Abstract

This paper concerns the reconstruction of the absorption and scattering parameters in a time-dependent linear transport equation from knowledge of angularly averaged measurements performed at the boundary of a domain of interest. We show that the absorption coefficient and the spatial component of the scattering coefficient are uniquely determined by such measurements. We obtain stability results on the reconstruction of the absorption and scattering parameters with respect to the measured albedo operator. The stability results are obtained by a precise decomposition of the measurements into components with different singular behavior in the time domain.

1 Introduction

Inverse transport theory has many applications in e.g. medical and geophysical imaging. It consists of reconstructing optical parameters in a domain of interest from measurements of the transport solution at the boundary of that domain. The optical parameters are the total absorption (extinction) parameter $\sigma(x)$ and the scattering parameter $k(x, v', v)$, which measures the probability of a particle at position $x \in X \subset \mathbb{R}^n$ to scatter from direction $v' \in S^{n-1}$ to direction $v \in S^{n-1}$, where $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$.

The domain of interest is probed as follows. A known flux of particles enters the domain and the flux of outgoing particles is measured at the domain’s boundary. Several inverse theories may then be envisioned based on available data. In this paper, we assume availability of time dependent measurements that are angularly averaged. Also the source term used to probe the domain is not resolved angularly in order to e.g. save time in the acquisition of data. More precisely, the incoming density of particles $\phi(t, x, v)$ as a function of time $t$, at position $x \in \partial X$ at the boundary of the domain of interest, and for incoming directions $v$, is of the form $\phi_S(t, x, v) = \phi(t, x)S(x, v)$, where $\phi(t, x)$ is arbitrary but $S(x, v)$ is fixed. This paper is concerned with the reconstruction of the optical parameters from such measurements. We show that the attenuation coefficient is uniquely determined and that the spatial structure of the scattering coefficient can be reconstructed provided that scattering vanishes in the vicinity of the domain’s boundary (except in dimension $n = 2$ and when $X$ is a disc, where our theory does not require $k$ to vanish in the vicinity of $\partial X$). For instance, when $k(x, v', v) = k_0(x)g(v', v)$ with $g(v', v)$ known a priori, then $k_0(x)$ is uniquely determined by the measurements. Similar
results were announced in [1] when measurements are available in the modulation frequency variable, which is the dual (Fourier) variable to the time variable.

Several other regimes have been considered in the literature. The uniqueness of the reconstruction of the optical parameters from knowledge of angularly resolved measurements both in the time-dependent and time-independent settings was proved in [9, 10]; see also [17] for a review. Stability in the time-independent case has been analyzed in dimension \( n = 2, 3 \) under smallness assumptions for the optical parameters in [14, 15] and in dimension \( n = 2 \) in [18]. Stability results in the presence of full, angularly resolved, measurements have been obtained in [3, 4, 19]. The intermediate case of angularly averaged measurements with angularly resolved sources was considered in [13]. The lack of stability of the reconstruction in the time independent setting with angularly averaged measurements and isotropic sources is treated in [6]. See also [2] for a recent review of results in inverse transport theory.

The rest of the paper is structured as follows. Section 2 recalls known results on the transport equation and the decomposition of the albedo operator. In section 3 we define and decompose the averaged albedo operator (Proposition 3.1) and we study its distributional kernel (Theorems 3.2–3.5). Our main results on uniqueness and stability are presented in section 4 (Theorems 4.1–4.2, Theorems 4.4–4.5 and Corollary 4.6). We show that the absorption coefficient and the spatial structure of the scattering coefficient (the phase function describing scattering from \( v \) to \( v' \) has to be known in advance) can be reconstructed stably from angularly averaged time dependent data. The reconstruction of the scattering coefficient requires inversion of a weighted Radon transform in the general case. In the specific case of a spherical geometry (measurements are performed at the boundary of a sphere), then the scattering coefficient may be obtained by inverting a classical Radon transform. In section 5 we prove Theorems 3.4–3.5. In section 6 we prove Theorems 4.1–4.2, Theorem 4.5 and Theorem 4.4 (4.13). In section 7 we prove Theorem 3.2. In section 8 we prove Theorem 3.3. In section 9 we prove Lemmas 7.1, 8.2–8.4 that are used in sections 7–8. In section 10 we prove Lemma 8.1 that is used in section 8. In section 11 we prove Proposition 3.1.

The derivation of the results is fairly technical and is based on a careful analysis of the temporal behavior of the decomposition of the albedo operator into components that are multilinear in the scattering coefficient. Our results are based on showing that the ballistic and single scattering components may be separated from the rest of the data. These two components are then used to obtain our uniqueness and stability results. It turns out that the structure of single scattering is different depending on whether \( k \) vanishes on \( \partial X \) or not. When \( k \) does not vanish on \( \partial X \), the main singularities of the single scattering component do not allow us to “see inside” the domain as they only depend on values of \( k \) at the domain’s boundary in dimension \( n \geq 3 \). The singular structure of single scattering and the resulting stability estimates are presented in detail when both \( k \) vanishes and does not vanish on \( \partial X \).

This is the extended version of a submitted paper [5].

2 The forward problem and albedo operator

2.1 The linear Boltzmann transport equation

We now introduce notation and recall some known results on the linear transport equation. Let \( X \) be a bounded open subset of \( \mathbb{R}^n \), \( n \geq 2 \), with a \( C^1 \) boundary \( \partial X \). Let \( \nu(x) \) denote the outward normal unit vector to \( \partial X \) at \( x \in \partial X \). Let \( \Gamma_\pm = \{(x, v) \in \partial X \times S^{n-1} | \pm \nu(x) \cdot v > 0\} \) be the sets of incoming and outgoing conditions. For \( (x, v) \in X \times \mathbb{S}^{n-1} \) we define \( \tau_\pm(x, v) \) and \( \tau(x, v) \) by \( \tau_\pm(x, v) := \inf\{s \in (0, +\infty) \mid x \pm sv \not\in X\} \) and \( \tau(x, v) := \tau_-(x, v) + \tau_+(x, v) \). For
Consider \( \sigma : X \times S^{n-1} \to \mathbb{R} \) and \( k : X \times S^{n-1} \times S^{n-1} \to \mathbb{R} \) two nonnegative measurable functions. We assume that \((\sigma, k)\) is admissible when

\[
0 \leq \sigma \in L^\infty(X \times S^{n-1}), \\
0 \leq k(x, v', .) \in L^1(S^{n-1}) \text{ for a.e. } (x, v') \in X \times S^{n-1}, \\
\sigma_p(x, v') = \int_{S^{n-1}} k(x, v', v)dv \text{ belongs to } L^\infty(X \times S^{n-1}).
\]

Let \( T > \eta > 0 \). We consider the following linear Boltzmann transport equation

\[
\begin{aligned}
\frac{\partial u}{\partial t}(t, x, v) + v \cdot \nabla_x u(t, x, v) + \sigma(x, v)u(t, x, v) \\
= \int_{S^{n-1}} k(x, v', v)u(t, x, v')dv', \quad (t, x, v) \in (0, T) \times X \times S^{n-1}, \\
u_{|0, T \times \Gamma_-}(t, x, v) = \phi(t, x, v), \\
u(0, x, v) = 0, \quad (x, v) \in X \times S^{n-1},
\end{aligned}
\]

where \( \phi \in L^1((0, T), L^1(\Gamma_-, d\xi)) \) and \( \text{supp} \phi \subseteq [0, \eta] \). Here, \( d\xi(x, v) = |v \cdot \nu(x)|dv d\mu(x) \), where \( d\mu \) is the surface measure on \( \partial X \) and \( dv \) is the surface measure on \( S^{n-1} \). In other words, we assume that the initial condition is concentrated in the \( \eta \)-vicinity of \( t = 0 \) and measurements are performed for time \( T \), which we will choose sufficiently large so that particles have the time to travel through \( X \) and be measured.

### 2.2 Semigroups and unbounded operators

We introduce the following space

\[
\begin{aligned}
\mathcal{Z} &:= \{ f \in L^1(X \times S^{n-1}) \mid v \cdot \nabla_x f \in L^1(X \times S^{n-1}) \}, \\
\| f \|_{\mathcal{Z}} &:= \| f \|_{L^1(X \times S^{n-1})} + \| v \cdot \nabla_x f \|_{L^1(X \times S^{n-1})},
\end{aligned}
\]

where \( v \cdot \nabla_x \) is understood in the distributional sense.

It is known (see [7, 8]) that the trace map \( \gamma_- \) from \( C^1(\bar{X} \times S^{n-1}) \) to \( C(\Gamma_-) \) defined by

\[
\gamma_-(f) = f|_{\Gamma_-}
\]

extends to a continuous operator from \( \mathcal{Z} \) onto \( L^1(\Gamma_-, \tau_+(x, v)d\xi(x, v)) \) and admits a continuous lifting. Note that \( L^1(\Gamma_-, d\xi) \) is a subset of the spaces \( L^1(\Gamma_-, \tau_+(x, v)d\xi(x, v)) \).

We introduce the following notation

\[
A_1f = -\sigma f, \quad A_2f = \int_{S^{n-1}} k(x, v', v)f(x, v')dv'.
\]

As \((\sigma, k)\) is admissible, the operators \( A_1 \) and \( A_2 \) are bounded operators in \( L^1(X \times S^{n-1}) \).

Consider the following unbounded operators

\[
\begin{aligned}
T_1f = -v \cdot \nabla_x f + A_1f, \quad D(T_1) = \{ f \in \mathcal{Z} \mid f|_{\Gamma_-} = 0 \}, \\
Tf = T_1f + A_2f, \quad D(T) = D(T_1).
\end{aligned}
\]

It is known that the unbounded operators \( T_1 \) and \( T \) are generators of strongly continuous semigroups in \( L^1(X \times S^{n-1}) \) \( U_1(t), U(t) \) respectively (see for example [11, Proposition 2 pp
In addition $U_1(t)$ and $U(t)$ preserve the cone of positive functions, and $U_1(t)$ is given explicitly by the following formula

$$U_1(t)f = e^{-\int_0^t \sigma(x-tv)dv} f(x-tv, v) \Theta(x, x-tv), \text{ for a.e. } (x, v) \in X \times S^{n-1},$$  \hspace{1cm} (2.9)

for $f \in L^1(X \times S^{n-1})$, where

$$\Theta(x, y) = \begin{cases} 1 & \text{if } x + p(y - x) \in X \text{ for all } p \in (0, 1], \\ 0 & \text{otherwise}, \end{cases}$$  \hspace{1cm} (2.10)

for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$.

We recall the Dyson-Phillips formula

$$U(t) = \sum_{m=0}^{+\infty} H_m(t)$$  \hspace{1cm} (2.11)

for $t \geq 0$, where

$$H_m(t) := \int_{s_1 \geq 0, \ldots, s_m \geq 0, s_1 + \ldots + s_m \leq t} U_1(t - s_1 - \ldots - s_m)A_2U_1(s_1) \ldots A_2U_1(s_m)ds_1 \ldots ds_m, \quad m \geq 1,$$  \hspace{1cm} (2.12)

$$H_m(t) = \int_0^t H_{m-1}(t - s)A_2U_1(s)ds, \quad m \geq 1,$$  \hspace{1cm} (2.13)

$$H_0(t) := U_1(t).$$  \hspace{1cm} (2.14)

### 2.3 Trace results

We introduce the following space

$$W := \{ u \in L^1((0, T) \times X \times S^{n-1}) \mid \left( \frac{\partial}{\partial t} + v \cdot \nabla_x \right) u \in L^1((0, T) \times X \times S^{n-1}) \},$$  \hspace{1cm} (2.15)

$$\| u \|_W := \| u \|_{L^1((0, T) \times X \times S^{n-1})} + \left\| \left( \frac{\partial}{\partial t} + v \cdot \nabla_x \right) u \right\|_{L^1((0, T) \times X \times S^{n-1})};$$  \hspace{1cm} (2.16)

where $\frac{\partial}{\partial t}$ and $v \cdot \nabla_x$ are understood in the distributional sense.

It is known (see [7, 8]) that the trace map $\gamma_-$ (respectively $\gamma_+$) from $C^1([0, T] \times \bar{X} \times S^{n-1})$ to $C((X \times S^{n-1}) \times C((0, T) \times \Gamma_{\pm})$ defined by

$$\gamma_-(\psi) = (\psi(0, \cdot), \psi|_{(0,T)\times\Gamma_-}) \quad \text{(respectively } \gamma_+(\psi) = (\psi(T, \cdot), \psi|_{(0,T)\times\Gamma_+}) \quad (2.17)$$

extends to a continuous operator from $W$ onto $L^1(X \times S^{n-1}, \tau_+(x,v)dx dv) \times L^1((0, T) \times \Gamma_{\pm}, \min(T-t, \tau_+(x,v))dt d\xi(x,v))$ (respectively $L^1(X \times S^{n-1}, \tau_-(x,v)dx dv) \times L^1((0, T) \times \Gamma_{\pm}, \min(t, \tau_-(x,v))dt d\xi(x,v))$). In addition $\gamma_\pm$ admits a continuous lifting. Note that $L^1(X \times S^{n-1})$ is a subset of $L^1(X \times S^{n-1}, \tau_+(x,v)dx dv)$. Note also that $L^1((0, T) \times \Gamma_{\pm}, d\xi(x,v))$ (resp. $L^1((0, T) \times \Gamma_+, \min(t, \tau_-(x,v))dt d\xi(x,v)$) is a subset of $L^1((0, T) \times \Gamma_{\pm}, \min(T-t, \tau_+(x,v))dt d\xi(x,v))$ (resp. $L^1((0, T) \times \Gamma_+, \min(t, \tau_-(x,v))dt d\xi(x,v)$).

We now introduce the space

$$\hat{W} := \{ u \in W \mid \gamma_-(u) \in L^1(X \times S^{n-1}) \times L^1((0, T) \times \Gamma_{\pm}, dt d\xi) \}.$$

We recall the following trace results (owed to [7, 8] in a more general setting).
Lemma 2.1. The following equality is valid
\[ \tilde{W} = \{ u \in W \mid \gamma_+(u) \in L^1(X \times S^{n-1}) \times L^1((0, T) \times \Gamma_+, dt \eta) \}. \] (2.19)

In addition the trace maps
\[ \gamma_{\pm} : \tilde{W} \rightarrow L^1(X \times S^{n-1}) \times L^1((0, T) \times \Gamma_{\pm}, dt \eta) \] are continuous, onto, and admit continuous lifting. (2.20)

2.4 Solution to equation (2.2)

We identify the space \( L^1((0, r), L^1((0, r) \times \Gamma_{\pm}, dt \eta)) \) with the space \( L^1((0, r) \times \Gamma_{\pm}, dt \eta) \) for any \( r > 0 \). We extend by 0 on \( \mathbb{R} \) outside the interval \((0, \eta)\) any function \( \phi \in L^1((0, \eta), L^1(\Gamma_{\pm}, d\xi)) \).

Let \( \phi \in L^1((0, \eta), L^1(\Gamma_{\pm}, d\xi)) \). Then we consider the lifting \( G_{-}(t)\phi \in \tilde{W} \) of \((0, \phi)\) defined by
\[ G_{-}(t)\phi(x, v) := e^{-\int_{0}^{t} \sigma_{t} d\xi} \phi_{-}(t-t_{-}(x, v), x-t_{-}(x, v), v), \quad \text{for a.e.} \ (t, x, v) \in (0, T) \times X \times S^{n-1}. \] (2.21)

Note that \( G_{-}(.)\phi \) is a solution in the distributional sense of the equation \( (\frac{\partial}{\partial t} + v \cdot \nabla_x)u + \sigma u = 0 \) in \((0, T) \times X \times S^{n-1} \) and
\[ \|G_{-}(\cdot)\phi\|_{\tilde{W}} \leq (1 + \|\sigma\|_{\infty}) \|G_{-}(\cdot)\phi\|_{L^{1}(0, T) \times X \times S^{n-1}} \leq (1 + \|\sigma\|_{\infty})T \|G_{-}\|_{L^{1}(0, \eta) \times \Gamma_{\pm}, dt \eta}. \] (2.22)

To prove this two latter statements, one can use the following change of variables (see [10]).

Lemma 2.2. We have
\[ \int_{X \times S^{n-1}} f(x, v)dxdv = \int_{\Gamma_{\pm}} \int_{0}^{\tau_{\pm}(x, v)} f(x \pm tv)dtd\xi(x, v), \] (2.23)
for \( f \in L^1(X \times V) \).

From (2.22) we obtain that the map \( i : L^1((0, \eta), L^1(\Gamma_{-}, d\xi)) \rightarrow \tilde{W} \) defined by
\[ i(\phi) = G_{-}(\cdot)\phi, \ \phi \in L^1((0, \eta), L^1(\Gamma_{-}, d\xi)), \] (2.24)
is continuous.

The following result holds (see [11, Theorem 3 p. 229]).

Lemma 2.3. The equation (2.2) admits a unique solution \( u \) in \( \tilde{W} \) which is given by
\[ u(t) = G_{-}(t)\phi + \int_{0}^{t} U(t-s)A_{2}G_{-}(s)\phi ds, \] (2.25)
where \( U(t) \) is the strongly continuous semigroup in \( L^1(X \times S^{n-1}) \) introduced in subsection 2.2.

Using (2.25) and the Dyson-Phillips expansion (2.11) we obtain that the solution \( u \) of (2.2) may be decomposed as
\[ u(t) = G_{-}(t)\phi + \sum_{m=0}^{\infty} \int_{-\infty}^{t} H_{m}(t-s)A_{2}G_{-}(s)\phi ds, \] (2.26)
for \( t \geq 0 \) and \( \phi \in L^1((0, \eta) \times \partial X, dtd\mu(x)) \). The first term in the above series \( G_{-}(t)\phi \) is the ballistic part of \( u(t) \) while the term corresponding to \( m \geq 1 \) is \( m \)-linear in the scattering kernel \( k \). The term corresponding to \( m = 1 \) is the single scattering term.

From (2.24), Lemma 2.3 and (2.20), we also obtain the existence of the albedo operator.
Lemma 2.4. The albedo operator $A$ given by the formula

$$A\phi = u_{(0,T) \times \Gamma_+}, \text{ for } \phi \in L^1((0, \eta), L^1(\Gamma_-, d\xi)) \text{ where } u \text{ is given by (2.25)},$$

(2.27)

is well-defined and is a bounded operator from $L^1((0, \eta), L^1(\Gamma_-, d\xi))$ to $L^1((0, T), L^1(\Gamma_+, d\xi))$.

We refer the reader to [9] for the reconstruction of the optical parameters when the full albedo operator is known. We assume here that only partial knowledge of the albedo operator is available from measurements.

3 The operator $A_{S,W}$ and its distributional kernel

3.1 Angularly averaged measurements

We now define more precisely the type of measurements we consider in this paper. The directional behavior of the source term is determined by a fixed function $S(x, v)$, which is bounded and continuous on $\Gamma_-$. We assume that the incoming conditions have the following structure

$$\phi_S(t', x', v') = S(x', v') \phi(t', x'), \quad t' \in (0, \eta), \quad (x', v') \in \Gamma_-,$$

(3.1)

where $\phi(t, x)$ is an arbitrary function in $L^1((0, \eta) \times \partial X)$. We model the detectors by the kernel $W(x, v)$, which we assume is a continuous and bounded function on $\Gamma_+$. The available measurements are therefore modeled by the availability of the averaged albedo operator $A_{S,W}$ from $L^1((0, \eta) \times \partial X, dtd\mu(x))$ to $L^1((0, T) \times \partial X, dtd\mu(x))$ and defined by

$$A_{S,W}\phi(t, x) = \int_{\mathbb{S}^{n-1}} A(\phi_S)(t, x, v) W(x, v) (\nu(x) \cdot v) dv, \text{ for a.e. } (t, x) \in (0, T) \times \partial X.$$  

(3.2)

The functions $S$ and $W$ are fixed throughout the paper. The case $W \equiv 1$ corresponds to measurements of the current of exiting particles at the domain’s boundary.

The decomposition of the transport solution (2.26) translates into a similar decomposition of the albedo operator of the form

$$A_{S,W}\phi(t, x) = \sum_{m=0}^{+\infty} A_{m,S,W}\phi(t, x),$$

(3.3)

for $(t, x) \in (0, T) \times \partial X$, where we have defined

$$A_{0,S,W}\phi(t, x) = \int_{\mathbb{S}^{n-1}} (\nu(x) \cdot v) W(x, v) (G_- (. ) \phi_S)_{|(0,T) \times \Gamma_+} (t, x, v) dv,$$

(3.4)

$$A_{m,S,W}\phi(t, x) = \int_{\mathbb{S}^{n-1}} (\nu(x) \cdot v) W(x, v) \left( \int_{-\infty}^{t} H_{m-1}(t - s) A_2 G_- (s) \phi_S ds \right)_{|(0,T) \times \Gamma_+} (t, x, v) dv,$$

(3.5)

for a.e. $(t, x) \in (0, T) \times \partial X$ where $\phi_S$ is defined by (3.1). The kernels of the operators $A_{m,S,W}$ can be written explicitly.
3.2 Distributional kernel of the operators $A_{m, S, W}$

Consider the nonnegative measurable function $E$ from $\partial X \times \partial X \to \mathbb{R}$ defined by

$$E(x_1, x_2) = \begin{cases} e^{-\int_0^{|x_1 - x_2|} \sigma(x_1 - s, x_2 - s, x_2 - x_1)} ds & \text{if } x_1 + p(x_2 - x_1) \in X \text{ for all } p \in (0, 1), \\ 0 & \text{otherwise,} \end{cases}$$

for a.e. $(x_1, x_2) \in \partial X \times \partial X$. For $m \geq 3$, we also define the nonnegative measurable real function $E(x_1, \ldots, x_m)$ by the formula

$$E(x_1, \ldots, x_m) = e^{-\sum_{i=1}^{m-1} \int_0^{|x_i - x_{i+1}|} \sigma(x_i - s, x_{i+1} - s, x_{i+1} - x_i)} ds \Theta(x_m, x_{m-1}) \Pi_{i=1}^{m-2} \Theta(x_i, x_{i+1}),$$

for a.e. $(x_1, \ldots, x_m) \in \partial X \times (\mathbb{R}^n)^{m-2} \times \partial X$, where $\Theta$ is defined by $(2.10)$. The function $E(x_1, \ldots, x_m)$ measures the total attenuation along the broken path $(x_1, \ldots, x_m) \in \partial X \times (\mathbb{R}^n)^{m-2} \times \partial X$ provided $(x_2, \ldots, x_{m-1}) \in X^{m-2}$.

For $m \in \mathbb{N}$, $m \geq 1$ and for any subset $U$ of $\mathbb{R}^m$ we denote by $\chi_U$ the characteristic function from $\mathbb{R}^m$ to $\mathbb{R}$ defined by $\chi_U(y) = 1$ when $y \in U$ and $\chi_U(y) = 0$ otherwise. Using (3.4)–(3.5), (2.21) and (2.13)–(2.14) we then obtain the following result on the structure of the kernels of the albedo operator.

Proposition 3.1. We have

$$A_{m, S, W}(\phi)(t, x) = \int_{(0, \eta) \times \partial X} \gamma_m(t - t', x, x') \phi(t', x') dt' d\mu(x'),$$

for $m \geq 0$ and for a.e. $(t, x) \in (0, T) \times \partial X$, where

$$\gamma_0(\tau, x, x') := \frac{E(x, x')}{|x - x'|^{n-1}} [W(x, v)S(x', v)(\nu(x) \cdot v)[\nu(x') \cdot v]]_{v = \frac{x + x'}{|x - x'|}} \delta(\tau - |x - x'|),$$

$$\gamma_1(\tau, x, x') := \chi(0, +\infty)(\tau - |x' - x|) \int_{S^{n-1}} (\nu(x) \cdot v) W(x, v) [E(x, x - sv, x') k(x - sv, v', v) \times \chi(0, \tau_{(x, v)}(s)) S(x', v') \nu(x') \cdot v')]_{v' = \frac{x - x'}{|x - x'|}, s = \frac{x - x'}{|x - x'| - 1}} 2^{n-2} (\tau - |x - x'|) \cdot v^{n-3} dv,$$

for $(\tau, x, x') \in \mathbb{R} \times \partial X \times \partial X$ and where $\gamma_m$ for $m \geq 2$ admits a similar, more complex, expression given in Section 8 (see (8.12)–(8.13)).

Because the above formulas are central in our uniqueness and stability results, we briefly present their derivation and refer the reader to Section 11 for the rest of the proof of Proposition 3.1.

Derivation of (3.9) and (3.10). From (3.4) and the definition of $G_-$, we obtain $A_{0, S, W}(\phi)(t, x) = \int_{S^{n-1}} (\nu(x) \cdot v) W(x, v) E(x, x - \tau_-(x, v)) S(x - \tau_-(x, v)) v \phi(t - \tau_-(x, v), x - \tau_-(x, v)) dv$, $(t, x) \in (0, T) \times \partial X$ and for $\phi \in L^1((0, \eta) \times \partial X)$. Therefore, performing the change of variables “$x' = x - \tau(x, v)$ and $\tau(x, v) = |x - x'|”, we obtain (3.9).

From the definition of $A_2$ and $G_-$ we note that $A_2 G_-(s) \phi_S(z, w) := \int_{S^{n-1}} k(z, v', w) E(z, z - \tau_-(z, v')) S(z - \tau_-(z, v')) v' \phi(s - \tau_-(z, v'), z - \tau_-(z, v')) dv'$, for a.e. $(z, w) \in X \times \mathbb{S}^{n-1}$ and for $\phi \in L^1((0, \eta) \times \partial X)$. Performing the change of variables “$x = z - \tau_-(z, v') v'$”, we obtain the equality $A_2 G_-(s) \phi_S(z, w) = \int \partial_X [k(z, v', w) S(x', v') \nu(x') \cdot v']_{v' = \frac{x' - x}{|x' - x|}} E(z, x') \phi(s - |z - x'|, x') d\mu(x')$, for a.e. $(z, w) \in X \times \mathbb{S}^{n-1}$ and $\phi \in L^1((0, \eta) \times \partial X)$. Using also the definition of
A_{1,s,W} (see (3.5) for m = 1) we obtain the following equality for any \( \phi \in L^1((0, \eta) \times \partial X) \) and for a.e. \( (t, x) \in (0, T) \times \partial X \)

\[
A_{1,s,W}(\phi)(t, x) = \int_{S^{n-1}} \int_{-\infty}^{t} \int_{\partial X} [k((t-s)v, v)p(x, v)S(x, v)|\nu(x') \cdot v|]_{\nu = \frac{x - (t-s)v - v'}{(\nu \cdot v)'}} (\nu(x) \cdot v) \\
\times \frac{E(x, x - (t-s)v, v')}{|x - (t-s)v - v'|^{n-1}} \Theta(x, x - (t-s)v)\phi(s - |x - (t-s)v - x'|, x')W(x, v)\,d\mu(x')\,ds\,dv.
\]

(3.11)

Then performing the changes of variables “s” = t − s and “t” = t − s − |x − sv − x’| (s = \( \frac{(t-t')^2 - |x-x'|^2}{2(t-t') - v(x-x')} \)), we obtain (3.10).

To simplify notation, we define the multiple scattering kernels

\[
\Gamma_k = \sum_{m=k}^{\infty} \gamma_m.
\]

(3.12)

### 3.3 Regularity of the albedo kernels

The reconstruction of the optical parameters is based on an analysis of the behavior in time of the kernels of the albedo operator. Our first result in this direction is the following.

**Theorem 3.2.** Assume that \( k \in L^\infty(X \times S^{n-1} \times S^{n-1}) \). Then the following holds:

\[
\sqrt{\tau - |x - x'|^2} \gamma_1(\tau, x, x') \in L^\infty((0, T) \times \partial X \times \partial X) \quad \text{when } n = 2;
\]

(3.13)

\[
\frac{\tau}{\ln \left( \frac{x + |x-x'|}{\tau - |x-x'|} \right)} \gamma_1(\tau, x, x') \in L^\infty((0, T) \times \partial X \times \partial X) \quad \text{when } n = 3;
\]

(3.14)

\[
\tau|\tau - |x - x'||^{n-2} \gamma_1(\tau, x, x') \in L^\infty((0, T) \times \partial X \times \partial X) \quad \text{when } n \geq 4.
\]

(3.15)

In addition, assume that \( k \in L^\infty(X \times S^{n-1} \times S^{n-1}) \) and that there exists \( \delta > 0 \) such that \( \text{supp} k \subseteq \{ y \in X \mid \sup_{x \in \partial X} |x - y| \geq \delta \} \), i.e., the scattering coefficient vanishes in the vicinity of \( \partial X \). Then, the following holds

\[
(\tau - |x - x'|)^{\frac{3-n}{2}} \gamma_1(\tau, x, x') \in L^\infty((0, T) \times \partial X \times \partial X) \quad \text{when } n \geq 2.
\]

(3.16)

Theorem 3.2 is proved in Section 7. The results (3.14) and (3.15) of Theorem 3.2 correspond to singularities of the single scattering contribution that depend on the values of \( k \) on \( \partial X \). The above theorem shows that the structure of the single scattering coefficient is quite different depending on whether \( k \) vanishes on \( \partial X \) or not.

The following result describes some regularity properties of the multiple scattering. It is because multiple scattering is more regular than single scattering, in an appropriate sense, that we can reconstruct the scattering coefficient in a stable manner.

**Theorem 3.3.** Assume that \( k \in L^\infty(X \times S^{n-1} \times S^{n-1}) \). Then the following holds:

\[
\Gamma_2(\tau, x, x') \in L^\infty((0, T) \times \partial X \times \partial X), \quad \text{when } n = 2;
\]

(3.17)

\[
\frac{\tau|\tau - |x - x'||^2 \Gamma_2(\tau, x, x')}{(\tau - |x - x'|)^2 \left(1 + \ln \left( \frac{\tau + |x - x'|}{\tau - |x - x'|} \right) \right)^2} \in L^\infty((0, T) \times \partial X \times \partial X), \quad \text{when } n = 3;
\]

(3.18)

\[
\frac{\tau|\tau - |x - x'||^{n-2} \Gamma_2(\tau, x, x')}{\tau - |x - x'|} \in L^\infty((0, T) \times \partial X \times \partial X), \quad \text{when } n \geq 4.
\]

(3.19)
In addition, assume that $k \in L^\infty(\mathcal{X} \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ and that there exists $\delta > 0$ such that $\text{supp} k \subseteq \{ y \in X \mid \text{sup}_{x \in \partial X} |x - y| \geq \delta \}$. Then the following holds:

\begin{equation}
(\tau - |x - x'|)\left(1 + \ln \left(\frac{\tau + |x - x'|}{\tau - |x - x'|}\right)\right)^{-1} \Gamma_2(\tau, x, x') \in L^\infty((0, T) \times \partial X \times \partial \mathcal{X}), \quad \text{when } n = 3;
\end{equation}

\begin{equation}
(\tau - |x - x'|)^{\frac{1}{n-2}} \Gamma_2(\tau, x, x') \in L^\infty((0, T) \times \partial X \times \partial \mathcal{X}), \quad \text{when } n \geq 4.
\end{equation}

Theorem 3.3 is proved in Section 8. These results quantify how “smoother” multiple scattering is compared to the single scattering contribution considered in Theorem 3.2.

### 3.4 Small time asymptotics of the single scattering term

In this subsection we assume that $X$ is also convex. We give limits for the single scattering term in two configurations given by:

- the nonnegative function $\sigma$ is bounded and continuous on $X \times \mathbb{R}^{n-1}$,
- the nonnegative function $k$ is continuous on $\mathcal{X} \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$,

or

there exists a convex subset $Y \subseteq X$ with $C^1$ boundary such that $\sigma(x, v) = 0$ for $(x, v) \in (X \setminus Y) \times \mathbb{R}^{n-1}$ and the nonnegative function $\sigma$ is bounded and continuous on $Y \times \mathbb{R}^{n-1}$; and there exists a convex subset $Z \subseteq Y \subseteq X$ with $C^1$ boundary such that $\delta := \inf_{(x, z) \in \partial X \times Z} |x - z| > 0$ and $k(x, v') = 0$ for $(x, v', v) \in (X \setminus Z) \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$, and the nonnegative function $k$ is bounded and continuous on $Z \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$.

When either (3.22) or (3.23) is satisfied, we want to analyze the behavior of the function $\gamma_1(\tau, x, x')$ given by the right hand side of (3.10) for all $(\tau, x, x') \in \mathbb{R} \times \partial X \times \partial \mathcal{X}$. We need to introduce some notation. Let $\vartheta_0 : \mathbb{R}^{n-1} \times X \to \mathbb{R}$ be the function defined by

\begin{equation}
\vartheta_0(v, x) = (\tau_-(x, v)\tau_+(x, v))^{-\frac{n-1}{2}}, \quad (v, x) \in \mathbb{R}^{n-1} \times X,
\end{equation}

and consider the weighted X-ray transform $P_{\vartheta_0}$ defined by

\begin{equation}
P_{\vartheta_0}f(v, x) = \int_{\tau_-(x, v)}^{\tau_+(x, v)} \vartheta_0(v, tv + x)f(tv + x, v)dt,
\end{equation}

for a.e. $(v, x) \in \mathbb{R}^{n-1} \times \partial X$ and $f \in L^2(X \times \mathbb{R}^{n-1}, \vartheta_0(v, x)dx dv)$. The first result analyzes the behavior of $\gamma_1$ under hypothesis (3.22).

**Theorem 3.4.** Assume that the open subset $X$ of $\mathbb{R}^n$ with $C^1$ boundary is convex. Let $(x, x'_0) \in \partial X^2$ be such that $x + s(x - x'_0) \in X$ for some $s \in (0, 1)$. Put $v_0 = \frac{x - x'_0}{|x - x'_0|}$ and $t_0 = |x - x'_0|$. Then under condition (3.22), we have the following results. When $n = 2$, then

\begin{equation}
\gamma_1(\tau, x, x'_0) = \frac{1}{\sqrt{\tau - t_0}} \sqrt{2}W(x, v_0)S(x'_0, v_0)(\nu(x) \cdot v_0)|\nu(x'_0) \cdot v_0|E(x, x'_0)
\end{equation}

\begin{equation}
\times P_{\vartheta_0}k_{v_0}(v_0, x) + o\left(\frac{1}{\sqrt{\tau - t_0}}\right), \quad \text{as } \tau \to t_0^+.
\end{equation}
where $P_{\partial_0}$ is defined by (3.25) and $k_v(y, v_0) := k(y, v, v_0)$ for $(y, v) \in X \times \mathbb{S}^{n-1}$.

When $n = 3$, then

$$
\gamma_1(\tau, x, x_0') = \ln\left(\frac{1}{\tau-t_0}\right)\frac{\pi}{t_0^2} W(x, v_0)S(x_0', v_0)(\nu(x) \cdot v_0)\nu(x_0') \cdot v_0 E(x, x_0')
$$

$$
\times (k(x, v_0, v_0) + k(x_0', v_0, v_0)) + o\left(\ln\left(\frac{1}{\tau-t_0}\right)\right), \text{ as } \tau \to t_0^+.
$$

(3.27)

When $n \geq 4$, then

$$
\gamma_1(\tau, x, x_0') = t_0^{1-n}E(x, x_0') \left[ S(x_0', v_0)\nu(x_0') \cdot v_0 \int_{\mathbb{S}^{n-1}_\nu} \frac{W(x, v)(\nu(x) \cdot v)k(x, v_0, v)}{1-v \cdot v_0} dv \right]
$$

$$+ W(x, v_0)(\nu(x) \cdot v_0) \int_{\mathbb{S}^{n-1}_\nu} \frac{k(x_0', v', v_0)S(x_0', v')(\nu(x_0') \cdot v')}{1-v' \cdot v_0} dv' + o(1), \text{ as } \tau \to t_0^+.
$$

(3.28)

Theorem 3.4 is proved in Section 5. Note that $\gamma_1$ depends on the value of $k$ on $\partial X$ in dimension $n \geq 3$. Under hypothesis (3.23), i.e., when the optical coefficients vanish in the vicinity of where measurements are collected, we have the quite different behavior:

**Theorem 3.5.** Assume that the open subset $X$ of $\mathbb{R}^n$ with $C^1$ boundary is also convex and assume that condition (3.23) is fulfilled. Let $(x, x_0') \in \partial X^2$ be such that $x_0' + s(x - x_0') \in Z$ for some $s \in (0, 1)$. Put $v_0 = \frac{s - x_0'}{|x - x_0'|}$ and $t_0 = |x - x_0'|$. Then we have the following.

When $n = 2$, then (3.26) still holds.

When $n \geq 3$, then

$$
\gamma_1(\tau, x, x_0') = (\tau-t_0)^{\frac{n-3}{2}}(2t_0)^{\frac{1-n}{2}}\text{Vol}_{n-2}(\mathbb{S}^{n-2})S(x_0', v_0)W(x, v_0)(\nu(x_0') \cdot v_0)(\nu(x) \cdot v_0)
$$

$$
\times E(x, x_0')P_{\partial_0}k_v(x, v_0) + o((\tau-t_0)^{\frac{n-3}{2}}), \text{ as } \tau \to t_0^+,
$$

(3.29)

where $P_{\partial_0}$ is defined by (3.25) and $k_v(y, v) := k(y, v, v_0)$ for $(y, v) \in X \times \mathbb{S}^{n-1}$.

Theorem 3.5 is proved in Section 5. Theorem 3.5 may remain valid under different conditions from those stated in (3.23). For instance, when $\sigma$ is bounded and continuous on $X$ and $k$ is continuous on $X \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ and $k(x, .., )$ decays sufficiently rapidly as $x$ get closer and closer to the boundary $\partial X$ for any $x \in X$, then the same asymptotics of $\gamma_1$ holds.

### 4 Uniqueness and stability results

We denote by $\gamma := \Gamma_0 = \sum_{m=0}^{+\infty} \gamma_m$ the distributional kernel of $A_{\mathbb{S}^1 \mathbb{W}}$. Then $\gamma - \gamma_0 = \Gamma_1$ denotes the distributional kernel of the multiple scattering of $A_{\mathbb{S}^1 \mathbb{W}}$. For the rest of the paper, we assume that the duration of measurement $T > \text{diam}(X) := \sup_{(x, y) \in X^2} |x - y|$ so that the singularities of the ballistic and single scattering contributions are indeed captured by the available measurements.

Let $(\tilde{\sigma}, \tilde{k})$ be a pair of absorption and scattering coefficients that also satisfy (2.1). We denote by a superscript any object (such as the albedo operator $\tilde{A}$ or the distributional kernels $\tilde{\gamma}$ and $\tilde{\gamma}_0$) associated to $(\tilde{\sigma}, \tilde{k})$. Moreover if $(\sigma, k)$ satisfies (3.23) for some $(Y, Z)$ and $(\tilde{\sigma}, \tilde{k})$ also satisfies (3.23) for some $(\tilde{Y}, \tilde{Z})$, then we always make the additional assumption $Y = \tilde{Y}$ and $Z = \tilde{Z}$. Let $\|\cdot\|_{\eta,T} := \|\cdot\|_{\text{L}^1((0,\eta)\times \partial X), \text{L}^1((0,T)\times \partial X))}$. 


4.1 Stability estimates under condition (3.22) or (3.23)

Theorem 4.1. Assume that the open subset $X$ of $\mathbb{R}^n$ with $C^1$ boundary is also convex. Let $(\sigma, k)$ and $(\tilde{\sigma}, \tilde{k})$ satisfy condition (3.22). Let $x_0' \in \partial X$. Then we have:

$$
\int_{\partial X} \frac{|E - \tilde{E}|(x, x_0')}{|x - x_0'|^{n-1}} \left|W(x, v_0)S(x', v_0)(\nu(x) \cdot v_0)|\nu(x_0') \cdot v_0\right| dv_0(x) \leq \|A_{S,W} - \tilde{A}_{S,W}\|_{\eta,T}.
$$

(4.1)

Let $x \in \partial X$ be such that $px_0' + (1-p)x \in X$ for some $p \in (0, 1)$. Put $v_0 = \frac{x - x_0'}{|x - x_0'|}$ and $t_0 = |x - x_0'|$.

When $n = 2$, we have

$$
W(x, v_0)S(x_0', v_0)(\nu(x) \cdot v_0)|\nu(x_0') \cdot v_0| E(x, x_0')P_{\theta_0}k_{v_0}(v_0, x) - \tilde{E}(x, x_0')P_{\theta_0}\tilde{k}_{v_0}(v_0, x)
\leq \frac{1}{2} \left\| \sqrt{\tau^2 - |z - z'|^2(\Gamma_1 - \tilde{\Gamma}_1)}(\tau, z, z') \right\|_{L^\infty}.
$$

(4.2)

where $\| \|_{L^\infty} := \| \|_{L^\infty([0, T] \times \partial X \times \partial X)}$, $P_{\theta_0}$ is defined by (3.25) and $k_{v_0}(y, v) := k(y, v, v_0)$ for $(y, v) \in X \times S^{n-1}$ ($\tilde{k}_{v_0}$ is defined similarly).

When $n = 3$, then

$$
E(x, x_0')(k(x, v_0, v_0) + k(x_0', v_0, v_0)) - \tilde{E}(x, x_0')(\tilde{k}(x, v_0, v_0) + \tilde{k}(x_0', v_0, v_0))
\times W(x, v_0)S(x_0', v_0)(\nu(x) \cdot v_0)|\nu(x_0') \cdot v_0| \leq \frac{1}{\pi} \left\| \frac{\tau |z - z'|}{\ln \left(\frac{\tau + |z - z'|}{\tau - |z - z'|}\right)}(\Gamma_1 - \tilde{\Gamma}_1)(\tau, z, z') \right\|_{L^\infty}.
$$

(4.3)

When $n \geq 4$, then

$$
S(x_0', v_0)|\nu(x_0') \cdot v_0| \int_{S^{n-1}_+} W(x, v)(\nu(x) \cdot v) \left|E(x, x_0')k(x, v_0, v) - \tilde{E}(x, x_0')\tilde{k}(x, v_0, v)\right| dv
+W(x, v_0)(\nu(x) \cdot v) \int_{S^{n-1}_0} S(x_0', v')(\nu(x_0') \cdot v') \left|E(x, x_0')k(x_0', v', v_0) - \tilde{E}(x, x_0')\tilde{k}(x_0', v', v_0)\right| dv'
\leq \left\| |z - z'|^{n-2}(\Gamma_1 - \tilde{\Gamma}_1)(\tau, z, z') \right\|_{L^\infty}.
$$

(4.4)

Theorem 4.1 is proved in Section 7. It shows that the spatial structure of $k$ may be stably reconstructed at the domain’s boundary. More interesting is the following theorem, which provides some stability of the reconstruction of the scattering coefficient when it vanishes in the vicinity of the boundary $\partial X$.

Theorem 4.2. Assume that the open subset $X$ of $\mathbb{R}^n$ with $C^1$ boundary is also convex. Assume also that $\inf_{(x', v') \in \Gamma_-} S(x', v') > 0$ and $\inf_{(x, v) \in \Gamma_+} W(x, v) > 0$. Let $(\sigma, k)$ and $(\tilde{\sigma}, \tilde{k})$ satisfy condition (3.23). Let $x_0' \in \partial X$. Then there exist constants $C_1 = C_1(S, W, X, Y)$ and $C_2 = C_2(S, W, X, Z)$ such that

$$
\int_{S^{n-1}_0} |E - \tilde{E}|(x_0' + \tau(x_0', v_0, v_0, x_0'))|\nu(x_0') \cdot v_0|dv_0 \leq C_1\|A_{S,W} - \tilde{A}_{S,W}\|_{\eta,T},
$$

(4.5)

$$
E(x, x_0')P_{\theta_0}k_{v_0}(v_0, x_0') - \tilde{E}(x, x_0')P_{\theta_0}\tilde{k}_{v_0}(v_0, x_0') \leq C_2\left\| (\tau - |z - z'|)^{\frac{n}{2}}(\Gamma_1 - \tilde{\Gamma}_1)(\tau, z, z') \right\|_{L^\infty}.
$$

(4.6)
for \( x \in \partial X \) such that \( px' + (1 - p)x \in Z \) for some \( p \in (0, 1) \) where \( v'_0 = \frac{x - x'}{|x - x'|} \), \( P_{\theta_0} \) is defined in (3.25), and \( k_{v'_0}(y, v) := k(y, v, v'_0) \) for \((y, v) \in X \times S^{n-1} \) (\( k_{v'_0} \) is defined similarly).

Theorem 4.2, which is one of the main results of this paper, is proved in Section 8.

4.2 The case when \( X \) is a ball of \( \mathbb{R}^n \)

When \( X \) is an open Euclidean ball of \( \mathbb{R}^n \), which is important from the practical point of view in medical imaging as it is relatively straightforward to place sources and detectors on a sphere, we are able to invert the weighted X-ray transform \( P_{\theta_0}f, f(x, v) := f(x) \in L^2(X, \sup_{v \in S^{n-1}} \theta_0(v, x)dx) \) using the classical inverse X-ray transform (inverse Radon transform in dimension \( n = 2 \)). In the next subsection we shall consider a larger class of domains \( X \), which requires one to solve more complex weighted X-ray transforms.

Up to rescaling, we assume \( X = B_n(0, 1) \), the ball in \( \mathbb{R}^n \) centered at 0 of radius 1. Consider the X-ray transform \( P \) defined by

\[
Pf(v, x) = \int_{\tau_+(x,v)}^{\tau_-(x,v)} f(sv + x)ds \text{ for a.e. } (v, x) \in S^{n-1} \times \partial X,
\]

for \( f \in L^2(X) \) (we extend \( f \) by 0 outside \( X \)). We have the following Proposition 4.3.

**Proposition 4.3.** When \( X = B_n(0, 1) \) we have

\[
P_{\theta_0}f(v, x) = P(gf)(v, x), \text{ for a.e. } (v, x) \in S^{n-1} \times \partial X,
\]

for \( f \in L^2(X, \sup_{v \in S^{n-1}} \theta_0(v, x)dx) \) where \( g(y) := (1 - |y|^2)^{-\frac{n-1}{2}}, y \in X \).

**Proof of Proposition 4.3.** It is easy to see that

\[
\tau_{\pm}(tv + qv^\perp, v) = \sqrt{1 - q^2} \mp t,
\]

\[
\theta_0(v, x) = (1 - q^2 - t^2)^{-\frac{n-1}{2}} = (1 - |x|^2)^{-\frac{n-1}{2}},
\]

for \( (t, q) \in \mathbb{R}^2, t^2 + q^2 \leq 1 \) and for \( (v, v^\perp) \in S^{n-1} \times S^{n-1}, v \cdot v^\perp = 0, \) where \( x = tv + qv^\perp \) (we remind that \( \theta_0 \) is defined by (3.24)). Then Proposition 4.3 follows from the definition (3.25). \( \square \)

Assume that \((\sigma, k)\) satisfies condition (3.22) when \( n = 2 \) or (3.23) when \( n \geq 2 \). Assume also that \( k(x, v, v') = k_0(x)g(v, v') \) for a.e. \((x, v, v') \in X \times S^{n-1} \times S^{n-1} \) where \( g \) is a given continuous function on \( S^{n-1} \times S^{n-1} \), \inf_{v \in S^{n-1}} g(v, v) > 0, \) and where \( k_0 \in L^\infty(X) \). Then from the decomposition of the angularly averaged albedo operator \( A_{SW} \) (Proposition 3.1) and from Theorems 3.2, 3.3, 3.4 and 3.5, and from Proposition 4.3 and methods of reconstruction of a function from its X-ray transform, it follows that \((\sigma, k_0)\) can be reconstructed from the small time-asymptotics of \( A_{SW} \) provided that \( \sigma = \sigma(x) \) and \( \inf_{(x, v') \in I_-} S(x, v') > 0 \) and \( \inf_{(x, v) \in I_+} W(x, v) > 0 \). In addition we have the following stability estimates.

**Theorem 4.4.** Assume \( X = B_n(0, 1) \) and \( \inf_{(x, v') \in I_-} S(x, v') > 0 \) and \( \inf_{(x, v) \in I_+} W(x, v) > 0 \). Let \((\sigma, k)\) and \((\bar{\sigma}, \bar{k})\) satisfy either condition (3.23) or (3.22). Assume that \( \sigma, \bar{\sigma} \) do not depend on the velocity variable \((\sigma(x, v) = \sigma(x))\) and \( \supp \sigma \cup \supp \bar{\sigma} \subseteq Y, \) where \( Y \subset X \) is convex with \( C^1 \) boundary, and let \( M = \max(\|\sigma\|_{L^\infty(Y)}, \|\bar{\sigma}\|_{L^\infty(Y)}) \). Assume \( k(x, v, v') = k_0(x)g(v, v') \) and \( \bar{k}(x, v, v') = \bar{k}_0(x)g(v, v') \), \( g(v, v') > 0, \) for \((x, v, v') \in X \times S^{n-1} \times S^{n-1} \) where \( g \) is an a priori known continuous function on \( S^{n-1} \times S^{n-1} \).
Then there exists $C_3 = C_3(S,W,X,Y,M)$ such that

$$
\|\sigma - \tilde{\sigma}\|_{H^{-\frac{1}{2}}(Y)} \leq C_3\|\sigma - \tilde{\sigma}\|_{L^\infty(Y)}^{\frac{3}{2}}\|A_{S,W} - \tilde{A}_{S,W}\|_{\eta,T}^{\frac{1}{2}}.
$$

(4.11)

When $n \geq 2$ and $(\sigma, k)$ satisfies (3.23), there exists $C_{4,1} = C_{4,1}(S,W,X,Y,Z,M,g)$ such that

$$
\|\varrho(k_0 - \tilde{k}_0)\|_{H^{-\frac{1}{2}}(Z)} \leq C_{4,1}\|k_0 - \tilde{k}_0\|_{L^\infty(Z)}^{\frac{3}{2}}\|A_{S,W} - \tilde{A}_{S,W}\|_{\eta,T}^{\frac{1}{2}}
$$

(4.12)

$$
+ \left\| \left( \tau - |z - z'| \right)^{\frac{3-n}{2}}(\Gamma_1 - \tilde{\Gamma}_1)(\tau, z, z') \right\|_{L^\infty}^{\frac{1}{2}}.
$$

When $n = 2$ and $(\sigma, k)$ satisfies (3.22), there exists $C_{4,2} = C_{4,2}(S,W,X,M,g)$ such that

$$
\|\varrho(k_0 - \tilde{k}_0)\|_{H^{-\frac{1}{2}}(X)} \leq C_{4,2}\|k_0 - \tilde{k}_0\|_{L^\infty(Z)}^{\frac{3}{2}}\|A_{S,W} - \tilde{A}_{S,W}\|_{\eta,T}^{\frac{1}{2}}
$$

(4.13)

$$
+ \left\| \sqrt{\tau^2 - |z - z'|^2}(\Gamma_1 - \tilde{\Gamma}_1)(\tau, z, z') \right\|_{L^\infty}^{\frac{1}{2}}.
$$

Theorem 4.4 can be proved by mimicking the proof of Theorem 4.5 given below for a larger class of domains $X$. However we give a proof of estimate (4.13) in Section 6.

Note that the left-hand side $\|\varrho(k_0 - \tilde{k}_0)\|_{H^{-\frac{1}{2}}(Z)}$ of (4.12) cannot be replaced by $\|k_0 - \tilde{k}_0\|_{H^{-\frac{1}{2}}(Z)}$ since $\varrho^{-1} \in C^\infty(\bar{Z})$ and the operator $f \rightarrow \varrho^{-1}f$ is bounded in $H^{-\frac{1}{2}}(Z)$ for any open convex subset $Z$ (with $C^1$ boundary) of $X$ which satisfies $\bar{Z} \subseteq X$.

Under the assumptions of Theorem 4.4 and additional regularity assumptions on $(\sigma, k)$ one obtains stability estimates similar to those given in Corollary 4.6 given below for a larger class of domains $X$.

### 4.3 Uniqueness and stability estimates for more general domains $X$

**Theorem 4.5.** Assume that the open subset $X$ of $\mathbb{R}^n$ is convex with a real analytic boundary and that $\inf_{(x',v') \in \Gamma_1} S(x',v') > 0$ and $\inf_{(x,v) \in \Gamma_+} W(x,v) > 0$. Let $(\sigma, k)$ and $(\tilde{\sigma}, \tilde{k})$ satisfy condition (3.23). Assume also that $\sigma, \tilde{\sigma}$ do not depend on the velocity variable $(\sigma(x, v) = \sigma(x))$ and $k(x,v,v') = k_0(x)g(x,v,v')$ and $\tilde{k}(x,v,v') = \tilde{k}_0(x)\tilde{g}(x,v,v')$, $g(x,v,v') > 0$, for $(x,v,v') \in X \times S^{n-1} \times S^{n-1}$ where $g$ is an a priori known real analytic function on $X \times S^{n-1} \times S^{n-1}$ and where $\text{supp} k_0 \cup \text{supp} \tilde{k}_0 \subseteq \bar{Z}$, $(k_0, \tilde{k}_0) \in L^\infty(Z)$. Then estimate (4.11) still holds and there exists $C_4 = C_4(S,W,X,Y,Z,M,g)$ such that

$$
\|k_0 - \tilde{k}_0\|_{H^{-\frac{1}{2}}(Z)} \leq C_4\|k - \tilde{k}\|_{L^\infty(Y)}^{\frac{1}{2}}\|A_{S,W} - \tilde{A}_{S,W}\|_{\eta,T}^{\frac{1}{2}}
$$

(4.14)

$$
+ \left\| \left( \tau - |z - z'| \right)^{\frac{3-n}{2}}(\Gamma_1 - \tilde{\Gamma}_1)(\tau, z, z') \right\|_{L^\infty}^{\frac{1}{2}},
$$

where $M = \max(\|\sigma\|_{L^\infty(Y)}, \|\tilde{\sigma}\|_{L^\infty(Y)})$.

Theorem 4.5 is proved in Section 6. Assume that $X$ is convex with a real analytic boundary and that $\inf_{(x',v') \in \Gamma_1} S(x',v') > 0$ and $\inf_{(x,v) \in \Gamma_+} W(x,v) > 0$. Let $Y$ and $Z$ be open convex subsets of $X$, $Z \subseteq X$, $Z \subseteq Y \subseteq X$, with a $C^1$ boundary. Let $g$ be an a priori known real analytic function on $X \times S^{n-1} \times S^{n-1}$, $g(x,v,v') > 0$ for $(x,v,v') \in X \times S^{n-1} \times S^{n-1}$. Let $r_1 > 0$, $r_2 > 0$. Consider the class

$$
N := \left\{ (\sigma, k) \in H^{\frac{3}{2}+r_1}(Y) \times L^\infty(Z \times S^{n-1} \times S^{n-1}) \mid \|\sigma\|_{H^{\frac{3}{2}+r_1}(Y)} \leq M_1, \quad k = k_0g, \quad \text{supp} k_0 \subseteq \bar{Z}, \quad \|k_0\|_{H^{\frac{3}{2}+r_2}(Z)} \leq M_2 \right\}.
$$

(4.15)
Note that there exist a function \(D_1: \mathbb{N} \times (0, +\infty) \to (0, +\infty)\) such that

\[
\|\sigma\|_{L^\infty(Y)} \leq D_1(n, r_1)\|\sigma\|_{H^{\frac{1}{2} + r_1}(Y)} \leq D_1(n, r_1)M_1,
\]
\[
\|k_0\|_{L^\infty(Z)} \leq D_1(n, r_2)\|k_0\|_{H^{\frac{1}{2} + r_2}(Z)} \leq D_1(n, r_2)M_2,
\]
\[
\|k\|_{L^\infty(Z)} \leq \|g\|_{L^\infty(Z)}\|k_0\|_{L^\infty(Z)} \leq D_1(n, r_2)\|g\|_{L^\infty(Z)},
\]
for \((\sigma, k) \in N\). We also use the interpolation formula

\[
\|f\|_{H^{\sigma}(O)} \leq \|f\|_{H^{\sigma}(O)}^{\frac{\sigma}{\sigma - \frac{3}{2} - s}}\|f\|_{H^{\rho}(O)}^{\frac{\rho - s}{\sigma - \frac{3}{2} - s}},
\]
for \(s_1 < s < s_2\) and for \((O, s_1, s_2) \in \{(Y, -\frac{1}{2}, \frac{n}{2} + r_1), (Z, -\frac{1}{2}, \frac{n}{2} + r_2)\}\). Using Theorem 4.5 and (4.16), and applying (4.17) on \(f = \sigma - \tilde{\sigma}\) and \(f = k_0 - \tilde{k}_0\) we obtain the following result.

**Corollary 4.6.** Let \((\sigma, k), (\tilde{\sigma}, \tilde{k}) \in N\). Then, for \(-\frac{1}{2} \leq s \leq \frac{n}{2} + r_1\) and for \(0 < r < r_1\), there exists \(C_5 = C_5(S, W, X, Y, M_1, r_1, s)\) such that

\[
\|\sigma - \tilde{\sigma}\|_{H^{\sigma}(Y)} \leq C_5\|\sigma - \tilde{\sigma}\|_{L^\infty(Y)}\|A_{S,W} - \tilde{A}_{S,W}\|_{H^{\sigma}(Y)},
\]
\[
\|\sigma - \tilde{\sigma}\|_{H^{\frac{1}{2} + \rho}(Y)} \leq C_6\|A_{S,W} - \tilde{A}_{S,W}\|_{H^{\rho}(Y)},
\]
where \((\kappa, \kappa') = \left(\frac{n + 2(r_1 - s)}{n + 1 + 2r_1}, \frac{2(r_1 - r)}{n + 1 + 2r_1}\right)\) and \(C_6 = C_5^{\frac{2}{\kappa - \kappa'}}D_1(n, r)^{\frac{\kappa'}{\kappa - \kappa'}}\) (\(D_1(n, r)\) is defined by (4.16)). In addition, there exists \(C_7 = C_7(S, W, X, Y, Z, g, M_1, r_1, M_2, r_2, s)\) such that

\[
\|k_0 - \tilde{k}_0\|_{H^{\sigma}(Z)} \leq C_7\|k_0 - \tilde{k}_0\|_{L^\infty(Z)}\left(\|A_{S,W} - \tilde{A}_{S,W}\|_{H^{\sigma}(Y)} + \left\|\left(\tau - |z - z'|\right)^{\frac{3 - \kappa}{\kappa}}(\Gamma_1 - \tilde{\Gamma})z\right\|_{L^\infty}\right)^{\frac{1}{\kappa' - \kappa}},
\]
\[
\|k_0 - \tilde{k}_0\|_{H^{\frac{1}{2} + \rho}(Z)} \leq C_8\left(\|A_{S,W} - \tilde{A}_{S,W}\|_{H^{\rho}(Y)} + \left\|\left(\tau - |z - z'|\right)^{\frac{3 - \kappa}{\kappa}}(\Gamma_1 - \tilde{\Gamma})z\right\|_{L^\infty}\right)^{\frac{1}{\kappa' - \kappa}},
\]
for \(-\frac{1}{2} \leq s \leq \frac{n}{2} + r_2\) and \(0 < r < r_2\), where \((\kappa, \kappa') = \left(\frac{n + 2(r_2 - s)}{n + 1 + 2r_2}, \frac{2(r_2 - r)}{n + 1 + 2r_2}\right)\) and \(C_8 = C_7^{\frac{2}{\kappa' - \kappa'}}D_1(n, r)^{\frac{\kappa'}{\kappa' - \kappa'}}\) (\(D_1(n, r)\) is defined by (4.16)).

**Remark 4.7.** (i) Theorem 4.5 and Corollary 4.6 remain valid when: \(X\) is only assumed to be convex with \(C^2\) boundary; the weight \(\theta_o\) defined by (3.24) (resp. the function \(g\) which appears in the assumptions of Theorem 4.5 and Corollary 4.6) is sufficiently close (in the \(C^2\) norm) to an analytic weight \(\theta_{o,a}\) on the vicinity of \(\tilde{Z} \times S^{n - 1}\) (resp. an analytic function \(g_a\) on the vicinity of \(\tilde{Z} \times S^{n - 1} \times S^{n - 1}\)); see proof of Theorem 4.5 and [12, Theorem 2.3].

(ii) When \(n = 3\) then under hypothesis (3.23), we have

\[
\left\|\left(\tau - |z - z'|\right)^{\frac{3 - \kappa}{\kappa}}(\Gamma_1 - \tilde{\Gamma})z\right\|_{L^\infty} = \left\|\sum_{m=1}^{+\infty}(A_{m,S,W} - \tilde{A}_{m,S,W})\right\|_{L^1(L^1((0, \eta) \times \partial X), L^\infty([0, T] \times \partial X))}.
\]
where the distributional kernel of the bounded operator \(\sum_{m=1}^{+\infty}(A_{m,S,W} - \tilde{A}_{m,S,W})\) from \(L^1((0, \eta) \times \partial X)\) to \(L^1((0, T) \times \partial X)\) is given by \(\Gamma_1 - \tilde{\Gamma}_1\). Therefore when \(n = 3\) and under condition (3.23), the right-hand side of the stability estimates (4.14) and (4.20) can be expressed with operator norms only (instead of using a norm on the distributional kernel of the multiple scattering).
5 Proof of Theorems 3.4, 3.5

Proof of Theorem 3.5. For the sake of simplicity and without loss of generality we assume 
$v_0 = (1, 0, \ldots, 0)$. Assume that condition (3.23) is satisfied. For $n \geq 2$ consider the following open subset of 
$(0, +\infty) \times S^{n-1} \times S^{n-1}$
\begin{equation}
\mathcal{D} := \{(s, v, v') \in (0, +\infty) \times S^{n-1}_{x,+} \times S^{n-1}_{x,0,-} \mid s \in (0, \tau_-(x, v))\}.
\end{equation}
(5.1)

Then we introduce the bounded function $\Psi_n$ on $\mathcal{D}$ defined by
\begin{equation}
\Psi_n(s, v, v') = 2^{n-2}W(x, v)(\nu(x) \cdot v)E(x, x - sv, x_0')k(x - sv, v')S(x_0, v')|\nu(x_0') \cdot v'|,
\end{equation}
(5.2)
for $(s, v, v') \in \mathcal{D}$. Note that from convexity of $X$ it follows that $\tau_\pm$ is continuous on $\Gamma_\mp$ and $E(x, x - sv, x_0') = e^{-\int_0^x \sigma(x - p, v') dp - \int_0^{s-x_0'} \sigma(x - sv - p) \frac{x-x_0'}{v'} dp}$ for $v \in S^{n-1}_{x,+}$ and $0 < s < \tau_-(x, v)$. Under (5.23) we obtain that
\begin{equation}
\Psi_n(s, v, v') = 0 \text{ for } (s, v, v') \in (0, +\infty) \times S^{n-1}_{x,+} \times S^{n-1}_{x,0,-} \text{ such that } x - sv \notin \bar{Z},
\end{equation}
and the function $\Psi_n$ is continuous at any point $(s, v, v') \in \mathcal{D}$ such that $x - sv \in Z$.

We first prove (3.26) for $n = 2$. Let $\tau > t_0$. From (5.2), (3.10), it follows that
\begin{equation}
\gamma_1(\tau, x, x_0') = \int_{-\alpha_0}^{\pi - \alpha_0} \frac{1}{\tau - t_0 \cos(\Omega)} \left[ \chi(0, \tau_-(x, v)) (s) \Psi_2(s, v, v') \right]_{v = (\cos \Omega, \sin \Omega)}^{v' = (\Omega, 0)} d\Omega
= \gamma_{1,1}(\tau, x, x_0') + \gamma_{1,2}(\tau, x, x_0'),
\end{equation}
(5.4)
where $S^{n-1}_{x,+} = \{(\cos(\Omega), \sin(\Omega)) \mid -\alpha_0 < \Omega < \pi - \alpha_0\}$ $\{0 < \alpha_0 < \pi\}$ and
\begin{equation}
\gamma_{1,1}(\tau, x, x_0') = \int_0^\tau \frac{\chi(0, \pi - \alpha_0)(\Omega)}{\tau - t_0 \cos(\Omega)} \left[ \chi(0, \tau_-(x, v)) (s) \Psi_2(s, v, v') \right]_{v = (\cos \Omega, \sin \Omega)}^{v' = (\Omega, 0)} d\Omega
= \frac{\tau - t_0}{2(\tau - t_0 \cos(\Omega))},
\end{equation}
(5.5)
\begin{equation}
\gamma_{1,2}(\tau, x, x_0') = \int_0^\tau \frac{\chi(0, \pi - \alpha_0)(\Omega)}{\tau - t_0 \cos(\Omega)} \left[ \chi(0, \tau_-(x, v)) (s) \Psi_2(s, v, v') \right]_{v = (\cos \Omega, \sin \Omega)}^{v' = (\Omega, 0)} d\Omega.
\end{equation}
(5.6)

We shall prove that
\begin{equation}
\sqrt{\tau - t_0} \gamma_{1,i}(\tau, x, x_0') \to \frac{W(x, v_0)S(x_0, v_0)(\nu(x) \cdot v_0)|\nu(x_0') \cdot v_0|E(x_0', x)}{\sqrt{2\tau}} \int_0^t k(x - sv, v_0, v_0) \int_0^s \frac{d}s(t_0 - s) ds,
\end{equation}
(5.7)
as $\tau \to t_0^+$,

for $i = 1, 2$. Then adding (5.7) for $i = 1$ and $i = 2$, we obtain (3.26). We only prove (5.7) for $i = 1$ since the proof for $i = 2$ is similar. Let $\tau > t_0$. Using the change of variables $s = \frac{\tau - t_0}{2(\tau - t_0 \cos(\Omega))} - \frac{\tau - t_0}{2}$, we obtain
\begin{equation}
\gamma_{1,1}(\tau, x, x_0') = \frac{1}{\sqrt{\tau - t_0^2}} \int_0^t \chi(0, \pi - \alpha_0)(\Omega(s, \tau)) \chi(0, \tau_-(x, v(s, \tau))) (s + \frac{\tau - t_0}{2}) \Psi_2(s, v(s, \tau), v'(s, \tau)) d\tau.
\end{equation}
(5.8)
where
\[ v(s, \tau) = (\cos \Omega(s, \tau), \sin \Omega(s, \tau)), \quad \Omega(s, \tau) = \arccos \left( \frac{\tau - t_0^2}{2\tau + t_0} \right), \]
\[ v'(s, \tau) = \frac{t_0(1, 0) - (s + \frac{\tau - t_0}{2}) v(s, \tau)}{\frac{\tau + t_0}{2} - s}. \tag{5.9} \]

Let \( s \in (0, t_0) \). From (5.9), it follows that
\[ v(s, \tau) \to (1, 0) \quad \text{as} \quad \tau \to t_0^+, \quad v'(s, \tau) \to (1, 0) \quad \text{as} \quad \tau \to t_0^+. \tag{5.10} \]

Note that using the definition of \( t_0 \) and using the assumption \( x_0' + \varepsilon(x - x_0') \in X \) for some \( \varepsilon \in (0, 1) \) we obtain \( t_0 = \tau_-(x, v_0) \). Note also that the function \( s \mapsto \frac{1}{\sqrt{t_0(s - s)}} \), \( s \in (0, t_0) \), is integrable in \((0, t_0)\). Therefore, using (5.3), the boundedness of \( \Psi_2 \) on \( D \) and the Lebesgue dominated convergence theorem, we obtain (5.7). This proves (3.26) when \( n = 2 \).

Let \( n \geq 3 \) and prove (3.29). From (5.2) and (3.10), it follows that
\[ \gamma_1(\tau, x, x_0') = \int_{S^{n-1}} (\tau - t_0 v_0 \cdot v)^{n-3} |t_0 v_0 - \tau v|^{2n-4} \chi_{(0, +\infty)}(\nu(x) \cdot v) \Psi_n(s, v, v') \left| v' = \frac{t_0 v_0 - sv}{s \sqrt{2(\tau - t_0)}}, \right. \tag{5.11} \]
for \( \tau > |x - x_0'| \).

Let \( \Phi(\Omega, \omega) = (\sin \Omega, \cos \Omega, \omega_1, \ldots, \cos \Omega, \omega_{n-1}) \) for \( \Omega \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \) and \( \omega = (\omega_1, \ldots, \omega_{n-1}) \in S^{n-2} \). Using spherical coordinates we obtain
\[ \gamma_1(\tau, x, x_0') = \int_{-\pi/2}^{\pi/2} \cos(\Omega)^{n-2} \left( \tau - t_0 \sin(\Omega) \right)^{n-3} \left( t_0^2 + \tau^2 - 2t_0 \tau \sin(\Omega) \right)^{n-2} \]
\[ \int_{S^{n-2}} \chi_{(0, +\infty)}(\nu(x) \cdot \Phi(\Omega, \omega)) \Psi_n(s, \Phi(\Omega, \omega), v', v') \left| v' = \frac{t_0 v_0 - sv \Phi(\Omega, \omega)}{s \sqrt{2(\tau - t_0) \sin(\Omega)}}, \right. \tag{5.12} \]
for \( \tau > t_0 \). Performing the change of variables \( \tau = \frac{\tau_1 - t_0}{2(\tau - t_0 \sin(\Omega))} - \frac{\tau_1 t_0}{2} \) on the first integral on the right-hand side of (5.12), we obtain
\[ \gamma_1(\tau, x, x_0') = 2^{2-n} t_0^{2-n} \left( \tau^2 - t_0^2 \right)^{n-3} t_0 \int_0^{t_0} \left[ \frac{\sqrt{\frac{\tau_0 - \tau}{2}} r}{\left( \frac{\tau_0 - \tau}{2} + r \right)^{n-2} \left( \frac{\tau_0 + \tau}{2} - r \right)^{n-2}} \right] \int_{S^{n-2}} \chi(\Phi(\Omega, \omega)) \Psi_n(r + \frac{\tau - t_0}{2}, \Phi(\Omega, \omega), v') d\omega \]
\[ \left| v' = \frac{\tau_0 v_0 - sv \Phi(\Omega, \omega)}{\sqrt{2(\tau - t_0) \sin(\Omega)}}, \right. \tag{5.13} \]
for \( \tau > t_0 \). Performing the change of variables \( \tau = \frac{\tau_1 - t_0}{2(\tau - t_0 \sin(\Omega))} - \frac{\tau_1 t_0}{2} \) on the first integral on the right-hand side of (5.12), we obtain
\[ \gamma_1(\tau, x, x_0') = 2^{2-n} t_0^{2-n} \left( \tau^2 - t_0^2 \right)^{n-3} t_0 \int_0^{t_0} \left[ \frac{\sqrt{\frac{\tau_0 - \tau}{2}} r}{\left( \frac{\tau_0 - \tau}{2} + r \right)^{n-2} \left( \frac{\tau_0 + \tau}{2} - r \right)^{n-2}} \right] \int_{S^{n-2}} \chi(\Phi(\Omega, \omega)) \Psi_n(r + \frac{\tau - t_0}{2}, \Phi(\Omega, \omega), v') d\omega \]

Therefore using (5.13), (5.3) and (5.2) and using Lebesgue dominated convergence theorem, we obtain (3.29). This concludes the proof of Theorem 3.5. \( \square \)

**Proof of Theorem 3.4.** For the sake of simplicity and without loss of generality we assume \( v_0 = (1, 0, \ldots, 0) \). Assume that condition (3.22) is satisfied. We consider the measurable function \( \Psi_n \) defined by (5.2) for all \((s, v, v') \in D \) where \( D \) is defined by (5.1). Under (3.22) we obtain that
\[ \text{the function } \Psi_n \text{ is continuous at any point } (s, v, v') \in D \]
(i.e. for any \((s, v, v') \in (0, +\infty) \times S_x^{n-1} \times S_{x_0}^{n-1}, x - sv \in X) \),
\[ \tag{5.14} \]
The proof of (3.26) under condition (3.22) is actually similar to the proof of (3.26) under condition (3.23). Note that (5.4)–(5.6) still hold so that we have to prove that (5.7) still holds for \( i = 1, 2 \). Again we only sketch the proof of (5.7) for \( i = 1 \). Note also that (5.8)–(5.10) still hold. Then using (5.8)–(5.10), (5.14) and (5.2) and using Lebesgue dominated convergence theorem, we obtain (5.7) for \( i = 1 \). This proves (3.26).

Let \( n \geq 3 \). Formula (5.12) still holds. Now assume that \( n = 3 \). We shall prove (3.27). Let \( \tau > t_0 \). Using the change of variables \( \varepsilon = \frac{\ln(t_0^2 + \tau^2 - 2\tau \sin(\Omega)) - \ln(\tau - t_0)^2}{\ln(\tau + t_0)^2 - \ln(\tau - t_0)^2} \), the equality (5.12) gives

\[
\gamma_1(\tau, x, x_0') = \frac{\ln\left(\frac{\tau + t_0}{\tau - t_0}\right)}{2t_0\tau} (\gamma_{1,1}(\tau, x, x_0') + \gamma_{1,2}(\tau, x, x_0'))
\]  

(5.15)

where

\[
\gamma_{1,1}(\tau, x, x_0') = \int_0^{1/2} \int_{S^1} \chi_{(0, +\infty)}(\nu(x) \cdot \Phi(\Omega(\tau, \varepsilon), \omega))\Psi_3(s(\tau, \varepsilon), \Phi(\Omega(\tau, \varepsilon), \omega), v'(\tau, \omega, \varepsilon))d\omega d\varepsilon
\]

(5.16)

\[
\gamma_{1,2}(\tau, x, x_0') = \int_{1/2}^1 \int_{S^1} \chi_{(0, +\infty)}(\nu(x) \cdot \Phi(\Omega(\tau, \varepsilon), \omega))\Psi_3(s(\tau, \varepsilon), \Phi(\Omega(\tau, \varepsilon), \omega), v'(\tau, \omega, \varepsilon))d\omega d\varepsilon
\]

(5.17)

and

\[
\Omega(\tau, \varepsilon) := \arcsin\left(\frac{\tau^2 + t_0^2 - (\tau - t_0)^2(1-\varepsilon)(\tau + t_0)^2\varepsilon}{2t_0\tau}\right),
\]

(5.18)

\[
s(\tau, \varepsilon) := \frac{\tau^2 - t_0^2}{2(\tau - t_0\sin(\Omega(\tau, \varepsilon)))},
\]

(5.19)

\[
v'(\tau, \omega, \varepsilon) := \frac{t_0v_0 - s(\tau, \varepsilon)\Phi(\Omega(\tau, \varepsilon), \omega)}{\tau - s(\tau, \varepsilon)},
\]

(5.20)

for \( 0 < \varepsilon < 1 \), \( \omega \in S^1 \).

We shall give some properties of \( \Omega(\tau, \varepsilon), s(\tau, \varepsilon) \) and \( v'(\tau, \omega, \varepsilon) \) for \( 0 < \varepsilon < 1 \) and \( \omega \in S^{n-1} \). From (5.18), it follows that

\[
\Omega(\tau, \varepsilon) \to \frac{\pi}{2}, \text{ as } \tau \to t_0^+, \text{ for all } \varepsilon \in (0, 1).
\]

(5.21)

From (5.18) and (5.19), it follows that

\[
s(\tau, \varepsilon) = \frac{\tau(t_0 + \tau)(\tau - t_0)^{2\varepsilon - 1}}{(t_0 + \tau)(\tau - t_0)^{2\varepsilon - 1} + (\tau + t_0)^{2\varepsilon}}, \quad s(\tau, \varepsilon) = \frac{\tau(t_0 + \tau)}{t_0 + \tau + (\tau - t_0)^{1-2\varepsilon}(\tau + t_0)^{2\varepsilon}}
\]

(5.22)

for \( \varepsilon \in (0, 1) \). From (5.22) it follows that

\[
s(\tau, \varepsilon) \to 0^+ \text{ as } \tau \to t_0^+, \text{ when } \varepsilon \in (1/2, 1), \quad s(\tau, \varepsilon) \to t_0^- \text{ as } \tau \to t_0^-, \text{ when } \varepsilon \in (0, 1/2).
\]

(5.23)

In addition, from (5.18)–(5.20), it follows that

\[
v'(\tau, \omega, \varepsilon) = \left(\frac{\tau(t_0 + \tau)^2(\tau - t_0)^{2\varepsilon} + 2t_0\tau(\tau + t_0)^{2\varepsilon} + \tau(\tau + t_0)^{1+2\varepsilon}(\tau - t_0)}{2t_0\tau^2(\tau + t_0)^{2\varepsilon}} \right)
\]

\[
\times \sqrt{-(\tau - t_0)^{4\varepsilon}(\tau + t_0)^2 - (t_0^2)^2(\tau + t_0)^{4\varepsilon} + 2(t_0^2 + t_0)(\tau - t_0)^{2\varepsilon}(\tau + t_0)^{2\varepsilon}} \frac{\tau + t_0 + (\tau - t_0)^{1-2\varepsilon}(\tau + t_0)^{2\varepsilon}}{2t_0\tau(\tau + t_0)^{2\varepsilon}}
\]

(5.24)
for $0 < \varepsilon < 1$. Therefore

$$ v'(\tau, \omega, \varepsilon) \to v_0 = (1, 0, 0) \text{ as } \tau \to t_0^+, \quad (5.25) $$

for $0 < \varepsilon < 1$.

Using (5.16), (5.21), (5.23), (5.25) and Lebesgue dominated convergence theorem, we obtain

$$ \gamma_1,1(\tau, x, x_0') \to \int_{t_0}^{\frac{1}{2}} \int_{\mathbb{S}^1} \lim_{s \to t_0} \Psi_3(s, v_0, v_0) d\omega d\varepsilon = 2\pi E(x_0', x) W(x, v_0) S(x_0, v_0)(\nu(x_0') v_0) (\nu(x) v_0) k(x_0', v_0, v_0) \quad (5.26) $$

(we also used (5.2)).

Similarly, using (5.16), (5.21), (5.23), (5.25) and Lebesgue dominated convergence theorem, we obtain

$$ \gamma_1,2(\tau, x, x_0') \to 2\pi E(x_0', x) W(x, v_0) S(x_0', v_0)(\nu(x_0') v_0)(\nu(x) v_0) k(x, v_0, v_0), \text{ as } \tau \to t_0^+. \quad (5.27) $$

Statement (3.27) follows from (5.15) and (5.26)–(5.27).

Let $n \geq 4$. We shall prove (3.28). From (5.12) it follows that

$$ \gamma_1(\tau, x, x_0') = \gamma_1,1(\tau, x, x_0') + \gamma_1,2(\tau, x, x_0'), \quad (5.28) $$

where

$$ \gamma_1,1(\tau, x, x_0') = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \chi_0(-\frac{\pi}{2}, \arcsin(\frac{\Psi}{\psi}))(\Omega) \cos(\Omega)^{n-2} \frac{(\tau - t_0 \sin(\Omega))^{n-3}}{(t_0^2 + \tau^2 - 2t_0 \tau \sin(\Omega))^{n-2}} \times \int_{\mathbb{S}^{n-2}} \chi(0, +\infty)(\Phi(\Omega, \omega) \cdot (\nu(x)) \chi(0, \tau, (x, \Phi(\Omega, \omega))))(s) \Psi_n(s, \Phi(\Omega, \omega), \nu) \nu' \to \int_{t_0}^{\infty} \chi(0, +\infty)(\Phi(\Omega, \omega) \cdot (\nu(x)) \chi(0, \tau, (x, \Phi(\Omega, \omega))))(s) \Psi_n(s, \Phi(\Omega, \omega), \nu) \nu' \to \frac{\tau - t_0 \sin(\Omega)}{2(\tau - t_0 \sin(\Omega))} d\omega d\Omega, \quad (5.29) $$

$$ \gamma_1,2(\tau, x, x_0') = \int_{\mathbb{S}^{n-2}} \chi(0, +\infty)(\nu(x) \cdot \Phi(\Omega, \omega)) \chi(0, \tau, (x, \Phi(\Omega, \omega)))(s) \Psi_n(s, \Phi(\Omega, \omega), \nu) \nu' \to \frac{\tau - t_0 \sin(\Omega)}{2(\tau - t_0 \sin(\Omega))} d\omega d\Omega. \quad (5.30) $$

First we study $\gamma_1,1$. Note that

$$ \frac{\tau - t_0 \sin(\Omega_1)}{t_0^2 + \tau^2 - 2t_0 \tau \sin(\Omega_1)} < \frac{\tau - t_0 \sin(\Omega_2)}{t_0^2 + \tau^2 - 2t_0 \tau \sin(\Omega_2)}, \quad (5.31) $$

for $-\frac{\pi}{2} \leq \Omega_1 < \Omega_2 \leq \frac{\pi}{2}$ and for $\tau > t_0$. Therefore using also the estimate $\cos(\Omega) \leq 1$ we obtain

$$ \frac{\tau - t_0 \sin(\Omega_1)}{t_0^2 + \tau^2 - 2t_0 \tau \sin(\Omega_1)} < \frac{\tau - t_0 \sin(\Omega_2)}{t_0^2 + \tau^2 - 2t_0 \tau \sin(\Omega_2)}, $$

$$ \leq \frac{\cos(\Omega)^{n-2}}{(t_0^2 + \tau^2 - 2t_0 \tau \sin(\Omega))^{n-2}} \left( \frac{\tau - t_0 \sin(\Omega)}{t_0^2 + \tau^2 - 2t_0 \tau \sin(\Omega)} \right)^{n-3} \leq \frac{C_0^2}{2t_0^2 \tau^{n-2}}, \quad (5.32) $$

for $\Omega \in (\frac{-\pi}{2}, \frac{\pi}{2})$ and $\sin(\Omega) \leq \frac{t_0}{2} \tau$ (we used (5.31) with “$\Omega_1$” = $\Omega$ and “$\Omega_2$” = $\frac{t_0}{2} \tau$, and we used the estimate $t_0^2 + \tau^2 - 2t_0 \tau \sin(\Omega) \geq 2t_0 \tau (1 - \sin(\Omega))$), where

$$ C_0 := \sup_{\phi \in [0, 2\pi]} \frac{\sin^2(\phi)}{1 - \cos(\phi)} = \sup_{\Omega \in (-\frac{\pi}{2}, \frac{\pi}{2})} \frac{\cos^2(\Omega)}{1 - \sin(\Omega)}. \quad (5.33) $$
Using (5.29), (5.32), (5.2) and Lebesgue dominated convergence theorem, we obtain

\[
\gamma_{1,1}(\tau, x, x') \rightarrow_{\tau \to t_0^+} 2^{2-n} t_0^{1-n} \int_{\mathbb{S}^{n-2}} \frac{2}{1 - \sin^2(\Omega)} \int_{\mathbb{S}^{n-2}} \chi\left(\nu(x) \cdot \Phi(\Omega, \omega)\right) \lim_{s \to 0^+} \Psi_n(s, \Phi(\Omega, \omega), v_0) d\omega d\Omega \\
= 2^{2-n} t_0^{1-n} \int_{\mathbb{S}^{n-1}} \frac{1}{1 - v \cdot v_0} \lim_{s \to 0^+} \Psi_n(s, v, v_0) dv \\
= t_0^{1-n} E(x'_0, x) S(x'_0, v_0)(\nu(x'_0) \cdot v_0) \int_{\mathbb{S}^{n-1}} \frac{1}{1 - v' \cdot v_0} W(x, v)(\nu(x) \cdot v) k(x, v, v_0) dv. (5.34)
\]

Now we shall study \(\gamma_{1,2}\) defined by (5.30). Note that using the convexity of \(X\) we obtain

\[
v' = \frac{x - sv - x'_0}{|x - sv - x'_0|} \in \mathbb{S}^{n-1}_x \quad \text{whenever} \quad v \in \mathbb{S}^{n-1}_x \quad \text{and} \quad s \in (0, \tau_-(x, v)).
\]

Therefore using the change of variables \(\sin(\Omega') = \frac{2\tau t_0 - (\tau^2 + t_0^2) \sin(\Omega)}{\tau^2 + t_0^2 - 2\tau t_0 \sin(\Omega')}, \Omega \in (-\frac{\pi}{2}, \frac{\pi}{2})\) \(\left(\cos(\Omega) \frac{d\Omega}{d\Omega'} = \frac{(\tau^2 - t_0^2)^2 \cos(\Omega')}{(\tau^2 + t_0^2 - 2\tau t_0 \sin(\Omega'))^2}\right)\), we obtain

\[
\gamma_{1,2}(\tau, x, x') := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \arcsin\left(\frac{\tau}{\tau - t_0^2} \sin(\Omega')\right) - \int_{\nu(x' \cdot \Phi(\Omega, \nu')) > 0} \frac{(\tau - t_0^2 \sin(\Omega'))^{n-2}}{t_0^2 + \tau^2 - 2\tau t_0 \sin(\Omega')} d(\Omega')
\]

where

\[
\Omega(\tau, \Omega') = \arcsin\left(\frac{2\tau t_0 - (\tau^2 + t_0^2) \sin(\Omega')}{\tau^2 + t_0^2 - 2\tau t_0 \sin(\Omega')}\right), \quad s(\tau, \Omega') = \frac{\tau^2 + t_0^2 - 2\tau t_0 \sin(\Omega')}{{2(\tau - t_0 \sin(\Omega'))}}, (5.36)
\]

for \(\Omega' \in (-\frac{\pi}{2}, \frac{\pi}{2})\). Note that

\[
\Omega(\tau, \Omega') \rightarrow_{\tau \to t_0^+} \frac{\pi}{2}, \quad \Phi(\Omega, \Omega', \omega) \rightarrow_{\tau \to t_0^+} v_0, \quad s(\tau, \Omega') \rightarrow_{\tau \to t_0^+} t_0, (5.37)
\]

for \((\Omega', \omega) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{S}^{n-2} \). Note also that from (5.37), it follows that at fixed \(v' = \Phi(\Omega', -\omega) \in \mathbb{S}^{n-1}_x\) the condition \(\chi(0, \tau_-(x, \Phi(\Omega(\tau, \Omega'), \omega))(s(\tau, \Omega')) = 1\) (which is equivalent to \(x - s(\tau, \Omega') \Phi(\Omega(\tau, \Omega'), \omega) \in X\) due to convexity of \(X\)) for \(\Phi(\Omega(\tau, \Omega'), \omega) \in \mathbb{S}^{n-1}_x\) is satisfied when \(\tau - t_0 > 0\) is sufficiently small. Therefore from (5.32) (with \(\Omega'\) replaced by \(\Omega''\)) and from (5.35) and Lebesgue dominated convergence theorem we obtain

\[
\gamma_{1,2}(\tau, x, x') \rightarrow_{\tau \to t_0^+} 2^{2-n} t_0^{1-n} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2}{1 - \sin^2(\Omega')} \int_{\mathbb{S}^{n-2}} \chi\left(0, +\infty\right)\left(-\nu(x'_0) \cdot \Phi(\Omega', -\omega)\right) \\
\times \lim_{s \to t_0^+} \Psi_n(s, v_0, \Phi(\Omega', -\omega)) d\omega d\Omega' = 2^{2-n} t_0^{1-n} \int_{\mathbb{S}^{n-1}} \frac{1}{1 - v' \cdot v_0} \lim_{s \to t_0^+} \Psi_n(s, v_0, v') dv' \\
= t_0^{1-n} E(x'_0, x) W(x, v_0)(\nu(x) \cdot v_0) \int_{\mathbb{S}^{n-1}} \frac{1}{1 - v' \cdot v_0} S(x'_0, v') |\nu(x'_0) \cdot v'| k(x, v', v_0) dv'. (5.39)
\]

Statement (3.28) follows from (5.28), (5.34) and (5.39). Theorem 3.4 is proved.
6 Proof of Theorems 4.1, 4.2, 4.5 and Theorem 4.4 (4.13)

Proof of Theorem 4.2. We now prove (4.5). Let \( x'_0 \in \partial X \). For \( \varepsilon = (\varepsilon_1, \varepsilon_2) \in (0, +\infty)^2 \) and \( \varepsilon_3 \in (0, +\infty) \) let \((f_{\varepsilon_1}, g_{\varepsilon_2}) \in C^1(\partial X) \times C^1(\mathbb{R})\) satisfy

\[
\begin{align*}
g_{\varepsilon_2} & \geq 0, f_{\varepsilon_1} \geq 0, \text{ supp } g_{\varepsilon_2} \subseteq (0, \min(\varepsilon_2, \eta)), \\
\text{ supp } f_{\varepsilon_1} & \subseteq \{x' \in \partial X \mid |x' - x'_0| < \varepsilon_1\}, \\
\int_0^\eta g_{\varepsilon_2}(t') dt' & = 1, \quad \int_{\partial X} f_{\varepsilon_1}(x') d\mu(x') = 1, \quad (6.3)
\end{align*}
\]

for \( \varepsilon = (\varepsilon_1, \varepsilon_2) \in (0, +\infty)^2 \). Therefore \( \phi_\varepsilon := g_{\varepsilon_2} f_{\varepsilon_1} \) is an approximation of the delta function at \( (0, x'_0) \in \mathbb{R} \times \partial X \) for \( \varepsilon := (\varepsilon_1, \varepsilon_2) \in (0, +\infty)^2 \). Let \( \psi_{\varepsilon_3} \in L^\infty((0, T) \times \partial X) \) be defined by

\[
\psi_{\varepsilon_3}(t, x) = \chi_{(-\varepsilon_3, \varepsilon_3)}(t - |x - x'_0|)(2\chi_{(0, +\infty)}((E - \tilde{E})(x, x'_0)) - 1), \quad (t, x) \in (0, T) \times \partial X,
\]

for \( \varepsilon_3 > 0 \). From (3.8) and (3.12) it follows that

\[
\begin{align*}
\int_{(0,T) \times \partial X} \psi_{\varepsilon_3}(t, x)(A_{S,W} - \tilde{A}_{S,W})\phi_\varepsilon(t, x)dt d\mu(x) & = I_0(\psi_{\varepsilon_3}, \phi_\varepsilon) \\
+ \int_{(0,T) \times \partial X \times (0,\eta) \times \partial X} \psi_{\varepsilon_3}(t, x)\phi_\varepsilon(t', x')(\Gamma_1 - \tilde{\Gamma}_1)(t - t', x, x')dt d\mu(x)d\mu(x'),
\end{align*}
\]

for \( \varepsilon = (\varepsilon_1, \varepsilon_2) \in (0, +\infty) \) and \( \varepsilon_3 \in (0, +\infty) \), where

\[
I_0(\psi_{\varepsilon_3}, \phi_\varepsilon) = \int_{(0,T) \times \partial X \times (0,\eta) \times \partial X} \psi_{\varepsilon_3}(t, x)\phi_\varepsilon(t - |x - x'|, x', E(x, x') - \tilde{E}(x, x'))\frac{1}{|x - x'|^{n-1}}
\]
\[
\times |W(x, v)S(x', v)(\nu(x) \cdot v)|\nu(x') \cdot v||v = \frac{x - x'}{|x - x'|} d\mu(x)d\mu(x').
\]

From (3.16), (3.17), (3.20) and (3.21) it follows that

\[
(\tau - |x - x'|)^{\frac{3 - n}{2}}(\Gamma_1 - \tilde{\Gamma}_1)(\tau, x, x') \in L^\infty((0, T) \times \partial X \times \partial X).
\]

Combining (6.5) and the equality \( \|\phi_\varepsilon\|_{L^1((0,\eta) \times \partial X)} = 1 \) and the estimate \( \|\psi_{\varepsilon_3}\|_{L^\infty((0,T) \times \partial X)} \leq 1 \) and (6.7) we obtain

\[
I_0(\psi_{\varepsilon_3}, \phi_\varepsilon) \leq \|A_{S,W} - \tilde{A}_{S,W}\|_{\eta,T} + C\Delta_1(\psi_{\varepsilon_3}, \phi_\varepsilon),
\]

for \( \varepsilon = (\varepsilon_1, \varepsilon_2) \in (0, +\infty) \) and \( \varepsilon_3 \in (0, +\infty) \), where \( C = \|(\tau - |x - x'|)^{\frac{3 - n}{2}}(\Gamma_1 - \tilde{\Gamma}_1)(\tau, x, x')\|_{L^\infty((0,T) \times \partial X \times \partial X)} \) and

\[
\Delta_1(\psi_{\varepsilon_3}, \phi_\varepsilon) = \int_{(0,T) \times \partial X \times (0,\eta) \times \partial X} \psi_{\varepsilon_3}(t, x)\phi_\varepsilon(t', x')(t - t' - |x - x'|)^{\frac{n - 3}{2}}d\mu(x)d\mu(x').
\]

Note that the function \( \Phi_{1,\varepsilon_3} : [0, \eta) \times \partial X \rightarrow \mathbb{R} \) defined by

\[
\Phi_{1,\varepsilon_3}(t', x') := \int_{(0,T) \times \partial X} \psi_{\varepsilon_3}(t, x)(t - t' - |x - x'|)^{\frac{n - 3}{2}}d\mu(x), \quad (t', x') \in [0, \eta) \times \partial X,
\]

is continuous on \([0, \eta) \times \partial X \) for \( \varepsilon_3 \in (0, +\infty) \). Therefore from (6.1)–(6.3) and the equality \( \Delta_1(\psi_{\varepsilon_3}, \phi_\varepsilon) = \int_{(0,\eta) \times \partial X} \phi_\varepsilon(t', x')\Phi_{1,\varepsilon_3}(t', x')dt' d\mu(x') \) it follows that

\[
\lim_{\varepsilon_3 \rightarrow 0^+} \lim_{\varepsilon_2 \rightarrow 0^+} \lim_{\varepsilon_1 \rightarrow 0^+} \Delta_1(\psi_{\varepsilon_3}, \phi_\varepsilon) = \lim_{\varepsilon_3 \rightarrow 0^+} \Phi_{1,\varepsilon_3}(0, x'_0) = 0
\]

(6.11)
is continuous on $\partial X$, for $(\varepsilon_2, \varepsilon_3) \in (0, +\infty)^2$. Therefore from the equality $I_0(\psi_{\varepsilon_3}, \phi_{\varepsilon}) = \int_{\partial X} \Phi_{0, \varepsilon_2, \varepsilon_3}(x') \times f_1(x')d\mu(x)$ (see (6.6)) it follows that

$$
\lim_{\varepsilon_1 \to 0^+} I_0(\psi_{\varepsilon_3}, \phi_{\varepsilon}) = \Phi_{0, \varepsilon_2, \varepsilon_3}(x_0'), \text{ for } (\varepsilon_2, \varepsilon_3) \in (0, +\infty)^2.
$$

Therefore using the Lebesgue dominated convergence theorem and (6.12) we obtain

$$
\lim_{\varepsilon_3 \to 0^+} \lim_{\varepsilon_2 \to 0^+} \lim_{\varepsilon_1 \to 0^+} I_0(\psi_{\varepsilon_3}, \phi_{\varepsilon}) = \int_{\partial X} \frac{E(x', x_0') - \tilde{E}(x', x_0')}{|x - x_0'|^{n-1}} |W(x, v)S(x_0, v)(\nu(x) \cdot v)| \nu(x_0') \cdot v|_{v = \frac{x-x_0'}{|x-x_0'|}} d\mu(x).
$$

Combining (6.14), (6.11) and (6.8) we obtain the formula (4.1). Using (4.1) and the estimates $\inf_{(x',v') \in \Gamma_-} S(x', v') > 0$ and $\inf_{(x,v) \in \Gamma_+} W(x, v) > 0$ and the change of variables $x = x_0' + \tau_+(x_0', v_0) v_0 \quad (\nu(x)_{x_0} \cdot d\nu(x) = d\mu(x))$ we obtain (4.5) where the constant $C_1$ which appears on the right-hand side of (4.5) is given by $C_1 = \left(\inf_{(x',v') \in \Gamma_-} S(x', v') \inf_{(x,v) \in \Gamma_+} W(x, v)\right)^{-1}$.

We now prove (4.6). Let $x \in \partial X$ be such that $px_0' + (1-p)x \in Z$ for some $p \in (0, 1)$. We set $t_0 = |x - x_0'|$ and $v_0 = \frac{x-x_0'}{|x-x_0'|}$. From (3.16), (3.17), (3.20) and (3.21) it follows that

$$
(\tau - |x - x_0'|)^{\frac{n-2}{2}} |\gamma_1 - \tilde{\gamma}_1|(|\tau, x, x_0| \leq (\tau - |x - x_0'|)^{\frac{n-2}{2}} |\Gamma_2 - \tilde{\Gamma}_2|(|\tau, z, z'| \leq (|s - z - z'|)^{\frac{n-2}{2}} (\Gamma_1 - \tilde{\Gamma}_1)(s, z, z') ||L^{\infty}(0, T), d\nu(x) ||L_{x, x'}),
$$

for $\tau > |x - x_0'|$. From (3.17), (3.20)–(3.21) it turns out that $\lim_{\tau \to |x - x_0'|^+} (\tau - |x - x_0'|)^{\frac{n-2}{2}} |\Gamma_2 - \tilde{\Gamma}_2|(|\tau, z, z'| = 0$. Therefore applying (3.26) and (3.29) on the left-hand side of (6.15) we obtain

$$
2^{\frac{n-2}{2}} |x - x_0'|^{\frac{n-2}{2}} C_n S(x_0', v_0) W(x, v_0) \nu(x_0') \cdot v_0 ||\nu(x) \cdot v_0||
\times \int_0^{t_0} e^{-f_0^0 \sigma(x_0', v_0, |s|) d\tilde{k}(x - p\tilde{v}_0, v_0, v_0) - e^{-f_0^0 \sigma(x_0', v_0, |s|) d\tilde{k}(x - p\tilde{v}_0, v_0, v_0) dp}}
\leq ||(s - z - z'|^{\frac{n-2}{2}} (\Gamma_1 - \tilde{\Gamma}_1)(s, z, z') ||L^{\infty}(0, T), d\nu(x) ||L_{x, x'}),
$$

where $C_n = 2$ if $n = 2$ and $C_n = \text{Vol}_{n-2}(S^{n-2})$ if $n \geq 3$. Then note that $C_X := \inf_{x_1 \in \partial X, \varepsilon \in \mathbb{R}^1} \nu(x_1) : \frac{x_1 - x}{|x_1 - x|} > 0$ since $X$ is a bounded convex subset of $\mathbb{R}^n$ with $C^1$ boundary and $\tilde{Z} \subset X$. Therefore (4.6) follows from (6.16) where the constant $C_2$ which appears on the right-hand side of (4.6) is given by

$$
C_2 = \frac{2^{\frac{n-1}{2}} \text{diam}(X)^{\frac{n-1}{2}}}{C_n C_X \inf_{(x',v') \in \Gamma_-} S(x', v') \inf_{(x,v) \in \Gamma_+} W(x, v)}.
$$

Theorem 4.2 is proved. □

**Proof of Theorem 4.1.** We prove (4.1). Let $x_0' \in \partial X$. For $\varepsilon = (\varepsilon_1, \varepsilon_2) \in (0, +\infty)^2$ and $\varepsilon_3 \in (0, +\infty)$ let $(f_{\varepsilon}, g_{\varepsilon}) \in C^1(\partial X) \times C^1(\mathbb{R})$ satisfy (6.1)–(6.3) and $\psi_{\varepsilon_3}$ be defined by (6.4). First note that (6.5)–(6.6) still hold. From Theorems 3.2 and 3.3 it follows that

$$
|x - x'|^{n-2} (\tau - |x - x'|)^{\frac{n-2}{2}} (\Gamma_1 - \tilde{\Gamma}_1)(\tau, x, x') \in L^\infty((0, T) \times \partial X \times \partial X).
$$

(6.17)
Combining (6.5), (6.17) and the equality \( \| \phi \|_{L^1((0,0) \times \partial X)} \) and the estimate \( \| \psi_{\varepsilon} \|_{L^\infty((0,T) \times \partial X)} \leq 1 \) we obtain

\[
I_0(\psi_{\varepsilon}, \phi_\varepsilon) \leq A_{SW} - \tilde{A}_{SW} \eta_T + C \Delta_2(\psi_{\varepsilon}, \phi_\varepsilon),
\]

for \( \varepsilon = (\varepsilon_1, \varepsilon_2) \in (0, +\infty), \varepsilon_3 \in (0, +\infty), \) where \( C = \| x - x' \|^{n-\frac{1}{2}} (t - |x - x'|)^2 (\Gamma_1 - \tilde{\Gamma}_1)(t, x, x') \|_{L^\infty((0,T) \times \partial X \times \partial X')} \) and

\[
\Delta_2(\psi_{\varepsilon}, \phi_\varepsilon) = \int_{(0,T) \times \partial X \times (0,\eta \mu \partial X')} \frac{\psi_{\varepsilon}(t, x) \phi_\varepsilon(t', x')}{|x - x'|^{n-\frac{1}{2}} (t - t' - |x - x'|)^2} dt \mu(x) dt' \mu(x').
\]

Note that the function \( \Phi_{1,\varepsilon_3} : [0, T] \times \partial X \rightarrow \mathbb{R} \) defined by

\[
\Phi_{3,\varepsilon_3}(t', x') := \int_{(0,T) \times \partial X \times (0,\eta \mu \partial X')} \frac{\psi_{\varepsilon_3}(t, x)}{|x - x'|^{n-\frac{1}{2}} (t - t' - |x - x'|)^2} dt \mu(x), \quad (t', x') \in [0, \eta) \times \partial X,
\]

is continuous on \([0, \eta) \times \partial X\). Therefore using (6.19), (6.1)–(6.3) and using the equality \( \Delta_2(\psi_{\varepsilon_3}, \phi_\varepsilon) = \int_{(0,\eta\mu \partial X')} \Phi_{3,\varepsilon_3}(t', x') \phi_\varepsilon(t', x') dt' \mu(x') \) we obtain

\[
\lim_{\varepsilon_3 \rightarrow 0^+} \lim_{\varepsilon_2 \rightarrow 0^+} \lim_{\varepsilon_1 \rightarrow 0^+} \Delta_2(\psi_{\varepsilon_3}, \phi_\varepsilon) = \lim_{\varepsilon_3 \rightarrow 0^+} \Phi_{3,\varepsilon_3}(0, x'_0) = 0.
\]

(we also used (6.20), (6.4) and Lebesgue dominated convergence theorem to prove \( \lim_{\varepsilon_3 \rightarrow 0^+} \Phi_{3,\varepsilon_3}(0, x'_0) = 0 \)). Note that under condition (3.22) the function \( \Phi_{4,\varepsilon_3, x_2} : \partial X \rightarrow \mathbb{R} \) defined by

\[
\Phi_{4,\varepsilon_3, x_2}(t, x_2) := \int_{(0,\eta \mu \partial X')} \psi_{\varepsilon_3}(t, x) g_\varepsilon(t - |x - x'|) \frac{E_{\partial X \times \partial X}(x, x', x'_0) - \tilde{E}_{\partial X \times \partial X}(x, x'_0)}{|x - x'|^{n-1}} \times |W(x, v)S(x', v)(\nu(x) \cdot v)| |v(x') \cdot v|_{v = \frac{x - x'}{|x - x'|}, d\mu(x)},
\]

is continuous on \( \partial X \). Therefore from (6.1)–(6.3) and the equality \( I_0(\psi_{\varepsilon_3}, \phi_\varepsilon) = \int_{\partial X} \Phi_{1,\varepsilon_3}(x') \Phi_{4,\varepsilon_3, x_2}(x') dt \mu(x') \) (see (6.6)) it follows that

\[
\lim_{\varepsilon_1 \rightarrow 0^+} I_0(\psi_{\varepsilon_3}, \phi_\varepsilon) = \Phi_{4,\varepsilon_3, x_2}(x'_0), \text{ for } (\varepsilon_2, \varepsilon_3) \in (0, +\infty).
\]

Then using Lebesgue dominated convergence theorem and (6.22) we obtain

\[
\lim_{\varepsilon_3 \rightarrow 0^+} \lim_{\varepsilon_2 \rightarrow 0^+} \lim_{\varepsilon_1 \rightarrow 0^+} I_0(\psi_{\varepsilon_3}, \phi_\varepsilon) = \int_{\partial X} \frac{|E_{\partial X \times \partial X}(x, x'_0) - \tilde{E}_{\partial X \times \partial X}(x, x'_0)|}{|x - x'_0|^{n-1}} \times |W(x, v)S(x'_0, v)(\nu(x) \cdot v)| |v(x'_0) \cdot v|_{v = \frac{x - x'_0}{|x - x'_0|}, d\mu(x)}.
\]

Combining (6.24), (6.21) and (6.18) we obtain (4.1).

We prove (4.2)–(4.4). Let \( x \in \partial X \) be such that \( px'_0 + (1 - p)x \in X \) for some \( p \in (0, 1) \). Let \( \beta_n : \{(\tau, z, z') \in (0, T) \times \partial X \times \partial X \mid z \neq z', \tau > |z - z'|\} \rightarrow \mathbb{R} \) be defined by

\[
\beta_n(\tau, z, z') = \begin{cases} \sqrt{\frac{\tau^2}{\tau |z - z'|^2}} & \text{if } n = 2, \\ \sqrt{\frac{\tau |z - z'|}{2 \tau |z - z'|^2}} & \text{if } n = 3, \\ \frac{\tau |z - z'|^{n-2}}{\beta_n(\tau, z, z')}. & \text{if } n \geq 4, \end{cases}
\]

for \( (\tau, z, z') \in (0, T) \times \partial X \times \partial X, z \neq z', \tau > |z - z'| \). From (3.13)–(3.15) and (3.17)–(3.19) it follows that

\[
\beta_n(\tau, x, x'_0)(\gamma_1 - \tilde{\gamma}_1)(\tau, x, x'_0) \leq \beta_n(\tau, x, x'_0)(\Gamma_2 - \tilde{\Gamma}_2)(\tau, z, z') \leq \beta_n(\tau, x, x'_0)(\Gamma_1 - \tilde{\Gamma}_1)(s, z, z').
\]
From (6.25) and (3.17)–(3.19) it turns out that \( \lim_{\tau \to \infty} |x - x_0| + \beta_n(\tau, x, x_0)(\Gamma_2 - \tilde{\Gamma}_2)(\tau, z, z') = 0 \). Therefore applying (3.26) (resp. (3.27), (3.28)) on the left-hand side of (6.26) we obtain (4.2) (resp. (4.3), (4.4)). Theorem 4.1 is proved. \( \square \)

**Proof of Theorem 4.5.** We first prove (4.11). We extend \( \sigma \) and \( \tilde{\sigma} \) by 0 outside \( Y \). For a bounded and continuous function \( f \) on \( Y \) consider the X-ray transform \( Pf : S^{n-1} \times \mathbb{R}^n \to \mathbb{R} \) defined by (4.7) (we extend \( f \) by 0 outside \( Y \)). We recall the following estimate

\[
\| f \|_{H^{-\frac{1}{2}}(Y)} \leq \left( \int_{S^{n-1}} \int_{\Pi_v} |Pf(v, x)|^2 dx dv \right)^{\frac{1}{2}},
\]  

(6.27)

where \( \Pi_v := \{ x \in \mathbb{R}^n \mid v \cdot x = 0 \} \) for \( v \in S^{n-1} \). Note that using the estimate \( \| \sigma \|_\infty \leq M \), we obtain

\[
\int_0^{\tau_+(x_0', v)} \sigma(x_0' + sv, v) ds \leq M \tau_+(x_0', v) \leq M \text{diam}(X), \quad \text{for } (x_0', v) \in \Gamma_.
\]  

(6.28)

Replacing \( \sigma \) by \( \tilde{\sigma} \) on the left-hand side of (6.28) we obtain an estimate similar to (6.28) for \( \tilde{\sigma} \). Therefore using the estimate \( |e^{t_1 - e^{t_2}}| \geq e^{-M \text{diam}(X)} |t_1 - t_2| \) for \( (t_1, t_2) \in [0, +\infty)^2 \), \( \max(t_1, t_2) \leq M \text{diam}(X) \), we obtain

\[
\left| e^{-\int_0^{\tau_+(x_0', v)} \sigma(x_0' + sv, v) ds} - e^{-\int_0^{\tau_+(x_0', v)} \tilde{\sigma}(x_0' + sv, v) ds} \right| \geq e^{-M \text{diam}(X)} |P(\sigma - \tilde{\sigma})(v, x_0')|,
\]  

(6.29)

for \((x_0', v) \in \Gamma_\cdot\). Integrating the left-hand side of (4.5) over \( \partial X \) and using (6.29), we obtain

\[
\int_{\Gamma_\cdot} |P(\sigma - \tilde{\sigma})(v, x_0')| d\xi(v, x_0') \leq e^{M \text{diam}(X)} \text{Vol}(\partial X) C_1 \| A_{S,W} - \tilde{A}_{S,W} \|_{\mathcal{N}, T},
\]  

(6.30)

where \( C_1 \) is the constant that appears on the right-hand side of (4.5). Note that using that \( X \) is a convex open subset of \( \mathbb{R}^n \) with \( C^1 \) boundary we obtain \( \int_{\Gamma_\cdot} |P(\sigma - \tilde{\sigma})(v, x_0')| d\xi(v, x_0') = \int_{S^{n-1}} \int_{\Pi_v} |P(\sigma - \tilde{\sigma})(v, x)| d\xi(v) d\sigma(\sigma) \) for \((v, x) \in T S^{n-1} \) (see (6.28) and the estimates \( \sigma \geq 0 \), \( \tilde{\sigma} \geq 0 \)) we obtain

\[
\left( \int_{S^{n-1}} \int_{\Pi_v} |P(\sigma - \tilde{\sigma})(v, x)|^2 d\sigma(\sigma) d\xi(v) \right)^{\frac{1}{2}} \leq C_3 \| \sigma - \tilde{\sigma} \|_{\mathcal{N}, T}^2 \| A_{S,W} - \tilde{A}_{S,W} \|_{\mathcal{N}, T}^2.
\]  

(6.31)

where \( C_3 = (\text{diam}(X) e^{M \text{diam}(X)} \text{Vol}(\partial X) C_1)^{\frac{1}{2}} \). Combining (6.31) and (6.27) we obtain (4.11).

We now prove (4.14). Let \( f \in L^2(X) \), supp\( f \subseteq \tilde{Z} \). We consider the weighted X-ray transform of \( f \), \( P_\theta f \), defined by

\[
P_\theta f(x, v) = \int_0^{\tau_+(x, v)} f(pv + x) \vartheta(pv + x) dp, \quad \text{for a.e. } (x, v) \in \Gamma_\cdot,
\]  

(6.32)

where \( \vartheta : X \times S^{n-1} \to (0, +\infty) \) is the analytic function given by

\[
\vartheta(x, v) = (\tau_-(x, v) \tau_+(x, v))^{-\frac{n-1}{2}} g(x, v, v), \quad \text{for } (x, v) \in X \times S^{n-1}.
\]  

(6.33)

From [12, theorem 2.2] and from [16, theorem 4] we obtain

\[
\| f \|_{H^{-\frac{1}{2}}(Z)} \leq C \| P_\theta f \|_{L^2(\Gamma_\cdot, d\xi)},
\]  

(6.34)
where \( C = C(X, Z, g) \) is a constant that does not depend on \( f \). Let \( x_0' \in \partial X \) and let \( x \in \partial X \) such that \( px_0' + (1 - p)x \in Z \) for some \( p \in (0, 1) \) where \( v_0 = \frac{x-x_0'}{|x-x_0'|} \) and \( t_0 = |x-x_0'| \). Note that using (3.23) (since \( \tilde{k} \in L^\infty(Z) \) and \( \text{supp} \tilde{k} \subseteq \tilde{Z} \subseteq \{ x \in X \mid \inf_{\delta(x) \in \partial X} |x-x'| \geq \delta \} \)), we obtain
\[
\int_0^\delta \frac{\tilde{k}(x_0' + pv_0', v_0', v_0')}{p^{\frac{n-1}{2}}(\tau_+ (x_0', v_0') - p)^{\frac{n-1}{2}}} \, dp \leq \| \tilde{k} \|_{L^\infty(Z)} \int_0^\delta \frac{1}{p^{\frac{n-1}{2}}(\tau_+ (x_0', v_0') - p)^{\frac{n-1}{2}}} \, dp \\
\leq \| \tilde{k} \|_{L^\infty(Z)} \delta^{-(n-1)} \tau_+ (x_0', v_0') \leq \| \tilde{k} \|_{L^\infty(Z)} \delta^{-(n-1)} \text{diam}(X). \tag{6.35}
\]
We use the estimate
\[
|P_\sigma(k_0 - \tilde{k}_0)(x_0', v_0')| \leq e^{P_\sigma(v_0', x_0')}|P_\sigma k_0(x_0', v_0')| e^{-P_\sigma(v_0', x_0')} - e^{-P_\sigma(v_0', x_0')} + e^{P_\sigma(v_0', x_0')} e^{-P_\sigma(v_0', x_0')} P_\sigma k_0(x_0', v_0') - e^{-P_\sigma(v_0', x_0')} P_\sigma k_0(x_0', v_0'). \tag{6.36}
\]
Integrating both sides of inequality (6.36) over \( v_0' \in S_{v_0}^{n-1} \) and using the estimate \( e^{P_\sigma(v_0', x_0')} \leq e^{M \text{diam}(X)} \), and using (6.35), (4.5)–(4.6), we obtain
\[
\int_{\mathbb{S}^{n-1}_{v_0}} |P_\sigma(k_0 - \tilde{k}_0)(x_0', v_0')| \nu(x_0') \cdot \nu |dv \leq \delta^{-(n-1)} \text{diam}(X)e^{M \text{diam}(X)} C_1 \| \tilde{k} \|_{L^\infty} \| A_{S,W} - \tilde{A}_{S,W} \|_{\eta,T} + \frac{\text{Vol}(\mathbb{S}^{n-1})e^{M \text{diam}(X)} C_2}{2} \left\| (\tau - |z - z'|)^{\frac{n-1}{2}}(\Gamma_1 - \tilde{\Gamma}_1)(\tau, z, z') \right\|_{L^\infty([0,T] \times \partial X \times \partial X)}, \tag{6.37}
\]
where \( C_1 \) and \( C_2 \) are the constants that appear on the right-hand side of (4.5) and (4.6).

From the estimate \( |P_\sigma(k_0 - \tilde{k}_0)(x_0', v_0')| \leq \| k - \tilde{k} \|_{L^\infty(Z)} \delta^{-(n-1)} \text{diam}(X) \) for a.e. \( (x_0', v_0') \in \Gamma_- \) (see (6.35)), it follows that
\[
\| P_\sigma(k_0 - \tilde{k}_0) \|_{L^2(\Gamma_-, \delta)} \leq \| k - \tilde{k} \|_{L^\infty(Z)} \delta^{-(n-1)} \text{diam}(X) \int_{\partial X} \int_{\mathbb{S}^{n-1}_{x_0}} |P_\sigma(k_0 - \tilde{k}_0)(x_0', v_0')| \nu(x_0') \cdot \nu |dv| \mu(x_0'). \tag{6.38}
\]
Combining (6.37)–(6.38) and (6.34) we obtain (4.14).

**Proof of Theorem 4.4** (4.13). We first prove (6.42) given below. Note that from (3.24)–(3.25), it follows that
\[
|P_\sigma f(v, x)| \leq \| f \|_{L^\infty} \int_0^\tau \frac{1}{\sqrt{t(x, v) - t}} \, dt = C \| f \|_{L^\infty}, \tag{6.39}
\]
for \((v, x) \in \Gamma_- \) and for \( f \in C(\Gamma \cap L^\infty(X)) \), where \( C = \int_0^1 \frac{1}{\sqrt{t(1-t)}} \, dt \).

Note that \( \nu(x) = x \) and \( \nu(x) \cdot (x-x_0') = |\nu(x_0') \cdot (x-x_0')| \) for \((x, x_0') \in \partial X = \mathbb{S}^1 \). Therefore from (4.2), it follows that
\[
|\nu(x_0') \cdot v_0'|^2 \left| E(x, x_0') P_{\sigma_0} k_0(v_0', x) - \tilde{E}(x, x_0') P_{\sigma_0} \tilde{k}(v_0', x) \right| \leq C' \left\| \sqrt{\tau^2 - |z - z'|^2} (\Gamma_1 - \tilde{\Gamma}_1)(\tau, z, z') \right\|_{L^\infty}, \tag{6.40}
\]
for \((x, x_0') \in \partial X^2, x \neq x_0' \) and \( \nu_0 = \frac{x-x_0'}{|x-x_0'|} \) (we also used \( P_{\sigma_0} k_0(v_0, x) = g(v_0, v_0) P_{\sigma_0} k_0(v_0, x) \) and the similar identity for \( \tilde{k} \)), where \( C' = \frac{1}{2 \inf_{\tau_-} \inf_{x \in \mathbb{S}^1} \text{W} \inf_{v \in \mathbb{S}^{n-1}} \frac{1}{g(v, v)} \). In addition, from (6.40), (6.36) (with \( P_{\sigma_0} \) in place of \( \text{"P}_{\sigma} \)) and from (6.28) and (6.39) (with \( \tilde{k}_0 \) in place of \( f \)), it follows that
\[
|\nu(x_0') \cdot v_0'|^2 \left| \text{"P}_{\sigma_0} (k_0 - \tilde{k}_0)(x_0', v_0') \right| \leq e^{M \text{diam}(X)} C \| k_0 \|_{L^\infty} \left| e^{-P_\sigma(v_0', x_0')} - e^{-P_\sigma(v_0', x_0')} \right| \nu(x_0') \cdot v_0' + e^{M \text{diam}(X)} \left| e^{-P_\sigma(v_0', x_0')} P_{\sigma_0} k_0(x_0', v_0') - e^{-P_\sigma(v_0', x_0')} P_{\sigma_0} \tilde{k}(x_0', v_0') \right| \nu(x_0') \cdot v_0'|, \tag{6.41}
\]
for \((x', v') \in \Gamma_-(\text{we also used the estimate } |\nu(x') \cdot v'| \leq 1). Performing the change of variables 
\(x = x' + \tau_+(x', v')v'\) \((\nu(x')v')^{-1} d\mu(x) = dv'\) on the left-hand side of (4.1), we obtain that the estimate (4.5) still holds. Using (4.5), (6.41) and (6.40) \((and (4.8)), we obtain that there exists a constant \(C''\) such that
\[
\int_{S_n}^n |\nu(x') \cdot v'|^2 |P(\rho(k_0 - \tilde{k}_0))(x', v')| dv' \leq C'' \left(\|k_0\|_\infty \|A_{S,W} - \tilde{A}_{S,W}\|_{v,T} \right) + \left\|\sqrt{\tau^2 - |z - z'|^2} (\Gamma_1 - \tilde{\Gamma}_1)(\tau, z, z')\right\|_{L^\infty} \quad (6.42)
\]
for \(x' \in \partial X. Moreover, using (6.39) \((with k_0 - \tilde{k}_0 in place of "f") and Cauchy-Bunyakowski-Schwarz estimate, we obtain
\[
\left(\int_{\Gamma_-} |P(\rho(k_0 - \tilde{k}_0))(v, x')d\xi(x', v')\right)\frac{1}{2} \leq C'\frac{1}{2} \left(\int_{\partial X} \int_{S_n}^n \left(\int_{\partial X} \int_{S_n}^n |P(\rho(k_0 - \tilde{k}_0))(v, x')|v' \cdot \nu(x')^2 dv' d\mu(x')\right)\right)^\frac{1}{2} = C'\left(\int_{\partial X} \int_{S_n}^n |P(\rho(k_0 - \tilde{k}_0))(v, x')|v' \cdot \nu(x')^2 dv' d\mu(x')\right)^\frac{1}{2} \quad (6.43)
\]
Finally combining (6.42)-(6.43), (6.27) \((and the identity \int_{\Gamma_-} |P(\rho(k_0 - \tilde{k}_0))(v, x')^2 d\xi(v, x') = \int_{S_n}^n \int_{\Pi_+} |P(\rho(k_0 - \tilde{k}_0))(v, x)|^2 dx dv), we obtain (4.13). \(\square\)

### 7 Proof of Theorem 3.2

For \(0 < b < a\) we remind that
\[
\int_0^{2\pi} \frac{1}{a - b \sin(\Omega)} d\Omega = \frac{2\pi}{\sqrt{a^2 - b^2}}. \quad (7.1)
\]
We will use the following Lemma 7.1 to prove Theorem 3.2 (3.13), (3.14) and (3.15). We will use the following Lemmas 7.2 and 7.3 to prove Theorem 3.2 (3.16).

**Lemma 7.1.** Let \(n \geq 2. Let N denote the nonnegative measurable function from \((0, T) \times \partial X \times \mathbb{R}^n to [0, +\infty[ defined by \)
\[
N(\tau, x, x') = \chi_{(0, +\infty)}(\tau - |x - x'|) \frac{(\tau - (x - x') \cdot v)^{n-3}}{|x - x' - \tau v|^{2n-4}} dv, \quad (7.2)
\]
for \((\tau, x, x') \in (0, T) \times \partial X \times \mathbb{R}^n. When \(n = 2, then \)
\[
N(\tau, x, x') = \chi_{(0, +\infty)}(\tau - |x - x'|) \frac{2\pi}{\sqrt{\tau^2 - |x - x'|^2}} \quad (7.3)
\]
for \((\tau, x, x') \in (0, T) \times \partial X \times \mathbb{R}^n. When \(n = 3, then \)
\[
N(\tau, x, x') = 2\pi \frac{\chi_{(0, +\infty)}(\tau - |x - x'|)}{\tau|x - x'|} \ln \left(\frac{\tau + |x - x'|}{\tau - |x - x'|}\right), \quad (7.4)
\]
for \( (\tau, x, x') \in (0, T) \times \partial X \times \mathbb{R}^n \). When \( n \geq 4 \), then
\[
\sup_{(\tau, x, x') \in (0, T) \times \partial X \times \mathbb{R}^n} \tau |x - x'|^{n-2} N(\tau, x, x') < \infty.
\] (7.5)

Lemma 7.1 is proved in Section 9.

**Lemma 7.2.** Assume that
\[
supp k \subseteq \{ x \in X \mid \sup_{y \in \partial X} |x - y| \geq \delta \} \text{ for some } 0 < \delta < \text{diam}(X).
\] (7.6)

Let \((\tau, x, x', v) \in (0, T) \times \partial X \times \partial X \times \mathbb{S}^{n-1}\) such that \( x \neq x' \) and \( \tau > |x - x'| \). Then the following statement is valid:
\[
\text{if } \tau < \delta, \text{ then } x - sv \notin supp k \text{ for } s = \frac{\tau^2 - |x - x'|^2}{2(\tau - (x - x') \cdot v)}.
\] (7.7)

**Proof of Lemma 7.2.** Let \( s := \frac{\tau^2 - |x - x'|^2}{2(\tau - (x - x') \cdot v)} \). Straightforward computations give
\[
s + |x - x' - sv| = \tau.
\] (7.8)

Therefore \( s \leq \tau \). Using (7.6) and (7.8) we obtain (7.7). Lemma 7.2 is proved. \( \square \)

**Lemma 7.3.** Assume that \( k \) satisfies (7.6). Let \((\tau, x, x', v) \in (0, T) \times \partial X \times \partial X \times \mathbb{S}^{n-1}\) such that \( x \neq x' \), \( \tau > |x - x'| \) and \( \tau \geq \delta \). Assume \( \frac{x - x'}{|x - x'|} = (1, 0, \ldots, 0) \) and put \( v = (v_1, \ldots, v_n) \). Let
\[
\zeta_-(\tau, x, x') = \frac{|x - x'|^2 - \tau^2 + \tau \delta}{\delta |x - x'|} \text{ and } \zeta_+(\tau, x, x') = \frac{\tau^2 + |x - x'|^2 - \delta \tau}{|x - x'|((2\tau - \delta)}.
\] (7.9)

Then the following statements are valid:
\[
\zeta_-(\tau, x, x') < 1 \text{ and } \zeta_+(\tau, x, x') > -1,
\] (7.10)
\[
\text{if } \zeta_-(\tau, x, x') \leq -1 \text{ or } \zeta_+(\tau, x, x') \geq 1, \text{ then } \tau \geq |x - x'| + \delta,
\] (7.11)
\[
\text{if } v_1 < \zeta_-(\tau, x, x') \text{ or } v_1 > \zeta_+(\tau, x, x'), \text{ then } x - sv \notin supp k \text{ for } s = \frac{\tau^2 - |x - x'|^2}{2(\tau - (x - x') \cdot v)}.
\] (7.12)

**Proof of Lemma 7.3.** We prove (7.10). Note that \( |x - x'|^2 - \tau^2 + \tau \delta - \delta |x - x'| = (|x - x'| - \frac{\delta}{2})^2 - (\tau - \frac{\delta}{2})^2 < 0 \) since we assume \( |x - x'| < \tau \) and \( \delta \leq \tau \). Therefore using the definition of \( \zeta_-(\tau, x, x') \) (7.9), we obtain \( \zeta_-(\tau, x, x') < 1 \). In addition note that \( \tau^2 + |x - x'|^2 - \delta \tau + 2|x - x'|\tau - |x - x'|\delta = (\tau + |x - x'|)(\tau + |x - x'| - \delta) > 0 \) since we assume \( \tau \geq \delta \). Therefore using the definition of \( \zeta_+(\tau, x, x') \) (7.9), we obtain \( \zeta_+(\tau, x, x') > -1 \).

Now we prove (7.11). First assume \( \zeta_-(\tau, x, x') \leq -1 \). Then from (7.9) it follows that \( (|x - x'| + \frac{\delta}{2})^2 \leq (\tau - \frac{\delta}{2})^2 \), which implies \( |x - x'| + \delta \leq \tau \). Now assume \( \zeta_+(\tau, x, x') \geq 1 \). Then from (7.9) it follows that \( (\tau - |x - x'| - \delta)(\tau - |x - x'|) \geq 0 \), which implies \( \tau \geq |x - x'| + \delta \) (since \( \tau > |x - x'| \)).

Finally we prove (7.12). As we assume \( \frac{x - x'}{|x - x'|} = (1, 0, \ldots, 0) \) we obtain
\[
s := \frac{\tau^2 - |x - x'|^2}{2(\tau - (x - x') \cdot v)} = \frac{\tau^2 - |x - x'|^2}{2(\tau - |x - x'| v_1)}
\] (7.13)
and
\[
|x - x' - sv| = \tau - s = \frac{\tau^2 - |x - x'|^2}{2(\tau - |x - x'| v_1)} + \tau.
\] (7.14)
Assume \( v_1 < \zeta_-(\tau, x, x') \). Then \( \zeta_-(\tau, x, x') > -|v| = -1 \) and we obtain \( s < \frac{\tau^2 - |x-x'|^2}{2(\tau - |x-x'|\zeta_-(\tau, x, x'))} = \frac{\delta}{2} < \delta \). Therefore, using assumption (7.6) and \((x, v) \in \partial X \times S^{n-1} \) we obtain \( x - sv \notin supp k \).

Now assume \( v_1 > \zeta_+(\tau, x, x') \). Then \( \zeta_+(\tau, x, x') < |v| = 1 \) and using (7.14) we obtain \( |x - x' - sv| < -\frac{\tau^2 - |x-x'|^2}{2(\tau - |x-x'|\zeta_+(\tau, x, x'))} + \tau = \frac{\delta}{2} < \delta \). Therefore, using the assumption (7.6) and \((x', v) \in \partial X \times S^{n-1} \) we obtain \( x - sv = x' + (x - x' - sv) \notin supp k \). Lemma 7.3 is proved.

We are ready to prove Theorem 3.2. First we give an estimate on the simple scattering term. From (3.10) it follows that

\[
|\gamma_1(\tau, x, x')| \leq 2^{n-2} \|W\|_{\infty} \|S\|_{\infty} \|k\|_{\infty} I_1(\tau, x, x') \tag{7.15}
\]

for a.e. \((\tau, x, x') \in \mathbb{R} \times \partial X \times \partial X\), where

\[
I_1(\tau, x, x') = \chi_{(0, \infty)}(\tau - |x - x'|) \int_{S^{n-1}} \chi_{supp k}(x - sv) |x = \frac{\tau^2 - |x-x'|^2}{2(\tau - |x-x'|\zeta_+(\tau, x, x'))} \times \frac{\tau - (x - x') \cdot v}{|x - x' - \tau v|^{2n-4}} dv. \tag{7.16}
\]

Let \((\tau, x, x') \in (0, T) \times \partial X \times \partial X\) be such that \( x \neq x' \) and \( \tau > |x - x'| \). Assume without loss of generality \( x' - x = |x' - x|(1, 0 \ldots 0) \).

First we prove (3.13)–(3.15). From (7.16) and (7.2), it follows that

\[
I_1(\tau, x, x') \leq N(\tau, x, x'). \tag{7.17}
\]

Combining (7.3) (respectively (7.4), (7.5)) with (7.15) and (7.17), we obtain (3.13) (respectively (3.14), (3.15)).

Now assume that \( k \) satisfies (7.6) for some \( \delta > 0 \). Using (7.16) and (7.7), we obtain

\[
\text{if } \tau < \delta \text{ then } I_1(\tau, x, x') = 0. \tag{7.18}
\]

We prove (3.16) for \( n = 2 \). Using (7.17), we obtain that \( I_1(\tau, x, x') \leq \frac{2\pi}{\sqrt{\delta \tau - |x-x'|}} \) for \( \tau \geq \delta \).

Combining (7.15) with this latter estimate and (7.18), we obtain (3.16) for \( n = 2 \).

We now prove (3.16) for \( n \geq 3 \). Let \( n \geq 3 \) and \( \tau \geq \delta \) (the case \( \tau < \delta \) is already considered in (7.18)). Performing the change of variables \( \tau = \frac{\tau^2 - |x-x'|^2}{2(\tau - |x-x'|\sin(\Omega))} = \frac{\tau - |x-x'|}{2} + \frac{\sin(\Omega)}{2|\Omega|^2} \) with \( \nu = (\sin(\Omega), \cos(\Omega)\omega), \Omega \in (-\frac{\pi}{2}, \frac{\pi}{2}), \omega \in S^{n-2} \) on the right-hand side of (7.16), we obtain

\[
I_1(\tau, x, x') = 2^{n-2} \left( \frac{\tau^2 - |x-x'|^2}{2(\tau - |x-x'|\sin(\Omega))} \right)^{n-3} \int_0^{\frac{|x-x'|}{\tau}} \omega \chi_{supp k}(x - sv) \omega = \frac{\tau^2 - |x-x'|^2}{2|\Omega|} \frac{\sin(\Omega)}{2|\Omega|^2} \frac{1}{\sqrt{\frac{\tau^2 - |x-x'|^2}{2(\tau - |x-x'|\sin(\Omega))}} + \sqrt{\frac{\tau^2 - |x-x'|^2}{2(\tau - |x-x'|\sin(\Omega))} + \frac{\tau - |x-x'|}{2}}} \omega \cdot d\omega. \tag{7.19}
\]

\( \Omega = \arcsin\left(\frac{|x-x'|}{\sqrt{\tau^2 - |x-x'|^2}}\right) \quad \Omega = \arccos\left(\frac{|x-x'|}{\sqrt{\tau^2 - |x-x'|^2}}\right) \quad \Omega = \arctan\left(\frac{|x-x'|}{\sqrt{\tau^2 - |x-x'|^2}}\right) \quad \Omega = \arctan\left(\frac{|x-x'|}{\sqrt{\tau^2 - |x-x'|^2}}\right) \)

Now assume \( \tau > \frac{\delta}{2} + |x - x'| \). Then

\[
|x - x'|^2 - n \int_0^{\frac{|x-x'|}{\tau}} \frac{\sqrt{\tau^2 - |x-x'|^2}}{\left(\frac{\tau^2 - |x-x'|^2}{2(\tau - |x-x'|\sin(\Omega))} + \frac{\tau - |x-x'|}{2}\right)^{n-2}} \frac{1}{\sqrt{\frac{\tau^2 - |x-x'|^2}{2(\tau - |x-x'|\sin(\Omega))} + \sqrt{\frac{\tau^2 - |x-x'|^2}{2(\tau - |x-x'|\sin(\Omega))} + \frac{\tau - |x-x'|}{2}}} \omega \cdot d\omega.
\]

\[
\leq \left( \frac{\delta}{4} \right)^{4-2n} |x - x'|^2 - n \int_0^{\frac{|x-x'|}{\tau}} \sqrt{\tau^2 - |x-x'|^2} \frac{1}{\sqrt{\tau^2 - |x-x'|^2} + \frac{\tau - |x-x'|}{2}} \omega \cdot d\omega \leq \left( \frac{\delta}{4} \right)^{4-2n}.
\]
Therefore using (7.19) we obtain

\[
(\tau - |x - x'|)^{-\frac{n-3}{2}} I_1(\tau, x, x') \leq 2^n \text{Vol}_{n-2}(S^{n-2})(T + \text{diam}(X))^{\frac{n-3}{2}} \left(\frac{\delta}{2}\right)^{4-2n}.
\] (7.20)

Finally assume \(\delta \leq \tau \leq \frac{\delta}{2} + |x - x'|\) and \(|x - x'| < \tau \leq T\). From (7.12) it follows that

\[
(\tau - |x - x'|)^{-\frac{n-3}{2}} I_1(\tau, x, x') \leq \frac{\text{Vol}_{n-2}(S^{n-2})(T + \text{diam}(X))^{\frac{n-3}{2}}}{2^{n-2}|x - x'|^{n-2}} \left(\frac{r_+(\tau, x, x')}{r_-(\tau, x, x')}\right)^{\frac{r_+(\tau, x, x')}{r_-(\tau, x, x')}} \left(\frac{\sqrt{r(|x - x'| - r)^{n-3}}}{r_-(\tau, x, x')} d\tau, \right.
\]

where

\[
\begin{align*}
  r_-(\tau, x, x') &:= \frac{\tau^2 - |x - x'|^2}{2(\tau - |x - x'|\zeta_-(\tau, x, x'))} - \frac{\tau - |x - x'|}{2} = \frac{|x - x'| + \delta - \tau}{2}, \\
r_+(\tau, x, x') &:= \frac{\tau^2 - |x - x'|^2}{2(\tau - |x - x'|\zeta_+(\tau, x, x'))} - \frac{\tau - |x - x'|}{2} = \frac{\tau - \delta + |x - x'|}{2}.
\end{align*}
\] (7.22) (7.23)

Note that

\[
\int_{r_-(\tau, x, x')}^{r_+(\tau, x, x')} \sqrt{r(|x - x'| - r)^{n-3}} d\tau = 2 \int_{r_-(\tau, x, x')}^{\frac{|x - x'|}{2}} \sqrt{r(|x - x'| - r)^{n-3}} \left(\frac{x - x'|}{\frac{|x - x'|}{2} + r}\right)^n d\tau
\]

\[
\leq 2 \left(\frac{\tau}{2}\right)^{2-n} |x - x'|^{n-3} \int_{r_-(\tau, x, x')}^{\frac{|x - x'|}{2}} \frac{1}{\left(\frac{|x - x'|}{2} + r\right)^n} d\tau = 2^{n-1} r^{2-n} |x - x'|^{-n} \int_{r_-(\tau, x, x')}^{\frac{|x - x'|}{2}} \frac{1}{\left(\frac{|x - x'|}{2} + r\right)^n} d\tau.
\] (7.24)

Using (7.22) we obtain

\[
\int_{r_-(\tau, x, x')}^{\frac{|x - x'|}{2}} \frac{1}{\left(\frac{|x - x'|}{2} + r\right)^n} d\tau = C(n, \tau) := \begin{cases} 
\ln \left(\frac{\tau}{2}\right), & \text{if } n = 3, \\
\frac{1}{n-3} \left(\frac{\delta}{\tau}\right)^{3-n} - \left(\frac{\delta}{\tau}\right)^{3-n}, & \text{otherwise}.
\end{cases} \] (7.25)

From (7.21), (7.24), (7.25) and the estimates \(\delta \leq \tau < \frac{\delta}{2} + |x - x'|\), it follows that

\[
(\tau - |x - x'|)^{-\frac{n-3}{2}} I_1(\tau, x, x') \leq 2^n \text{Vol}_{n-2}(S^{n-2})\delta^{-(n-1)}(T + \text{diam}(X))^{\frac{n-3}{2}} C(n, T),
\] (7.26)

where the constant \(C(n, T)\) is defined in (7.25). Combining (7.15) with (7.18), (7.20) and (7.26), we obtain (3.16) for \(n \geq 3\). \(\square\)

## 8 Proof of Theorem 3.3

We shall use the following Lemmas 8.1, 8.2, 8.3 and 8.4. Lemma 8.1 is proved in Section 10 and Lemmas 8.2, 8.3 and 8.4 are proved in Section 9.

We introduce some notation first. Let \(m \geq 1\) and \(x', x \in \partial X\) such that \(x \neq x'\). Let \(\tau \geq 0\). We denote by \(\mathcal{E}_{m,n}(\tau, x, x')\) the subset of \((\mathbb{R}^n)^m\) defined by

\[
\mathcal{E}_{m,n}(\tau, x, x') = \{ (z_1, \ldots, z_m) \in (\mathbb{R}^n)^m \mid \ |z_1| + \ldots + |z_m| + |x - x' - z_1 - \ldots - z_m| < \tau \}.
\] (8.1)

When \(\tau \leq |x - x'|\), then \(\mathcal{E}_{m,n}(\tau, x, x') = \emptyset\).
Lemma 8.1. Let \( J_2 \) be the function from \((0, T) \times \partial X \times \mathbb{R}^n \) defined by
\[
J_2(\mu, z, z') = \int_{E_{1,n}(\mu, z, z')} \frac{1}{|y|^{n-1}} N(\mu - |y|, z, z' + y) dy,
\]
where \( N \) is defined by (7.2). Then the following statements are valid:
\[
\sup_{(\mu, z, z') \in (0, T) \times \partial X \times \mathbb{R}^n} J_2(\mu, z, z') < \infty, \text{ when } n = 2; \tag{8.3}
\]
\[
\sup_{(\mu, z, z') \in (0, T) \times \partial X \times \mathbb{R}^n} (\mu - |z - z'|)^{-1} \mu |z - z'| \left( 1 + \ln \left( \frac{\mu + |z - z'|}{\mu - |z - z'|} \right) \right)^{-2} J_2(\mu, z, z') < \infty, \text{ when } n = 3; \tag{8.4}
\]
\[
\sup_{(\mu, z, z') \in (0, T) \times \partial X \times \mathbb{R}^n} (\mu - |z - z'|)^{-1} \mu |z - z'|^{n-2} J_2(\mu, z, z') < \infty, \text{ when } n \geq 4. \tag{8.5}
\]

Lemma 8.2. Let \( m \geq 3 \) and let \( \tilde{J}_m \) be the function from \( \{(\tau, x, x') \in (0, T) \times \mathbb{R}^n \times \mathbb{R}^n \mid 0 < |x - x'| < \tau \} \) defined by
\[
\tilde{J}_m(\tau, x, x') = \int_{E_{m-2,n}(\tau, x, x')} \frac{dy_2 \ldots dy_{m-1}}{|y_2|^{n-1} \ldots |y_{m-1}|^{n-1}|x - x' - y_2 - \ldots - y_{m-1}|^{n-2}}, \tag{8.6}
\]
for \((\tau, x, x') \in (0, T) \times \mathbb{R}^n \times \mathbb{R}^n, 0 < |x - x'| < \tau\). When \( n = 3 \), then there exists a constant \( \tilde{C} \) which does not depend on \( \tau \) such that
\[
\tilde{J}_m(\tau, x, x') \leq \tilde{C} \frac{\tau - |x - x'|}{|x - x'|} \left( 1 + \ln \left( \frac{\tau + |x - x'|}{\tau - |x - x'|} \right) \right) \frac{m^{-1}(\Vol_n(\mathbb{S}^{n-1})\tau)^{m-3}}{(m-3)!}, \tag{8.7}
\]
for \((\tau, x, x') \in (0, T) \times \mathbb{R}^n \times \mathbb{R}^n, 0 < |x - x'| < \tau\). When \( n \geq 4 \), then there exists a constant \( \tilde{C} \) which does not depend on \( \tau \) such that
\[
\tilde{J}_m(\tau, x, x') \leq \tilde{C} (\tau - |x - x'|)|x - x'|^{2-n} \frac{m^{-1}(\Vol_n(\mathbb{S}^{n-1}))^{m-3}}{(m-3)!}, \tag{8.8}
\]
for \((\tau, x, x') \in (0, T) \times \mathbb{R}^n \times \mathbb{R}^n, 0 < |x - x'| < \tau\).

Lemma 8.3. Let \( n \geq 2 \). Let \((\tau, x, x') \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \) be such that \( \tau > |x - x'| > 0 \), the following estimate is valid:
\[
\Vol_n(E_{1,n}(\tau, x, x')) \leq \frac{\Vol_n(\mathbb{S}^{n-2})\pi(\tau + |x - x'|)}{4} \left( \frac{\sqrt{\tau^2 - |x - x'|^2}}{2} \right)^{n-1}. \tag{8.9}
\]
where \( E_{1,n} \) is defined by (8.1).

Lemma 8.4. Let \( B \) be the function from \( \{(\mu, z, z') \in (0, T) \times \partial X \times \mathbb{R}^3 \mid \mu > |z - z'| > 0 \} \) to \( \mathbb{R} \) defined by
\[
B(\mu, z, z') := \int_{E_{1,3}(\mu, z, z')} \ln \left( \frac{\mu - |y| + |z - z' - y|}{\mu - |y| - |z - z' - y|} \right) dy = \int_{E_{1,3}(\mu, t_0(0,0,0),0)} \ln \left( \frac{\mu - |y| + (t_0, 0, 0) - y}{\mu - |y| - (t_0, 0, 0) - y} \right) dy, \tag{8.10}
\]
for \((\mu, z, z') \in (0, T) \times \partial X \times \mathbb{R}^n \) where \( t_0 = |z - z'|, z \neq z', \mu > |z - z'| \). Then we have:
\[
\sup_{(\mu, z, z') \in (0, T) \times \partial X \times \mathbb{R}^3} (\mu - |z - z'|)^{-1} \left( 1 + \ln \left( \frac{\mu + |z - z'|}{\mu - |z - z'|} \right) \right)^{-1} B(\mu, z, z') < \infty. \tag{8.11}
\]
We also need the explicit expression of $\gamma_m$, $m \geq 2$, to prove Theorem 3.3

\[
\gamma_2(\tau, x, x') := \int_{y \in \mathcal{E}_{1,n}(\tau, x, x')} \int_{x' \in X} (\nu(x') \cdot v) W(x, v) \left[ E(x, x - (\tau - |y| - s_1), v, x' + y, x') \right] dy \quad \text{for } \tau \in \mathbb{R} \text{ and } a.e. (x, x') \in \partial X \times \partial X \text{ and for } m \geq 3.
\]

We are ready to prove Theorem 3.3. We prove (3.17), (3.18) and (3.19). Let $\tau \in (0, T)$ and let $x \in \partial X$, $x' \in \partial X$ and $x \neq x'$. We put $t_0 = |x - x'|$. We first look for an upper bound on $|\gamma_2(\tau, x, x')|$. Using (8.12) and the fact that $\sigma$ is a nonnegative function, we obtain

\[
|\gamma_2(\tau, x, x')| \leq 2^{n-2} \|W\|_\infty \|S\|_\infty \|k\|_\infty^2 J_2(\tau, x, x'),
\]

where $J_2$ and $\mathcal{E}_{1,n}(\tau, x, x')$ are defined by (8.2) and (8.1). From (8.14) and (8.3)–(8.5) it follows that there exists a real constant $C$ such that

\[
|\gamma_2(\tau, x, x')| \leq C \|W\|_\infty \|S\|_\infty \|k\|_\infty^2 \sup_{(s, z, z') \in (0, T) \times \mathbb{R}^n \times \mathbb{R}^n} J_2(s, z, z'), \text{ when } n = 2,
\]

and

\[
\tau \frac{|x - x'|}{(\tau - |x - x'|) \left(1 + \ln \left(\frac{x^2 + |x - x'|^2}{\tau - |x - x'|^2}\right)\right)^2} \gamma_2(\tau, x, x') \leq C \|W\|_\infty \|S\|_\infty \|k\|_\infty^2, \text{ when } n = 3,
\]

and

\[
\frac{\tau |x - x'|^{n-2}}{\tau - |x - x'|} |\gamma_2(\tau, x, x')| \leq C \|W\|_\infty \|S\|_\infty \|k\|_\infty^2, \text{ when } n \geq 4.
\]

Let $m \geq 3$. Using (8.13) we obtain

\[
|\gamma_m(\tau, x, x')| \leq 2^{m-2} \|W\|_\infty \|S\|_\infty \|k\|_\infty^m J_m(\tau, x, x'),
\]
where
\[ J_m(\tau, x, x') = \int_{E_{m-2,n}(\tau, x, x')} \frac{J_2(\tilde{\tau}(\tilde{y}), w(\tilde{y}), 0)}{|y_2|^{n-1} \ldots |y_{m-1}|^{n-1}} dy_2 \ldots dy_{m-1}, \tag{8.19} \]
and \( J_2 \) (resp. \( E_{m-2,n}(\tau, x, x') \)) is defined by (8.2) (resp. (8.1)) and where \( \tilde{y} = (y_2, \ldots, y_{m-1}) \), \( \tilde{\tau}(\tilde{y}) = \tau - |y_2| - \ldots - |y_{m-1}| \), \( t_0(\tilde{y}) = |x - x' - y_2 - \ldots - y_{m-1}| \) and \( w(\tilde{y}) = x - x' - y_2 - \ldots - y_{m-1} \) for \( \tilde{y} \in (\mathbb{R}^n)^{m-2} \).

Assume \( n = 2 \). Then using (8.19), (8.3) and spherical coordinates (and (8.1)), we obtain
\[ J_m(\tau, x, x') \leq \sup_{s > |z - x'|} J_2(s, z, z') \int_{E_{m-2,n}(\tau, x, x')} \frac{1}{|y_3| \ldots |y_{m-1}|} d\tilde{y} \leq (2\pi)^{m-2} \sup_{s > |z - x'|} J_2(s, z, z') \int_{(s, z') \in (0, T) \times \mathbb{R}^n} d\tau \ldots d\tau_{m-1} = (2\pi)^{m-2} \frac{\tau^{m-2}}{(m - 2)!} J_2(s, z, z'), \tag{8.20} \]
Finally combining (8.20) and (8.18), we obtain
\[ |\gamma_m(\tau, x, x')| \leq (2\pi)^{m-2}||W||_\infty||S||_\infty||k||_\infty^m \frac{\tau^{m-2}}{(m - 2)!} \sup_{s \in (0, 1)} J_2(s, z, z'), \tag{8.21} \]
Statement (3.17) follows from (8.15), (8.21).

Assume \( n \geq 3 \). Note that
\[ \frac{\mu - |z - z'|}{\mu} = 1 - \frac{|z - z'|}{\mu} \leq 1, \tag{8.22} \]
and
\[ \frac{\mu - |z - z'|}{\mu} \left( 1 + \ln \left( \frac{\mu + |z - z'|}{\mu - |z - z'|} \right) \right)^2 \leq \sup_{s \in (0, 1)} (1 - s) \left( 1 + \ln \left( \frac{1 + s}{1 - s} \right) \right)^2, \tag{8.23} \]
for \((\mu, z, z') \in (0, T) \times \mathbb{R}^n \times \mathbb{R}^n\) such that \(|z - z'| < \mu\).

From (8.19), (8.22) and (8.4), (8.23) and (8.5), it follows that there exists a real constant \( C' \) such that
\[ J_m(\tau, x, x') \leq C J_m(\tau, x, x'), \tag{8.24} \]
where \( J_m \) is defined by (8.6).

Assume \( n = 3 \). Combining (8.18), (8.24), (8.7), we obtain that there exists a real constant \( C' \) (which does not depend on \( \tau, x, x' \) and \( m \)) such that
\[ |\gamma_m(\tau, x, x')| \leq C'||W||_\infty||S||_\infty||k||_\infty^m \frac{\tau - |x - x'|}{|x - x'|} \left( 1 + \ln \left( \frac{\tau + |x - x'|}{\tau - |x - x'|} \right) \right) \frac{m^{n-1} (\Vol_{n-1}(S^{n-1}) \tau)^{m-3}}{(m - 3)!}. \tag{8.25} \]
Statement (3.18) follows from (8.16) and (8.25).

Now assume \( n \geq 4 \). Combining (8.18), (8.24), (8.8), we obtain that there exists a real constant \( C' \) (which does not depend on \( \tau, x, x' \) and \( m \)) such that
\[ |\gamma_m(\tau, x, x')| \leq C'||W||_\infty||S||_\infty||k||_\infty^m (\tau - |x - x'|)|x - x'|^{2-n} \frac{m^{n-1} (\Vol_{n-1}(S^{n-1}) \tau)^{m-3}}{(m - 3)!}. \tag{8.26} \]
Statement (3.19) follows from (8.17) and (8.26).

We now prove (3.20)–(3.21). Let \( n \geq 3 \) and \( m \geq 2 \). From the expression of \( \gamma_m \) (see (8.12)–(8.13)), it follows that

\[
|\gamma_m(\tau, x, x')| \leq 2^{n-2}\|W\|_\infty\|S\|_\infty\|k\|_\infty I_m(\tau, x, x')
\]

where

\[
I_m(\tau, x, x') := \int_{(y_2,...,y_m)\in E_{m-1,n}(\tau,x,x')} \frac{N(\tau - \sum_{i=2}^{m} y_i, x, x' + \sum_{i=2}^{m} y_i) dy_m \cdots dy_2}{|y_2|^{n-1} \cdots |y_m|^{n-1}}, \tag{8.28}
\]

where \( N \) and \( E_{m-1,n}(\tau, x, x) \) are defined by (7.2) and (8.1). Note that

\[
|y_m| \geq \delta \quad \text{and} \quad \tau - \sum_{i=2}^{m} |y_i| \geq |x - x' - y_2 - \ldots - y_m| \geq \delta, \tag{8.29}
\]

for \((y_2,\ldots,y_m)\in E_{m-1,n}(\tau,x,x')\) such that \( x' + y_2 \in \text{suppk} \) and \( x' + y_2 + \ldots + y_m \in \text{suppk} \) since \( \text{suppk} \subseteq \{ z \in \mathbb{X} \mid \supp_{\gamma \in \partial \mathbb{X}} |y - z| \geq \delta \} \).

First we consider the case \( n \geq 4 \). Using (8.28) and (7.5), we obtain

\[
I_m(\tau, x, x') \leq \|s|z - z'|^{n-2} N(s, z, z')\|_{L^\infty(\mathbb{R}_x \times \partial \mathbb{X}_s \times \mathbb{R}^n_s)} \times \int_{(y_2,...,y_m)\in E_{m-1,n}(\tau,x,x')} \frac{dy_m \cdots dy_2}{|y_2|^{n-1} \cdots |y_m|^{n-1}|x - x' - \sum_{i=2}^{m} y_i|^{n-2}}. \tag{8.30}
\]

Assume \( m = 2 \). From (8.29)–(8.30), we obtain

\[
I_m(\tau, x, x') \leq \delta^{-2n+2}\|s|z - z'|^{n-2} N(s, z, z')\|_{L^\infty(\mathbb{R}_x \times \partial \mathbb{X}_s \times \mathbb{R}^n_s)} \Vol(E_{2,n}(\tau, x, x')). \tag{8.31}
\]

Therefore using (8.9), we obtain

\[
I_m(\tau, x, x') \leq \delta^{-2n+2}\|s|z - z'|^{n-2} N(s, z, z')\|_{L^\infty(\mathbb{R}_x \times \partial \mathbb{X}_s \times \mathbb{R}^n_s)} \Vol_{n-2}(S^{n-2}) \pi(\tau + |x - x'|) \left( \frac{\sqrt{\tau^2 - |x - x'|^2}}{2} \right)^{n-1}. \tag{8.32}
\]

Assume \( m \geq 3 \). From (8.29)–(8.30), we obtain

\[
I_m(\tau, x, x') \leq \delta^{-2n+2}\|s|z - z'|^{n-2} N(s, z, z')\|_{L^\infty(\mathbb{R}_x \times \partial \mathbb{X}_s \times \mathbb{R}^n_s)} \int_{(y_2,...,y_m)\in E_{m-1,n}(\tau,x,x')} \frac{dy_m \cdots dy_2}{|y_2|^{n-1} \cdots |y_m|^{n-1}}. \tag{8.33}
\]

Note that \( |y_m| + |x - x' - y_m| \leq |y_2| + \ldots + |y_m| + |x - x' - y_2 - \ldots - y_m| \) for \((y_2,...,y_m)\in(\mathbb{R}^n)^{m-1}\).

Hence

\[
|y_2| + \ldots + |y_{m-1}| < \tau - |y_m| \quad \text{and} \quad |y_m| + |x - x' - y_m| < \tau, \tag{8.34}
\]

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for \((y_2, \ldots, y_m) \in \mathcal{E}_{m-1, n}(\tau, x, x')\) (see (8.1)). Therefore

\[
I_m(\tau, x, x') \leq \delta^{-2n+2} \|s - z\|^{n-2} N(s, z, z') \left\| L^\infty(\mathbb{R}_+ \times \partial X_x \times \mathbb{R}_y) \right\| \int_{\mathcal{E}_{1, n}(\tau, x, x')} \frac{dy_m \cdots dy_2}{y_m^{n-1} \cdots y_{m-1}^{n-1}}
\]

\[
= \delta^{-2n+2} \text{Vol}(\mathbb{S}^{n-1})^{m-2} \|s - z\|^{n-2} N(s, z, z') \left\| L^\infty(\mathbb{R}_+ \times \partial X_x \times \mathbb{R}_y) \right\| \int_{\mathcal{E}_{1, n}(\tau, x, x')} \frac{(\tau - |y_m|)^{m-2}}{(m-2)!} dy_m
\]

\[
\leq \delta^{-2n+2} \text{Vol}(\mathbb{S}^{n-1})^{m-2} \|s - z\|^{n-2} N(s, z, z') \left\| L^\infty(\mathbb{R}_+ \times \partial X_x \times \mathbb{R}_y) \right\| \text{Vol}(\mathcal{E}_{1, n}(\tau, x, x')) \tau^{m-2} \frac{1}{(m-2)!}
\]

\[
= \delta^{-2n+2} \text{Vol}(\mathbb{S}^{n-1})^{m-2} \|s - z\|^{n-2} N(s, z, z') \left\| L^\infty(\mathbb{R}_+ \times \partial X_x \times \mathbb{R}_y) \right\| \text{Vol}_{m-2}(\mathbb{S}^{n-2}) \pi (\tau + |x - x'|)
\]

\[
\times \left( \frac{\sqrt{\tau^2 - |x - x'|^2}}{2} \right)^{n-1} \tau^{m-2} \frac{1}{(m-2)!}.
\]

(8.35)

Statement (3.21) follows from (8.27), (8.32) and (8.35).

Now assume \(n = 3\) and \(m = 2\). Using (8.28)–(8.29), (7.4) and (8.10) we obtain

\[
I_2(\tau, x, x') \leq 2\pi \delta^{-4} B(\tau, x, x').
\]

(8.36)

Therefore using (8.11) and (8.36) we obtain

\[
\sup_{(s, z, z') \in \{0, \tau\} \times \partial X_x \times \mathbb{R} \atop \tau^2 - |z - z'|^2 > 0} \left( s - |z - z'| \right)^{-1} \left( 1 + \ln \left( \frac{s + |z - z'|}{s - |z - z'|} \right) \right)^{-1} I_2(s, z, z') < \infty,
\]

(8.37)

Now assume \(n = 3\) and \(m \geq 3\). Using (8.28), we obtain

\[
I_m(\tau, x, x') \leq \int_{\mathcal{E}_{m-2, n}(\tau, x, x')} \frac{J_2(\tau - \sum_{i=3}^m |y_i|, x, x' + \sum_{i=3}^m y_i) dy_m \cdots dy_3}{|y_3|^{n-1} \cdots |y_m|^{n-1}},
\]

(8.38)

where \(J_2\) is defined by (8.2). Using (8.38) and (8.4) and the estimate \(\sup_{r \in (0, 1)} r^{1 - \ln(r)} < \infty\) we obtain

\[
I_m(\tau, x, x') \leq D \int_{\mathcal{E}_{m-2, n}(\tau, x, x')} \frac{|x - x' - y_3 - \ldots - y_m| + \tau - \sum_{i=3}^m |y_i| dy_m \cdots dy_3}{|y_3|^{n-1} \cdots |y_m|^{n-1}} \left( \tau - \sum_{i=3}^m |y_i| \right) |x - x' - y_3 - \ldots - y_m|,
\]

(8.39)

where \(D := \sup_{r \in (0, 1)} r^{1 - \ln(r)} \sup_{(s, z, z') \in \{0, \tau\} \times \partial X_x \times \mathbb{R} \atop \tau^2 - |z - z'|^2 > 0} \left( s - |z - z'| \right)^{-1} \left| s - |z - z'| \right|^{-1} \left( 1 + \ln \left( \frac{s + |z - z'|}{s - |z - z'|} \right) \right)^{-2} \text{Vol}(\mathcal{E}_{1, 3}(\tau, x, x')) \right)^{-2} J_2(s, z, z').\)

If \(m = 3\), then using (8.29) with \(\left( y_2, \ldots, y_m \right) = (y_2, \ldots, y_m)\) replaced by \(\left( y_3, \ldots, y_m \right)\), we obtain

\[
I_3(\tau, x, x') \leq 2\tau \delta^{-4} \text{Vol}(\mathcal{E}_{1, 3}(\tau, x, x'))
\]

(8.40)

(we also used the estimate \(|x - x' - y_3| + \tau - |y_3| \leq 2\tau\) for \(y_3 \in \mathcal{E}_{1, 3}(\tau, x, x')\)). If \(m \geq 4\), then using (8.29) with \(\left( y_2, \ldots, y_m \right) = (y_2, \ldots, y_m)\) replaced by \(\left( y_3, \ldots, y_m \right)\), we obtain

\[
I_3(\tau, x, x') \leq 2\tau \delta^{-4} \text{Vol}(\mathcal{E}_{1, 3}(\tau, x, x')) \int_{\mathcal{E}_{m-3, n}(\tau, x, x')} \frac{dy_3 \cdots dy_{m-1}}{|y_3|^{n-1} \cdots |y_m|^{n-1}}
\]

\[
\leq 2\tau \delta^{-4} \text{Vol}(\mathcal{E}_{1, 3}(\tau, x, x')) \text{Vol}(\mathbb{S}^{n-1})^{m-3} \int_{(y_3, \ldots, y_{m-1}) \in \{0, \infty\}^{m-3}} ds_3 \cdots ds_{m-1} \text{Vol}(\mathcal{E}_{1, 3}(\tau, x, x'))
\]

\[
= 2\tau \delta^{-4} \text{Vol}(\mathbb{S}^{n-1})^{m-3} \frac{\tau^{m-3}}{(m-3)!} \text{Vol}(\mathcal{E}_{1, 3}(\tau, x, x'))
\]

(8.41)
(we also used the estimate $|x - x' - y_3 - \ldots - y_m| + \tau - |y_3| - \ldots - |y_m| \leq 2\tau$ for $(y_3, \ldots, y_m) \in E_{m-2,3}(\tau, x, x')$ and we performed the changes of variables $y_i = s_i\omega_i$, $(s_i, \omega_i) \in (0, +\infty) \times S^{n-1}$. Statement (3.20) follows from (8.27), (8.37) and (8.40)-(8.41) and (8.9)).

\[ \square \]

9 Proof of Lemmas 7.1, 8.2, 8.3 and 8.4

We remind the following change of variables for the proof of Lemmas 8.1, 8.3, 8.4 and 8.2.

\[
\int_{E_{l,n}(\tau, t_0, v, 0)} f(y)dy = \begin{cases} 
\int_{(0,2\pi) \times (0,\tau)} f \left( \frac{t_0 + s \cos(\varphi)}{2}, \frac{\sqrt{s^2 - t_0^2}}{2} \sin \varphi \right) \times \left( \frac{\cos^2(\varphi) \sqrt{s^2 - t_0^2}}{4} + \frac{s^2 \sin^2(\varphi)}{4\sqrt{s^2 - t_0^2}} \right) d\varphi dx, & \text{if } n = 2, \\
\int_{S^{n-2} \times (0,\pi) \times (0,\tau)} f \left( \frac{t_0 + s \cos(\varphi)}{2}, \frac{\sqrt{s^2 - t_0^2}}{2} \sin \varphi \right) \times \left( \frac{\sin(\varphi) \sqrt{s^2 - t_0^2}}{2} \sin(\varphi) \right)^{n-2} \left( \frac{\cos^2(\varphi) \sqrt{s^2 - t_0^2}}{4} + \frac{s^2 \sin^2(\varphi)}{4\sqrt{s^2 - t_0^2}} \right) d\varphi dx, & \text{if } n \geq 3,
\end{cases}
\]

(9.1)

for $f \in L^1(\mathbb{R}^n)$ and $(\tau, t_0, v) \in (0, +\infty) \times (0, +\infty) \times S^{n-1}$ such that $\tau > t_0$.

**Proof of Lemma 7.1.** Let $(\tau, x, x') \in (0, T) \times \partial X \times \mathbb{R}^n$. We first prove (7.3). Let $n = 2$. Note that

\[
N(\tau, x, x') = \chi_{(0, +\infty)}(\tau - |x - x'|) \int_0^{2\pi} \frac{1}{\tau - |x - x'| \sin(\Omega)} d\Omega.
\]

Therefore using (7.1) we obtain (7.3). We prove (7.4). Let $n = 3$. Note that

\[
N(\tau, x, x') = 2\pi \frac{\chi_{(0, +\infty)}(\tau - |x - x'|)}{2\tau |x - x'|} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d}{d\Omega} \ln \left( \tau^2 + |x - x'|^2 - 2\tau |x - x'| \sin(\Omega) \right) d\Omega,
\]

which gives (7.4).

We prove (7.5). Let $n \geq 4$ and let $(\tau, x, x') \in (0, T) \times \partial X \times \mathbb{R}^n$ be such that $\tau > |x - x'|$ (we remind that $N(\tau, x, x') = 0$ if $\tau \leq |x - x'|$). Using spherical coordinates, we obtain

\[
N(\tau, x, x') = \text{Vol}_{n-2}(S^{n-2}) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(\tau - |x - x'| \sin(\Omega))^{n-3}}{|(x - x'|^2 + \tau^2 - 2\tau |x - x'| \sin(\Omega))^{n-2}} \cos(\Omega)^{-n} d\Omega.
\]

(9.2)

Performing the change of variables $r = \frac{\tau^2 - |x - x'|^2}{2\tau |x - x'| \sin(\Omega)} - \frac{\tau - |x - x'|}{2}$, we obtain

\[
N(\tau, x, x') = \begin{cases} 
\text{Vol}_{n-2}(S^{n-2}) (\tau^2 - |x - x'|^2) \frac{n-3}{2} \int_0^{|x - x'|} \sqrt{r(x - x'|^2 - r)^{n-3}} \frac{1}{(\frac{\tau - |x - x'|}{2} + r)^n} dr, & n \geq 4,
\end{cases}
\]

\[
\leq \begin{cases} 
2\text{Vol}_{n-2}(S^{n-2}) (\tau^2 - |x - x'|^2) \frac{n-3}{2} \int_0^{|x - x'|} \sqrt{r(x - x'|^2 - r)^{n-3}} \frac{1}{(\frac{\tau - |x - x'|}{2} + r)^n} dr, & n \geq 4,
\end{cases}
\]

\[34\]
\[
\leq \frac{2\Vol_{n-2}(S^{n-2}) (\tau^2 - |x - x'|^2)^{\frac{n-3}{2}}}{\tau^{n-1}} \int_0^{\frac{\sqrt{\tau^2 - t_0^2}}{2}} \frac{1}{(\frac{\tau - |x - x'|}{2r} + r)^{\frac{n-3}{2}}} dr
\]
\[
\leq \frac{2^{n-1}}{n-3} \Vol_{n-2}(S^{n-2}) |x - x'|^{2-n} \left( \frac{\tau + |x - x'|}{2\tau} \right)^{\frac{n-3}{2}} \tau^{-1}
\]
\[
\leq \frac{2^{n-1}}{n-3} \Vol_{n-2}(S^{n-2}) |x - x'|^{2-n} \tau^{-1}, \quad (9.3)
\]
which proves (7.5). \qed

**Proof of Lemma 8.3.** Let \( n \geq 2 \). Using a rotation and (8.1), we have
\[
\Vol_n(\mathcal{E}_{1,n}(\tau, x, x')) = \Vol_n(\mathcal{E}_{1,n}(\tau, t_0 e_1, 0)), \quad (9.4)
\]
where \( t_0 = |x - x'| \) and \( e_1 = (0, \ldots, 0) \in \mathbb{R}^n \).

From (9.1), it follows that
\[
\Vol_n(\mathcal{E}_{1,n}(\tau, t_0 e_1, 0)) = \Vol_{n-2}(S^{n-2}) \int_0^\pi \left( \frac{\sin(\varphi)\sqrt{s^2 - t_0^2}}{2} \right)^{n-2}
\times \left( \cos^2(\varphi)\sqrt{s^2 - t_0^2} + \frac{s^2 \sin^2(\varphi)}{4\sqrt{s^2 - t_0^2}} \right) d\varphi. \quad (9.5)
\]

From (9.5) and the estimate \( \sin(\varphi)\sqrt{s^2 - t_0^2} \leq \sqrt{\tau^2 - t_0^2} \) for \( s \in (t_0, \tau) \), we obtain
\[
\Vol_n(\mathcal{E}_{1,n}(\tau, t_0 e_1, 0)) \leq \Vol_{n-2}(S^{n-2}) \left( \frac{\sqrt{\tau^2 - t_0^2}}{2} \right)^{n-2}
\times \int_0^\pi \left( \cos^2(\varphi)\sqrt{s^2 - t_0^2} + \frac{s^2 \sin^2(\varphi)}{4\sqrt{s^2 - t_0^2}} \right) d\varphi
\leq \frac{1}{2} \Vol_{n-2}(S^{n-2}) \left( \frac{\sqrt{\tau^2 - t_0^2}}{2} \right)^{n-2} \Vol(\mathcal{E}_{1,2}(\tau, t_0 e_1, 0)). \quad (9.6)
\]

We remind that \( \Vol(\mathcal{E}_{1,2}(\tau, t_0 e_1, 0)) = \frac{\pi (\tau + r) \sqrt{\tau^2 - t_0^2}}{4} \). Therefore (8.9) follows from (9.6). Lemma 8.3 is proved. \qed

**Proof of Lemma 8.4.** Let \( (\mu, z, z') \in (0, T) \times \partial X \times \mathbb{R}^3 \) be such that \( \mu > |z - z'| > 0 \) and \( \mu \geq \delta \).

Using the change of variables \( y = \frac{t_0}{2}(1, 0) - (s \cos(\varphi), \sqrt{s^2 - t_0^2} \sin(\varphi) \omega) \) (see (9.1)), \( \varphi \in (0, \pi), s \in (t_0, \mu), \omega \in S^1 \), we obtain
\[
B(\mu, z, z') = \frac{\pi}{4} \int_{t_0}^\mu \int_0^\pi B_1(\mu, s, \varphi) d\varphi ds + \frac{\pi}{4} \int_{t_0}^\mu \int_0^\pi B_2(\mu, s, \varphi) d\varphi ds, \quad (9.7)
\]
where
\[
B_1(\mu, s, \varphi) = (s^2 - t_0^2) \sin(\varphi) \cos^2(\varphi) \ln \left( \frac{\mu - t_0 \cos(\varphi)}{\mu - s} \right), \quad (9.8)
\]
\[
B_2(\mu, s, \varphi) = s^2 \sin^3(\varphi) \ln \left( \frac{\mu - t_0 \cos(\varphi)}{\mu - s} \right), \quad (9.9)
\]
for $\varphi \in (0, 2\pi)$ and $s \in (t_0, \mu)$. Using (9.8) and the estimates $\ln\left(\frac{\mu - t_0 \cos(\varphi)}{\mu - s}\right) \leq \ln\left(\frac{\mu + t_0}{\mu - s}\right)$, $s^2 - t_0^2 \leq \mu^2 - t_0^2$, we obtain

$$
\int_0^\pi B_1(\mu, s, \varphi) d\varphi \leq (\mu^2 - t_0^2) \ln\left(\frac{\mu + t_0}{\mu - t_0}\right) \int_0^\pi \sin(\varphi) \cos^2(\varphi) d\varphi.
$$

(9.10)

Using (9.9) and the estimates $\ln\left(\frac{\mu - t_0 \cos(\varphi)}{\mu - s}\right) \leq \ln\left(\frac{\mu + t_0}{\mu - s}\right)$, $s \leq \mu$, we obtain

$$
\int_0^\mu \ln\left(\frac{\mu + t_0}{\mu - s}\right) ds = (\mu - t_0) \ln\left(\frac{\mu + t_0}{\mu - t_0}\right) + \mu - t_0.
$$

(9.12)

Combining (9.7), (9.10), (9.11) and (9.12) we obtain

$$
B(\mu, z, z') = \frac{\pi}{4}(\mu - t_0) \left(\ln\left(\frac{\mu + t_0}{\mu - t_0}\right) + 1\right)
\times \left(\mu^2 \int_0^\pi \sin^3(\varphi) d\varphi + (\mu^2 - t_0^2) \int_0^\pi \sin(\varphi) \cos^2(\varphi) d\varphi\right),
$$

(9.13)

which proves (8.11).

\[\square\]

Proof of Lemma 8.2. Let $(\tau, x, x') \in (0, T) \times \mathbb{R}^n \times \mathbb{R}^n$ such that $\tau > |x - x'| > 0$. We put $t_0 = |x - x'|$. Let $(y_2, \ldots, y_{m-1}) \in E_{m-2, n}(\tau, x, x')$. Then $t_0 \leq |y_2| + \ldots + |y_{m-1}| + |x - x' - y_2 - \ldots - y_{m-1}|$. Therefore either $|x - x' - y_2 - \ldots - y_{m-1}| \geq \frac{\ln t_0}{m-1}$ or $|x - x' - y_2 - \ldots - y_{m-1}| < \frac{\ln t_0}{m-1}$ and there exists $j \in \mathbb{N}$, $j = 1 \ldots n$ such that $|y_j| \geq \frac{\ln t_0}{m-1}$. Therefore using (8.6) we obtain

$$
\tilde{J}_m(\tau, x, x') \leq \sum_{j=2}^{m-1} \int_{\frac{|y_j| \geq \frac{\ln t_0}{m-1}}{|y_j|} \frac{|y_{j+1}| \leq \frac{\ln t_0}{m-1}}{|y_{j+1}|} \ldots \frac{|y_{m-1}| \leq \frac{\ln t_0}{m-1}}{|y_{m-1}|}} dy_j \ldots dy_{m-1} \frac{dy_j \ldots dy_{m-1} \left(y_j \ldots y_{m-1} \right)}{\left|y_j \ldots y_{m-1} \right|^n}. 
$$

(9.14)

We first reduce the estimate of $\tilde{J}_{m,0}$ and $\tilde{J}_{m,1}$ to an estimate on

$$
P_m(\tau, x, x') := \int_{\frac{|y_j| \geq \frac{\ln t_0}{m-1}}{|y_j|} \frac{|y_{j+1}| \leq \frac{\ln t_0}{m-1}}{|y_{j+1}|} \ldots \frac{|y_{m-1}| \leq \frac{\ln t_0}{m-1}}{|y_{m-1}|}} dy_j \ldots dy_{m-1} \frac{dy_j \ldots dy_{m-1} \left(y_j \ldots y_{m-1} \right)}{\left|y_j \ldots y_{m-1} \right|^n}.
$$

(9.17)
From (9.15) and the estimate $|x - x' - y_2 - \ldots - y_{m-1}| > \frac{t_0}{m-1}$ it follows that
\[ J_{m,0}(\tau, x, x') \leq \left( \frac{m-1}{t_0} \right)^{n-2} P_m(\tau, x, x'). \] (9.18)

From (9.16) and the estimates $|y_2| \geq \frac{t_0}{m-1} \geq |x - x' - y_2 - \ldots - y_{m-1}|$ it follows that
\[ J_{m,1}(\tau, x, x') \leq \int_{\substack{(y_2, \ldots, y_{m-1}) \in E_{m-2,n}(\tau, x, x') \\ |x - x' - \sum_{i=2}^{m-1} y_i| < \frac{t_0}{m-1}}} dy_2 \ldots dy_{m-1} \frac{dy_2 \ldots dy_{m-1}}{|y_2|^{n-2} \ldots |y_{m-1}|^{n-1} |x - x' - \sum_{i=2}^{m-1} y_i|^{n-1}}. \] (9.19)

Therefore performing the change of variables “$y_2^2 = x - x' - y_2 - \ldots - y_{m-1}$” we obtain
\[ J_{m,1}(\tau, x, x') \leq \tilde{J}_{m,0} \leq \left( \frac{m-1}{t_0} \right)^{n-2} P_m(\tau, x, x'). \] (9.20)

Now we estimate $P_3(\tau, x, x')$. From (9.1) and (9.17) it follows that
\[ P_3(\tau, x, x') = \text{Vol}_{n-2}(S^{n-2}) \int_0^\pi \int_0^\tau \sin^{n-2}(\varphi) \left( s^2 - t_0^2 \right)^{\frac{n-3}{2}} (s - t_0 \cos(\varphi)) \frac{ds}{2(s + t_0 \cos(\varphi))^{n-2}}. \] (9.21)

Let $n = 3$. Then using the estimate $\cos(\varphi) \geq -1$ (and the fact that $\frac{s - t_0 \cos(\varphi)}{s + t_0 \cos(\varphi)} = \frac{s - t_0}{s + t_0 \cos(\varphi)} + \frac{t_0 (1 - \cos(\varphi))}{s + t_0 \cos(\varphi)} \leq 1 + \frac{2t_0}{s + t_0 \cos(\varphi)}$) we obtain
\[ P_3(\tau, x, x') \leq \pi \int_0^\tau \int_0^\pi \left( \sin(\varphi) - 2 \frac{d}{d\varphi} \ln(s + t_0 \cos(\varphi)) \right) d\varphi ds \]
\[ = 2\pi \left( \tau - t_0 + \int_0^\tau \ln(s + t_0) ds - \int_0^\tau \ln(s - t_0) ds \right) \]
\[ = 2\pi (\tau - t_0) \left( 2 + \ln \left( \frac{\tau + t_0}{\tau - t_0} \right) \right) \] (9.22)

(we used the estimate $\ln(s + t_0) \leq \ln(\tau + t_0)$ for $s \in (t_0, \tau)$ and we used the integral value (9.12)). Let $n \geq 4$. Using (9.21) and using the estimates $\sqrt{s^2 - t_0^2} \sin(\varphi) \leq s + t_0 \cos(\varphi)$, $s + t_0 \cos(\varphi) \geq s(1 + \cos(\varphi))$ and $s - t_0 \cos(\varphi) \leq s + t_0 \leq 2s$ for $(s, \varphi) \in (t_0, \tau) \times (0, \pi)$, we obtain
\[ P_3(\tau, x, x') \leq \text{Vol}_{n-2}(S^{n-2}) \int_0^\pi \int_0^\tau \frac{\sin^2(\varphi) \sqrt{s^2 - t_0^2} (s - t_0 \cos(\varphi))}{2(s + t_0 \cos(\varphi))^2} ds d\varphi \]
\[ \leq C_0 \text{Vol}_{n-2}(S^{n-2}) \int_0^\tau \int_0^\pi \frac{\sqrt{s^2 - t_0^2}}{s + t_0 \cos(\varphi)} d\varphi ds \leq \pi C_0 \text{Vol}_{n-2}(S^{n-2})(\tau - t_0), \] (9.23)

where $C_0$ is defined by (5.33) (we also used (7.1)).

Finally let $m \geq 4$ then
\[ (y_2, \ldots, y_{m-1}) \in E_{m-2,n}(\tau, x, x') \Rightarrow \left( y_2 \in E_{1,n}(\tau, x, x') \right. \text{ and } \left. \sum_{i=3}^{m-1} |y_i| < \tau \right). \] (9.24)

Therefore using (9.17), spherical coordinates (“$y_i^m = s_i \omega_i$, $s_i \in (0, +\infty)$, $\omega_i \in S^{n-1}$ for $i = 3, \ldots, m - 1$”) we obtain
\[ P_m(\tau, x, x') \leq P_3(\tau, x, x') \int_{|y_3| + \ldots + |y_{m-1}| < \tau} dy_3 \ldots dy_{m-1} \]
\[ \leq P_3(\tau, x, x') \text{Vol}_{n-1}(S^{n-1})^{m-3} \int_{0 < s_i, i = 3, \ldots, m-1} \Gamma_{\frac{m-1}{2}}^{m-3} ds_i = P_3(\tau, x, x') \text{Vol}_{n-1}(S^{n-1})^{m-3} \frac{\tau^{m-3}}{(m-3)!}. \] (9.25)
Finally statement (8.7) follows from (9.14), (9.18), (9.20), (9.22) and (9.25), and statement (8.8) follows from (9.14), (9.18), (9.20), (9.23) and (9.25).

10 Proof of Lemma 8.1

10.1 Proof of (8.3)

Let \((\mu, z, z') \in (0, T) \times \partial X \times \mathbb{R}^2\) be such that \(\mu > |z - z'|.\) From (8.2) and (7.3) it follows that

\[
J_2(\mu, z, z') = \int_{\mathcal{E}_{1,2}(\mu, z, z')} \frac{2\pi}{|y|\sqrt{(\mu - |y|)^2 - |z - z' - y|^2}} dy \\
= \int_{\mathcal{E}_{1,2}(\mu, t_0(1,0),0)} \frac{2\pi}{|y|\sqrt{(\mu - |y|)^2 - |t_0(1,0) - y|^2}} dy,
\]

where \(t_0 = |z - z'|.\)

Using (7.3) and the change of variables \(y = \frac{\mu}{2}(1,0) - s \cos(\varphi), \frac{\sqrt{s^2 - t_0^2}}{2} \sin(\varphi))\) (see (9.1)), \(\varphi \in (0, 2\pi), s \in (t_0, \mu),\) we obtain

\[
J_2(\mu, z, z') = 4\pi \int_{t_0}^{\mu} \int_{0}^{\pi} J_{2,1}(\mu, s, \varphi) d\varphi ds + 4\pi \int_{t_0}^{\mu} \int_{0}^{\pi} J_{2,2}(\mu, s, \varphi) d\varphi ds,
\]

where

\[
J_{2,1}(\mu, s, \varphi) = \frac{\sqrt{s^2 - t_0^2} \cos^2(\varphi)}{(s + t_0 \cos(\varphi)) \sqrt{\mu - s \sqrt{\mu - t_0 \cos(\varphi)}}}, \quad J_{2,2}(\mu, s, \varphi) = \frac{s^2 \sin^2(\varphi)}{(s + t_0 \cos(\varphi)) \sqrt{s^2 - t_0^2 \mu - s \sqrt{\mu - t_0 \cos(\varphi)}}},
\]

for \(\varphi \in (0, 2\pi)\) and \(s \in (t_0, \mu).\)

We give estimate on \(J_{2,1}\) and \(J_{2,2}.\) Let \(\varphi \in (0, \frac{\pi}{2})\) and \(s \in (t_0, \mu).\) From (10.3) and the estimates \(\cos(\varphi) \geq 0, \cos^2(\varphi) \leq 1, s + t_0 \cos(\varphi) \geq s,\) it follows that

\[
J_{2,1}(\mu, s, \varphi) \leq \frac{\sqrt{s^2 - t_0^2}}{s \sqrt{\mu} - s \sqrt{\mu - t_0 \cos(\varphi)}} \leq \frac{\sqrt{2}}{\sqrt{s} \sqrt{\mu - s}},
\]

(we also used the estimate \(\sqrt{s^2 - t_0^2} = \sqrt{s + t_0 \sqrt{s - t_0 \cos(\varphi)}} \leq \frac{\sqrt{2}}{\sqrt{s}} \sqrt{s} \sqrt{\mu - t_0 \cos(\varphi)}\)). From (10.3) and the estimates \(\cos(\varphi) \geq 0, s + t_0 \cos(\varphi) \geq s, s \sin(\varphi) \leq \sqrt{s + t_0 \sqrt{\mu}},\) it follows that

\[
J_{2,2}(\mu, s, \varphi) \leq \sqrt{C_0} \frac{\sin(\varphi) s}{\sqrt{\mu \sqrt{s^2 - t_0^2 \mu - s}}} \leq \frac{\sqrt{C_0}}{\sqrt{s - t_0 \sqrt{\mu - s}}},\]

where \(C_0\) is defined by (5.33).

Let \(\varphi \in (0, \frac{\pi}{2}, \pi)\) and \(s \in (t_0, \mu).\) From (10.3) and the estimates \(\mu - t_0 \cos(\varphi) \geq \mu, \cos^2(\varphi) \leq 1,\) it follows that

\[
J_{2,1}(\mu, s, \varphi) \leq \frac{\sqrt{s^2 - t_0^2}}{(s + t_0 \cos(\varphi)) \sqrt{\mu} - s \sqrt{\mu}}.
\]

Therefore using (7.1), we obtain

\[
\int_{t_0}^{\mu} \int_{0}^{\pi} J_{2,1}(\mu, s, \varphi) d\varphi ds \leq -\frac{1}{\sqrt{\mu}} \int_{t_0}^{\mu} \frac{d}{ds} \sqrt{\mu - s ds} \leq 1.
\]
From (10.4) and the estimates \( \mu - t_0 \cos(\varphi) \geq \mu, s + t_0 \cos(\varphi) \geq s(1 + \cos(\varphi)) \), it follows that

\[
J_{2,2}(\mu, s, \varphi) \leq C_0 \frac{s}{\sqrt{s^2 - t_0^2 \mu - s \mu}} \leq \frac{C_0}{\sqrt{s - t_0 \sqrt{\mu - s}}}.
\]  
(10.8)

Note that using the estimate \( s \geq s - t_0 \) and performing the change of variables \( s = t_0 + \varepsilon(\mu - t_0) \) we have

\[
\int_{t_0}^{\mu} \frac{1}{\sqrt{s^2 - t_0^2 \mu - s \mu}} ds \leq \int_{t_0}^{\mu} \frac{1}{\sqrt{s - t_0 \sqrt{\mu - s}} + s} ds = \int_{0}^{1} \frac{1}{\varepsilon(1 - \varepsilon)} d\varepsilon < +\infty,
\]  
(10.9)

for \( s \in (t_0, \mu) \). Combining (10.9), (10.2), (10.5)–(10.8), we obtain

\[
\sup_{(\mu, z, z') \in [0, T] \times \mathbb{R}^2 \times \mathbb{R}^2 \, : \, \mu > |z - z'|} J_2(\mu, z, z') \leq 4\pi + 2\pi^2(\sqrt{2} + \sqrt{C_0} + C_0) \int_{0}^{1} \frac{1}{\varepsilon(1 - \varepsilon)} d\varepsilon < \infty.
\]  
(10.10)

Statement (8.3) follows from (10.10).

\[
\square
\]

10.2 Proof of (8.4)

Let \( (\mu, z, z') \in (0, T) \times \partial X \times \mathbb{R}^3 \) be such that \( \mu > |z - z'| \). From (8.2) and (7.4) it follows that

\[
J_2(\mu, z, z') \leq \int_{\mathcal{E}_{1,3}(\mu, z, z')} 2\pi \ln \left( \frac{\mu - |y| + |z - z'| - |y|}{\mu - |y| - |z - z'| - |y|} \right) dy 
\leq \int_{\mathcal{E}_{1,3}(\mu, t_0(1,0,0), 0)} 2\pi \ln \left( \frac{\mu - |y| + |t_0(1,0,0) - y|}{\mu - |y| - |t_0(1,0,0) - y|} \right) dy,
\]  
where \( t_0 = |z - z'| \).

Using the change of variables \( y = \frac{\mu}{\mu}(1, 0, 0) - (s \cos(\varphi), \sqrt{s^2 - t_0^2} \sin(\varphi) \omega) \) (see (9.1)), \( \varphi \in (0, \pi), s \in (t_0, \mu), \omega \in \mathbb{S}^1 \), we obtain

\[
J_2(\mu, z, z') = 8\pi^2 \int_{t_0}^{\mu} \int_{0}^{\pi} J_{2,1}(\mu, s, \varphi) d\varphi d\varepsilon + 8\pi^2 \int_{t_0}^{\mu} \int_{0}^{\pi} J_{2,2}(\mu, s, \varphi) d\varphi ds,
\]  
(10.12)

where

\[
J_{2,1}(\mu, s, \varphi) = \frac{(s^2 - t_0^2) \sin(\varphi) \cos^2(\varphi) \ln \left( \frac{\mu - t_0 \cos(\varphi)}{\mu - s} \right)}{(s + t_0 \cos(\varphi))^2 (2\mu - s - t_0 \cos(\varphi)) (s - t_0 \cos(\varphi))},
\]  
(10.13)

\[
J_{2,2}(\mu, s, \varphi) = \frac{s^2 \sin^3(\varphi) \ln \left( \frac{\mu - t_0 \cos(\varphi)}{\mu - s} \right)}{(s + t_0 \cos(\varphi))^2 (2\mu - s - t_0 \cos(\varphi)) (s - t_0 \cos(\varphi))},
\]  
(10.14)

for \( \varphi \in (0, 2\pi) \) and \( s \in (t_0, \mu) \).

We give estimate on \( J_{2,1} \) and \( J_{2,2} \). Let \( \varphi \in (0, \frac{\pi}{2}) \) and \( s \in (t_0, \mu) \). From (10.13) and the estimates \( \cos^2(\varphi) \leq 1, s + t_0 \cos(\varphi) \geq s, \sqrt{s^2 - t_0^2} \sin(\varphi) \leq s - t_0 \cos(\varphi), \ln \left( \frac{\mu - t_0 \cos(\varphi)}{\mu - s} \right) \leq \ln \left( \frac{\mu + t_0}{\mu - s} \right) \) and the estimate \( 2\mu - s - t_0 \cos(\varphi) \geq \mu - t_0 \cos(\varphi) \), it follows that

\[
J_{2,1}(\mu, s, \varphi) \leq \frac{\sqrt{s^2 - t_0^2} \ln \left( \frac{\mu + t_0}{\mu - s} \right)}{s^2 (\mu - t_0 \cos(\varphi))}.
\]
Therefore using (7.1) we obtain
\[
\int_0^{2\pi} J_{2,1}(\mu, s, \varphi) d\varphi \leq \frac{\pi \sqrt{s^2 - t_0^2} \ln \left( \frac{\mu + t_0}{\mu - s} \right)}{2s^2 \sqrt{\mu^2 - t_0^2}} \leq \frac{\pi \ln \left( \frac{\mu + t_0}{\mu - s} \right)}{2s^2}.
\]  
(10.15)

From (10.14) and the estimates \(\cos^2(\varphi) \leq 1\), \(s + t_0 \cos(\varphi) \geq s\), \(\ln \left( \frac{\mu - t_0 \cos(\varphi)}{\mu - s} \right) \leq \ln \left( \frac{\mu + t_0}{\mu - s} \right)\) and the estimate \(2\mu - s - t_0 \cos(\varphi) \geq \mu(1 - \cos(\varphi))\), it follows that
\[
J_{2,2}(\mu, s, \varphi) \leq C_0 \frac{\sin(\varphi) \ln \left( \frac{\mu + t_0}{\mu - s} \right)}{\mu(s - t_0 \cos(\varphi))}.
\]
where \(C_0\) is defined by (5.33). Therefore
\[
\int_0^{2\pi} J_{2,2}(\mu, s, \varphi) d\varphi \leq C_0 \frac{\ln \left( \frac{s - t_0}{s - t_0} \right) \ln \left( \frac{\mu + t_0}{\mu - s} \right)}{t_0 \mu} \leq C_0 \frac{\ln \left( \frac{\mu + t_0}{\mu - s} \right) \ln \left( \frac{\mu + t_0}{\mu - s} \right)}{t_0 \mu}.
\]  
(10.16)

Let \(\varphi \in \left( \frac{\pi}{2}, \pi \right)\) and \(s \in (t_0, \mu)\). From (10.13) and the estimates \(\cos^2(\varphi) \leq 1\), \(s - t_0 \cos(\varphi) \geq s\), \(\sqrt{s^2 - t_0^2} \sin(\varphi) \leq s + t_0 \cos(\varphi)\), \(\ln \left( \frac{\mu - t_0 \cos(\varphi)}{\mu - s} \right) \leq \ln \left( \frac{\mu + t_0}{\mu - s} \right)\) and the estimate \(2\mu - s - t_0 \cos(\varphi) \geq \mu\), it follows that
\[
J_{2,1}(\mu, s, \varphi) \leq \frac{\sqrt{s^2 - t_0^2} \ln \left( \frac{\mu + t_0}{\mu - s} \right)}{(s + t_0 \cos(\varphi)) \mu s}.
\]

Therefore using (7.1), we obtain
\[
\int_0^{\pi} J_{2,1}(\mu, s, \varphi) d\varphi d\varphi \leq \frac{\pi \ln \left( \frac{s - t_0}{s - t_0} \right) \ln \left( \frac{\mu + t_0}{\mu - s} \right)}{2 \mu s} \leq \frac{\pi \ln \left( \frac{\mu + t_0}{\mu - s} \right)}{2 \mu t_0}.
\]  
(10.17)

From (10.14) and the estimates \(s - t_0 \cos(\varphi) \geq s\), \(s + t_0 \cos(\varphi) \geq s(1 + \cos(\varphi))\), \(\ln \left( \frac{\mu - t_0 \cos(\varphi)}{\mu - s} \right) \leq \ln \left( \frac{\mu + t_0}{\mu - s} \right)\), the estimate \(2\mu - s - t_0 \cos(\varphi) \geq \mu\), it follows that
\[
\int_0^{\pi} J_{2,2}(\mu, s, \varphi) d\varphi \leq C_0 \int_0^{\pi} \frac{\sin(\varphi) \ln \left( \frac{\mu + t_0}{\mu - s} \right)}{(s + t_0 \cos(\varphi)) \mu} d\varphi \leq C_0 \frac{\ln \left( \frac{s - t_0}{s - t_0} \right) \ln \left( \frac{\mu + t_0}{\mu - s} \right)}{t_0 \mu},
\]  
(10.18)

where \(C_0\) is defined by (5.33) (we also used the estimate \(s \geq t_0\) and \(\sin(\varphi) \leq 1\)).

Using the estimate \(s^{-1} \leq \left( \frac{t_0 + \mu}{2} \right)^{-1}\) for \(s \in (\frac{t_0 + \mu}{2}, \mu)\) and using the estimate \(\ln \left( \frac{\mu + t_0}{\mu - s} \right) \leq \ln \left( \frac{2(\mu + t_0)}{\mu - t_0} \right)\) for \(s \in (t_0, \frac{t_0 + \mu}{2})\), we obtain
\[
\int_{t_0}^{\mu} \ln \left( \frac{\mu + t_0}{\mu - s} \right) \frac{ds}{s^2} \leq \ln \left( \frac{2(\mu + t_0)}{\mu - t_0} \right) \int_{t_0}^{\frac{t_0 + \mu}{2}} \frac{1}{s^2} ds + \frac{4}{(t_0 + \mu)^2} \int_{\frac{t_0 + \mu}{2}}^{\mu} \ln \left( \frac{\mu + t_0}{\mu - s} \right) ds
\leq \ln \left( \frac{2(\mu + t_0)}{\mu - t_0} \right) \frac{\mu - t_0}{t_0(\mu + t_0)} + 4(\mu - t_0) \left( \ln \left( \frac{\mu + t_0}{\mu - t_0} \right) + 1 \right)
\leq \frac{\mu - t_0}{t_0 \mu} \left( 5 \ln \left( \frac{\mu + t_0}{\mu - t_0} \right) + 4 + \ln(2) \right).
\]  
(10.19)
Using the estimate \( \ln \left( \frac{\mu + t_0}{\mu - s} \right) \leq \ln \left( \frac{2(\mu + t_0)}{\mu - t_0} \right) \) for \( s \in (t_0, \frac{t_0 + \mu}{2}) \), we obtain

\[
\int_{t_0}^{\mu} \ln \left( \frac{\mu + t_0}{s - t_0} \right) \ln \left( \frac{\mu + t_0}{\mu - s} \right) ds = 2 \int_{t_0}^{\frac{t_0 + \mu}{2}} \ln \left( \frac{\mu + t_0}{s - t_0} \right) \ln \left( \frac{\mu + t_0}{\mu - s} \right) ds
\]

\[
\leq \ln \left( \frac{2(\mu + t_0)}{\mu - t_0} \right) \int_{t_0}^{\frac{t_0 + \mu}{2}} \ln \left( \frac{\mu + t_0}{s - t_0} \right) ds
\]

\[
\leq (\mu - t_0) \left( \ln \left( \frac{\mu + t_0}{\mu - t_0} \right) + \ln(2) \right) \ln \left( \frac{\mu + t_0}{\mu - t_0} + 1 \right).
\]

Combining (10.15)–(10.18), (10.19)–(10.20), we obtain

\[
J_2(\mu, z, z') \leq D \frac{\mu - |z - z'|}{\mu |z - z'|} \left( 1 + \ln \left( \frac{\mu + |z - z'|}{\mu - |z - z'|} \right) \right)^2,
\]

where the constant \( D \) does not depend on \( \mu, z, z' \). Statement (8.4) follows from (10.21). \( \square \)

10.3 Proof of (8.5)

Let \( n \geq 4 \). Let \( (\mu, z, z') \in (0, T) \times \partial X \times \mathbb{R}^3 \) be such that \( \mu > |z - z'| \). From (8.2) and (7.5) it follows that

\[
J_2(\mu, z, z') \leq C \int_{\partial X} |y|^{1-n} - |y| \int |z - z' - y|^{2-n} dy
\]

where \( t_0 = |z - z'| \) and \( C = \sup_{(\mu, z, z') \in (0, T) \times \partial X \times \mathbb{R}^3} \mu^2 \int |z - z'|^{n-2} N(\mu, z, z') \).

Using the change of variables \( y = \frac{\omega}{\mu}(1, 0) - (s \cos(\varphi), \sqrt{s^2 - t_0^2} \sin(\varphi), \omega) \) (see (9.1)), \( \varphi \in (0, \pi) \), \( s \in (t_0, \mu) \), \( \omega \in S^{n-2} \), we obtain

\[
J_2(\mu, z, z') \leq C' \int_{t_0}^{\mu} \int_0^\pi J_{2,1}(\mu, s, \varphi) d\varphi ds + \int_{t_0}^{\mu} \int_0^\pi J_{2,2}(\mu, s, \varphi) d\varphi ds,
\]

where \( C' = 2^{n-2} \text{Vol}_{n-2}(S^{n-2})C \).

\[
J_{2,1}(\mu, s, \varphi) = \frac{(s^2 - t_0^2)^{n-1}}{(s + t_0 \cos(\varphi))^{n-1}((2s - t_0 \cos(\varphi)) (s - t_0 \cos(\varphi))^{n-2}},
\]

\[
J_{2,2}(\mu, s, \varphi) = \frac{(s^2 - t_0^2)^{n-1}}{(s + t_0 \cos(\varphi))^{n-1}((2s - t_0 \cos(\varphi)) (s - t_0 \cos(\varphi))^{n-2}},
\]

for \( \varphi \in (0, 2\pi) \) and \( s \in (t_0, \mu) \).

We give estimate on \( J_{2,1} \) and \( J_{2,2} \). Let \( \varphi \in (0, \frac{\pi}{2}) \) and \( s \in (t_0, \mu) \). From (10.24) and the estimates \( \cos^2(\varphi) \leq 1 \), \( s + t_0 \cos(\varphi) \geq s, \sqrt{s^2 - t_0^2} \sin(\varphi) \leq s - t_0 \cos(\varphi) \), it follows that

\[
J_{2,1}(\mu, s, \varphi) \leq \frac{\sqrt{s^2 - t_0^2}}{s^{n-1}(s - t_0 \cos(\varphi))}.
\]

Therefore using (7.1) we obtain

\[
\int_0^{\frac{\pi}{2}} J_{2,1}(\mu, s, \varphi) d\varphi \leq \frac{\pi \sqrt{s^2 - t_0^2}}{2s^{n-1}\sqrt{\mu^2 - t_0^2}} \leq \frac{\pi}{2s^{n-1}}.
\]
From (10.25) and the estimate \( \cos^2(\varphi) \leq 1, s + t_0 \cos(\varphi) \geq s, \sqrt{s^2 - t_0^2} \sin(\varphi) \leq s - t_0 \cos(\varphi) \)
and the estimate \( 2\mu - s - t_0 \cos(\varphi) \geq \mu - t_0 \cos(\varphi) \), it follows that

\[
J_{2,2}(\mu, s, \varphi) \leq \frac{s^n}{\sqrt{s^2 - t_0^2} \sin^4(\varphi)} \leq C_0 \frac{s^n}{\sqrt{s^2 - t_0^2}}.
\]

where \( C_0 \) is defined by (5.33) (we also used the estimate \( s - t_0 \cos(\varphi) \geq s(1 - \cos(\varphi)) \)). Therefore

\[
\int_0^\pi J_{2,2}(\mu, s, \varphi) d\varphi ds \leq \frac{\pi C_0^2}{2s^{n-1}}.
\]

(10.27)

Let \( \varphi \in \left(\frac{\pi}{2}, \pi\right) \) and \( s \in (t_0, \mu) \). From (10.24) and the estimates \( \cos^2(\varphi) \leq 1, s - t_0 \cos(\varphi) \geq s, \sqrt{s^2 - t_0^2} \sin(\varphi) \leq s + t_0 \cos(\varphi) \) and the estimate \( 2\mu - s - t_0 \cos(\varphi) \geq \mu \), it follows that

\[
J_{2,1}(\mu, s, \varphi) \leq \frac{\sqrt{s^2 - t_0^2}}{(s + t_0 \cos(\varphi)) \mu s^{n-2}} \leq \frac{\sqrt{s^2 - t_0^2}}{(s + t_0 \cos(\varphi)) \mu t_0^{n-2}}.
\]

Therefore using (7.1), we obtain

\[
\int_0^\pi J_{2,1}(\mu, s, \varphi) d\varphi ds \leq \frac{\pi}{2\mu t_0^{n-2}}.
\]

(10.28)

From (10.25) and the estimates \( s - t_0 \cos(\varphi) \geq s, \sqrt{s^2 - t_0^2} \sin(\varphi) \leq s + t_0 \cos(\varphi) \) and the estimate \( 2\mu - s - t_0 \cos(\varphi) \geq \mu \), it follows that

\[
J_{2,2}(\mu, s, \varphi) \leq \frac{\sqrt{s^2 - t_0^2} \sin^4(\varphi)}{(s + t_0 \cos(\varphi)) \mu s^{n-4}} \leq C_0^2 \frac{\sqrt{s^2 - t_0^2}}{(s + t_0 \cos(\varphi)) \mu s^{n-2}},
\]

where \( C_0 \) is defined by (5.33) (we also used the estimate \( s + t_0 \cos(\varphi) \geq s(1 + \cos(\varphi)) \)). Therefore using (7.1), we obtain

\[
\int_0^\pi J_{2,2}(\mu, s, \varphi) \leq C_0^2 \frac{\pi}{2\mu s^{n-2}} \leq C_0^2 \frac{\pi}{2\mu t_0^{n-2}}.
\]

Note that

\[
\int_{t_0}^\mu \frac{1}{s^{n-1}} ds = \frac{1}{n - 2} \left( \frac{\mu^{n-2} - t_0^{n-2}}{t_0^{n-2} \mu^{n-2}} \right) = \frac{\mu - t_0}{n - 2} \sum_{i=0}^{n-3} \mu^{-1-i} t_0^{i+2-n} \leq \frac{\mu - t_0}{\mu t_0^{n-2}}.
\]

(10.29)

(we used the estimate \( t_0 < \mu \) which gives \( \mu^{-1-i} t_0^{i+2-n} \leq \mu^{-1} t_0^{n-2} \) for \( i = 0 \ldots n - 3 \)).

Combining (10.23), (10.26)–(10.29), we obtain

\[
J_2(\mu, z, z') \leq C' \frac{\mu - t_0}{\mu t_0^{n-2}}.
\]

(10.30)

where \( C' \) does not depend on \( \mu \) and \( z, z' \). Statement (8.5) follows from (10.30). \( \square \)
11 The distributional kernel of the operators $H_m$ and the proof of Proposition 3.1

Before we prove Proposition 3.1 we shall introduce and prove Proposition 11.1 given below, which gives the distributional kernel of the operators $H_m$ defined by (2.12).

Let $E$ denote the nonnegative measurable function from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$ defined by

$$
\bar{E}(x_1, x_2) = e^{-\|x_1 - x_2\|^2} \sigma(x_1 - s x_2/s | x_1 - s x_2/s | x_1 - s x_2/s \|) ds \Theta(x_1, x_2), \text{ for a.e. } (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n,
$$

(11.1)

where $\Theta$ is defined by (2.10). For $m \geq 3$, we define recursively the nonnegative measurable real function $\bar{E}(x_1, \ldots, x_m)$ by the formula

$$
\bar{E}(x_1, \ldots, x_m) = \bar{E}(x_1, \ldots, x_{m-1})\bar{E}(x_{m-1}, x_m),
$$

(11.2)

for $(x_1, \ldots, x_m) \in (\mathbb{R}^n)^m$.

Concerning the distributional kernel of the $H_m$, $m \geq 2$, we have the following result.

Proposition 11.1. We have

$$
H_2(t)\phi(x, v) = \int_{X \times S^{n-1}} \beta_2(t, x, v, x')\phi(x', v')dx' dv',
$$

(11.3)

for $t \in (0, T)$ and a.e. $(x, v) \in X \times S^{n-1}$, where

$$
\beta_2(t, x, v, x', v') = \int_0^t \int_{0 \leq i \leq m} \left( |x' - (x - s_2 v')| \right)^{2n-2} \left( t - s_2 - (x - s_2 v' - x') \cdot v \right)^{n-3}
$$

$$
\times \left[ \bar{E}(x, x - t - s_1 - s_2 v', x' + s_2 v', x')k(x - (t - s_1 - s_2)v, v_1, v)
\right.
$$

$$
\left. \times k(x' + s_2 v', v, v_1) \right|_{v_1 = \frac{x' - s_2 v' - x' - (t - s_1 - s_2)v}{s_1}} ds_2,
$$

(11.4)

for $t \in (0, T)$ and a.e. $(x, v, x', v') \in X \times S^{n-1} \times X \times S^{n-1}$.

In addition for $m \geq 3$ we have

$$
H_m(t)\phi(x, v) = \int_{X \times S^{n-1}} \beta_m(t, x, v, x', v')\phi(x', v')dx' dv',
$$

(11.5)

for $t \in (0, T)$ and a.e. $(x, v) \in X \times S^{n-1}$, where

$$
\beta_m(t, x, v, x', v') = \int_{(S^{n-1})^{m-2}} \int_{s_1 \geq 0, \ldots, s_m \leq 1} \left( |x' + s_m v' + \ldots + s_1 v_2 - x| \right)^{2n-2} \left( t - s_2 - \ldots - s_m - (x - x' - s_2 v_2 - \ldots - s_m v_m - s_m v') \cdot v \right)^{n-3}
$$

$$
\times \left[ \bar{E}(x, x - t - s_1 - s_2 v_2 - \ldots - s_m v_m - s_m v') - (t - s_2 - \ldots - s_m) v \right]^{2n-4} \left[ \bar{E}(x, x - (t - s_1 - \ldots - s_m)v, v', v_1, v)
\right.
$$

$$
\left. \times k(x - (t - s_1 - \ldots - s_m)v, v_1, v)k(x' + s_m v' + s_m v_m + \ldots + s_1 v_2, v_1)
\right.
$$

$$
\left. \ldots k(x' + s_m v', s_m v_m + \ldots + s_1 v_2, v_1, v_1) \right|_{v_1 = \frac{x' - s_2 v' - \ldots - s_m v_m - (t - s_1 - \ldots - s_m)v}{s_1}} ds_2 \ldots ds_m dv_2 \ldots dv_m,
$$

(11.6)

for $t \in (0, T)$ and a.e. $(x, v, x', v') \in X \times S^{n-1} \times X \times S^{n-1}$.
Proof of Proposition 11.1. Note that

\[
H_2(t)\phi(x, v) = \left( \int_0^t \int_0^{t-s_1} U_1(t-s_1-s_2)A_2U_1(s_1)A_2U_1(s_2)\phi ds_2 ds_1 \right)(x, v)
\]

\[
= \left( \int_0^t \int_0^{t-s_2} U_1(t-s_1-s_2)A_2U_1(s_1)A_2ds_1 \right)U_1(s_2)\phi ds_2 (x, v)
\]

\[
= \int_0^t \int_0^{t-s_2} \bar{E}(x, x - (t-s_1-s_2)v) \int_{S^{n-1}} k(x - (t-s_1-s_2)v, v, v_1) \times \bar{E}(x - (t-s_1-s_2)v, x - (t-s_1-s_2)v - s_1v_1) \times k(x - (t-s_1-s_2)v - s_1v_1, v_1, v_2) \times \phi(x - (t-s_1-s_2)v - s_1v_1 - s_2v_2, v_2) dv_2 dv_1 ds_1 ds_2,
\]

for \( t \in (0, T) \) and \( (x, v) \in X \times S^{n-1} \), where functions \( \bar{E} \) are defined by (11.1)–(11.2).

Using the change of variables "\( y(s_1, v_1) = (t-s_2-s_1)v + s_1v_1 \)" we obtain

\[
H_2(t)\phi(x, v) = \int_0^t \int_{S^{n-1}} \left[ \bar{E}(x, x - (t-s_1-s_2)v, x - y, x - y - s_2v_2)k(x - (t-s_1-s_2)v, v_1, v) \times k(y, y, v_2, v_1) \right]_{v_1 = \frac{y - (t-s_1-s_2)v}{s_1}} \\
\times \frac{2^{n-2} (t-s_2 - y \cdot v)^{n-3}}{|y - (t-s_2)v|^{2n-4}} \phi(x - y - s_2v_2, v_2) dy dv_2 ds_2.
\]

Hence we obtain (11.3).

Note that

\[
(H_3(t)\phi)(x, v) = \int_0^t H_2(t-s_3)A_2U_1(s_3)\phi ds_3
\]

\[
= \int_0^t \int_{X \times S^{n-1}} \beta_2(t-s_3, x, v, x_2, v_2)(A_2U_1(s_3))\phi(x_2, v_2) dx_2 dv_2 ds_3
\]

\[
= \int_{X \times S^{n-1}} \int_0^t \beta_2(t-s_3, x, v, x_2, v_2) \int_{S^{n-1}} k(x_2, v_1, v_2) \bar{E}(x_2, x_2 - s_3v') \times \phi(x_2 - s_3v', v') dv' ds_3 dx_2 dv_2.
\]

Hence

\[
(H_3(t)\phi)(x, v) = \int_{X \times S^{n-1}} \beta_3(t, x, v, x', v')\phi(x', v') dx' dv', \quad (11.7)
\]

where

\[
\beta_3(t, x, v, x', v') = \int_{S^{n-1}} \int_0^t \int_0^{t-s_3} \chi(0, t-s_3-s_2)(|x' + s_3v' - x + s_2v_2|) \times \frac{2^{n-2} (t-s_2-s_3 - (x-s_2v_2 - x' - s_3v') \cdot v)^{n-3}}{|x - x' - s_2v_2 - s_3v' - (t-s_2-s_3)v|^{2n-4}} \times \left[ \bar{E}(x, x - (t-s_1-s_2)v, x' + s_2v_2 + s_3v', x') \times k(x - (t-s_3-s_1)v, v_1, v_1) \times k(x' + s_3v', v'_{s_1}) \right]_{v_1 = \frac{x-x'-s_2v_2-s_3v'}{(t-s_2-s_3)v}} ds_3 dv_2. \quad (11.8)
\]

The proof of (11.6) follows by induction from (11.7) and (2.13).
Proof of (8.12)–(8.13). We remind that

\[(A_2 G_-(s) \phi_S) (z, w) = \int_{\partial X} [k(z, v', w) S(x', v') |\nu(x') \cdot v'|] v' = z - x' |z - x'|^{n-1} E(z, x') \phi(s - |z - x'|, x') \, d\mu(x'), \]

for a.e. \((z, w) \in X \times \mathbb{S}^{n-1}\) and \(\phi \in L^1((0, \eta) \times \partial X)\) (see the derivation of (3.9) and (3.10) given in Section 3).

Let \(m \geq 3\). From (3.5), (11.5) and (11.9) it follows that

\[A_{m+1, S}(\phi)(t, x) = \int_{\mathbb{S}^{n-1}} \int_{\partial X} (\nu(x) \cdot v) W(x, v) \int_X \int_{(t, t - |x_m - x'|) \times \mathbb{S}^{n-1}} \beta_m (t - t' - |x_m - x'|, x, v, x_m, v_m) \]

\[\quad \times [k(x_m, v', v_m) S(x', v') |\nu(x') \cdot v'|] v' = \frac{x_m - x'}{|x_m - x'|} \frac{x_m - x'}{\phi(t', x', \phi(t', x') dt' \, dx_m \, dv_m \, dv} \]

\[= \int_{(0, \eta) \times \partial X} \gamma_{m+1}(t - t', x, x') \phi(t', x') \, dt' \, d\mu(x'), \]

where

\[\gamma_{m+1}(\tau, x, x') := \int_{\mathbb{S}^{n-1}} (\nu(x) \cdot v) W(x, v) \int_X X(0, +\infty)(\tau - |x_m - x'|) \]

\[\int_{(\mathbb{S}^{n-1})^m} \int_{\mathbb{S}^{n-1}} \left. \right| (x_m + m v_m + \ldots + s_2 v_2 - x) \]

\[\times \frac{2^{n-2} (\tau - |x_m - x'| - s_2 - \ldots - s_m - (x - x_m - s_2 v_2 - \ldots - s_m v_m) \cdot v')^{n-3} \frac{x_m - x'}{|x_m - x'|}}{\frac{x_m - x'}{|x_m - x'|} |x_m - x'|^{n-1} |x_m - x'| - s_2 - \ldots - s_m w_v} \]

\[\times E(x, x - (\tau - |x_m - x'| - s_1 - \ldots - s_m) v, x_m + s_1 v_m + s_2 v_2 + \ldots + s_3 v_3 + \ldots, x_m + s_1 v_m, x_m, x') \]

\[k(x_m + s_1 v_m, v_m, v_m - 1) k(x_m + s_1 v_m, v_m, v_m - 1) S(x', v') |\nu(x') \cdot v'| \]

\[d s_2 \ldots d s_m d v_2 \ldots d v_m - 1 d x_m d v_m d v. \]

Performing the change of variables \(y_i = s_i v_i, i = 2 \ldots m, \) and \(y_m + 1 = x_m + 1 - x'\), we obtain (8.13) for \(m \geq 4\).

Let \(m = 2\). Then from (2.12) and (2.13) it follows that

\[A_{2, S}(\phi)(t, x) = \int_{\mathbb{S}^{n-1}} (\nu(x) \cdot v) W(x, v) \int_{t - s}^t \int_{\mathbb{S}^{n-1}} \int_{\partial X} k(x - (t - s - 1) v, v_1, v) \]

\[\times k(x - (t - s - 1) v - s_1 v_1, v', v_1) S(x', v') |\nu(x') \cdot v'| \]

\[E(x, x - (t - s - 1) v, x - (t - s - 1) v) \]

\[\times x - (t - s - 1) v - s_1 v_1, x') \phi(s - |x - (t - s - 1) v - s_1 v_1 - x'|, x') \]

\[d \mu(x') d v_1 d s_1 d s_2 d v. \]
Performing the change of variables \( y(s_1, v_1) = (t - s - s_1)v + s_1v_1 \), we obtain

\[
A_{2,S,W}(\phi)(t, x) = \int_{S_{t-s}^{-1} \times \partial X \times \mathbb{R}^n} (\nu(x) \cdot v) W(x, v) \int_{-\infty}^{t-s} \chi_{(0,t-s)}(\|y\|) \times [E(x, x - (t - s - s_1)v, x - y, x')]k(x - (t - s - s_1)v, v, v_1)k(x - y, v', v_1)S(x', v')
\]

\[
\times |\nu(x') \cdot v'||_{s_1 = \frac{2(t-s)v - y - (t-s-s_1)v}{2(t-s)}; v' = \frac{x' - y}{|x' - y|}}
\]

Performing the change of variables \( "y" = x - x' - y \) and \( t' = s - |y| \) we obtain (8.12).

Let \( m = 3 \). Then from (3.5), (11.3) and (11.9) it follows that

\[
A_{3,S,W}(\phi)(t, x) = \int_{S_{t-s}^{-1} \times \partial X \times \mathbb{R}^n} (\nu(x) \cdot v) W(x, v) \int_{-\infty}^{t-s} \beta_2(t - s - s_2, v_2)
\]

\[
\int_{\partial X} [k(x_2, v_2)S(x', v')|\nu(x') \cdot v'||_{v' = \frac{x_2 - x'}{|x_2 - x'|}}]E(x_2, x')|\nu(x) \cdot v'||_{v = \frac{x_2 - x}{|x_2 - x|}}\phi(s - |x_2 - x'|, x')d\mu(x')dx_2dv_2dsdv,
\]

for \( t \in (0, T) \) and \( x \in \partial X \). From (11.13) and (11.4) we obtain

\[
A_{3,S,W}(\phi)(t, x) = \int_{S_{t-s}^{-1} \times \partial X \times \mathbb{R}^n} (\nu(x) \cdot v) W(x, v) \int_{-\infty}^{t-s} \chi_{(0,t-s-s_2)}(\|x_2 - (x - s_2v_2)\|)
\]

\[
2n-2 \frac{(t - s - s_2 - (x - s_2v_2 - x_2) \cdot v)^{n-3}}{|x_2 - x'|^{n-1}} \frac{|x - s_2v_2 - x_2 - (t - s - s_2)v|^{2n-4}}{|s_1 = \frac{2(t-s)v - y - (t-s-s_1)v}{2(t-s-v_2 - x_2)}; v' = \frac{x_2 - x}{|x_2 - x|}}}
\]

Performing the change of variables \( y_2 = s_2v_2 \) and \( y_3 = x_2 - x' \) we obtain (8.13) for \( "m = 3" \).

\[ \square \]

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References


