\mathbf{x}',\mathbf{k}',\mathbf{p}')a^1_{in}(\mathbf{x}',\mathbf{p}')\right).\end{aligned}\tag{1.5}$$

	Flat surface	Rough surface, reflection coefficient correction	Rough surface, scattering cross-section
Dirichlet	1	(4.27) entry $2,2$	(4.28) entry $2,2$
Neumann	1	(4.27) entry $1,1$	(4.28) entry $1,1$
Interface	(3.11)	(4.23)	(4.22)

Table 1.1: References to the explicit formulas for the reflection and transmission coefficients and scattering cross-sections

Here the two first terms represent specular (coherent) reflection and transmission, and the integral ones correspond to diffusive (incoherent) scattering, that occurs because waves at the boundary may scatter in any direction. The net normal flux across the boundary is conserved so for the waves that are reflected and transmitted from the first medium we have the energy flux conservation law

$$1 \;=\; |R_{11}(\mathbf{k}')|^2 \;+\; |T_{21}(\mathbf{k}')|^2 \;+\; \int d\mathbf{p}'\,[\tau_{11}(\mathbf{p}',\mathbf{k}') + \tau_{21}(\mathbf{p}',\mathbf{k}')] \tag{1.6}$$

with a symmetric relation holding for R_{22}, T_{12}, τ_{12} and τ_{22}. This is the analog of (1.2) for surface scattering. Here $v^{1,2}$ are the acoustic wave speeds in the two media, and $\hat{k}_n^{1,2}$ is the normal component of normalized wave vector $\hat{\mathbf{k}}^{1,2} = \mathbf{k}^{1,2}/|\mathbf{k}^{1,2}|$ where $\mathbf{k}^{1,2}$ have tangential component $\mathbf{k}'$.

We derive in this paper explicit formulas for the surface differential cross sections τ_{ij} and the corrected reflection and transmission coefficients in the case when the height of the interface is smaller than the wave length: $\eta = \dfrac{H}{\lambda} \ll 1$. We consider for simplicity a homogeneous background medium. A generalization of our results for an inhomogeneous medium, where no plane wave decomposition is available is presented in [2]. The results remain essentially unchanged but their derivation is substantially more involved. The equation numbers for the explicit formulas are given in Table 1.1.

In this paper we assume that interface conditions (1.4) and (1.5) are derived when the adjacent media are homogeneous. We expect, however, that they will remain valid when the adjacent media are random and the transport approximation (1.1) holds. This is because waves are not trapped near the boundary when the fluctuations of the interface are small. Nevertheless, a mathematical justification of this assumption is difficult and not available at present.

2 Acoustic Wave Transport

We recall how transport equations for the phase space energy densities are constructed [23, 14, 24]. For simplicity we assume here that the space domain is $\mathbb{R}^3$ ($n = 3$). The acoustic equations for the velocity $\mathbf{v}$ and pressure p are

$$\begin{aligned} \rho\frac{\partial \mathbf{v}}{\partial t} + \nabla p &= 0 \\ \kappa\frac{\partial p}{\partial t} + \nabla\cdot\mathbf{v} &= 0. \end{aligned} \tag{2.1}$$

This system may be written in a general form of a symmetric hyperbolic system (with convention of summation over repeated indices):

$$A(\mathbf{x})\frac{\partial \mathbf{u}}{\partial t} + D^j \frac{\partial \mathbf{u}}{\partial x^j} = 0, \tag{2.2}$$

where $\mathbf{u} = (\mathbf{v}, p)$, and $\mathbf{x} \in \mathbb{R}^n$. The matrix $A(\mathbf{x}) = \mathrm{diag}(\rho, \rho, \rho, \kappa)$ is symmetric and positive definite and the matrices D^j are symmetric and independent of $\mathbf{x}$ and t. We consider high frequency solutions of (2.2). Physically that means that the typical wave length λ of the initial data is much smaller than the overall propagation distance L with $\varepsilon = \dfrac{\lambda}{L} \ll 1$. The spatial energy density for the solutions of (2.2) is given by

$$\mathcal{E}(t, \mathbf{x}) = \frac{\rho \mathbf{v}^2}{2} + \frac{\kappa p^2}{2} = \frac{1}{2}(A(\mathbf{x})\mathbf{u}(t, \mathbf{x}) \cdot \mathbf{u}(t, \mathbf{x})) = \frac{1}{2} A_{ij}(\mathbf{x}) u_i(t, \mathbf{x}) \bar{u}_j(t, \mathbf{x}) \tag{2.3}$$

and the flux $\boldsymbol{\mathcal{F}}(\mathbf{x})$ by

$$\boldsymbol{\mathcal{F}}_i(t, \mathbf{x}) = p\mathbf{v} = \frac{1}{2}(D^i \mathbf{u}(t, \mathbf{x}) \cdot \mathbf{u}(t, \mathbf{x})). \tag{2.4}$$

We have the energy conservation law

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \boldsymbol{\mathcal{F}} = 0. \tag{2.5}$$

and thus the total energy is conserved:

$$\frac{d}{dt} \int \mathcal{E}(t, \mathbf{x}) d\mathbf{x} = 0. \tag{2.6}$$

The high frequency limit $\varepsilon \to 0$ of the energy density $\mathcal{E}(t, \mathbf{x})$ is described in terms of the Wigner transform, which is defined by

$$W_\varepsilon(t, \mathbf{x}, \mathbf{k}) = \left(\frac{1}{2\pi}\right)^n \int e^{i\mathbf{k}\cdot\mathbf{y}} \mathbf{u}_\varepsilon(t, \mathbf{x} - \varepsilon\mathbf{y}/2) \mathbf{u}_\varepsilon^*(\mathbf{x} + \varepsilon\mathbf{y}/2) d\mathbf{y}, \tag{2.7}$$

where $\mathbf{u}_\varepsilon(t, \mathbf{x})$ is the solution of (2.2). The Wigner transform W_ε is a 4×4 Hermitian matrix. Its limit as $\varepsilon \to 0$ is called the Wigner distribution and is denoted by $W(t, \mathbf{x}, \mathbf{k})$. The limit Wigner matrix is not only Hermitian but also positive definite. The limit energy density and flux are expressed in terms of $W(t, \mathbf{x}, \mathbf{k})$ by

$$\mathcal{E}(t, \mathbf{x}) = \frac{1}{2} \int \mathrm{Tr}(A(\mathbf{x}) W(t, \mathbf{x}, \mathbf{k})) d\mathbf{k}$$

and

$$\mathcal{F}_i(t, \mathbf{x}, \mathbf{k}) = \frac{1}{2} \int \mathrm{Tr}(D^i W(t, \mathbf{x}, \mathbf{k})) d\mathbf{k}.$$

The limit Wigner distribution may be decomposed over different wave modes in a way that generalizes the plane wave decomposition in a homogeneous medium. The dispersion matrix of the system (2.2) is defined by

$$L(\mathbf{x}, \mathbf{k}) = A^{-1}(\mathbf{x}) k_i D^i = \begin{pmatrix} 0 & 0 & 0 & k_1/\rho \\ 0 & 0 & 0 & k_2/\rho \\ 0 & 0 & 0 & k_3/\rho \\ k_1/\kappa & k_2/\kappa & k_3/\kappa & 0 \end{pmatrix}. \tag{2.8}$$

It has one double eigenvalue $\omega_1 = \omega_2 = 0$ and two simple eigenvalues

$$\omega_f = v|\mathbf{k}| \ , \quad \omega_b = -v|\mathbf{k}| \ , \tag{2.9}$$

where $|\mathbf{k}| = \sqrt{k_1^2 + k_2^2 + k_3^2}$ and v is the sound speed

$$v = \frac{1}{\sqrt{\kappa\rho}}. \tag{2.10}$$

The corresponding basis of eigenvectors is

$$\begin{aligned} \mathbf{b}_1 &= \frac{1}{\sqrt{\rho}}(\mathbf{z}^{(1)}(\mathbf{k}), 0)^t, \quad \mathbf{b}_2 = \frac{1}{\sqrt{\rho}}(\mathbf{z}^{(2)}(\mathbf{k}), 0)^t, \\ \mathbf{b}_f &= (\frac{\hat{\mathbf{k}}}{\sqrt{2\rho}}, \frac{1}{\sqrt{2\kappa}})^t, \quad \mathbf{b}_b = (\frac{\hat{\mathbf{k}}}{\sqrt{2\rho}}, -\frac{1}{\sqrt{2\kappa}})^t, \end{aligned} \tag{2.11}$$

where the vectors $\hat{\mathbf{k}}$, $\mathbf{z}^{(1)}(\mathbf{k})$ and $\mathbf{z}^{(2)}(\mathbf{k})$ form an orthonormal triplet. The eigenvectors $\mathbf{b}_1(\mathbf{k})$ and $\mathbf{b}_2(\mathbf{k})$ correspond to transverse advection modes, orthogonal to the direction of propagation. These modes do not propagate because $\omega_{1,2} = 0$. The eigenvectors $\mathbf{b}_f(\mathbf{k})$ and $\mathbf{b}_b(\mathbf{k})$ represent forward and backward acoustic waves, which are longitudinal , and which propagate with the sound speed v given by (2.10).

The limit Wigner distribution matrix $W(t, \mathbf{x}, \mathbf{k})$ has the form [23]:

$$W(t,\mathbf{x},\mathbf{k}) = \sum_{\tau=1}^{2} W_{ij}^{\tau}(t,\mathbf{x},\mathbf{k})\mathbf{b}^i(\mathbf{k})\mathbf{b}^{j*}(\mathbf{k}) + a_f(t,\mathbf{x},\mathbf{k})\mathbf{b}_f(\mathbf{k})\mathbf{b}_f^*(\mathbf{k}) + a_b(t,\mathbf{x},\mathbf{k})\mathbf{b}_b(\mathbf{k})\mathbf{b}_b^*(\mathbf{k}). \tag{2.12}$$

The first term corresponds to the non-propagating modes and may be set to zero here without any loss of generality. The last two terms correspond to forward and backward propagating sound waves. The scalar functions $a_{f,b}$ are related by $a_f(t, \mathbf{x}, \mathbf{k}) = a_b(t, \mathbf{x}, -\mathbf{k})$, and a_f satisfies the Liouville equation

$$\frac{\partial a}{\partial t} + \nabla_{\mathbf{k}}\omega \cdot \nabla_{\mathbf{x}} a - \nabla_{\mathbf{x}}\omega \cdot \nabla_{\mathbf{k}} a = 0. \tag{2.13}$$

They may be interpreted as phase space energy densities since they are non-negative (because the matrix $W(t, \mathbf{x}, \mathbf{k})$ is non-negative) and

$$\mathcal{E}(\mathbf{x}) = \frac{1}{2}\int d\mathbf{k}[a_f(t,\mathbf{x},\mathbf{k}) + a_b(t,\mathbf{x},\mathbf{k})] = \int d\mathbf{k} a_f(t,\mathbf{x},\mathbf{k}).$$

The flux is given by

$$\mathcal{F} = \frac{v}{2}\int d\mathbf{k}[\hat{\mathbf{k}}a_f(t,\mathbf{x},\mathbf{k}) - \hat{\mathbf{k}}a_b(t,\mathbf{x},\mathbf{k})] = v\int d\mathbf{k}\hat{\mathbf{k}}a_f(t,\mathbf{x},\mathbf{k}). \tag{2.14}$$

The radiative transport equation (1.1) arises when the density ρ and compressibility κ are random and oscillating on the scale of the wave length, so we assume they have the form

$$\rho \to \rho(1 + \sqrt{\varepsilon}\rho_1(\frac{\mathbf{x}}{\varepsilon})), \quad \kappa \to \kappa(1 + \sqrt{\varepsilon}\kappa_1(\frac{\mathbf{x}}{\varepsilon})).$$

The random processes ρ_1 and κ_1 are mean zero space homogeneous with power spectral densities $\hat{R}_{\rho\rho}$, $\hat{R}_{\kappa\kappa}$, and cross spectral density $\hat{R}_{\kappa\rho}$. The radiative transport equation for $a(t,\mathbf{x},\mathbf{k}) = a_f(t,\mathbf{x},\mathbf{k})$ has the form

$$\frac{\partial a}{\partial t} + v\hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}} a - |\mathbf{k}|\nabla_{\mathbf{x}} v \cdot \nabla_{\mathbf{k}} a = \frac{\pi v^2 |\mathbf{k}|^2}{2}\int \delta(v|\mathbf{k}| - v|\mathbf{k}'|)[a(\mathbf{k}') - a(\mathbf{k})] \cdot \left\{(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')^2 \hat{R}_{\rho\rho}(\mathbf{k}-\mathbf{k}') + 2(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')\hat{R}_{\rho\kappa}(\mathbf{k}-\mathbf{k}') + \hat{R}_{\kappa\kappa}(\mathbf{k}-\mathbf{k}')\right\} d\mathbf{k}'. \tag{2.15}$$

This equation is of the form (1.1). A systematic derivation using the Wigner distribution is given in [23].

The radiative equation (1.1) has been derived from equations governing particular wave motions by various authors, such as Stott [26], Watson et.al. [32], [33], [19], Barabanenkov et.al. [3], Besieris and Tappert [6], Howe [16], Ishimaru [17] and Besieris et. al. [7], with a recent survey presented in [4]. See also [12] for similar results. These derivations also determine the functions $\omega(\mathbf{x},\mathbf{k})$ and $\sigma(\mathbf{x},\mathbf{k},\mathbf{k}')$ and show how a is related to the wave field. In [23] (1.1) and these functions are derived as a special case of a more general theory.

3 Flat Boundary

3.1 Reflection and Transmission of Waves in Homogeneous Media

We will derive transport theoretic boundary conditions assuming that the media adjacent to the interface are homogeneous, with the interface flat, as in this section, or random, as in later sections.

In this section we recall briefly the transmission and reflection of time harmonic, acoustic plane waves at a plane interface. The time harmonic solutions of (2.1) with frequency ω satisfy

$$\begin{aligned} \nabla p - i\omega\rho\mathbf{v} &= 0 \\ \nabla\cdot\mathbf{v} - i\kappa p &= 0. \end{aligned} \tag{3.1}$$

We consider the high frequency regime, and replace accordingly $\omega \to \frac{\omega}{\varepsilon}$ with $\varepsilon \ll 1$. Then (3.1) becomes

$$\begin{aligned} \varepsilon\nabla p - i\omega\rho\mathbf{v} &= 0 \\ \varepsilon\nabla\cdot\mathbf{v} - i\kappa p &= 0. \end{aligned} \tag{3.2}$$

Consider a media composed of two homogeneous half spaces in $\mathbb{R}^n$, characterized by constant densities and compressibilities ρ^i and κ^i, $i = 1, 2$. Here $i = 1$ is the index of the upper half space $x_n > 0$ and $i = 2$ of the lower one $x_n < 0$. The velocity of propagation is defined by $v^i = 1/\sqrt{\rho^i\kappa^i}$ in each half space.

The wave number $|\mathbf{k}^i|$ of the acoustic plane wave is related to the frequency ω by the dispersion relation

$$\omega \;=\; v^i|\mathbf{k}^i| \quad \text{for } i = 1, 2. \tag{3.3}$$

A general solution of the acoustic wave equation (3.2) in a homogeneous region with frequency ω can be decomposed into a sum of waves incoming and outgoing from the interface $x_n = 0$,

$$\mathbf{u}^i_\varepsilon(\mathbf{x}) \;=\; \mathbf{u}^i_{I,\varepsilon}(\mathbf{x}) + \mathbf{u}^i_{R,\varepsilon}(\mathbf{x}) \tag{3.4}$$

with

$$\mathbf{u}^i_{I,\varepsilon}(\mathbf{x}) = \int \frac{d\mathbf{k}'}{(2\pi\varepsilon)^{(n-1)/2}} \exp[i\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon}] \exp[-i\frac{\epsilon^i k^i_n x_n}{\varepsilon}]\ \alpha^i_\varepsilon(\mathbf{k}')\ \mathbf{b}^i(\mathbf{k}^i_-) \tag{3.5}$$

$$\mathbf{u}^i_{R,\varepsilon}(\mathbf{x}) = \int \frac{d\mathbf{k}'}{(2\pi\varepsilon)^{(n-1)/2}} \exp[i\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon}] \exp[i\frac{\epsilon^i k^i_n x_n}{\varepsilon}]\ \beta^i_\varepsilon(\mathbf{k}')\ \mathbf{b}^i(\mathbf{k}^i_+). \tag{3.6}$$

Here $\mathbf{b} = \mathbf{b}_f$ since the frequency $\omega > 0$ is fixed. The wave vector $\mathbf{k}'$ is the horizontal component of $\mathbf{k}$ and

$$\mathbf{k}^i_\pm \ = \ (\mathbf{k}', \pm\epsilon^i k^i_n), \qquad k^i_n(\mathbf{k}') = \sqrt{|\mathbf{k}^i|^2 - |\mathbf{k}'|^2}, \qquad \epsilon^1 = 1, \qquad \epsilon^2 = -1 \tag{3.7}$$

with k^i_n, $i = 1,2$ the normal component of the wave number. We consider only propagating waves here. The energy of evanescent waves is exponentially small in the high frequency limit and vanishes as $\varepsilon \to 0$. Therefore we assume that the normal components of the wave vectors k^1_n and k^2_n are real. Then the support of the amplitudes α^i_ε and β^i_ε is uniformly inside the ball $\{|\mathbf{k}'| < \omega/v^i\}$. Notice that $\hat{\mathbf{u}}^i_{I,\varepsilon}$, the Fourier transform of $\mathbf{u}^i_{I,\varepsilon}$, and the corresponding amplitudes $\alpha^i_\varepsilon(\mathbf{k}')$ are related by

$$\hat{\mathbf{u}}^i_{I,\varepsilon}(\frac{\mathbf{k}}{\varepsilon}) \ = \ (2\pi\varepsilon)^{(n+1)/2}\ \alpha^i_\varepsilon(\mathbf{k}')\ \mathbf{b}^i(\mathbf{k}^i_-)\ \delta(k_n + \epsilon^i k^i_n) \tag{3.8}$$

An analogous relation holds for the reflected waves.

The amplitudes α^i_ε and β^i_ε are not independent from one another. They are related through the interface conditions at $x_n = 0$, expressed by the continuity of the normal velocity and pressure

$$\begin{aligned} u^1_n(\mathbf{x}', 0) &= u^2_n(\mathbf{x}', 0) \\ p^1(\mathbf{x}', 0) &= p^2(\mathbf{x}', 0). \end{aligned} \tag{3.9}$$

Equations (2.11) and (3.5-3.6) give the following relations for the wave amplitudes (for $i \neq j$)

$$(k^i_n)^{1/2}(k^i)^{-1}\beta^i_\varepsilon(\mathbf{k}') \ = \ \mathcal{R}^i(\mathbf{k}')\ (k^i_n)^{1/2}(k^i)^{-1}\alpha^i_\varepsilon(\mathbf{k}') \ + \ \mathcal{T}^j(\mathbf{k}')\ (k^j_n)^{1/2}(k^j)^{-1}\alpha^j_\varepsilon(\mathbf{k}'). \tag{3.10}$$

Here $\mathcal{R}^i$ is the Fresnel reflection coefficient for medium i, and $\mathcal{T}^j$ is the Fresnel transmission coefficient from medium j into medium i. They are given by (see e.g. [18, 8])

$$\mathcal{R}^i(\mathbf{k}') \ = \ \frac{\rho^j k^i_n - \rho^i k^j_n}{\rho^j k^i_n + \rho^i k^j_n}, \qquad \mathcal{T}^i(\mathbf{k}') \ = \ \frac{2(\rho^i\rho^j k^i_n k^j_n)^{1/2}}{\rho^j k^i_n + \rho^i k^j_n} \tag{3.11}$$

with $i \neq j$. Note that according to the chosen normalization $\mathcal{R}^2(\mathbf{k}') = -\mathcal{R}^1(\mathbf{k}')$ and $\mathcal{T}^1(\mathbf{k}') = \mathcal{T}^2(\mathbf{k}')$. Moreover, the conservation of energy flux implies that

$$(\mathcal{R}^i(\mathbf{k}'))^2 + (\mathcal{T}^i(\mathbf{k}'))^2 = 1 \tag{3.12}$$

with $i = 1, 2$. The angles of incidence and refraction θ^i and θ^j are related by Snell's law

$$\frac{\sin\theta^i}{v^i} \ = \ \frac{\sin\theta^j}{v^j}. \tag{3.13}$$

3.2 High Frequency Limit: Reflection and Transmission of Energy

In order to derive boundary conditions for the phase space energy density in a domain with flat boundaries we consider the Wigner transform of incoming and outgoing waves. We denote by $W[\mathbf{u}]$ the limit as $\varepsilon \to 0$ of $W_\varepsilon[\mathbf{u}_\varepsilon]$, the Wigner transform of $\mathbf{u}_\varepsilon$ defined by (2.7). The directions of the incident and reflected waves are not overlapping, and thus according to the general theory of the Wigner distributions [14, 24] they decouple in the high frequency limit $\varepsilon \to 0$ so that we obtain

$$W[\mathbf{u}^i](\mathbf{x},\mathbf{k}) \;=\; W[\mathbf{u}^i_I](\mathbf{x},\mathbf{k}) \;+\; W[\mathbf{u}^i_R](\mathbf{x},\mathbf{k}). \tag{3.14}$$

This may be also checked directly using (3.5-3.6). We will now show that

$$\begin{aligned} W[\mathbf{u}^i_I](\mathbf{x},\mathbf{k}) &= W^{n-1}[\alpha^i]\ (\mathbf{k}', -[\mathbf{x}' - \frac{x_n}{k_n}\mathbf{k}'])\ \delta(k_n + \epsilon^i k^i_n)\ \mathbf{b}^i(\mathbf{k}^i_-)\mathbf{b}^{i*}(\mathbf{k}^i_-) \\ W[\mathbf{u}^i_R](\mathbf{x},\mathbf{k}) &= W^{n-1}[\beta^i]\ (\mathbf{k}', -[\mathbf{x}' - \frac{x_n}{k_n}\mathbf{k}'])\ \delta(k_n - \epsilon^i k^i_n)\ \mathbf{b}^i(\mathbf{k}^i_+)\mathbf{b}^{i*}(\mathbf{k}^i_+), \end{aligned} \tag{3.15}$$

where $W^{n-1}[\alpha^i_\varepsilon]$ is the limit as $\varepsilon \to 0$ of the $(n-1)$-dimensional Wigner transform of the family α^i_ε. Let us compute first $W[\mathbf{u}^i_I](\mathbf{x},\mathbf{k})$. By virtue of the relation [14, 24]

$$W_\varepsilon[f_\varepsilon(.)](\mathbf{x},\mathbf{k}) \;=\; W_\varepsilon[\frac{1}{(2\pi\varepsilon)^{n/2}}\hat{f}_\varepsilon(\frac{\cdot}{\varepsilon})](\mathbf{k},-\mathbf{x}),$$

and equation (3.8), we have

$$\begin{aligned} W_\varepsilon[\mathbf{u}^i_{I,\varepsilon}](\mathbf{x},\mathbf{k}) = \frac{2\pi\varepsilon}{(2\pi)^n}\int d\mathbf{p}\, \alpha^i_\varepsilon(\mathbf{k}' + \frac{\varepsilon\mathbf{p}'}{2})\alpha^{i*}_\varepsilon(\mathbf{k}' - \frac{\varepsilon\mathbf{p}'}{2})\delta(k_n + \varepsilon\frac{p_n}{2} + \epsilon^i k^i_n(\mathbf{k}' + \varepsilon\frac{\mathbf{p}'}{2})) \\ \times\delta(k_n - \varepsilon\frac{p_n}{2} + \epsilon^i k^i_n(\mathbf{k}' - \varepsilon\frac{\mathbf{p}'}{2}))\exp{(i\mathbf{p}\cdot\mathbf{x})}\mathbf{b}^i((\mathbf{k} + \varepsilon\frac{\mathbf{p}}{2})^i_-)\mathbf{b}^{i*}((\mathbf{k} - \varepsilon\frac{\mathbf{p}}{2})^i_-). \end{aligned} \tag{3.16}$$

According to the dispersion relation (3.3) we deduce from the arguments in the delta functions of the latter equation that

$$\begin{aligned} (k_n + \varepsilon\frac{p_n}{2})^2 &= \frac{\omega^2}{(v^i)^2} - (\mathbf{k}' + \varepsilon\frac{\mathbf{p}'}{2})^2 \\ (k_n - \varepsilon\frac{p_n}{2})^2 &= \frac{\omega^2}{(v^i)^2} - (\mathbf{k}' - \varepsilon\frac{\mathbf{p}'}{2})^2. \end{aligned}$$

¿From this we get

$$|\mathbf{k}|^2 + |\frac{\varepsilon\mathbf{p}}{2}|^2 \;=\; \frac{\omega^2}{(v^i)^2} \quad \text{and} \quad k_n p_n \;=\; \mathbf{k}'\cdot\mathbf{p}'.$$

On the other hand one readily checks that for constants c_1 and c_2 one has:

$$\delta(k_n + \frac{\varepsilon p_n}{2} + c_1)\ \delta(k_n - \frac{\varepsilon p_n}{2} + c_2) \;=\; \delta(k_n + \frac{c_1 + c_2}{2})\ \delta(p_n - \frac{c_2 - c_1}{\varepsilon}).$$

Therefore by the change of variables $p_n \to \varepsilon p_n$ we obtain in the limit $\varepsilon \to 0$ that

$$\begin{aligned} W[\mathbf{u}^i_I](\mathbf{x},\mathbf{k}) \;\sim\; & \frac{1}{(2\pi)^{n-1}}\int d\mathbf{p}'\alpha^i_\varepsilon(\mathbf{k}' + \frac{\varepsilon\mathbf{p}'}{2})\alpha^{i*}_\varepsilon(\mathbf{k}' - \frac{\varepsilon\mathbf{p}'}{2})\exp i[\mathbf{p}'\cdot(\mathbf{x}' - \frac{x_n}{k_n}\mathbf{k}')] \\ & \times\mathbf{b}^i(\mathbf{k}^i_-)\mathbf{b}^{i*}(\mathbf{k}^i_-)\ \delta(k_n + \epsilon^i k^i_n) \end{aligned} \tag{3.17}$$

where $\sim$ means that both terms have the same limit as $\varepsilon \to 0$. The latter expression converges to the first equation in (3.15) as $\varepsilon \to 0$. The Wigner transform for the outgoing waves is computed similarly. The explicit representation (3.15) implies that the Wigner matrix of the incoming waves has the form

$$W_I^i = a_I^i(\mathbf{x},\mathbf{k})\mathbf{b}^i(\mathbf{k})\mathbf{b}^{i*}(\mathbf{k})$$

with a_I^i supported on the wave vectors $\mathbf{k}$ with $k_n = -\epsilon^i k_n^i$. In the interior, the function a_I^i satisfies the transport equation

$$\hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} a_I^i = 0. \tag{3.18}$$

Similarly we have

$$W_R^i = a_R^i(\mathbf{x},\mathbf{k})\mathbf{b}^i(\mathbf{k})\mathbf{b}^{i*}(\mathbf{k}).$$

The function a_R^i is supported on the hemisphere of outgoing wave vectors with $k_n = \epsilon^i k_n^i$ and satisfies the same transport equation (3.18).

Since the coefficients $\mathcal{R}^i$ and $\mathcal{T}^i$ do not depend on ε we deduce from (3.10) that

$$\begin{aligned} \frac{k_n^i}{(k^i)^2} W^{n-1}[\beta^i] \;\;&=\;\; \frac{k_n^i}{(k^i)^2}|\mathcal{R}^i(\mathbf{k}')|^2\; W^{n-1}[\alpha^i] \;+\; \frac{k_n^j}{(k^j)^2}|\mathcal{T}^j(\mathbf{k}')|^2\; W^{n-1}[\alpha^j] \\ &+\frac{\sqrt{k_n^i}}{k^i}\frac{\sqrt{k_n^j}}{k^j}[\mathcal{R}^i(\mathbf{k}')\;\overline{\mathcal{T}^j}(\mathbf{k}') + \overline{\mathcal{R}^i}(\mathbf{k}')\;\mathcal{T}^j(\mathbf{k}')]\; W^{n-1}[\alpha^i,\alpha^j]. \end{aligned} \tag{3.19}$$

By virtue of (3.15), the two first terms of the right-hand side of (3.19) can be related to the incident energy $W[\mathbf{u}_I^i]$. The third term involving $W^{n-1}[\alpha^i,\alpha^j]$ however corresponds to the cross-Wigner between amplitudes of waves traveling in different directions.

Such cross-Wigner terms do not appear in the interior in our formulation, since we deal with transport theory that does not track partial coherence. In order to obtain linear boundary conditions for the transport equation we have to assume that cross-Wigner terms generated at the boundary vanish.

$$W^{n-1}[\alpha^i,\alpha^j] \;=\; 0. \tag{H1}$$

By linear we mean that the outgoing energy into domain i generated by incoming waves α^i and α^j is the sum of the two outgoing energies into domain i generated by only α^i and α^j respectively.

Condition (H1) is not satisfied in general; and in particular not in the case of two incoming plane waves (where the phase is fixed). However (H1) is satisfied when the wave packets arriving at the interface from different sides have uncorrelated random phases. This is the case in the radiative transport regime when waves arriving to the boundary have undergone independent multiple scattering and hence phase correlations vanish in the high frequency limit.

To summarize, the Wigner transform has in the high frequency limit the form $W[\mathbf{u}^i](\mathbf{x},\mathbf{k}) = a^i(\mathbf{x},\mathbf{k})\mathbf{b}^i(\mathbf{x},\mathbf{k})\mathbf{b}^{i*}(\mathbf{x},\mathbf{k})$ where a^i is a solution to a transport equation in the interior. When (H1) is satisfied, we deduce from (3.19) that a^i satisfy the boundary conditions

$$v^i\hat{k}_3^i a^i(\mathbf{x}',0,\mathbf{k}',\epsilon^i k_n^i) = |\mathcal{R}^i(\mathbf{k}')|^2\; v^i\hat{k}_3^i a^i(\mathbf{x}',0,\mathbf{k}',-\epsilon^i k_n^i) \;+\; |\mathcal{T}^j(\mathbf{k}')|^2\; v^j\hat{k}_3^j a^j(\mathbf{x}',0,\mathbf{k}',-\epsilon^j k_n^j) \tag{3.20}$$

These relation hold also if the two adjacent media are deterministic but not homogeneous, as shown in [24]. We also expect that they hold when the adjacent media are random with weak fluctuations but this has not been analyzed in any detail.

4 Rough Boundaries in the High Frequency Limit

We will now derive transport theoretic boundary conditions when the interface is randomly varying, with small fluctuations. We begin in the next section, however, with a more general formulation and specialize to small fluctuations afterwards.

4.1 Surface Reflection Operator

Let us assume that the boundary of the domain or the interface between two subdomains is a surface given by $(\mathbf{x}', h(\mathbf{x}'))$ where $h(\mathbf{x}')$ is a bounded scalar function. In this section we first obtain the reflected and transmitted waves generated by an incident wave, and then derive boundary conditions for the phase space energy density in the high frequency limit.

The first part of this problem has been studied at length in the past. We refer to [5, 11, 29] for detailed presentations. For simplicity we deal first only with reflected waves, without transmission. The superscript corresponding to the medium is then dropped. Consider an incident plane wave, with horizontal component of the wave vector $\mathbf{k}'_0$, given by

$$\psi_{in}(\mathbf{x}) \;=\; \exp\left[i\mathbf{k}'_0\cdot\mathbf{x}' - ik_n(\mathbf{k}'_0)x_n\right]\,\mathbf{b}(\mathbf{k}_-)$$

and with amplitude $\alpha_\varepsilon = 1$. The scattered wave can be decomposed into a sum of outgoing plane waves:

$$\psi_{sc}(\mathbf{x}) \;=\; \int d\mathbf{k}'\; S(\mathbf{k}', \mathbf{k}'_0)\exp\left[i\mathbf{k}'\cdot\mathbf{x}' + ik_n(\mathbf{k}')x_n\right]\,\mathbf{b}(\mathbf{k}_+)$$

in the region $x_n > h_{max}$. The kernel $S(\mathbf{k}', \mathbf{k}'_0)$ is the *scattering amplitude* or (SA) of incident horizontal wave number $\mathbf{k}'_0$ to scattered horizontal wave $\mathbf{k}'$. It is determined by the random height $h(\mathbf{x}')$ of the surface. In general S can be obtained by solving an integral equation that follows from the boundary conditions imposed on the acoustic equations (3.2). We assume that S or an approximation to it is given.

Let the boundary vary on the scale of the wave length, so as to have full interaction between the waves and the roughness of the boundary. We rescale the boundary accordingly, in the high frequency regime. Let ε be proportional to the wave length and let the scaled boundary h_ε be given by

$$h_\varepsilon(\mathbf{x}') \;=\; \varepsilon h\left(\frac{\mathbf{x}'}{\varepsilon}\right). \tag{4.1}$$

where h is a bounded function. This scaling implies that the height and the variations of the boundary, and the wave length are of the same order. The operator S is then independent of ε. In the high frequency limit an incident wave has scaled horizontal wave vector $\mathbf{k}_0/\varepsilon$. Then the change of variables $\mathbf{x} \to \frac{\mathbf{x}}{\varepsilon}$ yields the same equation for S as when $\varepsilon = 1$. This, however, does not imply that problem (3.2) is independent of ε since the incoming waves still depend on ε through the amplitude α_ε.

In order to express conservation of energy flux at the boundary in a natural way we introduce the amplitudes for energy fluxes:

$$\tilde{\alpha}_\varepsilon \;=\; \sqrt{v\hat{k}_n}\;\alpha_\varepsilon, \qquad \tilde{\beta}_\varepsilon \;=\; \sqrt{v\hat{k}_n}\;\beta_\varepsilon. \tag{4.2}$$

These are energy flux densities, as can be seen from the expression (2.14) of the total flux.

Consider the general incoming and outgoing plane waves given by (3.5-3.6) with the incident and reflected amplitudes related by

$$\tilde{\beta}_\varepsilon(\mathbf{k}') \;=\; \int d\mathbf{k}'_0\; S(\mathbf{k}', \mathbf{k}'_0)\tilde{\alpha}_\varepsilon(\mathbf{k}'_0).$$

The incoming and outgoing Wigner transform are given in (3.15) with $W^{n-1}[\tilde{\alpha}_\varepsilon]$ being the known incoming energy flux. The unknown $W^{n-1}[\tilde{\beta}]$ is given by

$$\begin{aligned} W^{n-1}[\tilde{\beta}](\mathbf{k}', -\mathbf{x}') \;&=\; \lim_{\varepsilon\to 0}\int d\mathbf{p}'\;\exp\left[i\mathbf{p}'\cdot\mathbf{x}'\right]\int d\mathbf{k}'_0 S(\mathbf{k}' - \varepsilon\frac{\mathbf{p}'}{2}, \mathbf{k}'_0)\tilde{\alpha}_\varepsilon(\mathbf{k}'_0)\int d\tilde{\mathbf{k}}'_0\overline{S}(\mathbf{k}' + \varepsilon\frac{\mathbf{p}}{2}, \tilde{\mathbf{k}}'_0)\overline{\tilde{\alpha}}_\varepsilon(\tilde{\mathbf{k}}'_0) \\ &=\; \int d\mathbf{p}'\;\exp\left[i\mathbf{p}'\cdot\mathbf{x}'\right]\left|\int d\mathbf{k}'_0 S(\mathbf{k}', \mathbf{k}'_0)\tilde{\alpha}_\varepsilon(\mathbf{k}'_0)\right|^2 + O(\varepsilon), \end{aligned}$$

since S does not depend on ε. It is clear from this expression that we can not write $W^{n-1}[\tilde{\beta}]$ only in terms of $W^{n-1}[\tilde{\alpha}]$ for a general scattering kernel S.

We consider random boundaries that are stationary, or statistically invariant under translation. The physical problem is not changed by translation and the incident and reflected waves change phase only. Then the scattering operator for a surface shifted by a vector $\mathbf{d}'$ must satisfy the relation

$$S_{h(\mathbf{x}'-\mathbf{d}')}(\mathbf{k}',\mathbf{k}'_0) \;=\; \exp\left[i(\mathbf{k}'-\mathbf{k}'_0)\cdot\mathbf{d}'\right]S_{h(\mathbf{x}')}(\mathbf{k}',\mathbf{k}'_0). \tag{4.3}$$

Since the mean over all realizations $\langle h(\mathbf{x}')\rangle$ is a constant, we conclude from (4.3) that

$$\langle S\rangle(\mathbf{k}',\mathbf{k}'_0) \;=\; \exp\left[i(\mathbf{k}'-\mathbf{k}'_0)\cdot\mathbf{d}'\right]\langle S\rangle(\mathbf{k}',\mathbf{k}'_0).$$

Since this relation is true for all shifts $\mathbf{d}'$ we conclude that $\langle S\rangle$ has the form

$$\langle S\rangle(\mathbf{k}',\mathbf{k}'_0) \;=\; R(\mathbf{k}')\delta(\mathbf{k}'-\mathbf{k}'_0). \tag{4.4}$$

Here $R(\mathbf{k}')$ is the coherent reflection coefficient.

Consider now the fluctuations of the scattering amplitude

$$S(\mathbf{k}',\mathbf{k}_0) \;=\; R(\mathbf{k}')\delta(\mathbf{k}'-\mathbf{k}'_0) \;+\; \Delta S(\mathbf{k}',\mathbf{k}'_0). \tag{4.5}$$

The correlations of the fluctuations are given by

$$\Pi(\mathbf{k}',\mathbf{k}'_0;\mathbf{a}',\mathbf{a}'_0) \;=\; \Delta S(\mathbf{k}'-\frac{\mathbf{a}'}{2},\mathbf{k}'_0-\frac{\mathbf{a}'_0}{2})\Delta S(\mathbf{k}'+\frac{\mathbf{a}'}{2},\mathbf{k}'_0+\frac{\mathbf{a}'_0}{2}).$$

Again, because of translation invariance, we see that the average of Π is proportional to $\delta(\mathbf{a}'-\mathbf{a}'_0)$. Thus, there exists a kernel $\Delta\mathcal{E}$ such that:

$$\langle\Pi\rangle(\mathbf{k}',\mathbf{k}'_0;\mathbf{a}',\mathbf{a}'_0) \;=\; \Delta\mathcal{E}(\mathbf{k}',\mathbf{k}'_0;\mathbf{a}')\delta(\mathbf{a}'-\mathbf{a}'_0). \tag{4.6}$$

Squaring the amplitudes and averaging yields the energy conservation relation

$$|R(\mathbf{k}')|^2 + \int_{|\mathbf{k}'_0|<\omega/v} \tau(\mathbf{k}'_0,\mathbf{k}')\; d\mathbf{k}'_0 \;=\; 1, \qquad |\mathbf{k}'| < \omega/v, \tag{4.7}$$

where

$$\tau(\mathbf{k}'_0,\mathbf{k}') \;=\; \Delta\mathcal{E}(\mathbf{k}'_0,\mathbf{k}';0) \tag{4.8}$$

is the differential cross section for the incoherent or diffuse surface scattering.

We are now in a position to calculate the average energy associated with the reflected waves in terms of the incident (and deterministic) energy. We have

$$\begin{aligned}
&\langle W^{n-1}[\tilde{\beta}]\rangle(\mathbf{k}',-\mathbf{x}')\\
=\;& \lim_{\varepsilon\to 0}\int d\mathbf{p}'\;\exp\left[i\mathbf{p}'\cdot\mathbf{x}'\right]\langle\int d\mathbf{k}'_0 S(\mathbf{k}'-\varepsilon\frac{\mathbf{p}'}{2},\mathbf{k}'_0)\tilde{\alpha}_\varepsilon(\mathbf{k}'_0)\int d\tilde{\mathbf{k}'}_0\overline{S}(\mathbf{k}'+\varepsilon\frac{\mathbf{p}'}{2},\tilde{\mathbf{k}'}_0)\overline{\tilde{\alpha}}_\varepsilon(\tilde{\mathbf{k}'}_0)\rangle\\
=\;& \lim_{\varepsilon\to 0}\int d\mathbf{p}'\;\exp\left[i\mathbf{p}'\cdot\mathbf{x}'\right]\int d\mathbf{k}'_0\int d\tilde{\mathbf{k}'}_0\langle S(\mathbf{k}'-\varepsilon\frac{\mathbf{p}'}{2},\mathbf{k}'_0)\overline{S}(\mathbf{k}'+\varepsilon\frac{\mathbf{p}'}{2},\tilde{\mathbf{k}'}_0)\rangle\tilde{\alpha}_\varepsilon(\mathbf{k}'_0)\overline{\tilde{\alpha}}_\varepsilon(\tilde{\mathbf{k}'}_0).
\end{aligned}$$

Because of (4.4-4.6) we have

$$\begin{aligned}
< S(\mathbf{k}'-\varepsilon\tfrac{\mathbf{p}'}{2},\mathbf{k}'_0)\overline{S}(\mathbf{k}'+\varepsilon\tfrac{\mathbf{p}'}{2},\tilde{\mathbf{k}'}_0)\rangle \;&=\; |R(\mathbf{k}')|^2\delta(\mathbf{k}'-\varepsilon\tfrac{\mathbf{p}'}{2}-\mathbf{k}'_0)\delta(\mathbf{k}'+\varepsilon\tfrac{\mathbf{p}'}{2}-\tilde{\mathbf{k}'}_0)\\
&+\;\; \Delta\mathcal{E}(\mathbf{k}',\tfrac{\mathbf{k}'_0+\tilde{\mathbf{k}'}_0}{2};\varepsilon\mathbf{p}')\;\delta(\mathbf{k}'_0-\tilde{\mathbf{k}'}_0-\varepsilon\mathbf{p}')\;+\;O(\varepsilon)
\end{aligned}$$

since R is independent of ε. Since $\Delta\mathcal{E}$ does not depend on ε, we get therefore

$$\begin{aligned}\langle W^{n-1}[\tilde{\beta}]\rangle(\mathbf{k}',-\mathbf{x}') &= |R(\mathbf{k}')|^2\langle W^{n-1}[\tilde{\alpha}]\rangle(\mathbf{k}',-\mathbf{x}')\\ &+ \int d\mathbf{p}'\ \exp[i\mathbf{p}'\cdot\mathbf{x}']\int d\mathbf{k}_0'\Delta\mathcal{E}(\mathbf{k}',\mathbf{k}_0'+\varepsilon\mathbf{p}';0)\tilde{\alpha}_\varepsilon(\mathbf{k}_0')\overline{\tilde{\alpha}}_\varepsilon(\mathbf{k}_0'-\varepsilon\mathbf{p}') + O(\varepsilon)\\ &= |R(\mathbf{k}')|^2\langle W^{n-1}[\tilde{\alpha}]\rangle(\mathbf{k}',-\mathbf{x}') + \int d\mathbf{k}_0'\ \tau(\mathbf{k}',\mathbf{k}_0')\langle W^{n-1}[\tilde{\alpha}]\rangle(\mathbf{k}_0',-\mathbf{x}')\end{aligned}$$

From this follows the boundary condition for the average scalar energy $a(\mathbf{x},\mathbf{k})$ at $x_n=0$, going back from energy fluxes to energies:

$$\begin{aligned}v\hat{k}_n(\mathbf{k}')\ a(\mathbf{x}',0,\mathbf{k}',k_n) &= |R(\mathbf{k}')|^2 v\hat{k}_n(\mathbf{k}')\ a(\mathbf{x}',0,\mathbf{k}',-k_n)\\ &+ \int d\mathbf{k}_0'\ \tau(\mathbf{k}',\mathbf{k}_0')\ v\hat{k}_n(\mathbf{k}_0')\ a(\mathbf{x}',0,\mathbf{k}_0',-k_n(\mathbf{k}_0'))\end{aligned} \tag{4.9}$$

with $k_n>0$. Here $R(\mathbf{k})$ defined by (4.4) is the coherent reflection coefficient and $\tau(\mathbf{k},\mathbf{k}_0)$ defined by (4.8) is the incoherent differential scattering cross-section at the boundary.

4.2 Reflection and Transmission from Rough Boundaries

Let us consider now the more general case of two homogeneous half spaces separated by a rough interface h. In this setting the scalar scattering operator S has to be replaced by a two by two matrix-valued operator $(S^{ij})_{1\le i,j\le 2}$ where S^{ij} represents the scattering of a wave incident from medium j on the surface h and reflected (or transmitted) into medium i. In order to symmetrize the two continuity equations for the normal velocity and pressure we introduce as previously the flux amplitudes $\tilde{\alpha}^i_\varepsilon$ and $\tilde{\beta}^i_\varepsilon$:

$$\tilde{\alpha}^i_\varepsilon = \sqrt{v^i\hat{k}^i_n}\ \alpha^i_\varepsilon, \qquad \tilde{\beta}^i_\varepsilon = \sqrt{v^i\hat{k}^i_n}\ \beta^i_\varepsilon = \frac{\sqrt{\omega}}{k^i}\sqrt{k^i_n}\ \beta^i_\varepsilon. \tag{4.10}$$

This change of variables allows for a natural expression of energy flux conservation at the interface.

Consider general incoming and outgoing plane waves given by (3.5-3.6), with the incident and reflected amplitudes related by

$$\tilde{\beta}^i_\varepsilon(\mathbf{k}') = \int d\mathbf{k}_0'\ S^{ii}(\mathbf{k}',\mathbf{k}_0')\tilde{\alpha}^i_\varepsilon(\mathbf{k}_0') + \int d\mathbf{k}_0'\ S^{ij}(\mathbf{k}',\mathbf{k}_0')\tilde{\alpha}^j_\varepsilon(\mathbf{k}_0').$$

As in the case of a homogeneous half space with rough boundary we deduce from translation invariance the form

$$S^{ij}(\mathbf{k}',\mathbf{k}_0') = R^{ij}(\mathbf{k}')\delta(\mathbf{k}'-\mathbf{k}_0') + \Delta S^{ij}(\mathbf{k}',\mathbf{k}_0'), \tag{4.11}$$

for the scattering amplitudes. Moreover,

$$\langle\Pi^{i;kl}\rangle(\mathbf{k}',\mathbf{k}_0';\mathbf{a}',\mathbf{a}_0') = \Delta\mathcal{E}^{i;kl}(\mathbf{k}',\mathbf{k}_0';\mathbf{a}')\delta(\mathbf{a}'-\mathbf{a}_0'), \tag{4.12}$$

where

$$\Pi^{i;kl}(\mathbf{k}',\mathbf{k}_0';\mathbf{a},\mathbf{a}_0) = \Delta S^{ik}(\mathbf{k}'-\frac{\mathbf{a}'}{2},\mathbf{k}_0'-\frac{\mathbf{a}_0'}{2})\Delta S^{il}(\mathbf{k}'+\frac{\mathbf{a}'}{2},\mathbf{k}_0'+\frac{\mathbf{a}_0'}{2}).$$

Therefore we have

$$\begin{aligned}\langle S^{ik}(\mathbf{k}'-\varepsilon\tfrac{\mathbf{p}'}{2},\mathbf{k}_0')\overline{S^{il}}(\mathbf{k}'+\varepsilon\tfrac{\mathbf{p}'}{2},\tilde{\mathbf{k}}'_0)\rangle &= R^{ik}(\mathbf{k}')\overline{R}^{il}(\mathbf{k}')\delta(\mathbf{k}'-\varepsilon\tfrac{\mathbf{p}'}{2}-\mathbf{k}_0')\delta(\mathbf{k}'+\varepsilon\tfrac{\mathbf{p}'}{2}-\tilde{\mathbf{k}}'_0)\\ &+ \Delta\mathcal{E}^{i;kl}(\mathbf{k}',\tfrac{\mathbf{k}_0'+\tilde{\mathbf{k}}'_0}{2};\varepsilon\mathbf{p}')\ \delta(\mathbf{k}_0'-\tilde{\mathbf{k}}'_0-\varepsilon\mathbf{p}') + O(\varepsilon)\end{aligned}$$

Here $R^{ik}(\mathbf{k}')$ are the coherent reflection coefficients.

Denote the incoherent scattering cross-sections by

$$\tau^{i;kl}(\mathbf{k}',\mathbf{k}_0') \;=\; \Delta\mathcal{E}^{i;kl}(\mathbf{k}',\mathbf{k}_0';0), \tag{4.13}$$

we obtain, as above,

$$\begin{aligned}\langle W^{n-1}[\tilde{\beta}^i]\rangle(\mathbf{k}',-\mathbf{x}') \;&=\; \sum_{k,l=1}^{2} R^{ik}(\mathbf{k}')\overline{R}^{il}(\mathbf{k}')W^{n-1}[\tilde{\alpha}^k,\tilde{\alpha}^l](\mathbf{k}',-\mathbf{x}')\\ &+\; \sum_{k,l=1}^{2}\int d\mathbf{k}_0'\;\tau^{i;kl}(\mathbf{k}',\mathbf{k}_0')W^{n-1}[\tilde{\alpha}^k,\tilde{\alpha}^l](\mathbf{k}_0',-\mathbf{x}').\end{aligned}$$

Assuming here again the decorrelation condition (H1), stated in the previous section, and letting $\tau^{ij}=\tau^{i;jj}$ we get the incoming-outgoing energy flux relations for a^i

$$\begin{aligned}v^i\hat{k}_n^i(\mathbf{k}')\;a^i(\mathbf{x}',0,\mathbf{k}',\epsilon^i k_n^i) \;&=\; \sum_{j=1}^{2}|R^{ij}(\mathbf{k}')|^2\;v^j\hat{k}_n^j(\mathbf{k}')\;a^j(\mathbf{x}',0,\mathbf{k}',-\epsilon^j k_n^j)\\ &+\; \sum_{j=1}^{2}\int d\mathbf{k}_0'\;\tau^{ij}(\mathbf{k}',\mathbf{k}_0')\;v^j\hat{k}_n^j(\mathbf{k}_0')\;a^j(\mathbf{x}',0,\mathbf{k}_0',-\epsilon^j k_n^j(\mathbf{k}_0')),\end{aligned} \tag{4.14}$$

with R^{ij} defined by (4.11) and τ^{ij} by (4.13).

4.3 Reflection and Transmission from Small Amplitude Rough Boundaries

We now specialize to transmission and reflection of energy at a rough boundary with small random fluctuations. The boundary is given by $x_n=\varepsilon\eta h(\frac{\mathbf{x}'}{\varepsilon})$, where η is another small parameter. The random process $h(\mathbf{y}')$ has mean zero and is stationary, with covariance function $Q(\mathbf{y}')$ defined by

$$\langle h(\mathbf{x}'+\mathbf{y}')h(\mathbf{x}')\rangle \;=\; Q(\mathbf{y}').$$

The power spectral density of the surface fluctuations is the Fourier transform of Q:

$$Q(\mathbf{y}') \;=\; \int\frac{d\mathbf{p}'}{(2\pi)^{n-1}}e^{i\mathbf{p}'\cdot\mathbf{y}'}\hat{Q}(\mathbf{p}'). \tag{4.15}$$

We begin with a remark that will simplify the calculations. Assume that the reflection and transmission operator S, as a 2×2 matrix, has an expansion of the form

$$S \;=\; S_0 \;+\; \eta\;S_1 \;+\; \eta^2 S_2 + o(\eta^2) \tag{4.16}$$

with $S_0(\mathbf{k}',\mathbf{k}_0')=R(\mathbf{k}')\delta(\mathbf{k}'-\mathbf{k}_0')$ being the reflection and transmission operator of the unperturbed flat surface. Clearly S_1 has zero mean because it is linear in surface fluctuations. Then

$$\begin{aligned}R(\mathbf{k}') \;&=\; R_0(\mathbf{k}')\delta(\mathbf{k}'-\mathbf{k}_0')+\eta^2R_2(\mathbf{k}')\delta(\mathbf{k}'-\mathbf{k}_0')+o(\eta^2)\\ \Delta S \;&=\; \eta S_1+O(\eta^2).\end{aligned} \tag{4.17}$$

Therefore (4.13) implies that the differential scattering cross section τ depends only on S_1 up to $o(\eta^2)$. Moreover R depends only on S_2 through its statistical average R_2.

The details of the small η computation of the reflection and transmission coefficients is given in Appendix A. The results are as follows. The leading order terms are from the flat surface case. They are given by

$$R_0 = \begin{pmatrix} \mathcal{R} & \mathcal{T} \\ \mathcal{T} & -\mathcal{R} \end{pmatrix}, \tag{4.18}$$

where the Fresnel coefficients $\mathcal{R}$ and $\mathcal{T}$ are as in (3.11) and can also be written in the form

$$\mathcal{R} = \frac{\delta_1^2 - \delta_2^2}{\delta_1^2 + \delta_2^2}, \qquad \mathcal{T} = \frac{2\delta_1\delta_2}{\delta_1^2 + \delta_2^2}, \tag{4.19}$$

where

$$\delta_i = \sqrt{\frac{k_n^i}{\rho^i}}.$$

Let us define some auxiliary matrices. We have

$$P = \begin{pmatrix} \delta_1 & \delta_2 \\ \delta_1^{-1} & -\delta_2^{-1} \end{pmatrix}, \quad M(\mathbf{p}', \mathbf{k}') = \begin{pmatrix} -\frac{(k^1)^2 - \mathbf{p}' \cdot \mathbf{k}'}{k_n^1(\mathbf{k}')}\delta_1(\mathbf{k}') & \frac{(k^2)^2 - \mathbf{p}' \cdot \mathbf{k}'}{k_n^2(\mathbf{k}')}\delta_2(\mathbf{k}') \\ k_n^1(\mathbf{k}')\delta_1^{-1}(\mathbf{k}') & k_n^2(\mathbf{k}')\delta_2^{-1}(\mathbf{k}') \end{pmatrix} \tag{4.20}$$

We define the matrix $N(\mathbf{p}', \mathbf{k}') = P^{-1}(\mathbf{p}')V_0 M(\mathbf{p}', \mathbf{k}')$ where

$$V_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let us also define the matrix $B(\mathbf{p}', \mathbf{k}') = \frac{1}{2}(N(\mathbf{p}', \mathbf{k}')R_0(\mathbf{k}') + N(\mathbf{p}', \mathbf{k}')R_0(\mathbf{k}'))$, given by

$$B_{ij} = \frac{\Gamma(\mathbf{p}' \cdot \mathbf{k}')\delta_i(\mathbf{p}')\delta_j(\mathbf{k}') \; + \; \epsilon^i \epsilon^j (\rho_1 - \rho_2)\delta_1\delta_2(\mathbf{p}')\delta_1\delta_2(\mathbf{k}')\delta_{i'}(\mathbf{k}')\delta_{j'}(\mathbf{p}')}{(\delta_1^2 + \delta_2^2)(\mathbf{p}')(\delta_1^2 + \delta_2^2)(\mathbf{k}')}, \tag{4.21}$$

where $i' = 2$ if $i = 1$ and vice-versa and where

$$\Gamma(\mathbf{p}' \cdot \mathbf{k}') = \frac{(k^2)^2 - \mathbf{k}' \cdot \mathbf{p}')}{\rho^2} - \frac{((k^1)^2 - \mathbf{k}' \cdot \mathbf{p}')}{\rho^1}.$$

The boundary conditions at the interface are continuity of the normal velocity and pressure at the interface (3.9). Then, from (4.13) and (4.17) we find after a long computation, given in Appendix A, that the incoherent scattering cross-section tensor has the form

$$\tau^{ij}(\mathbf{p}', \mathbf{k}') = 4B_{ij}^2(\mathbf{p}', \mathbf{k}')\hat{Q}(\mathbf{p}' - \mathbf{k}'). \tag{4.22}$$

The part R_2 of the coherent reflection coefficient may be split into a sum of two terms

$$R_2 = R_{2,0} + R_{2,1},$$

which are given by

$$\begin{aligned} R_{2,0} &= \frac{\mathcal{T}}{2}((k^1)^2 - (k^2)^2) \; Q(0) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ R_{2,1}(\mathbf{p}') &= -2\int d\mathbf{k}' N(\mathbf{p}', \mathbf{k}')\, B(\mathbf{k}', \mathbf{p}')\hat{Q}(\mathbf{p}' - \mathbf{k}'). \end{aligned} \tag{4.23}$$

Conservation of energy flux (4.7) has to be satisfied across the interface. Physically this means that energy incident upon the interface is either reflected or transmitted into coherent or incoherent energy. According to (4.17) we have

$$|R^{ij}|^2 \;=\; |R_0^{ij} + \eta^2 R_2^{ij}|^2 \;=\; (R_0^{ij})^2 + 2\eta^2 R_0^{ij} R_2^{ij} + o(\eta^2). \tag{4.24}$$

Therefore the conservation of energy flux up to $o(\eta^2)$ is given by

$$\begin{aligned} 1 &= |R^{11}(\mathbf{p}')|^2 + |R^{21}(\mathbf{p}')|^2 + \eta^2 \int d\mathbf{k}'(\tau^{11}(\mathbf{k}',\mathbf{p}') + \tau^{21}(\mathbf{k}',\mathbf{p}')) \\ 1 &= |R^{22}(\mathbf{p}')|^2 + |R^{12}(\mathbf{p}')|^2 + \eta^2 \int d\mathbf{k}'(\tau^{22}(\mathbf{k}',\mathbf{p}') + \tau^{12}(\mathbf{k}',\mathbf{p}')), \end{aligned} \tag{4.25}$$

for each $\mathbf{p}'$. In the limit $\eta = 0$ this energy conservation simplifies to

$$\mathcal{R}^2 + \mathcal{T}^2 \;=\; 1$$

which is (3.12). The order η^2 terms in (4.25) give the identity

$$\begin{aligned} 0 &= 2(-\mathcal{T}R_{2,0}^{21} + \mathcal{R}R_{2,1}^{11} + \mathcal{T}R_{2,1}^{21}) + \int d\mathbf{k}'(\tau^{11}(\mathbf{k}',\mathbf{p}') + \tau^{21}(\mathbf{k}',\mathbf{p}')) \\ 0 &= 2(\mathcal{T}R_{2,0}^{12} - \mathcal{R}R_{2,1}^{22} + \mathcal{T}R_{2,1}^{12}) + \int d\mathbf{k}'(\tau^{22}(\mathbf{k}',\mathbf{p}') + \tau^{12}(\mathbf{k}',\mathbf{p}')). \end{aligned}$$

A straightforward though tedious computation shows that this is indeed true when we use the formulas given above.

We have derived boundary conditions for the transport equation (3.18) in a homogeneous half space from the plane wave decomposition (3.5) and (3.6) for both flat and rough boundaries, with small fluctuations. For slowly varying heterogeneous media and flat or smoothly varying surfaces this is analyzed in detail in [24]. For weakly fluctuating random media or heterogeneous media with random rough interfaces no such decomposition is available. However in the high frequency limit the boundary conditions for the transport equation remain unchanged. This is difficult to justify and there is no mathematical analysis for it at present.

Since the fluctuations of the interface are taken to be small, the assumption that the volume fluctuations do not interact with the boundary fluctuations is reasonable.

4.4 Derivation of the Dirichlet and Neumann Boundary Conditions

We can get expressions for the incoherent scattering cross sections for Dirichlet and Neumann boundary conditions as a limiting cases of the reflection–transmission problem. We consider the limit $\rho^1/\rho^2 \to 0$ that corresponds to a hard boundary for the waves incident from the upper medium (Neumann boundary condition with vanishing displacement) and to a soft boundary for the waves incident from the lower medium (Dirichlet boundary conditions corresponding to a vanishing pressure).

We easily check that $\mathcal{R} \to 1$ and $\mathcal{T} \to 0$. Therefore

$$R_0 \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Moreover

$$S_1(\mathbf{p}',\mathbf{k}') \;=\; \begin{pmatrix} -2i\dfrac{(k^1)^2 - \mathbf{p}'\cdot\mathbf{k}'}{(k_n^1(\mathbf{p}')k_n^1(\mathbf{k}'))^{1/2}}\;\hat{h}(\mathbf{p}'-\mathbf{k}') & 0 \\ 0 & -2i(k_n^2(\mathbf{p}')\;k_n^2(\mathbf{k}'))^{1/2}\;\hat{h}(\mathbf{p}'-\mathbf{k}') \end{pmatrix}. \tag{4.26}$$

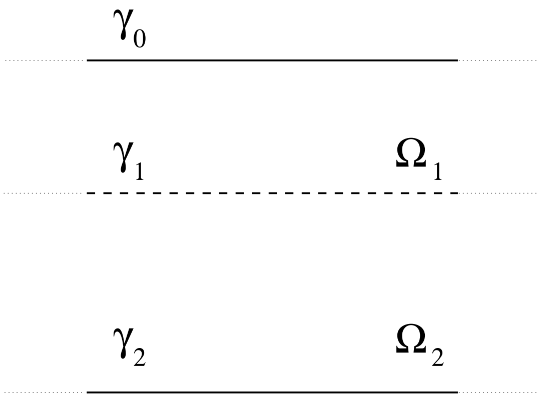

Figure 5.1: Geometry of the media

As expected $R_{2,0} = 0$ in the limit and $R_2 = R_{2,1}$ is given by

$$R_2(\mathbf{p}') \;=\; \begin{pmatrix} -2\int d\mathbf{k}' \dfrac{((k^1)^2 - \mathbf{k}'\cdot\mathbf{p}')^2}{k_n^1(\mathbf{k}')k_n^1(\mathbf{p}')}\hat{Q}(\mathbf{p}'-\mathbf{k}') & 0 \\ 0 & 2\int d\mathbf{k}' k_n^2(\mathbf{k}')k_n^2(\mathbf{p}')\hat{Q}(\mathbf{p}'-\mathbf{k}') \end{pmatrix}. \quad (4.27)$$

The incoherent scattering tensor τ is deduced from the expression of S_1

$$\tau(\mathbf{p}',\mathbf{k}') \;=\; \begin{pmatrix} 4\dfrac{((k^1)^2 - \mathbf{k}'\cdot\mathbf{p}')^2}{k_n^1(\mathbf{k}')k_n^1(\mathbf{p}')}\hat{Q}(\mathbf{p}'-\mathbf{k}') & 0 \\ 0 & 4k_n^2(\mathbf{k}')k_n^2(\mathbf{p}')\hat{Q}(\mathbf{p}'-\mathbf{k}') \end{pmatrix}. \quad (4.28)$$

Here the entry τ_{11} is the differential scattering cross-section for the Neumann problem for the first medium, and the entry τ_{22} is the corresponding cross-section for the Dirichlet problem in the second medium. Conservation of energy flux is also valid, for $i = 1, 2$

$$0 \;=\; 2R_0^{ii}R_{2,1}^{ii}(\mathbf{p}') + \int d\mathbf{k}'\tau^{ii}(\mathbf{p}',\mathbf{k}')$$

as expected.

5 The Initial Boundary Value Problem for the Radiative Transport Equation

We summarize in this section the initial, boundary and interface conditions for the radiative transport equation for acoustic waves in two adjacent plane layers Ω_1 and Ω_2 with rough boundaries, separated by a rough interface, as shown in Figure 5.1. Radiative transport theory is valid in the interior of the layers. The boundary and interface conditions are the ones for high frequency time harmonic waves that were derived in the previous sections. We use them in the time domain here assuming that time variations in the transport regime are slow compared to the period of the oscillations. That is, the frequency spread of the waves about a carrier frequency is narrow. The boundary conditions on the interface γ_1 come from continuity of pressure and acoustic velocity. At the outside boundaries we consider a Neumann boundary condition at γ_0 and a Dirichlet one at γ_2. We may also consider transmission into a homogeneous medium at γ_0 and γ_2.

The equation for the transport of energy in the interior of the two regions Ω_1 and Ω_2 is

$$\frac{\partial a^i(t,\mathbf{x},\mathbf{k})}{\partial t} + v^i \hat{\mathbf{k}}^i \cdot \nabla_{\mathbf{x}} a^i(t,\mathbf{x},\mathbf{k}) = \int_{R^3} \sigma^i(\mathbf{x},\mathbf{k},\mathbf{k}') \left[a^i(t,\mathbf{x},\mathbf{k}') - a^i(t,\mathbf{x},\mathbf{k})\right] d\mathbf{k}' \tag{5.1}$$

with the scattering cross-sections given by

$$\begin{aligned}\sigma^i(\mathbf{x},\mathbf{k},\mathbf{k}') = {} & \frac{M(v^i)^2|\mathbf{k}|^2}{2}\delta(v^i|\mathbf{k}| - v^i|\mathbf{k}'|) \\ & \cdot \left\{(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')^2 \hat{R}^i_{\rho\rho}(\mathbf{k}-\mathbf{k}') + 2(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')\hat{R}^i_{\rho\kappa}(\mathbf{k}-\mathbf{k}') + \hat{R}^i_{\kappa\kappa}(\mathbf{k}-\mathbf{k}')\right\}\end{aligned} \tag{5.2}$$

for $i = 1, 2$.

The boundaries γ_0, γ_1 and γ_2 are fluctuating around the flat surfaces $x_n = L_0$, $x_n = 0$ and $x_n = -L_2$ respectively. We consider a soft boundary condition at γ_0, so that the pressure $p = 0$, as at the interface between air and the earth, for instance. We consider a hard boundary condition at γ_2, so that the normal velocity $v_n = 0$.

Then the boundary and interface conditions at the boundaries γ_0 and γ_2 are given respectively by

$$\begin{aligned} a^1(t,\mathbf{x}',L_0,\mathbf{k}',k_n) &= \left(1 - 2R^{11}_{2,1}(\mathbf{k}')\eta^2\right) a^1(t,\mathbf{x}',L_0,\mathbf{k}',-k_n) \\ &+ \eta^2 \int d\mathbf{p}' \tau^{11}(\mathbf{k}',\mathbf{p}') a^1(t,\mathbf{x}',L_0,\mathbf{p}',p^1_n) \end{aligned} \tag{5.3}$$

$$\begin{aligned} a^2(t,\mathbf{x}',-L_2,\mathbf{k}',k_n) &= \left(1 - 2R^{22}_{2,1}(\mathbf{k}')\eta^2\right) a^2(t,\mathbf{x}',-L_2,\mathbf{k}',-k_n) \\ &+ \eta^2 \int d\mathbf{p}' \tau^{22}(\mathbf{k}',\mathbf{p}') a^2(t,\mathbf{x}',-L_2,\mathbf{p}',p^2_n). \end{aligned} \tag{5.4}$$

Here we have $k_n < 0$, $p_n > 0$ at the boundary γ_1, and $k_n > 0$, $p_n < 0$ at the boundary γ_2, so that the part on the left represents the outgoing scattered waves, and the part on the right corresponds to the incoming waves. The matrices $R_{2,1}$ and τ appearing in (5.3) and (5.4) are given by (4.27) and (4.28).

The interface conditions at γ_1 for the outgoing field $a^1(t,\mathbf{x}',0,\mathbf{k}',k^1_n)$, with $k_n > 0$, are given by

$$\begin{aligned} a^1(t,\mathbf{x}',0,\mathbf{k}',k^1_n) = {} & \left(|\mathcal{R}|^2 + 2\mathcal{R}R^{11}_{2,1}\eta^2\right) a^1(t,\mathbf{x}',0,\mathbf{k}',-k^1_n) + \left(|\mathcal{T}|^2 + 2\mathcal{T}(R^{12}_{2,0} + R^{12}_{2,1})\eta^2\right) a^2(t,\mathbf{x}',0,\mathbf{k}',k^2_n) \\ & + \eta^2 \int d\mathbf{p}' \tau^{11}(\mathbf{k}',\mathbf{p}') a^1(t,\mathbf{x}',0,\mathbf{p}',-p^1_n) + \eta^2 \int d\mathbf{p}' \tau^{12}(\mathbf{k}',\mathbf{p}') a^2(t,\mathbf{x}',0,\mathbf{p}',p^2_n), \end{aligned} \tag{5.5}$$

where $-p^1_n < 0$ and $p^2_n > 0$ corresponds to waves incoming to the interface. The boundary conditions at the interface for a^2 are similar to (5.5). Here the matrices $R_{2,0}$, $R_{2,1}$ and τ are given by (4.23) and (4.22).

Recall that these interface conditions are valid under an assumption that waves incident on the interface from two sides are uncorrelated. This assumption is valid provided that the widths of the layers L_1, L_2 are much larger than the corresponding scattering mean free paths given by

$$l_j = \frac{v^j}{\Sigma^j}, \quad j = 1, 2$$

where the total scattering cross-sections Σ^j are the integrals over wave number of the differential scattering cross-sections σ^j, (1.2). When this condition is violated, waves incident on the interface from the two sides are correlated and the computations of Section 4 are not correct since they do not account for these correlations.

6 Concluding Remarks

We have derived boundary conditions for acoustic radiative transport equations in a domain with rough boundaries. We have also obtained transport theoretic interface conditions for acoustic waves in adjacent scattering media. These conditions are valid in domains that are large compared to the local mean free path.

Transport theoretic boundary and interface conditions can also be derived for electromagnetic waves, accounting correctly for polarization effects. Transport theory for electromagnetic waves in random media has been derived systematically in [23], where many other papers are cited. The phenomenological theory of transport of polarized waves is presented in detail in Chandrasekhar's book [9]. A lot is known about the reflection and transmission of electromagnetic waves from rough surfaces [5, 11, 21, 29]. However, a detailed treatment of transport theoretic boundary conditions seems to be unavailable at present.

Transport theoretic boundary conditions for the system of radiative transport equations for elastic waves is also unavailable at present. Radiative transport theory for elastic waves in random media is considered in [23]. A phenomenological introduction to radiative transport for elastic waves with applications to seismology is given in [13]. Transport theoretic boundary conditions for elastic waves, and their diffusion approximation for small mean free path, are of great interest in seismology [20].

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A Derivation of the Transport Theoretic Interface Conditions

In this appendix, we present in detailed the computation of the coherent reflection coefficients given by (4.18) and (4.23), and of the incoherent scattering cross section tensor given by (4.22).

Continuity of the pressure and of the normal component of velocity at the interface gives two boundary conditions. We note first that the normal vector to the surface $(\frac{\mathbf{x}'}{\varepsilon}, x_n = h(\frac{\mathbf{x}'}{\varepsilon}))$ is given by

$$\nu(\frac{\mathbf{x}'}{\varepsilon}, h(\frac{\mathbf{x}'}{\varepsilon})) \;=\; C(\mathbf{e}_z - \nabla h^\varepsilon(\frac{\mathbf{x}'}{\varepsilon})) = \; C(\mathbf{e}_z - \eta(\nabla h)(\frac{\mathbf{x}'}{\varepsilon})), \tag{A.1}$$

where C is a normalization constant. Using the plane wave decomposition (3.5-3.6), the definition of the eigenvector $\mathbf{b}_f$, (2.11), and the flux amplitudes (4.10), we obtain the following equations

$$\begin{aligned}
&\int d\mathbf{k}' e^{i\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon}} e^{-i\epsilon^1 k_n^1 \eta h(\frac{\mathbf{x}'}{\varepsilon})} \frac{\mathbf{k}'\cdot\nu' - \epsilon^1 k_n^1 \nu_n}{k_n^1}\; \delta_1 \tilde{\alpha}_\varepsilon^1 + \int d\mathbf{k}' e^{i\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon}} e^{i\epsilon^1 k_n^1 \eta h(\frac{\mathbf{x}'}{\varepsilon})} \frac{\mathbf{k}'\cdot\nu' + \epsilon^1 k_n^1 \nu_n}{k_n^1}\; \delta_1 \tilde{\beta}_\varepsilon^1 \\
= &\int d\mathbf{k}' e^{i\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon}} e^{-i\epsilon^2 k_n^2 \eta h(\frac{\mathbf{x}'}{\varepsilon})} \frac{\mathbf{k}'\cdot\nu' - \epsilon^2 k_n^2 \nu_n}{k_n^2}\; \delta_2 \tilde{\alpha}_\varepsilon^2 + \int d\mathbf{k}' e^{i\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon}} e^{i\epsilon^2 k_n^2 \eta h(\frac{\mathbf{x}'}{\varepsilon})} \frac{\mathbf{k}'\cdot\nu' + \epsilon^2 k_n^2 \nu_n}{k_n^2}\; \delta_2 \tilde{\beta}_\varepsilon^2
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
&\int d\mathbf{k}' e^{i\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon}} e^{-i\epsilon^1 k_n^1 \eta h(\frac{\mathbf{x}'}{\varepsilon})}\; \delta_1^{-1} \tilde{\alpha}_\varepsilon^1 + \int d\mathbf{k}' e^{i\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon}} e^{i\epsilon^1 k_n^1 \eta h(\frac{\mathbf{x}'}{\varepsilon})}\; \delta_1^{-1} \tilde{\beta}_\varepsilon^1 \\
= &\int d\mathbf{k}' e^{i\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon}} e^{-i\epsilon^2 k_n^2 \eta h(\frac{\mathbf{x}'}{\varepsilon})}\; \delta_2^{-1} \tilde{\alpha}_\varepsilon^2 + \int d\mathbf{k}' e^{i\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon}} e^{i\epsilon^2 k_n^2 \eta h(\frac{\mathbf{x}'}{\varepsilon})}\; \delta_2^{-1} \tilde{\beta}_\varepsilon^2.
\end{aligned}$$

Here δ_i is defined by

$$\delta_i \ = \ \sqrt{\frac{k_n^i}{\rho^i}}.$$

The incoming energy flux amplitudes $\tilde{\alpha}_\varepsilon^i$ are deterministic. We want now to calculate the reflected wave energy flux amplitudes $\tilde{\beta}_\varepsilon^i$ using a Born expansion in the height of the rough boundary. We need the following intermediate expansions

$$\begin{aligned}
\exp\left[ik_n\eta h(\frac{\mathbf{x}'}{\varepsilon})\right] \ &= \ 1 \ + \ ik_n h(\frac{\mathbf{x}'}{\varepsilon})\ \eta \ - \ \frac{1}{2}(k_n)^2 h^2(\frac{\mathbf{x}'}{\varepsilon})\ \eta^2 + o(\eta^2) \\
\exp\left[ik_n\eta h(\frac{\mathbf{x}'}{\varepsilon})\right](\mathbf{k}\cdot\mathbf{n})C^{-1} \ &= \ k_n \ + \ (i(k_n)^2 h(\frac{\mathbf{x}'}{\varepsilon}) - \mathbf{k}'\cdot\nabla h(\frac{\mathbf{x}'}{\varepsilon}))\ \eta \\
&\quad +(-\frac{1}{2}(k_n)^3 h^2(\frac{\mathbf{x}'}{\varepsilon}) - ik_n h(\frac{\mathbf{x}'}{\varepsilon})\mathbf{k}'\cdot\nabla h(\frac{\mathbf{x}'}{\varepsilon}))\ \eta^2 + o(\eta^2) \\
\tilde{\beta}_\varepsilon^i \ &= \ \tilde{\beta}_{\varepsilon,0} \ + \ \eta\ \tilde{\beta}_{\varepsilon,1} \ + \ \eta^2\tilde{\beta}_{\varepsilon,2} \ + \ o(\eta^2).
\end{aligned} \tag{A.3}$$

Plugging these expressions into (A.2) we find at the zeroth order in the η expansion the following relations

$$\begin{aligned}
\tilde{\beta}_{\varepsilon,0}^1\delta_1 \ - \ \tilde{\alpha}_\varepsilon^1\delta_1 \ &= \ -\tilde{\beta}_{\varepsilon,0}^2\delta_2 \ + \ \tilde{\alpha}_\varepsilon^2\delta_2 \\
\tilde{\beta}_{\varepsilon,0}^1\delta_1^{-1} \ + \ \tilde{\alpha}_\varepsilon^1\delta_1^{-1} \ &= \ \tilde{\beta}_{\varepsilon,0}^2\delta_2^{-1} \ + \ \tilde{\alpha}_\varepsilon^2\delta_2^{-1}.
\end{aligned} \tag{A.4}$$

We need to invert the 2×2 system

$$\begin{aligned}
\delta_1\beta_1 + \delta_2\beta_2 \ &= \ \gamma_1 \\
\delta_1^{-1}\beta_1 - \delta_2^{-1}\beta_2 \ &= \ \gamma_2.
\end{aligned} \tag{A.5}$$

Its solution is given by

$$\begin{aligned}
\beta_1 \ &= \ \frac{\delta_1}{\delta_1^2+\delta_2^2}\gamma_1 + \frac{\delta_1\delta_2^2}{\delta_1^2+\delta_2^2}\gamma_2 \\
\beta_2 \ &= \ \frac{\delta_2}{\delta_1^2+\delta_2^2}\gamma_1 - \frac{\delta_2\delta_1^2}{\delta_1^2+\delta_2^2}\gamma_2.
\end{aligned} \tag{A.6}$$

We can now invert (A.4) to obtain

$$\tilde{\beta}_{\varepsilon,0} \ = \ R_0\tilde{\alpha}_\varepsilon \ = \ \begin{pmatrix} \mathcal{R} & \mathcal{T} \\ \mathcal{T} & -\mathcal{R} \end{pmatrix}\tilde{\alpha}_\varepsilon, \tag{A.7}$$

where $\tilde{\alpha}_\varepsilon = (\tilde{\alpha}_\varepsilon^i)$ and $\tilde{\beta}_{\varepsilon,k} = (\tilde{\beta}_{\varepsilon,k}^i)$, with $k = 0,1,2$, are two-component vectors giving the energy flux amplitudes for both half spaces, and where the Fresnel coefficients $\mathcal{R}$ and $\mathcal{T}$ are given by

$$\mathcal{R} \ = \ \frac{\delta_1^2-\delta_2^2}{\delta_1^2+\delta_2^2}\,, \qquad \mathcal{T} \ = \ \frac{2\delta_1\delta_2}{\delta_1^2+\delta_2^2}. \tag{A.8}$$

At the first order in η in equation (A.2) we obtain

$$\begin{aligned}
&\int d\mathbf{k}' \exp[i\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon}]\left\{\left(ik_n^1 h(\frac{\mathbf{x}'}{\varepsilon}) - \frac{\mathbf{k}'\cdot(\nabla h)(\frac{\mathbf{x}'}{\varepsilon})}{k_n^1}\right)\delta_1(\tilde\alpha_\varepsilon^1 + \tilde\beta_{\varepsilon,0}^1) + \delta_1\tilde\beta_{\varepsilon,1}^1\right\}\\
= &\int d\mathbf{k}' \exp[i\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon}]\left\{\left(ik_n^2 h(\frac{\mathbf{x}'}{\varepsilon}) - \frac{\mathbf{k}'\cdot(\nabla h)(\frac{\mathbf{x}'}{\varepsilon})}{k_n^2}\right)\delta_2(\tilde\alpha_\varepsilon^2 + \tilde\beta_{\varepsilon,0}^2) - \delta_2\tilde\beta_{\varepsilon,1}^2\right\}\\
&\int d\mathbf{k}' \exp[i\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon}]\left\{ik_n^1 h(\frac{\mathbf{x}'}{\varepsilon})\delta_1^{-1}(-\tilde\alpha_\varepsilon^1 + \tilde\beta_{\varepsilon,0}^1) + \delta_1^{-1}\tilde\beta_{\varepsilon,1}^1\right\}\\
= &\int d\mathbf{k}' \exp[i\frac{\mathbf{k}'\cdot\mathbf{x}'}{\varepsilon}]\left\{ik_n^2 h(\frac{\mathbf{x}'}{\varepsilon})\delta_2^{-1}(\tilde\alpha_\varepsilon^2 - \tilde\beta_{\varepsilon,0}^2) + \delta_2^{-1}\tilde\beta_{\varepsilon,1}^2\right\}.
\end{aligned} \tag{A.9}$$

Denoting by $\hat h$ the Fourier transform of h, we have

$$h(\frac{\mathbf{x}'}{\varepsilon}) = \int \exp(i\frac{\xi'\cdot\mathbf{x}'}{\varepsilon})\hat h(\xi')d\xi', \qquad (\nabla h)(\frac{\mathbf{x}'}{\varepsilon}) = \int i\xi' \exp(i\frac{\xi'\cdot\mathbf{x}'}{\varepsilon})\hat h(\xi')d\xi'.$$

Replacing $\tilde\beta_{\varepsilon,0}$ in (A.9) by its expression (A.7) and computing the Fourier transform in $\mathbf{x}'$ (with conjugate variable $\mathbf{p}'$), we obtain

$$\begin{aligned}
&\delta_1\tilde\beta_{\varepsilon,1}^1(\mathbf{p}') + \int d\mathbf{k}'\hat h(\mathbf{p}'-\mathbf{k}')\ \delta_1\ (ik_n^1 - i\frac{\mathbf{k}'\cdot(\mathbf{p}'-\mathbf{k}')}{k_n^1})\ ((1+\mathcal{R})\tilde\alpha_\varepsilon^1 + \mathcal{T}\tilde\alpha_\varepsilon^2)\\
= \quad &-\delta_2\tilde\beta_{\varepsilon,1}^2(\mathbf{p}') + \int d\mathbf{k}'\hat h(\mathbf{p}'-\mathbf{k}')\ \delta_2\ (ik_n^2 - i\frac{\mathbf{k}'\cdot(\mathbf{p}'-\mathbf{k}')}{k_n^2})\ (\mathcal{T}\tilde\alpha_\varepsilon^1 + (1-\mathcal{R})\tilde\alpha_\varepsilon^2)\\
&\delta_1^{-1}\tilde\beta_{\varepsilon,1}^1(\mathbf{p}') + \int d\mathbf{k}'\hat h(\mathbf{p}'-\mathbf{k}')\ \delta_1^{-1}\ ik_n^1\ ((-1+\mathcal{R})\tilde\alpha_\varepsilon^1 + \mathcal{T}\tilde\alpha_\varepsilon^2)\\
= \quad &\delta_2^{-1}\tilde\beta_{\varepsilon,1}^2(\mathbf{p}') + \int d\mathbf{k}'\hat h(\mathbf{p}'-\mathbf{k}')\ \delta_2^{-1}\ ik_n^2\ (-\mathcal{T}\tilde\alpha_\varepsilon^1 + (1+\mathcal{R})\tilde\alpha_\varepsilon^2).
\end{aligned} \tag{A.10}$$

For $m = 1, 2$ we have

$$ik_n^m - i\frac{\mathbf{k}'\cdot(\mathbf{p}'-\mathbf{k}')}{k_n^m} = i\frac{(k^m)^2 - \mathbf{k}'\cdot\mathbf{p}'}{k_n^m}.$$

Using the definition of M given in (4.20) we can rewrite system (A.10) in vector form

$$P(\mathbf{p}')\tilde\beta_{\varepsilon,1}(\mathbf{p}') = \int d\mathbf{k}'\ i\hat h(\mathbf{p}'-\mathbf{k}')(M(\mathbf{p}',\mathbf{k}')\tilde\alpha_\varepsilon(\mathbf{k}') + V_0 M(\mathbf{p}',\mathbf{k}')\tilde\beta_{\varepsilon,0})(\mathbf{k}'). \tag{A.11}$$

¿From (A.8) we have

$$\delta_1\ (1+\mathcal{R}) = \frac{2\delta_1^3}{\delta_1^2+\delta_2^2}, \quad \delta_2\ (1-\mathcal{R}) = \frac{2\delta_2^3}{\delta_1^2+\delta_2^2}$$

along with similar expressions for $\mathcal{T}$. Therefore, from (A.10) we have

$$\begin{aligned}
(\delta_1\tilde\beta_{\varepsilon,1}^1 + \delta_2\tilde\beta_{\varepsilon,1}^2)(\mathbf{p}') &= 2i\int d\mathbf{k}'\hat h(\mathbf{p}'-\mathbf{k}')\Gamma(\mathbf{p}'\cdot\mathbf{k}')\ \frac{(\delta_1\tilde\alpha_\varepsilon^1 + \delta_2\tilde\alpha_\varepsilon^2)}{(\delta_1^2+\delta_2^2)}(\mathbf{k}')\\
(\delta_1^{-1}\tilde\beta_{\varepsilon,1}^1 - \delta_2^{-1}\tilde\beta_{\varepsilon,1}^2)(\mathbf{p}') &= 2i\int d\mathbf{k}'\hat h(\mathbf{p}'-\mathbf{k}')(\rho_1-\rho_2)\ \frac{(\delta_1\delta_2^2\tilde\alpha_\varepsilon^1 - \delta_2\delta_1^2\tilde\alpha_\varepsilon^2)}{(\delta_1^2+\delta_2^2)}(\mathbf{k}')
\end{aligned} \tag{A.12}$$

where

$$\Gamma(\mathbf{p}' \cdot \mathbf{k}') \;=\; \frac{(k^2)^2 - \mathbf{k}' \cdot \mathbf{p}')}{\rho^2} - \frac{((k^1)^2 - \mathbf{k}' \cdot \mathbf{p}')}{\rho^1}.$$

Using the solution (A.6) of system (A.5) we obtain in vector form

$$(\tilde{\beta}_{\varepsilon,1})(\mathbf{p}') \;=\; \int d\mathbf{k}' S_1(\mathbf{p}',\mathbf{k}')(\tilde{\alpha}_\varepsilon)(\mathbf{k}') \;=\; 2i \int d\mathbf{k}' \hat{h}(\mathbf{p}' - \mathbf{k}') B(\mathbf{p}',\mathbf{k}')(\tilde{\alpha}_\varepsilon)(\mathbf{k}') \qquad \text{(A.13)}$$

where B is given by (4.21).

Equating the terms of order η^2 in (A.2) we find the interface relations

$$\int \exp\,(\frac{\mathbf{k}' \cdot \mathbf{x}'}{\varepsilon})\gamma^1 d\mathbf{k}' = \int \exp\,(\frac{\mathbf{k}' \cdot \mathbf{x}'}{\varepsilon})\gamma^2 d\mathbf{k}'$$

where the vectors γ^m are given by

$$\begin{aligned}\gamma_1^m &= \delta_m[-(\frac{1}{2}(k_n^m)^2 h^2(\frac{\mathbf{x}'}{\varepsilon}) + ih(\frac{\mathbf{x}'}{\varepsilon})\mathbf{k}' \cdot \nabla h(\frac{\mathbf{x}'}{\varepsilon}))\epsilon^m(-\tilde{\alpha}_\varepsilon + \tilde{\beta}^m_{\varepsilon,0}) + (ik_n^m h(\frac{\mathbf{x}'}{\varepsilon}) - \frac{\mathbf{k}' \cdot \nabla h(\frac{\mathbf{x}'}{\varepsilon})}{k_n^m})\tilde{\beta}^m_{\varepsilon,1} + \epsilon^m \tilde{\beta}^m_{\varepsilon,2}] \\ \gamma_2^m &= \delta_m^{-1}[-\frac{1}{2}(k_n^m)^2 h^2(\frac{\mathbf{x}'}{\varepsilon})(\tilde{\alpha}^m_\epsilon + \tilde{\beta}^m_{\varepsilon,0}) + (i\epsilon^m k_n^m h(\frac{\mathbf{x}'}{\varepsilon}))\tilde{\beta}^m_{\varepsilon,1} + \tilde{\beta}^m_{\varepsilon,2}].\end{aligned} \qquad \text{(A.14)}$$

As in (4.17), we need to compute the statistical average of S_2. We average the interface conditions (A) before inverting them.

We note first that

$$\begin{aligned}\langle h^2(\frac{\mathbf{x}'}{\varepsilon})\rangle \;&=\; \int \hat{Q}(\xi')d\xi' \;=\; Q(0) \\ \langle h(\frac{\mathbf{x}'}{\varepsilon})\ \mathbf{k}' \cdot \nabla h(\frac{\mathbf{x}'}{\varepsilon})\rangle \;&=\; \int i\mathbf{k}' \cdot \xi'\ \hat{Q}(\xi')d\xi' \;=\; \mathbf{k}' \cdot \nabla Q(0).\end{aligned}$$

We also note the following Fourier transforms for $m = 1, 2$

$$\begin{aligned}&\mathcal{F}^{-1}_{\mathbf{x}' \to \mathbf{p}'}\langle \int d\mathbf{k}' \delta_m(\mathbf{k}') \exp\,[i\mathbf{k}' \cdot \mathbf{x}'/\varepsilon](ik_n^m h(\frac{\mathbf{x}'}{\varepsilon}) - \frac{\mathbf{k}' \cdot \nabla h(\frac{\mathbf{x}'}{\varepsilon})}{k_n^m})\tilde{\beta}^m_{\varepsilon,1}\rangle \\ =\; &-2\int d\mathbf{k}' \delta_m(\mathbf{k}') \frac{(k^m)^2 - \mathbf{p}' \cdot \mathbf{k}'}{k_n^m(\mathbf{k}')} \hat{Q}(\mathbf{k}' - \mathbf{p}')(B(\mathbf{k}',\mathbf{p}')\tilde{\alpha}_\varepsilon(\mathbf{p}'))_m \\ &\mathcal{F}^{-1}_{\mathbf{x}' \to \mathbf{p}'}\langle \int d\mathbf{k}' \delta_m(\mathbf{k}') \exp\,[i\mathbf{k}' \cdot \mathbf{x}'/\varepsilon] ik_n^m h(\frac{\mathbf{x}'}{\varepsilon})\tilde{\beta}^m_{\varepsilon,1}\rangle \\ =\; &-2\int d\mathbf{k}' \delta_m(\mathbf{k}') k_n^m(\mathbf{k}') \hat{Q}(\mathbf{k}' - \mathbf{p}')(B(\mathbf{k}',\mathbf{p}')\tilde{\alpha}_\varepsilon(\mathbf{p}'))_m\end{aligned}$$

Here $\mathcal{F}^{-1}_{\mathbf{x}' \to \mathbf{p}'}$ is the inverse Fourier transform with conjugate variables $\mathbf{x}'$ and $\mathbf{p}'$.

Taking the Fourier transform of the interface conditions (A), we get the following 2×2 system

$$
\begin{aligned}
& \delta_1(\mathbf{p}')[-(\frac{1}{2}(k_n^1)^2 Q(0) + i\mathbf{p}' \cdot \nabla Q(0))(-\tilde{\alpha}_\varepsilon^1 + \tilde{\beta}_{\varepsilon,0}^1)(\mathbf{p}')] \\
& -2\int d\mathbf{k}' \delta_1(\mathbf{k}') \frac{(k^1)^2 - \mathbf{p}' \cdot \mathbf{k}'}{k_n^1(\mathbf{k}')} \hat{Q}(\mathbf{p}' - \mathbf{k}')(B(\mathbf{k}', \mathbf{p}')\tilde{\alpha}_\varepsilon(\mathbf{p}'))_1 + \delta_1(\mathbf{p}')\langle \tilde{\beta}_{\varepsilon,2}^1 \rangle \\
= \; & \delta_2(\mathbf{p}')[-(\frac{1}{2}(k_n^2)^2 Q(0) + i\mathbf{p}' \cdot \nabla Q(0))(\tilde{\alpha}_\varepsilon^2 - \tilde{\beta}_{\varepsilon,0}^2)(\mathbf{p}')] \\
& -2\int d\mathbf{k}' \delta_2(\mathbf{k}') \frac{(k^2)^2 - \mathbf{p}' \cdot \mathbf{k}'}{k_n^2(\mathbf{k}')} \hat{Q}(\mathbf{p}' - \mathbf{k}')(B(\mathbf{k}', \mathbf{p}')\tilde{\alpha}_\varepsilon(\mathbf{p}'))_2 - \delta_2(\mathbf{p}')\langle \tilde{\beta}_{\varepsilon,2}^2 \rangle
\end{aligned}
\tag{A.15}
$$

$$
\begin{aligned}
& \delta_1^{-1}(\mathbf{p}')[-\frac{1}{2}(k_n^1)^2 Q(0)(\tilde{\alpha}_\varepsilon^1 + \tilde{\beta}_{\varepsilon,0}^1)] \\
& -2\int d\mathbf{k}' \delta_1^{-1}(\mathbf{k}') k_n^1(\mathbf{k}') \; \hat{Q}(\mathbf{p}' - \mathbf{k}')(B(\mathbf{k}', \mathbf{p}')\tilde{\alpha}_\varepsilon(\mathbf{p}'))_1 + \delta_1^{-1}(\mathbf{p}')\langle \tilde{\beta}_{\varepsilon,2}^1 \rangle \\
= \; & \delta_2^{-1}(\mathbf{p}')[-\frac{1}{2}(k_n^2)^2 Q(0)(\tilde{\alpha}_\varepsilon^2 + \tilde{\beta}_{\varepsilon,0}^2)] \\
& +2\int d\mathbf{k}' \delta_2^{-1}(\mathbf{k}') k_n^2(\mathbf{k}') \; \hat{Q}(\mathbf{p}' - \mathbf{k}')(B(\mathbf{k}', \mathbf{p}')\tilde{\alpha}_\varepsilon(\mathbf{p}'))_2 + \delta_2^{-1}(\mathbf{p}')\langle \tilde{\beta}_{\varepsilon,2}^2 \rangle.
\end{aligned}
$$

This is a 2×2 system of the form (A.5) that we have to invert. Splitting the tensor R_2 into two parts $R_2 = R_{2,0} + R_{2,1}$, we obtain the formulas (4.23).

B Grazing Angles and the Smoothing Method

As pointed out by several authors [28, 10, 30, 31] the reflection coefficient may blow up as the incident angle becomes parallel to the mean surface in the case of Neumann boundary conditions. The corrector $R_{2,1}^{11}$ is not defined for grazing angles, when $k_n(\mathbf{p}') = 0$, as can be seen from (4.27). Therefore the range of validity of the Born expansion is reduced to angles whose normal component is much larger than η. To overcome this shortcoming, we can use the smoothing method, as will be explained below. In simple situations, like Neumann boundary conditions, the smoothing method gives a reflection coefficient of the form

$$
R(\mathbf{k}) \;=\; \frac{1 + \eta^2 \mathcal{Q}}{1 - \eta^2 \mathcal{Q}} \tag{B.1}
$$

where $\mathcal{Q}$ is given by

$$
\mathcal{Q} \;=\; -\int d\mathbf{k}' \frac{((k^1)^2 - \mathbf{k}' \cdot \mathbf{p}')^2}{k_n^1(\mathbf{k}') k_n^1(\mathbf{p}')} \hat{Q}(\mathbf{p}' - \mathbf{k}'). \tag{B.2}
$$

The Born expansion for the reflection coefficient gives

$$
R(\mathbf{k}) \;=\; 1 + 2\eta^2 \mathcal{Q} + O(\eta^3)
$$

which comes from (4.27). Thus, the renormalized expression (B.1) is consistent to $O(\eta^3)$ with the Born expansion and it is not singular at grazing angles. Note that $\mathcal{Q}$ remains negative since we consider only propagating waves for which k_n^i is real, and the reflection coefficient R ranges between 0 and 1.

We must also renormalize the incoherent energy flux scattering cross-section in order to ensure conservation of energy flux. This is done by replacing (4.28) by

$$
\tau_{ren}^{11}(\mathbf{p}', \mathbf{k}') \;=\; \frac{4}{(1 - \eta^2 \mathcal{Q}(\mathbf{k}'))^2} \frac{((k^1)^2 - \mathbf{k}' \cdot \mathbf{p}')^2}{k_n^1(\mathbf{k}') k_n^1(\mathbf{p}')} \hat{Q}(\mathbf{p}' - \mathbf{k}'). \tag{B.3}
$$

This renormalized diffuse reflection coefficient is nonsingular at grazing angles.

A physical explanation for this renormalization is that part of the multiple scattering at the boundary is now included. The second order Born expansion does not account for multiple scattering. The part of the multiple scattering that is included with the smoothing method corresponds to the ladder diagrams in the diagrammatic resummation methods [25, 27].

B.1 Smoothing Method for Dirichlet and Neumann Boundary Conditions

The smoothing method is a general way to renormalize Born expansions so as to include ladder diagrams. We will present it in a form suitable for reflection from a random surface with Dirichlet or Neumann boundary conditions.

Vanishing pressure (soft boundary condition) and vanishing normal component of the velocity at the rough boundary (hard boundary condition) give boundary conditions of the form

$$K(-\eta)\tilde{\beta} \;=\; \pm K(\eta)\tilde{\alpha} \tag{B.4}$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are the incident and reflected wave amplitudes and the sign $\pm$ is $-$ for Dirichlet and $+$ for Neumann boundary conditions. The operator K is known in the Fourier domain and is given for Dirichlet and Neumann boundary conditions by

$$\begin{aligned}
(K_D(\eta)\tilde{g})(\mathbf{p}') &= (2M)^{-n}\int d\mathbf{x}' e^{-i\mathbf{p}'\cdot\mathbf{x}'}\int d\mathbf{k}' e^{i\mathbf{k}'\cdot\mathbf{x}'} e^{-ik_n\eta h(\mathbf{x}')}\delta^{-1}(\mathbf{k}')\tilde{g}(\mathbf{k}') \\
(K_N(\eta)\tilde{g})(\mathbf{p}') &= (2M)^{-n}\int d\mathbf{x}' e^{-i\mathbf{p}'\cdot\mathbf{x}'}\int d\mathbf{k}' e^{i\mathbf{k}'\cdot\mathbf{x}'} e^{-ik_n\eta h(\mathbf{x}')}(-1+\eta\frac{\mathbf{k}'\cdot\nabla h(\mathbf{x}')}{k_n})\delta(\mathbf{k}')\tilde{g}(\mathbf{k}')
\end{aligned}$$

Expansion in powers of η gives

$$K(\eta) \;=\; 1+\eta K_1+\eta^2 K_2+O(\eta^3) \tag{B.5}$$

where the integral operators K_1 and K_2 do not depend on the sign of η. Plugging (B.5) into (B.4) yields

$$(I-\eta K_1+\eta^2 K_2)\tilde{\beta} = \pm(I+\eta K_1+\eta^2 K_2)\tilde{\alpha}+O(\eta^3). \tag{B.6}$$

We denote by $< f >$ the average over realizations of a random quantity f. We denote the fluctuation by $f' = f- < f >$. We now use this notation to invert approximately (B.6). Note first that the incoming energy flux amplitude $\tilde{\alpha}$ is independent of h. We also note from (4.26) that $< K_1 >= 0$.

Averaging (B.6) over the realizations yields

$$(I+\eta^2 < K_2 >) < \tilde{\beta} > -\eta < K_1'\tilde{\beta}' > +\eta^2 < K_2'\tilde{\beta}' >= \pm(I+\eta^2 < K_2 >)\tilde{\alpha}+O(\eta^3). \tag{B.7}$$

Subtracting (B.7) from (B.6) and keeping terms through $O(\eta)$ we get an expression for the fluctuating part of the reflected wave amplitude

$$\tilde{\beta}' \;=\; \eta(K_1' < \tilde{\beta} > \pm K_1'\tilde{\alpha})+O(\eta^2). \tag{B.8}$$

Finally we use (B.8) for $\tilde{\beta}'$ in (B.7) and obtain

$$([I+\eta^2 < K_2 >]-\eta^2 < K_1'K_1' >) < \tilde{\beta} > \;=\; \pm([I+\eta^2 < K_2 >]+\eta^2 < K_1'K_1' >)\tilde{\alpha}+O(\eta^3). \tag{B.9}$$

Dropping the error and noting that $< K_2 >$ and $< K_1'K_1' >$ are scalars, we get for the mean reflected amplitude

$$< \tilde{\beta} >= \frac{\pm(1+\eta^2 < K_2 > +\eta^2 < K_1'K_1' >)}{1+\eta^2 < K_2 > -\eta^2 < K_1'K_1' >}\tilde{\alpha}$$

The factor $< K_2 >$ is not singular at grazing angles, while $< K_1' K_1' >$ is singular in the Neumann case. We can therefore divide both the numerator and denominator in (B.1) by $1 + \eta^2 < K_2 >$, use the approximation $[1 + \eta^2 < K_2 >]^{-1} \sim 1 - \eta^2 < K_2 >$ and obtain

$$< \tilde{\beta} >= \frac{\pm(1 + \eta^2 < K_1' K_1' >)}{1 - \eta^2 < K_1' K_1' >} \tilde{\alpha} \tag{B.10}$$

to order η^3. Thus defining $\mathcal{Q} =< K_1' K_1' >$ we recover (B.2) for Neumann boundary conditions. For Dirichlet boundary conditions the smoothing is of no real interest since the formula before renormalization is bounded independently of the angle of incidence.

Using expression (B.8) for $\tilde{\beta}'$ and (B.10) for $< \tilde{\beta} >$ allows us to compute the renormalized correlation function $< W^{n-1}[\tilde{\beta}] >$. By doing so we check that the renormalized scattering cross-section for Neumann Boundary conditions is (B.3).

B.2 Extension to Interface Problems

We will now use the smoothing method to renormalize the interface reflection and transmission coefficients in section 4.3.

Continuity of pressure and normal component of the velocity give interfacial boundary conditions of the form

$$\begin{pmatrix} K^{11}(\eta)\delta_1 & K^{12}(-\eta)\delta_2 \\ K^{21}(\eta)\delta_1^{-1} & -K^{22}(-\eta)\delta_2^{-1} \end{pmatrix} \tilde{\beta} = \begin{pmatrix} K^{11}(-\eta)\delta_1 & K^{12}(\eta)\delta_2 \\ -K^{21}(-\eta)\delta_1^{-1} & K^{22}(\eta)\delta_2^{-1} \end{pmatrix} \tilde{\alpha} \tag{B.11}$$

in vector form for $\tilde{\alpha} = (\tilde{\alpha}^i)$ and $\tilde{\beta} = (\tilde{\beta}^i)$, $i = 1, 2$, where we have defined

$$\begin{aligned} K^{ij}(\eta) &= 1 + \eta K_1^{ij} + \eta^2 K_2^{ij} + O(\eta^3) \\ (K_1^{ij}(\eta)\gamma)(\mathbf{p}') &= \int \hat{h}(\mathbf{p}' - \mathbf{k}') m_i^j(\mathbf{p}', \mathbf{k}')\gamma(\mathbf{p}')d\mathbf{p}' \\ (< K_2^{ij}(\eta) > (\gamma))(\mathbf{k}') &= \frac{1}{2}(m_1^j)^2 Q(0)\gamma(\mathbf{k}') \end{aligned} \tag{B.12}$$

where only the real part of K^{1j} has been written since the imaginary part does not contribute eventually, and where

$$m_1^j = i\epsilon^j k_n^j, \qquad m_2^j(\mathbf{p}', \mathbf{k}') = i\frac{(k^j)^2 - \mathbf{p}' \cdot \mathbf{k}'}{\epsilon^j k_n^j(\mathbf{k}')}.$$

These expressions are obtained from (A.2).

We note that $R_0 = P^{-1} V_0 P$ where P is defined in (4.20). Therefore, if we multiply both sides of (B.11) by P^{-1}, we can write it in the form

$$(I + \eta R_1 + \eta^2 R_2)\ \tilde{\beta} = R_0\ (I - \eta R_1 + \eta^2 R_2)\ \tilde{\alpha}, \tag{B.13}$$

where R_1 and R_2 are defined by using (B.12) in (B.11) and adopting matrix notation.

We can now use the smoothing method, as in the previous section, to invert approximately (B.13). With the notation of the previous section for average and fluctuation, we have

$$\tilde{\beta}' = -\eta[R_1' < \tilde{\beta} > + R_0 R_1' \tilde{\alpha}] + O(\eta^2), \tag{B.14}$$

and then

$$< I + \eta^2 R_2 >< \tilde{\beta} > -\eta^2 < R_1' R_1' >< \tilde{\beta} > = \; R_0 < I + \eta^2 R_2 > \tilde{\alpha} + \eta^2 < R_1' R_0 R_1' > \tilde{\alpha} \qquad \text{(B.15)}$$

up to terms of order η^3. Again $< R_2 >$ is a local multiplication operator and we can multiply both sides of the equation by an approximation of order $O(\eta^3)$ to $(I + \eta^2 R_2)^{-1}$. This yields

$$< \tilde{\beta} > = [I - \eta^2 < R_1' R_1' >]^{-1} [R_0 + \eta^2 < R_1' R_0 R_1' > + \eta^2 (R_0 < R_2 > - < R_2 > R_0)] \tilde{\alpha} \qquad \text{(B.16)}$$

which is correct up to order η^3 uniformly in angles, including grazing angles. Expanding the inverse to order $O(\eta^3)$ in this equation yields the standard Born expansion

$$< \tilde{\beta} > = \; [R_0 + \eta^2 ((R_0 < R_2 > - < R_2 > R_0) + < R_1' R_0 R_1' > + < R_1' R_1' > R_0)] \; \tilde{\alpha}.$$

The term $< R_1' R_0 R_1' > + < R_1' R_1' R_0 >$ correspond to $R_{2,1}$ in (4.23) for the Born expansion. These coefficients blow up at grazing angles, as in the case of Neumann boundary conditions, restricting the domain of validity of the Born expansion away from the grazing angles. We note that the term $(R_0 < R_2 > - < R_2 > R_0)$, which is proportional to $Q(0) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and corresponds to the term $R_{2,0}$, does not vanish as in the case of a half space. Since it does not depend on the angle of incidence, this term is always bounded and does not need to be renormalized.

The matrices appearing in (B.16) can be related to the quantities already computed in section 4.3 as follows. We deduce from (A.11) given in Appendix A that the operator $R_1 = R_1'$ has the form

$$(R_1' g)(\mathbf{p}') \;=\; \int d\mathbf{k}' \; i\hat{h}(\mathbf{p}' - \mathbf{k}') N(\mathbf{p}', \mathbf{k}') g(\mathbf{k}').$$

Therefore we have

$$\begin{aligned} (I - \eta^2 < R_1' R_1' >) &= I + \eta^2 W \\ (R_0 + \eta^2 < R_1' \; R_0 \; R_1' >) &= R_0 (I - \eta^2 V), \end{aligned}$$

where

$$\begin{aligned} V(\mathbf{p}') &= \int d\mathbf{k}' P^{-1}(\mathbf{p}') M(\mathbf{p}', \mathbf{k}') P^{-1}(\mathbf{k}') M(\mathbf{k}', \mathbf{p}') \hat{Q}(\mathbf{p}' - \mathbf{k}') \\ W(\mathbf{p}') &= \int d\mathbf{k}' N(\mathbf{p}', \mathbf{k}') N(\mathbf{k}', \mathbf{p}') \hat{Q}(\mathbf{p}' - \mathbf{k}'). \end{aligned}$$

We rewrite expression (B.16) for the coherent reflection coefficient in the form

$$< \tilde{\beta} > = \; [\tilde{R}_0 \; + \; \eta^2 (R_0 < R_2 > - < R_2 > R_0) \;] \; \tilde{\alpha} + O(\eta^3), \qquad \text{(B.17)}$$

where

$$\tilde{R}_0 \;=\; (I + \eta^2 W)^{-1} R_0 (I - \eta^2 V). \qquad \text{(B.18)}$$

The incoherent reflection and transmission cross-sections (4.22) also need to be renormalized. From (B.14) and (B.17) we find that

$$\tilde{\beta}' \;=\; -\eta [R_1' \tilde{R}_0 + R_0 R_1'] \alpha + O(\eta^2).$$

Therefore replacing the matrix B by

$$\tilde{B}(\mathbf{p}', \mathbf{k}') \;=\; \frac{1}{2} (N(\mathbf{p}', \mathbf{k}') \tilde{R}_0(\mathbf{k}') + R_0(\mathbf{k}') N(\mathbf{p}', \mathbf{k}')), \qquad \text{(B.19)}$$

the renormalized cross-sections are given by

$$\tau^{ij}_{ren}(\mathbf{p}', \mathbf{k}') = = 4 \tilde{B}^2_{ij}(\mathbf{p}', \mathbf{k}') \hat{Q}(\mathbf{p}' - \mathbf{k}'). \qquad \text{(B.20)}$$

To summarize, the reflection and transmission coefficients given in Table 1.1, valid in the Born expansion, are now replaced by the ones in Table B.1 given by the smoothing method.

	Born expansion	Smoothing Method
Dirichlet B.C.	(4.27) entry $2,2$ (4.28) entry $2,2$	(4.27) entry $2,2$ (4.28) entry $2,2$
Neumann B.C.	(4.27) entry $1,1$ (4.28) entry $1,1$	(B.1) (B.3)
Interface Condition	(4.23) (4.22)	(B.17) (B.20)

Table B.1: Reference to the explicit formulas for the Smoothing Method

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