

# Introduction to Inverse Problems

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January 29, 2012

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# Contents

<b>1</b>	<b>What constitutes an Inverse Problem</b>	<b>1</b>
1.1	Elements of an <i>Inverse Problem</i> (IP)	1
1.1.1	Injectivity and stability of the Measurement Operator	1
1.1.2	“Noise”, Modeling, and Prior Information	2
1.1.3	Numerical simulations	4
1.2	Examples of Measurement Operator	6
1.3	IP and Modeling. Application to MRI	8
1.4	Inverse Problems and Smoothing: Hilbert scale	12
1.4.1	Fourier transforms and well-posedness	12
1.4.2	Hilbert scale and degrees of ill-posedness	13
<b>2</b>	<b>Integral Geometry. Radon transforms</b>	<b>19</b>
2.1	Transmission Tomography	19
2.2	Two dimensional Radon transform	21
2.3	Three dimensional Radon transform	28
2.4	Attenuated Radon Transform	29
2.4.1	Single Photon Emission Computed Tomography	30
2.4.2	Riemann Hilbert problem	31
2.4.3	Inversion of the Attenuated Radon Transform	32
2.4.4	Step (i): The $\bar{\partial}$ problem, an elliptic equation	33
2.4.5	Step (ii): jump conditions	35
2.4.6	Step (iii): reconstruction formulas	37
<b>3</b>	<b>Integral Geometry. Generalized Ray Transform</b>	<b>39</b>
3.1	Generalized Ray Transform: Setting in two dimensions.	40
3.1.1	Family of curves.	40
3.1.2	Generalized Ray Transform.	40
3.1.3	Adjoint operator and rescaled Normal operator	41
3.2	Oscillatory integrals and Fourier Integral Operators	43
3.2.1	Symbols, phases, and oscillatory integrals.	43
3.2.2	Parameterized oscillatory integrals	45
3.2.3	Definition of Fourier Integral Operators.	46
3.3	Pseudo-differential operators and GRT	47
3.3.1	Absence of singularities away from the diagonal $x = y$	48
3.3.2	Change of variables and phase $(x - y) \cdot \xi$	49
3.3.3	Choice of a parametrix.	50

3.3.4	Proof of smoothing by one derivative . . . . .	51
3.3.5	Boundedness of $\Psi$ DOs of order 0 in the $L^2$ sense . . . . .	51
3.3.6	Injectivity and implicit inversion formula. . . . .	53
3.4	Kinematic Inverse Source Problem . . . . .	55
3.4.1	Transport equation . . . . .	56
3.4.2	Variational form and energy estimates . . . . .	56
3.4.3	Injectivity result . . . . .	58
3.4.4	Summary on GRT. . . . .	58
3.5	Propagation of singularities for the GRT. . . . .	59
3.5.1	Wave Front Set and Distributions. . . . .	59
3.5.2	Propagation of singularities in FIOs . . . . .	61
<b>4</b>	<b>Inverse wave problems</b> . . . . .	<b>65</b>
4.1	One dimensional inverse scattering problem . . . . .	66
4.2	Linearized Inverse Scattering problem . . . . .	68
4.2.1	Setting and linearization . . . . .	68
4.2.2	Far field data and reconstruction . . . . .	70
4.2.3	Comparison to $X$ -ray tomography . . . . .	73
4.3	Inverse source problem in PAT . . . . .	74
4.3.1	An explicit reconstruction formula for $\Sigma$ the unit sphere . . . . .	74
4.3.2	An explicit reconstruction for detectors on a plane . . . . .	75
4.4	One dimensional inverse coefficient problem . . . . .	77
<b>5</b>	<b>Inverse Kinematic and Inverse Transport Problems</b> . . . . .	<b>81</b>
5.1	Inverse Kinematic Problem . . . . .	82
5.1.1	Spherical symmetry . . . . .	83
5.1.2	Abel integral and Abel transform . . . . .	85
5.1.3	Kinematic velocity Inverse Problem . . . . .	86
5.2	Forward transport problem . . . . .	88
5.3	Inverse transport problem . . . . .	92
5.3.1	Decomposition of the albedo operator and uniqueness result . . . . .	92
5.3.2	Stability in inverse transport . . . . .	96
<b>6</b>	<b>Inverse diffusion</b> . . . . .	<b>99</b>
6.1	Cauchy Problem and Electrocardiac potential . . . . .	99
6.2	Half Space Problem . . . . .	100
6.2.1	The well posed problem . . . . .	101
6.2.2	The electrocardiac application . . . . .	102
6.2.3	Prior bounds and stability estimates . . . . .	102
6.2.4	Analytic continuation . . . . .	104
6.3	General two dimensional case . . . . .	105
6.3.1	Laplace equation on an annulus . . . . .	105
6.3.2	Riemann mapping theorem . . . . .	106
6.4	Backward Heat Equation . . . . .	106

<b>7</b>	<b>Calderón problem</b>	<b>109</b>
7.1	Introduction	109
7.2	Uniqueness and Stability	110
7.2.1	Reduction to a Schödinger equation	110
7.2.2	Proof of injectivity result	112
7.2.3	Proof of the stability result	113
7.3	Complex Geometric Optics Solutions	115
7.4	The Optical Tomography setting	118
<b>8</b>	<b>Coupled-physics IP I: PAT and TE</b>	<b>121</b>
8.1	Introduction to PAT and TE	122
8.1.1	Modeling of photoacoustic tomography	122
8.1.2	First step: Inverse wave source problem	124
8.1.3	Second step: Inverse problems with internal functionals	125
8.1.4	Reconstruction of one coefficient.	126
8.1.5	Introduction to Transient Elastography	127
8.2	Theory of quantitative PAT and TE	127
8.2.1	Uniqueness and stability results in QPAT	127
8.2.2	Application to Quantitative Transient Elastography	134
8.3	<i>Well-chosen</i> illuminations in PAT and TE	134
8.3.1	The two dimensional case	134
8.3.2	The $n$ dimensional case	135
<b>9</b>	<b>Coupled-physics IP II: UMT</b>	<b>137</b>
9.1	Ultrasound Modulation Tomography	137
9.2	Inverse problems in ultrasound modulation.	139
9.3	Eliminations and redundant systems of ODEs	142
9.3.1	Elimination of $F$	142
9.3.2	System of ODEs for $S_j$	144
9.3.3	ODE solution and stability estimates	146
9.4	<i>Well-chosen</i> illuminations	146
9.4.1	The case $n = 2$	146
9.4.2	The case $n \geq 3$	146
9.5	Remarks on hybrid inverse problems	148
<b>10</b>	<b>Priors and Regularization</b>	<b>151</b>
10.1	Smoothness Regularization	152
10.1.1	Ill-posed problems and compact operators	152
10.1.2	Regularity assumptions and error bound	153
10.1.3	Regularization methods	156
10.2	Sparsity and other Regularization Priors	163
10.2.1	Smoothness Prior and Minimizations	163
10.2.2	Sparsity Prior and Minimizations	164
10.3	Bayesian framework and regularization	165
10.3.1	Penalization methods and Bayesian framework	166
10.3.2	Computational and psychological costs of the Bayesian framework	167

<b>11 Geometric Priors and Parameterizations</b>	<b>171</b>
11.1 Reconstructing the domain of inclusions	171
11.1.1 Forward Problem	172
11.1.2 Factorization method	173
11.1.3 Reconstruction of $\Sigma$	177
11.2 Reconstructing small inclusions	178
11.2.1 First-order effects	179
11.2.2 Stability of the reconstruction	181
<b>12 Inverse Problems and Modeling</b>	<b>183</b>
12.1 Imaging in Highly Heterogeneous Media	184
12.1.1 Wave model	184
12.1.2 Kinetic Model	185
12.1.3 Statistical Stability	186
12.1.4 Inverse transport problem	188
12.1.5 Random media and Correlations	189
12.1.6 Imaging with waves in random media	190
12.2 Random fluctuations and Noise models	190
<b>Bibliography</b>	<b>195</b>

# Chapter 1

## What constitutes an Inverse Problem

### 1.1 Elements of an *Inverse Problem* (IP)

The definition of an *inverse problem* starts with that of a mapping between objects of interest, which we call *parameters*, and acquired information about these objects, which we call *data* or *measurements*. The mapping, or *forward problem*, is called the *measurement operator* (MO). We denote it by  $\mathfrak{M}$ .

The MO maps parameters in a functional space  $\mathfrak{X}$ , typically a Banach or Hilbert space, to the space of *data*  $\mathfrak{Y}$ , typically another Banach or Hilbert space. We write

$$y = \mathfrak{M}(x) \quad \text{for } x \in \mathfrak{X} \text{ and } y \in \mathfrak{Y}, \quad (1.1)$$

the correspondence between the parameter  $x$  and the data  $y$ . Solving the inverse problem amounts to finding point(s)  $x \in \mathfrak{X}$  from knowledge of the data  $y \in \mathfrak{Y}$  such that (1.1) or an approximation of (1.1) holds.

The MO describes our best effort to construct a *model* for the available data  $y$ , which we assume here depend only on the sought parameters  $x$ . The choice of  $\mathfrak{X}$  describes our best effort to characterize the space where we believe the parameters belong.

#### 1.1.1 Injectivity and stability of the Measurement Operator

The first question to ask about the MO is whether we have acquired enough data to uniquely reconstruct the parameters. In other words, whether the MO is *injective*. Injectivity means that

$$\mathfrak{M}(x_1) = \mathfrak{M}(x_2) \quad \implies \quad x_1 = x_2 \text{ for all } x_1, x_2 \in \mathfrak{X}. \quad (1.2)$$

Then the data  $y$ , if given in the range of  $\mathfrak{M}$ , uniquely characterize the parameter  $x$ .

Measurement operators used in practice are typically discretized and available data typically contain noise as we shall see below. Such measurement operators are often not (quite) injective. Yet, most practical measurement operators can be seen as approximations to measurement operators that are indeed injective. The study of the MO provides very significant information about the structure of the inverse problem considered.

When  $\mathfrak{M}$  is injective, we can construct an inversion operator  $\mathfrak{M}^{-1}$  mapping the range of  $\mathfrak{M}$  to a uniquely defined element in  $\mathfrak{X}$ . This inverse operation is what is solved numerically in practice. The main features of the inverse operator are captured by what are called *stability estimates*. Such estimates quantify how errors in the available measurements translate into errors in the reconstructions.

Stability estimates typically take the following form:

$$\|x_1 - x_2\|_{\mathfrak{X}} \leq \omega(\|\mathfrak{M}(x_1) - \mathfrak{M}(x_2)\|_{\mathfrak{Y}}), \quad (1.3)$$

where  $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is an increasing function such that  $\omega(0) = 0$  quantifying the modulus of continuity of the inversion operator  $\mathfrak{M}^{-1}$ . This function gives an estimate of the reconstruction error  $\|x_1 - x_2\|_{\mathfrak{X}}$  based on what we believe is the error in the data acquisition  $\|\mathfrak{M}(x_1) - \mathfrak{M}(x_2)\|_{\mathfrak{Y}}$ .

When noise is not amplified too drastically so that the error on the reconstructed parameters is *acceptable*, for instance *typically* when  $\omega(x) = Cx$  for some constant  $C$ , then we say that the inverse problem is *well-posed*. When noise is strongly amplified and the reconstruction is contaminated by *too large* a noisy component, for instance *typically* when  $\omega(x) = |\log|x||^{-1}$  so that measurement errors of  $10^{-10}$  translate into reconstruction errors of  $\frac{1}{10}$ , then we say that the inverse problem is *ill-posed*. The notion of the ill-posedness of an inverse problem is therefore *subjective*.

Let us make two observations about the advantages and limitations of stability estimates. First, the modulus of continuity  $\omega$  depends on the choice of  $\mathfrak{X}$  and  $\mathfrak{Y}$ . Let us assume that the metric space  $\mathfrak{Y}$  is endowed with a metric  $d(y_1, y_2) = \omega(\|y_1 - y_2\|_{\mathfrak{Y}})$ . Then the distance between the two reconstructions  $x_1$  and  $x_2$  is bounded by the distance between the two measurements  $y_1$  and  $y_2$ , which seems to correspond to a “nice”, well-posed, inverse problem. The stability estimates and corresponding moduli of continuity are thus also *subjective*.

Second, we were careful to look at the difference between  $y_1 = \mathfrak{M}(x_1)$  and  $y_2 = \mathfrak{M}(x_2)$ , that is to consider errors of measurements in the *range* of the measurement operator. This is legitimate since all that we have constructed so far is measurements in  $\mathfrak{Y}$  that are in the range of the measurement operator  $\mathfrak{M}$ . In practice, however, noise in the data acquisition may cause the measurements to leave the natural set  $\mathfrak{M}(\mathfrak{X})$  where measurements are currently defined. Data then need to be projected onto the range of the MO first, or an entirely new set of stability estimates need to be developed.

In spite of the aforementioned caveats, the notions of injectivity and stability of a measurement operator are very fruitful concepts that provide practically useful information about the structure of the inverse problem of interest. Most of this book is devoted to the analysis of injectivity of different MO and the derivation of (typically several) corresponding stability estimates. Let us reiterate that most practical inverse problems are approximations to injective measurement operators with well-characterized stability properties.

### 1.1.2 “Noise”, Modeling, and Prior Information

The MO is typically not sufficient to describe an IP *satisfactorily*, again, a subjective notion. Often, “noisy” contributions need to be added to realistically model the available

data, for instance contributions of parameters not included in  $\mathbf{x}$  because we have no chance of reconstructing them. They need to be incorporated because they may have *undesirable* effects on the reconstructions. This undesirable effect will in turn require *action* from the user by means of imposition of prior assumptions on the parameters. But first, let us be more specific about what we mean by noise.

**Modeling and Measurement Errors.** Rather than modeling the IP as  $\mathbf{y} = \mathfrak{M}(\mathbf{x})$ , it is customary to define an IP as  $\mathbf{y} = \mathfrak{M}(\mathbf{x}) + \mathbf{n}$ , where  $\mathbf{n}$  is “noise” and its definition is precisely

$$\mathbf{n} := \mathbf{y} - \mathfrak{M}(\mathbf{x}), \quad (1.4)$$

the *discrepancy* between the model  $\mathfrak{M}(\mathbf{x})$  and the available data  $\mathbf{y}$ .  $\mathbf{n}$  typically has two contributions. One standard contribution is the *detector* noise, since measurements are performed by instruments that are imperfect. Often, a more important contribution is what we shall call a *modeling* error, for instance reflecting that  $\mathfrak{M}$  is an imperfect modeling of the physical underlying process mapping parameters  $\mathbf{x}$  to data  $\mathbf{y}$ .

We have seen that the analysis of the MO gave some intuition on the effect of noise in the reconstructions. Let us assume that we “know” how to project the available data  $\tilde{\mathbf{y}}$  onto the range of  $\mathfrak{M}$  so as to decompose it as  $\mathbf{y} = \mathfrak{M}(\mathbf{x}_2)$  plus  $\tilde{\mathbf{y}} - \mathfrak{M}(\mathbf{x}_2)$ . Let us then discard  $\tilde{\mathbf{y}} - \mathbf{y}$  for the moment. Then  $\mathbf{n} = \mathfrak{M}(\mathbf{x}_2) - \mathfrak{M}(\mathbf{x})$  and the stability estimates provide a measure of the error  $\mathbf{x} - \mathbf{x}_2$  in the reconstruction. Estimating the above error term is therefore an important part of solving an IP in practice. Several guiding principles to do so will be considered in Chapter 12. For the moment, let us simply mention that it often makes sense to model  $\mathbf{n}$  as a “random” process. Indeed, if a simple (deterministic) model for  $\mathbf{n}$  was available, that is a model involving well-characterized parameters, then we would modify the MO to account for said model. The randomness comes from the aforementioned sources: the detector and modeling errors. What are the statistics of  $\mathbf{n}$ , and in particular the ensemble average  $\mathfrak{m}(\mathbf{x}) = \mathbb{E}\{\mathbf{n}(\mathbf{x})\}$  and the two point correlation function  $\mathfrak{c}(\mathbf{x}, \mathbf{y}) = \{\mathbf{n}(\mathbf{x})\mathbf{n}(\mathbf{y})\}$  are practically important information whose knowledge can very significantly improve the solution to a given IP.

**Prior assumptions.** We now have two items characterizing an IP: a MO and a noise model. The very reason “noise” is modeled is because it has an effect on the reconstruction, which is estimated by the stability estimates associated to the MO. Sometimes, the effect is *undesirable*. One is then faced with essentially (a combination of) three strategies: (i) acquire more accurate data to lower the size of  $\|\mathbf{n}\|_{\mathfrak{Y}}$  and hence of  $\|\mathbf{x}_1 - \mathbf{x}_2\|_{\mathfrak{X}}$ . When the stability estimates imply a strong amplification of the noise level in the reconstruction, this strategy has limited value; (ii) change the MO and acquire different data. This ideal scenario may not always be feasible; and (iii) restrict the class in which the unknown parameters are sought. We shall call the latter strategy incorporating *prior information*. Several major methodologies to do so can be classified as follows:

1. *Penalization theory* is a deterministic methodology that restricts the domain of definition of the parameters. It includes two subcategories, one being *regularization theory*, which assumes that the parameters of interest are “sufficiently smooth”, and the other one being *sparsity theory*, which assumes that the parameters are sparsely represented in a given, specific, basis.

2. The *Bayesian framework* is an alternative, very versatile, methodology to incorporate prior assumptions in a statistical fashion. A *prior* probability density describes the potential values that the parameters can take. A *posterior* probability density describes how these potential values are affected by the measurements.
3. Several *geometric constraints* give rise to simplified reconstructions (of small inclusions for instance) or qualitative methods of reconstructions (of support of inclusions for instance).

The structure (MO, noise model, prior information) completes the description of an inverse problem. The main focus of this book is the analysis of typical measurement operators that appear in practice and on the mathematical tools that are useful in such analyses. Yet, explicit solutions of inverse problems often involve numerical simulations. We now briefly mention several types of simulations that appear frequently in the solution of inverse problems. Depending on the type of aforementioned structure, the computational cost may vary greatly. For additional information, we refer the reader to, e.g., [22, 38, 63, 66].

### 1.1.3 Numerical simulations

Let us consider the case of a linear operator  $\mathfrak{M}$  and of an inverse problem of the form  $y = \mathfrak{M}x + n$ . Any nonlinear problem  $y = \mathfrak{M}(x) + n$  is typically linearized before being solved, for instance by iteratively solving linear problems in the form  $y - \mathfrak{M}(x_0) = A(x - x_0) + n$ , where  $A$  is some differential of  $\mathfrak{M}$  at  $x_0$ . Then the computational cost of solving the inverse problem modeled by  $y = \mathfrak{M}x + n$  typically falls into one of the following categories.

(a) Penalization theories such as regularization theories typically replace  $y = \mathfrak{M}x + n$  by

$$\mathfrak{M}^*y = (\mathfrak{M}^*\mathfrak{M} + \delta B)x_\delta, \quad (1.5)$$

where  $\delta$  is a regularization parameter and  $B$  is a positive definite operator such that  $(\mathfrak{M}^*\mathfrak{M} + \delta B)$  is an invertible operator with bounded inverse. Here  $\mathfrak{M}^*$  is the adjoint operator to  $\mathfrak{M}$ . The inverse problem then involves solving a *linear system* of equations.

(b) Penalization theories such as those based on sparsity constraints typically replace  $y = \mathfrak{M}x + n$  by

$$x_\delta = \operatorname{argmin} \|y - \mathfrak{M}x\|_{\mathfrak{Y}_1} + \delta \|x\|_{\mathfrak{X}_2}, \quad (1.6)$$

where  $\mathfrak{Y}_1$  is typically a  $L^2$  norm and  $\mathfrak{X}_2$  typically a  $L^1$  norm to promote sparsity. Again,  $\delta$  is a small, regularizing parameter. Solving such a problem therefore requires solving an optimization (minimization) problem, which is algorithmically more challenging than the linear problem in (1.5).

(c) Bayesian frameworks are typically much more challenging computationally. The premise of such a framework is that we *know* a *prior distribution*  $\pi(x)$  which assigns a probability (density) to all potential candidates  $x$  before any data are acquired. We also assume that we *know* the *likelihood function*, which is the *conditional* distribution  $\pi(y|x)$  of the data  $y$  conditioned on knowing the parameter  $x$ . This is equivalent to knowing the distribution of the noise  $n$ . Bayes theorem then states that

$$\pi(x|y) = C\pi(y|x)\pi(x). \quad (1.7)$$

Here  $C$  is a normalizing constant so that  $\pi(x|y)$  is a probability density, i.e., so that  $\int_{\mathfrak{X}} \pi(x|y) d\mu(x) = 1$  for  $d\mu(x)$  the measure of integration on  $\mathfrak{X}$ . In other words, what we know a priori,  $\pi(y|x)$  and  $\pi(x)$  before acquiring any data, plus the additional knowledge obtained from measuring  $y$ , allows us to calculate  $\pi(x|y)$  the probability distribution of the unknown parameter  $x$  knowing the data  $y$ .

There are many advantages and caveats associated to this formalism, which will be presented in more detail in Chapter 10. However, from a computational view point, the Bayesian framework poses extraordinary challenges. If we know the distribution of the noise level  $n$ , then estimating  $\pi(y|x)$  requires solving a forward problem  $x \mapsto \mathfrak{M}(x)$ . This calculation has to be performed for a large number of values of  $x$  in order to sample  $\pi(x|y)$ . Moreover, sampling a distribution  $\pi(x|y)$  that does not necessarily admit a closed form expression is in itself computationally very intensive.

One of the main advantages of the Bayesian framework is that it insists on an important aspect of inverse problems in general: data are often not sufficiently informative to provide exact reconstructions of  $x$ . The Bayesian framework recognizes this fact by providing as an output a distribution of possible parameters  $x$  with probability  $\pi(x|y)$ . That said, this probability density needs to be processed and presented in a way that we can analyze and understand.

Two main methodologies are used to do so. The first one corresponds to estimating the *maximum likelihood* of the posterior distribution

$$x = \operatorname{argmax} \pi(x|y) = \operatorname{argmin} -\pi(x|y). \quad (1.8)$$

This MAP (maximum a posteriori) estimation is a minimization problem that bears very strong resemblances with (1.6), including its computational cost. Most models of the form (1.6) can be recast as MAP estimators of a Bayesian posterior.

A second method consists of estimating the first few statistical moments of  $\pi(x|y)$ , for instance

$$\bar{x} = \int_{\mathfrak{X}} x \pi(x|y) d\mu(x), \quad c = \int_{\mathfrak{X}} (x - \bar{x}) \otimes (x - \bar{x}) \pi(x|y) d\mu(x), \quad (1.9)$$

with  $\bar{x}$  the ensemble average of  $x_i$  and  $c$  the correlation “matrix” of  $x$ . For  $x = (x_i)$  a vector, think of  $x \otimes \bar{x}$  as the matrix with elements  $x_i \bar{x}_j$ . Here,  $d\mu(x)$  is a measure of integration on  $\mathfrak{X}$ . Other important moments, for instance for *uncertainty quantification* involve the estimation of the quantiles of  $x$ :

$$\varphi_\delta = \int_{\mathfrak{X}} H(\varphi(x) - \delta) \pi(x|y) d\mu(x). \quad (1.10)$$

Here  $\varphi$  is a functional from  $\mathfrak{X}$  to  $\mathbb{R}$ , and could be for instance a norm of  $x$ , or the value of one component of a vector valued  $x$ , or the evaluation of the parameter  $x$  at a spatial point and/or a specific time. Also,  $H(t)$  is the Heaviside function equal to  $H(t) = 1$  for  $t \geq 0$  and  $H(t) = 0$  for  $t < 0$ . Then  $\varphi_\delta$  above is the probability that  $\varphi(x)$  be larger than a given parameter  $\delta > 0$ .

All of these moments require that we *sample*  $\pi(x|y)$ . This is an extremely difficult and computationally expensive task. Standard methodologies to do so involve the Markov Chain Monte Carlo method that will be briefly presented in Chapter 10.

(d) Geometric constraints and qualitative methods, like the Bayesian framework, realize that available data are not nearly sufficient to uniquely determine  $x$ , even approximately. They severely restrict the set of potential parameters or try to recover not all of  $x$  but rather partial information about  $x$ .

Many geometric constraints and qualitative methods have been developed in the mathematical literature. Their main advantage is that they are computationally much more tractable than Bayesian reconstructions; see Chapter 11.

## 1.2 Examples of Measurement Operator

We now present several examples of measurement operators. The first three MO are there for the reader to gain some familiarity with the notion if necessary, while the last two examples of MO find important applications in, e.g., medical imaging.

**Example 1.** Let  $\mathfrak{X} = \mathcal{C}([0, 1])$ , the space of continuous functions. Let  $\mathfrak{Y} = \mathfrak{X}$  and define

$$\mathfrak{M}(f)(x) = \int_0^x f(y)dy.$$

Here, a “point”  $x$  in  $\mathfrak{X}$  is the function denoted by  $f$ , traditionally denoted by  $f(x)$ . The operator  $\mathfrak{M}$  is certainly injective since the equality of data  $\mathfrak{M}(f) = \mathfrak{M}(g)$  implies that  $f = \frac{d}{dx}\mathfrak{M}(f) = \frac{d}{dx}\mathfrak{M}(g) = g$ , i.e., the equality of parameters.

However, the operator  $\mathfrak{M}$  is “smoothing” since  $\mathfrak{M}(f)$  is one derivative more regular than  $f$ . So inverting the operator, as indicated above, involves differentiating the data. If “noise” is added to the data and “noise” is high frequency, the the derivative of “noise” will be large and may overwhelm the reconstruction of  $f(x)$ . The objective of penalization theory is precisely to make sure this *undesirable* fact does not happen.

**Example 2.** On  $\mathfrak{X} = \mathcal{C}_0^1([0, 1])$ , the space of continuously differentiable functions with value 0 at  $x = 0$  and  $\mathfrak{Y} = \mathcal{C}([0, 1])$ , we can define

$$\mathfrak{M}(f)(x) = f'(x), \quad \text{the derivative of } f.$$

Note that the derivative of a continuous function is not continuous (and may not exist as a function although it can be defined as a distribution). So here, we found it convenient to define the domain of definition of  $\mathfrak{M}$  as a subset of the space of continuous functions.

Why do we also insist on  $f(0) = 0$  in  $\mathfrak{X}$ ? It is because otherwise,  $\mathfrak{M}$  would *not* be injective. Indeed, antiderivatives are defined up to a constant. By enforcing  $f(0) = 0$ , the constant is fixed. In  $\mathfrak{X}$ , we have from the fundamental theory of calculus that

$$f(x) = f(0) + \int_0^x f'(y)dy = \int_0^x f'(y)dy = \int_0^x \mathfrak{M}(f)(y)dy.$$

Now obviously,  $\mathfrak{M}(f) = \mathfrak{M}(g)$  implies  $f = g$ . From the point of view of inverse problems, this is a very favorable situation. If “noise” is added to  $\mathfrak{M}(f)$ , it will be *integrated* during the reconstruction, which is a much more stable process than differentiating.

**Example 3.** With the same setting as in Example 1, consider  $\mathfrak{M}(f)(x) = \int_0^1 f(y)dy$ . This operator is well defined. But it is clearly not injective. All we learn about  $f$  is its mean on  $(0, 1)$ . This operator corresponds to data that are not very informative.

**Example 4.** Let  $\mathfrak{X} = \mathcal{C}_c(\mathbb{R}^2)$  the space of continuous functions with compact support (functions supported in a bounded domain in  $\mathbb{R}^2$ , i.e., vanishing outside of that domain). Let  $\mathfrak{Y} = \mathcal{C}(\mathbb{R} \times (0, 2\pi))$ . We define  $l(s, \theta)$  for  $s \in \mathbb{R}$  and  $\theta \in (0, 2\pi)$  as the *line* with direction perpendicular to  $\vartheta = (\cos \theta, \sin \theta)$  and at a distance  $|s|$  from the origin  $(0, 0)$ . More precisely, let  $\vartheta^\perp = (-\sin \theta, \cos \theta)$  the rotation of  $\vartheta$  by  $\frac{\pi}{2}$ . Then

$$l(s, \theta) = \{x \in \mathbb{R}^2 \text{ such that } x = s\vartheta + t\vartheta^\perp \text{ for } t \in \mathbb{R}\}. \quad (1.11)$$

We define

$$\mathfrak{M}(f)(s, \theta) = \int_{l(s, \theta)} f(x) dl = \int_{\mathbb{R}} f(s\vartheta + t\vartheta^\perp) dt,$$

where  $dl$  is the line measure along  $l(s, \theta)$ . In other words,  $\mathfrak{M}$  maps a function to the value of its integrals along *any* line. The operator  $\mathfrak{M}$  is the **two-dimensional Radon Transform**. We shall see that  $\mathfrak{M}$  is injective and admits an explicit inversion.

The Radon transform and its inversion form the mathematical back-bone of Computerized Tomography (CT), one of the most successful medical imaging techniques available to-date.

**Example 5.** Let us conclude with a more involved, though practically very relevant, example: the Calderón problem. We first introduce the following elliptic partial differential equation

$$\begin{aligned} -\nabla \cdot \gamma(x) \nabla u(x) &= 0, & x \in X \\ u(x) &= f(x) & x \in \partial X, \end{aligned} \quad (1.12)$$

where  $X$  is a smooth domain in  $\mathbb{R}^n$ ,  $\partial X$  its boundary,  $\gamma(x)$  a smooth coefficient in  $X$  bounded above and below by positive constants, and  $f(x)$  is a prescribed Dirichlet data for the elliptic problem. This equation is a standard forward problem and it is known that it admits a unique solution. Moreover, the outgoing current

$$j(x) = \gamma(x) \frac{\partial u}{\partial \nu}(x), \quad \text{with } \nu(x) \text{ the outward unit normal to } X \text{ at } x \in \partial X,$$

is a well defined function. This allows us to define the Dirichlet-to-Neumann (a.k.a. Poincaré-Steklov) operator

$$\begin{aligned} \Lambda_\gamma : \quad H^{\frac{1}{2}}(\partial X) &\rightarrow H^{-\frac{1}{2}}(\partial X) \\ f(x) &\mapsto \Lambda_\gamma[f](x) = j(x) = \gamma(x) \frac{\partial u}{\partial \nu}(x). \end{aligned} \quad (1.13)$$

Here,  $H^s(\partial X)$  are standard Hilbert spaces of distributions defined at the domain's boundary.

Let now  $\mathfrak{X} = \mathcal{C}^2(\bar{X})$  and  $\mathfrak{Y} = \mathcal{L}(H^{\frac{1}{2}}(\partial X), H^{-\frac{1}{2}}(\partial X))$ , where the space  $\mathcal{L}(\mathfrak{X}_1, \mathfrak{X}_2)$  means the space of linear bounded (continuous) operators from  $\mathfrak{X}_1$  to  $\mathfrak{X}_2$ . We define the measurement operator

$$\mathfrak{M} : \quad \mathfrak{X} \ni \gamma \mapsto \mathfrak{M}(\gamma) = \Lambda_\gamma \in \mathfrak{Y}. \quad (1.14)$$

So the measurement operator maps the unknown conductivity  $\gamma$  to the Dirichlet-to-Neumann operator  $\Lambda_\gamma$ , which is by construction a functional of  $\gamma$  since the solution  $u$  of (1.12) is a functional of  $\gamma$ . The measurement operator therefore lives in a "huge" (admittedly, this is subjective) space. Acquiring data means acquiring  $\Lambda_\gamma$ , which means for each and every function  $g \in H^{\frac{1}{2}}(\partial X)$  at the domain's boundary, perform an experiment that measures  $j(x)$ .

As we shall see, the Calderón problem is a typical example of what is usually regarded as an *ill-posed* inverse problem. It finds important applications in Electrical Impedance Tomography and Optical Tomography, two medical imaging modalities. A milestone of inverse problems theory is the proof that  $\mathfrak{M}$  is an injective operator [60].

### 1.3 Inverse Problems and Modeling: application to Magnetic Resonance Imaging

This section presents an extremely simplified version of the extremely successful medical imaging modality called Magnetic Resonance Imaging (MRI). While doing so, we observe that MRI reconstructions may be modeled by (at least) three different measurement operators, and hence three different inverse problems. This serves as an example of the importance of modeling when dealing with practical inverse problems.

MRI exploits the precession of the spin of protons in a magnetic field  $\mathbf{H}(\mathbf{x})$ , which is a vector in  $\mathbb{R}^3$  for each position  $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ . The axis of the precession is that of the magnetic field and the frequency of the precession is  $\omega(\mathbf{x}) = \gamma|\mathbf{H}|(\mathbf{x})$ , where  $\gamma = e/(2m)$  is called the gyromagnetic ratio,  $e$  is the electric charge of the proton and  $m$  its mass.

In a nutshell, MRI works as follows. Assume first that we impose a strong static magnetic field  $\mathbf{H}_0 = H_0\mathbf{e}_z$  along the  $z$  axis. All protons end up with their spin parallel to  $\mathbf{H}_0$  and slightly more so in the direction  $\mathbf{H}_0$  than in  $-\mathbf{H}_0$ . This difference is responsible for a macroscopic *magnetization*  $\mathbf{M}$  pointing in the same direction as  $\mathbf{H}_0$ .

In a second step, we generate radio frequency magnetic waves at the Larmor frequency  $\omega_0 = \gamma|\mathbf{H}_0|$ . In clinical MRI, the frequency is typically between 15 and 80 MHz (for hydrogen imaging), which corresponds to wavelengths between 20 and 120 m (since  $\omega = ck = 2\pi c/\lambda$  and  $c \approx 3 \cdot 10^8$ ). So the wavelength is not what governs spatial resolution in MRI, which is as most successful medical imaging modalities, sub-millimetric. For instance the pulse (assumed to be independent of  $\mathbf{x}$  to start with) may be of the form  $\mathbf{H}_1(t) = 2H_1 \cos(\omega_0 t)\mathbf{e}_x$  and turned on for a duration  $t_p$ . Because the field oscillates at the Larmor frequency, the spins of the protons are affected. The resulting effect on the macroscopic magnetization is that it precesses around the axis  $\mathbf{e}_z$  at frequency  $\omega_0$ . The spins make an angle with respect to the direction  $\mathbf{e}_z$  given at time  $t_p$  by

$$\theta = \gamma H_1 t_p.$$

Generally,  $t_p$  is chosen such that  $\theta = \pi/2$  or  $\theta = \pi$ . The corresponding pulses are called  $90^\circ$  and  $180^\circ$  pulses, respectively. Thus, after a  $90^\circ$  pulse, the magnetization oscillates in the  $xy$  plane and after a  $180^\circ$  pulse, the magnetization is pointing in the direction  $-\mathbf{H}_0$ .

Once the radio frequency is turned off (but not the static field  $\mathbf{H}_0$ ), protons tend to realign with the static field  $\mathbf{H}_0$ . By doing so, they emit radio frequency waves at the Larmor frequency  $\omega_0$  that can be *measured*. This wave is called the free induction decay (FID) signal. The FID signal after a  $90^\circ$  pulse will have the form

$$S(t) = \rho e^{i\omega_0 t} e^{-t/T_2}. \quad (1.15)$$

Here  $\rho$  is the density of the magnetic moments and  $T_2$  is the spin-spin relaxation time. (There is also a spin-lattice relaxation time  $T_1 \ll T_2$ , which cannot be imaged with  $90^\circ$  pulses and which we ignore.) The main reason for doing all this is that the density  $\rho$  and the relaxation time  $T_2$  depend on the tissue sample. We restrict ourselves to the reconstruction of  $\rho$  here, knowing that similar experiments can be devised to image  $T_2$  (and  $T_1$ ) as well. To simplify, we assume that measurements are performed over a period of time that is small compared to  $T_2$  so that the exponential term  $e^{-t/T_2}$  can be neglected.

Now human tissues are not spatially homogeneous, which makes imaging a lot more useful. The density of magnetic moments  $\rho = \rho(\mathbf{x})$  depends on type of tissue at  $\mathbf{x} \in \mathbb{R}^3$ . This is the parameter we wish to reconstruct.

The physical mechanism that allows for the good spatial resolution of MRI (sub-millimeter resolution for brain imaging) is that only tissue samples under a static magnetic field  $\mathbf{H}$  such that  $|\mathbf{H}| = \gamma\omega_0$  will be affected by the radio frequency pulse  $\mathbf{H}_1(t) = 2H_1 \cos(\omega_0 t)\mathbf{e}_x$ . We thus need to make sure that the static field has the correct amplitude in as small a spatial area as possible. To do so, we impose the static field

$$\mathbf{H}(z) = \mathbf{H}_0 + G_z z \mathbf{e}_z.$$

Only those protons in the slice with  $z$  is close to 0 will be affected by the pulse  $\mathbf{H}_1$  since we have assumed that  $|\mathbf{H}_0| = \gamma\omega_0$ . As a consequence, the measured signal takes the form

$$S(t) = e^{i\omega_0 t} \int_{\mathbb{R}^2} \rho(x, y, 0) dx dy \quad \text{so that} \quad e^{-i\omega_0 t} S(t) = \int_{\mathbb{R}^2} \rho(x, y, 0) dx dy. \quad (1.16)$$

The above right-hand-side thus describes the average density in the plane  $z = 0$ . MRI is thus a tomographic technique (tomos meaning section or slice in Greek).

By changing  $\mathbf{H}_0$  or  $\omega_0$ , we can obtain the average density in the plane  $z = z_0$  for all values of  $z_0 \in \mathbb{R}$ . Moreover, by rotating the generated magnetic field  $\mathbf{H}(z)$ , we are ideally able to obtain the average density in any plane in  $\mathbb{R}^3$ . Planes may be parameterized by their normal vector  $\phi \in \mathbb{S}^2$ , with  $\mathbb{S}^2$  the unit sphere in  $\mathbb{R}^3$ , and their distance  $s$  to the origin  $(0, 0, 0) \in \mathbb{R}^3$ . Let  $P(s, \phi)$  be the corresponding plane. Then what we have obtained is that MRI experiments allows us to obtain the plane integrals of the density

$$R(s, \phi) = \int_{P(s, \phi)} \rho(\mathbf{x}) d\sigma(\mathbf{x}), \quad (1.17)$$

where  $d\sigma(\mathbf{x})$  is the surface (Euclidean) measure on the plane  $P(s, \phi)$ . Here,  $R(s, \phi)$  is the **three-dimensional Radon transform** of  $\rho(\mathbf{x})$ . This is the first inverse problem we encounter. The measurement operator maps functions defined on  $\mathbb{R}^3$  (for instance compactly supported continuous functions) to the Radon transform, which is a function (for instance compactly supported continuous function) of  $(s, \phi) \in \mathbb{R} \times \mathbb{S}^2$ . We thus have the Inverse Problem:

**3D Radon Transform:** Reconstruct density  $\rho(\mathbf{x})$  from knowledge of  $R(s, \phi)$   
 Equivalently: Reconstruct a function from its plane integrals.

There are several issues with the above inverse problem. First of all, the Radon transform integrates over planes, which is a smoothing operation. This smoothing has to be undone in the reconstruction procedure. With more modeling, we will obtain another MO that does not involve any smoothing. More fundamentally from a practical point of view, rotating the whole magnetic field from the direction  $e_z$  to an arbitrary direction  $\phi$  is very challenging technologically. One could also rotate the object of interest rather than the heavy magnet. However, for the imaging of human patients, this is not feasible either for rather obvious reasons. Additional modeling is therefore necessary.

So far, we have a vertical discrimination of the proton density. The transversal discrimination is obtained by imposing a static field linearly varying in the  $x$  and  $y$  directions. Remember that after the  $90^\circ$  pulse, the magnetization  $\mathbf{M}(x, y, 0)$  rotates with frequency  $\omega_0$  in the  $xy$  plane (i.e., is orthogonal to  $\mathbf{e}_z$ ), and is actually independent of  $x$  and  $y$ . Let us now impose a static field  $\mathbf{H}(y) = \mathbf{H}_0 + G_y y \mathbf{e}_z$  for a duration  $T$ . Since the frequency of precession is related to the magnetic field, the magnetization at position  $y$  will rotate with frequency  $\omega(y) = \omega_0 + \gamma G_y y$ . Therefore, compared to the magnetization at  $z = 0$ , the magnetization at  $z$  will accumulate a phase during the time  $T$  the field  $G_y y \mathbf{e}_z$  is turned on given by  $T(\omega(y) - \omega_0) = T\gamma G_y y$ . Once the field  $G_y y \mathbf{e}_z$  is turned off, the magnetization will again rotate everywhere with frequency  $\omega_0$ . However, the phase depends on position  $y$ . This part of the process is called *phase encoding*. A measurement of the FID would then give us a radio frequency signal of the form

$$S(t; T) = e^{i\omega_0 t} \int_{\mathbb{R}^2} e^{i\gamma G_y T y} \rho(x, y, 0) dx dy. \quad (1.18)$$

By varying the time  $T$  or the gradient  $G_y$ , we see that we can obtain the frequency content in  $y$  of the density  $\rho(x, y, 0)$ . More precisely,

$$\int_{\mathbb{R}} \rho(x, y, 0) dx = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ik_y y} S(t; \frac{k_y}{\gamma G_y}) dk_y. \quad (1.19)$$

This provides us with the *line integrals* of  $\rho(x, y, 0)$  at  $z = 0$  for all the lines that are parallel to the  $x$ -axis. Note that the phase encoding was performed by using a field that was linear in the  $y$  variable. We can use a field that is linear in the variable  $\cos \theta x + \sin \theta y$  instead. Denoting by  $\vartheta = (\cos \theta, \sin \theta) \in \mathbb{S}^1$  a unit vector in  $\mathbb{R}^2$ , by  $\vartheta^\perp = (-\sin \theta, \cos \theta)$  its rotation by  $\frac{\pi}{2}$ , and by  $l(s, \theta)$  the line with normal  $\vartheta$  at a distance  $s$  from the origin  $(0, 0) \in \mathbb{R}^2$  defined in (1.11), we are thus able to measure all line integrals of the function  $\rho(x, y, 0)$ :

$$R(s, \theta) = \int_{l(s, \theta)} \rho(x, y, 0) dl(x, y) = \int_{\mathbb{R}} \rho(s\vartheta + t\vartheta^\perp, 0) dt, \quad (1.20)$$

where  $dl(x, y)$  is the line (Euclidean) measure on  $l(s, \theta)$ . This is the second inverse problem we encounter: we wish to reconstruct  $\rho(x, y)$  from knowledge of its line integrals  $R(s, \theta)$ . This is the **two-dimensional Radon transform** of  $\rho(x, y, 0)$ ; see Example 4 above. The measurement operator maps functions defined on  $\mathbb{R}^2$  (for instance compactly supported continuous functions) to the Radon transform, which is a function (for

instance compactly supported continuous function) of  $(s, \phi) \in \mathbb{R} \times \mathbb{S}^1$ . We thus have the Inverse Problem:

**2D Radon Transform:** Reconstruct density  $\rho(x, y)$  from knowledge of  $R(s, \theta)$   
 Equivalently: Reconstruct a function from its line integrals.

The 2D Radon transform still involves smoothing (integration along lines), as does the three-dimensional Radon transform. Note, however, that there is no need to rotate the magnet or the patient to acquire the data. Although the inverse Radon transform is useful and used in practical MRI, the missing information in the  $x$  variable in measurements of the form (1.18) can in fact be obtained by additional modeling. Indeed, nothing prevents us from adding an  $x$ -dependent static field during the FID measurements. Let us assume that after time  $T$  (where we reset time to be  $t = 0$ ), we impose a static field of the form  $\mathbf{H}(x) = \mathbf{H}_0 + G_x x \mathbf{e}_z$ . The magnetization will now precess around the  $z$  axis with  $x$ -dependent frequency  $\omega(x) = \omega_0 + \gamma G_x x$ . This implies that the measured signal will be of the form

$$S(t; T) = \int_{\mathbb{R}^2} e^{i\gamma G_y T y} e^{i(\omega_0 + \gamma G_x x)t} \rho(x, y, 0) dx dy. \quad (1.21)$$

We have thus access to the *measured data*

$$d(k_x, k_y) = e^{-i\omega_0 k_x / (\gamma G_x)} S\left(\frac{k_x}{\gamma G_x}; \frac{k_y}{\gamma G_y}\right) = \int_{\mathbb{R}^2} e^{ik_y y} e^{ik_x x} \rho(x, y, 0) dx dy. \quad (1.22)$$

By varying  $T$  (or  $G_y$ ) and  $t$  and  $G_x$ , we can obtain the above information for essentially all values of  $k_x$  and  $k_y$ . This is our third Inverse Problem:

**2D Fourier Transform:** Reconstruct density  $\rho(x, y, 0)$  from knowledge of  $d(k_x, k_y)$   
 Equivalently: Reconstruct a function from its plane wave decomposition.

This is a well-known problem whose solution involves applying the *Inverse Fourier Transform*

$$\rho(x, y, 0) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-i(k_x x + k_y y)} d(k_x, k_y) dk_x dk_y. \quad (1.23)$$

Several approximations have been made to obtain this reconstruction formula. Within this framework, we see however that density reconstructions are relatively simple: all we have to do is to invert a Fourier transform. The above procedure can be repeated for all values of  $z$  providing the density  $\rho(x, y, z)$  everywhere in the domain of interest.

We do not consider the difficulties of MRI further. The above derivation shows that MRI can be modeled by at least three different inverse problems. A first inverse problem, based on the three dimensional Radon transform, is not very practical. The second and third inverse problems, based on the inverse Radon transform (RT) and the inverse Fourier transform (FT), are used in practical reconstructions in MRI. Note that the IP based on the FT requires the acquisition of more data than the IP based on the RT. The reason for acquiring more data is that the IP based on the FT is *better posed* than than based on the FT. The next section introduces a simple Hilbert scale that allows one to quantify the notion of *well-* or *ill-*posedness in some simple cases.

## 1.4 Inverse Problems and Smoothing: Hilbert scale

The inverse problems derived above all involve reconstructions in  $\mathbb{R}^n$  for dimension  $n = 2$  or  $n = 3$  in geometries that are invariant by translation. The Fourier transform is therefore an ideal candidate to devise reconstruction algorithms and we briefly recall its definition below.

We have also mentioned on several occasions already that inverse problems could be coined as well-posed or ill-posed. Such notions are, we say it again, subjective. The main reason a problem is ill-posed is because noise is amplified *too much* during the reconstruction, which is a subjective statement. Nonetheless, there is one important reason why noise is amplified in the reconstruction: it is because high frequencies are damped during the acquisition of the measurements, or in other words, the MO is a smoothing operator. The more smoothing (regularizing) the MO, the more amplified (deregularized) the noise is going to be during the reconstruction. Below we introduce a Hilbert scale that allows us to quantify the aforementioned regularization and to be more specific about the notions of well-posedness and ill-posedness.

### 1.4.1 Fourier transforms and well-posedness

Let  $f(x)$  be a complex-valued function in  $L^2(\mathbb{R}^n)$  for some  $n \in \mathbb{N}^*$ , which means a (measurable) function on  $\mathbb{R}^n$  that is square integrable in the sense that

$$\|f\|^2 = \int_{\mathbb{R}^n} |f(x)|^2 dx < \infty. \quad (1.24)$$

Here  $\|f\|$  is the  $L^2(\mathbb{R}^n)$ -norm of  $f$  and  $dx$  the Lebesgue (volume) measure on  $\mathbb{R}^n$ . We define the Fourier transform of  $f$  as

$$\hat{f}(k) = [\mathcal{F}_{x \rightarrow k} f](k) = \int_{\mathbb{R}^n} e^{-ik \cdot x} f(x) dx. \quad (1.25)$$

It is a well-known results about Fourier transforms that  $\hat{f}(k) \in L^2(\mathbb{R}^n)$  and the Fourier transform admits an inverse on  $L^2(\mathbb{R}^n)$  given by

$$f(x) = [\mathcal{F}_{k \rightarrow x}^{-1} \hat{f}](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ik \cdot x} \hat{f}(k) dk. \quad (1.26)$$

More precisely we have the Parseval relation

$$(\hat{f}, \hat{g}) = (2\pi)^n (f, g) \quad \text{and} \quad \|\hat{f}\| = (2\pi)^{\frac{n}{2}} \|f\|. \quad (1.27)$$

where the Hermitian product is given by

$$(f, g) = \int_{\mathbb{R}^n} f(x) \bar{g}(x) dx. \quad (1.28)$$

Here  $\bar{g}$  is the complex conjugate to  $g$ . So up to the factor  $(2\pi)^{\frac{n}{2}}$ , the Fourier transform and its inverse are isometries.

Important properties of the Fourier transform for us here are how they interact with differentiation and convolutions. Let  $\alpha = (\alpha_1, \dots, \alpha_n)$  be a multi-index of non-negative

components  $\alpha_j \geq 0$ ,  $1 \leq j \leq n$  and let  $|\alpha| = \sum_{i=1}^n \alpha_i$  be the length of the multi-index. We then define the differentiation  $D^\alpha$  of degree  $|\alpha|$  as

$$D^\alpha = \prod_{i=1}^n \frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}}. \quad (1.29)$$

We then deduce from the definition (1.25) that

$$\mathcal{F}_{x \rightarrow k}[D^\alpha f](k) = \left( \prod_{j=1}^n (ik_j)^{\alpha_j} \right) [\mathcal{F}_{x \rightarrow k} f](k). \quad (1.30)$$

Let us now define the convolution as

$$f \star g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy. \quad (1.31)$$

We then verify that

$$\begin{aligned} \mathcal{F}_{x \rightarrow k}(f \star g) &= \mathcal{F}_{x \rightarrow k} f \mathcal{F}_{x \rightarrow k} g, & \text{i.e.} & \widehat{f \star g} = \hat{f} \hat{g}, \\ \mathcal{F}_{k \rightarrow x}^{-1}(\hat{f} \star \hat{g}) &= (2\pi)^n f g & \text{i.e.} & \hat{f} \star \hat{g} = (2\pi)^d \widehat{f g}. \end{aligned} \quad (1.32)$$

So the Fourier transform *diagonalizes* differential operators (replaces them by multiplication in the Fourier domain). However Fourier transforms replace products by non-local convolutions.

The Fourier transform is a well-posed operator from  $L^2(\mathbb{R}^n)$  to  $L^2(\mathbb{R}^n)$  since the inverse Fourier transform is also defined from  $L^2(\mathbb{R}^n)$  to  $L^2(\mathbb{R}^n)$  and is bounded as shown in (1.27). Let us assume that we measure

$$\hat{d}(k) = \hat{f}(k) + \hat{N}(k),$$

where we believe that  $\delta = \|\hat{N}\|$  is relatively small. Then the error in the reconstruction will also be of order  $\delta$  in the  $L^2(\mathbb{R}^n)$  norm. Indeed let  $d(x)$  be the reconstructed function from the data  $d(k)$  and  $f(x)$  be the real function we are after. Then we have

$$\|d - f\| = \|\mathcal{F}_{k \rightarrow x}^{-1} \hat{d} - \mathcal{F}_{k \rightarrow x}^{-1} \hat{f}\| = \|\mathcal{F}_{k \rightarrow x}^{-1}(\hat{d} - \hat{f})\| = (2\pi)^{-\frac{n}{2}} \delta. \quad (1.33)$$

In other words, measurements errors can still be seen in the reconstruction. The resulting image is not perfect. However the error due to the noise has not been amplified too drastically.

## 1.4.2 Hilbert scale and degrees of ill-posedness

**Well-posed Problems and Lipschitz Stability.** Let us revisit and make more explicit the notion of well-posedness and ill-posedness mentioned in section 1.1. We assume that  $\mathfrak{M}$  is a linear operator  $A$  from  $\mathfrak{X}$  to  $\mathfrak{Y}$  for  $\mathfrak{X}$  and  $\mathfrak{Y}$  Banach spaces associated with their natural norms. For a given data  $y \in \mathfrak{Y}$ , we would like to solve the linear problem

$$\text{Find } x \text{ such that} \quad Ax = y. \quad (1.34)$$

As we mentioned earlier, a well-posed problem is a problem where data noise is not amplified too drastically during the reconstruction. Mathematically, this subjective notion may be described by the property that the (bounded) operator  $A$  is invertible ( $A^{-1}y$  is defined for all  $y \in \mathfrak{Y}$ ) and (hence by the open mapping theorem) of bounded inverse, i.e.,  $\|A^{-1}y\|_{\mathfrak{X}} \leq C\|y\|_{\mathfrak{Y}}$  for a constant  $C$  that depends on  $A$  but not on  $y \in \mathfrak{Y}$ . The error between two solutions  $x_1$  and  $x_2$  corresponding to two data  $y_1$  and  $y_2$  satisfies that

$$\|x_1 - x_2\|_{\mathfrak{X}} \leq C\|y_1 - y_2\|_{\mathfrak{Y}}. \quad (1.35)$$

This stability estimate is consistent with the statement that “Noise”  $y_1 - y_2$  is not amplified by more than a multiplicative factor  $C$  during the reconstruction. Moreover, when noise levels are reduced by a factor two, which may be performed by adding detectors or obtaining higher quality measurements, then (1.35) states that in the worst possible case (since (1.35) for arbitrary  $x_1 - x_2$ , and hence  $C$  reflects the amplification of noise in the worst scenario), the error in the reconstruction will also be reduced by a factor two.

Note that the choice of the spaces  $\mathfrak{X}$  and  $\mathfrak{Y}$  and their norms  $\|\cdot\|_{\mathfrak{X}}$  and  $\|\cdot\|_{\mathfrak{Y}}$  matters. The definition and the boundedness of the operator  $A^{-1}$  obviously depends upon these choices and hence so does the validity of (1.35). An estimate of the form (1.35) is a *stability* estimate with Lipschitz constant  $C$  and we then say that the inverse problem (1.34) is *Lipschitz-stable*.

**Ill-posed problems and unbounded operators.** The inverse Fourier transform is an example of a *well-posed* inverse problem from  $L^2(\mathbb{R}^n)$  to  $L^2(\mathbb{R}^n)$ . We will see in the following chapter that the two-dimensional and three-dimensional Radon transforms are also well posed with appropriate choices of  $\mathfrak{X}$  and  $\mathfrak{Y}$ . Many inverse problems are, however, *considered* to be *ill-posed* for instance because the application of  $A^{-1}$  to the noisy data  $y$  produces a result that is deemed *inadequate* or because it is not possible to define an operator  $A^{-1}$ . Being ill-posed *does not* mean that a problem cannot be solved. However, it means that additional information needs to be incorporated into the inversion.

Let us attempt to mathematize these subjective notions. Typically, we can distinguish two notions of ill-posedness. The first one corresponds to operators  $A$  that are not injective. In this case, the data do not uniquely determine the parameters. This situation is typically remedied by acquiring additional information. We will not consider such settings much in these notes except to say that the Bayesian framework considered in Chapter 10 is adapted to such problems. Many practical inverse problems may be seen as discretization of injective operators.

The second notion of ill-posedness involves operators  $A$  that are injective but not surjective on the whole space  $\mathfrak{Y}$  (i.e., the range of  $A$  defined by  $\text{Range}(A) = A(\mathfrak{X})$  is a proper subset of  $\mathfrak{Y}$ ; that is to say, is not equal to  $\mathfrak{Y}$ ). Because  $A$  is injective,  $A^{-1}$  can be defined from  $\text{Range}(A)$  to  $\mathfrak{X}$ . However, it is *not a bounded* operator for the norms of  $\mathfrak{Y}$  and  $\mathfrak{X}$  in the sense that a Lipschitz equality such as (1.35) does not hold. From a practical point of view, applying  $A^{-1}$  to the available (noisy) data  $y$  provides a results that the user *feels* is too different from the expected  $x$ .

Mathematically, the unbounded character of  $A^{-1}$  very much depends on the choice of functional space. The operator  $A^{-1}$  could very well be defined and bounded from the

other space  $\mathfrak{Y}'$  to the other space  $\mathfrak{X}'$ , in which case the same inverse problem based on the MO  $A$  could be *ill posed* from  $\mathfrak{X}$  to  $\mathfrak{Y}$  but *well-posed* from  $\mathfrak{X}'$  to  $\mathfrak{Y}'$ . In other words, a user comfortable with the modeling of  $A$  from  $\mathfrak{X}'$  to  $\mathfrak{Y}'$  deals with a well-posed problem, whereas the user insisting on a modeling of  $A$  from  $\mathfrak{X}$  to  $\mathfrak{Y}$  needs to add information to the inversion to obtain a more satisfactory answer.

In spite of the subjectivity of the notion of ill-posedness, one of the main reasons why an inverse problem is deemed ill-posed *in practice* is because the MO  $A$  is “smoothing”. Smoothing means that  $Ax$  is “more regular” than  $x$ , in the sense that details (small scale structures) are attenuated by the MO. Again, this does not mean that the details cannot be reconstructed. When the MO is injective, they can. This means, however, that the reconstruction has to undo this smoothing. As soon as the data are noisy (i.e., always in practice), and that the noise contribution has small scale structures (i.e., often in practice), then the deregularization process has the effect of amplifying the noise in a way that can potentially be very *harmful* (which is, as we already said, *subjective*).

The answer to ill-posedness is to impose prior assumptions on the parameters we wish to reconstruct. As we mentioned earlier in this chapter, the simplest example of such an assumption is to assume that the function is sufficiently smooth. In order to define what we mean by ill-posedness and quantify the degree of ill-posedness, we need a scale of spaces in which the “smoothing” of  $A$  can be measured. We will use what is probably the simplest scale of function spaces, namely the scale of Hilbert spaces.

**The scale of Hilbert spaces.** Let  $s \geq 0$  be a non-negative real-number. We define the scale of Hilbert spaces  $H^s(\mathbb{R}^n)$  as the space of measurable functions  $f(x)$  such that

$$\|f\|_{H^s(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} (1 + |k|^2)^s |\mathcal{F}_{x \rightarrow k} f|^2(k) dk < \infty. \quad (1.36)$$

We verify that  $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$  since the Fourier transform is an isometry. We also verify that

$$\{f \in H^1(\mathbb{R}^n)\} \iff \left\{ f \in L^2(\mathbb{R}^n) \text{ and } \frac{\partial f}{\partial x_i} \in L^2(\mathbb{R}^n), 1 \leq i \leq n \right\}. \quad (1.37)$$

This results from (1.30). More generally the space  $H^m(\mathbb{R}^n)$  for  $m \in \mathbb{N}$  is the space of functions such that all partial derivatives of  $f$  of order up to  $m$  are square integrable. The advantage of the definition (1.36) is that it holds for real values of  $s$  as well. So  $H^{\frac{1}{2}}(\mathbb{R}^n)$  is the space of functions such that “half-derivatives” of  $f$  are square integrable. Notice also that  $s$  characterizes the degree of smoothness of a function  $f(x)$ . The larger  $s$ , the smoother the function  $f \in H^s(\mathbb{R}^n)$ , and the faster the decay of its Fourier transform  $\hat{f}(k)$  as can be seen from the definition (1.36).

It is also useful to define the Hilbert scale for functions supported on subdomains of  $\mathbb{R}^n$ . Let  $X$  be a sufficiently smooth subdomain of  $\mathbb{R}^n$ . We define two scales. The first scale is  $H_0^s(X)$ , defined as the closure of  $C_0^\infty(X)$ , the space of functions of class  $C^\infty$  with support in  $X$  (so these functions and all their derivatives vanish at the boundary of  $X$ ), for the norm in  $H^s(\mathbb{R}^n)$ . Thus,  $f \in H_0^s(X)$  can be described as the limit of functions  $f_n \in C_0^\infty(\mathbb{R}^n)$  uniformly bounded in  $H^s(\mathbb{R}^n)$ . We also define  $H^s(X)$  as the space of functions  $f$  on  $X$  that can be extended to functions  $f^*$  in  $H^s(\mathbb{R}^n)$  (i.e.,  $f = f^* \chi_X$ , where

$\chi_X$  is the characteristic function of  $X$ ) and  $\|f\|_{H^s(X)}$  is the lower bound of  $\|f\|_{H^s(\mathbb{R}^n)}$  over all possible extensions. There are several (sometimes not exactly equivalent) ways to define the scale of Hilbert spaces  $H^s(X)$ .

Finally, it is also convenient to define  $H^s$  for negative values of  $s$ . We define  $H^{-s}(\mathbb{R}^n)$  for  $s \geq 0$  as the subspace of  $\mathcal{S}'(\mathbb{R}^n)$ , the space of *tempered distributions*, such that (1.36) holds. For bounded domains we define  $H^{-s}(X)$  as the dual to  $H_0^s(X)$  equipped with the norm

$$\|f\|_{H^{-s}(X)} = \sup_{g \in H_0^s(X)} \frac{\int_X fg dx}{\|g\|_{H_0^s(X)}}. \quad (1.38)$$

We can verify that the two definitions agree when  $X = \mathbb{R}^n$ , in which case  $H_0^s(\mathbb{R}^n) = H^s(\mathbb{R}^n)$ .

**A well-posed, ill-posed inverse problem.** Let us illustrate on a simple example how the Hilbert scale can be used to understand the ill-posedness of inverse problems. Let  $f(x)$  be a function in  $L^2(\mathbb{R})$  and let  $u$  be the solution in  $L^2(\mathbb{R})$  of the following ODE

$$-u'' + u = f, \quad x \in \mathbb{R}. \quad (1.39)$$

There is a unique solution in  $L^2(\mathbb{R})$  to the above equation given by

$$u(x) = \frac{1}{2} \int_{\mathbb{R}} e^{-|y-x|} f(y) dy = (g * f)(x), \quad g(x) = \frac{1}{2} e^{-|x|},$$

as can be verified by inspection. In the Fourier domain, this is

$$(1 + k^2)\hat{u}(k) = \hat{f}(k).$$

This implies that  $u$  is not only in  $L^2(\mathbb{R})$  but also in  $H^2(\mathbb{R})$  as is easily verified.

**The problem is ill-posed...** Let us define the operator  $A$  as follows

$$A : \begin{array}{l} L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \\ f \mapsto Af = u, \end{array} \quad (1.40)$$

where  $u$  is the solution to (1.39). As such, the operator  $A$  is not invertible on  $L^2(\mathbb{R})$ . Indeed the inverse of the operator  $A$  is formally defined by  $A^{-1}u = -u'' + u$ . However for  $u \in L^2(\mathbb{R})$ ,  $-u''$  is not a function in  $L^2(\mathbb{R})$  but a distribution in  $H^{-2}(\mathbb{R})$ . The inverse problem consisting of reconstructing  $f(x) \in L^2(\mathbb{R})$  from  $u(x) \in L^2(\mathbb{R})$  is therefore *ill-posed*. The reason is that the operator  $A$  is regularizing.

**... or is it?** However let us define the “same” operator

$$\tilde{A} : \begin{array}{l} L^2(\mathbb{R}) \rightarrow H^2(\mathbb{R}) \\ f \mapsto \tilde{A}f = u. \end{array} \quad (1.41)$$

Now  $\tilde{A}$  is invertible from  $H^2(\mathbb{R})$  to  $L^2(\mathbb{R})$  and its inverse is indeed given by  $\tilde{A}^{-1}u = -u'' + u$ . So  $\tilde{A}$  is *well-posed* from  $L^2(\mathbb{R})$  to  $H^2(\mathbb{R})$  as can easily be verified. Yet a third instantiation of the “same” operator is

$$\begin{aligned} \check{A}: \quad H^{-2}(\mathbb{R}) &\rightarrow L^2(\mathbb{R}) \\ f &\mapsto \check{A}f = u. \end{aligned} \tag{1.42}$$

Now  $\check{A}$  is invertible from  $L^2(\mathbb{R})$  to  $H^{-2}(\mathbb{R})$  and its inverse is indeed given by  $\check{A}^{-1}u = -u'' + u$  and thus  $\check{A}$  is *well-posed* from  $H^{-2}(\mathbb{R})$  to  $L^2(\mathbb{R})$ .

If we assume that our noise (the error between measurement  $u_1$  and measurement  $u_2$ ) is small in the  $H^2$ -norm, so that  $\|u_1 - u_2\|_{H^2(\mathbb{R})} \leq \delta$ , and we are happy with a small error in the parameter in the  $L^2(\mathbb{R})$  sense, then there is “no problem”. The reconstruction will be accurate in the sense that  $\|f_1 - f_2\|_{L^2(\mathbb{R})} \leq C\delta$ , where  $f_j = \tilde{A}^{-1}u_j$ ,  $j = 1, 2$ .

The same occurs if we assume that noise is small in the  $L^2$ -norm and we are happy with a small error in the parameter in the  $H^{-2}(\mathbb{R})$  sense. This would typically mean that we are satisfied if spatial moments of  $u$  (i.e., quantities of the form  $\int_{\mathbb{R}} u(x)\varphi(x)dx$  for  $\varphi \in H^2(\mathbb{R})$ ) are accurate, rather than insisting that  $u$  be accurate in the  $L^2$  sense.

However, in many instance, noise is not small in the strong norm  $H^2(\mathbb{R})$ , but rather in the weaker norm  $L^2(\mathbb{R})$ . At least this is our *perception*. Moreover, we do not want small errors in the  $H^{-2}(\mathbb{R})$  sense, but rather insist that the reconstructions “look good” in some  $L^2(\mathbb{R})$  sense. Then the problem is certainly ill-posed as  $A$  is not boundedly invertible the spaces  $\mathfrak{X} = \mathfrak{Y} = L^2(\mathbb{R})$ .

**A possible definition of mildly and severely ill-posed inverse problems.** We now introduce a useful practical distinction among ill-posed problems. As heuristic and subjective as the notion may be, an ill-posed inverse problems that require relatively mild action from the user (in terms of the introduction of prior information) will be called *mildly ill-posed*. Some inverse problems may require much more stringent action from the user. They will be called *severely ill-posed* problems.

Using the Hilbert scale introduced earlier, a possible mathematical definition of *mildly* and *severely* ill-posed problems is as follows. We assume here that  $A$  is *injective*. It is not that the inverse of  $A$  does not exist that causes problems. It is because  $A$  is smoothing that action is necessary. When the smoothing is *mild*, then that action may be limited to an appropriate penalization such as those that will be described in Chapter 10. When the smoothing is *severe*, then typically much more drastic prior information needs to be introduced, for instance by using the Bayesian framework introduced in Chapter 10. We somewhat arbitrarily separate *mild* smoothing by a finite number of derivatives from *severe* smoothing by an “infinite” number of derivatives.

The problem (1.34) with  $\mathfrak{X} = \mathfrak{Y} = L^2(\mathbb{R}^n)$  is said to be *mildly ill-posed* provided that there exists a positive constant  $C$  and  $\alpha > 0$  such that

$$\|Af\|_{H^\alpha(\mathbb{R}^n)} \geq C\|f\|_{L^2(\mathbb{R}^n)}. \tag{1.43}$$

We define  $\|Af\|_{H^\alpha(\mathbb{R}^n)} = +\infty$  if  $f$  does not belong to  $H^\alpha(\mathbb{R}^n)$ . We say that  $A$  is *mildly ill-posed of order*  $\alpha$  if  $\alpha$  is the smallest real number such that (1.43) holds for some  $C = C(\alpha)$ . Notice that we can choose any  $\alpha \geq 2$  for  $\tilde{A}$  so the operator that maps  $f$  to

$u$  solution of (1.39) is a mildly ill-posed problem of order 2. The operator  $\tilde{A}$  in (1.41) essentially integrates twice the function  $f$ . Any injective operator that corresponds to a finite number of integrations is therefore mildly ill-posed.

We call the inversion a *severely ill-posed* problems when no such  $\alpha$  exists. Unfortunately, there are many practical instances of such inverse problems. A typical example is the following operator

$$Bf(x) = \mathcal{F}_{k \rightarrow x}^{-1}[e^{-k^2} \mathcal{F}_{x \rightarrow k} f](x). \quad (1.44)$$

Physically this corresponds to solving the heat equation forward in time: a very smoothing operation. We easily verify that the operator  $B$  maps  $L^2(\mathbb{R})$  to  $L^2(\mathbb{R})$ . However it damps high frequencies exponentially strongly, and more strongly than any finite number of integrations ( $m$  integrations essentially multiply high frequencies by  $|k|^{-m}$ ) so no  $\alpha > 0$  in (1.43) exists for  $B$ . Note that it does not mean that  $B$  is never invertible. Indeed for sufficiently smooth functions  $g(x)$  in the range of  $B$  (for instance for functions  $g$  such that  $\hat{g}(k)$  has compact support), we can easily define the inverse operator

$$B^{-1}g(x) = \mathcal{F}_{k \rightarrow x}^{-1}[e^{k^2} \mathcal{F}_{x \rightarrow k} g](x).$$

Physically, this corresponds to solving the heat equation backwards in time. It is clear that on a space of sufficiently smooth functions, we have  $BB^{-1} = B^{-1}B = Id$ . Yet, if noise is present in the data, it will be amplified by  $e^{k^2}$  in the Fourier domain. Unless noise is low frequency, this has devastating effects on the reconstruction. It is instructive to numerically add the inverse Fourier transform of  $\varepsilon \hat{n}(k)e^{k^2}$  to an image  $f(x)$  for  $\hat{n}(k) = (1 + k^2)^{-1}$ , say, so that noise is square integrable. Even when  $\varepsilon$  is machine precision, the image is quickly drowned by noise even if a relatively small number of Fourier modes is used to perform the Fourier transform.

# Chapter 2

## Integral Geometry. Radon transforms

This chapter and the next are devoted to Integral Geometry, which involves the reconstruction of an object for instance described by a function from knowledge of integral quantities of the object such as for instance its integrals along lines or planes.

In this chapter we consider the simplest example of integral geometry: the integration of a two-dimensional function along all possible lines in the plane, which is called the Radon transform, and the inversion of such a transform. This forms the mathematical backbone for one of the most successful medical imaging techniques: computed (or computerized) tomography (CT).

Later in the chapter, we consider the three dimensional Radon transform, which concerns the integral of functions over planes in three dimensions of space, as well as a specific example of a weighted two dimensional Radon transform, the attenuated Radon transform, which finds an application a the important medical imaging modality called Single Photon Emission Computerized Tomography (SPECT).

### 2.1 Transmission Tomography

In transmission tomography, objects to be imaged are probed with non-radiating sources such as X-rays. X-rays are composed of high energy photons (on the order of 60keV, which corresponds to a wavelength of about 0.02nm) that propagate through the object along straight lines unless they interact with the underlying medium and get absorbed. Let  $x = (x_1, x_2)$  denote two-dimensional spatial position and  $\vartheta \in \mathbb{S}^1$  orientation. We denote by  $u(x, \vartheta)$  the density of X-rays with position  $x$  and orientation  $\vartheta$ , and by  $\sigma(x)$  a linear attenuation coefficient. Velocity of X-rays is normalized to 1 so that locally the density  $u(x, \vartheta)$  satisfies the following transport equation:

$$\vartheta \cdot \nabla_x u(x, \vartheta) + \sigma(x)u(x, \vartheta) = 0, \quad x \in X, \quad \vartheta \in \mathbb{S}^1. \quad (2.1)$$

Here  $X$  is the physical domain (assumed to be convex) of the object and  $\mathbb{S}^1$  is the unit circle. We identify any point  $\vartheta \in \mathbb{S}^1$  with the angle  $\theta \in [0, 2\pi)$  such that  $\vartheta = (\cos \theta, \sin \theta)$ . The advection operator is given by

$$\vartheta \cdot \nabla_x = \cos \theta \frac{\partial}{\partial x_1} + \sin \theta \frac{\partial}{\partial x_2}$$

and models free transport of X-rays while  $\sigma(x)u(x, \vartheta)$  models the number of absorbed photons per unit distance at  $x$ .

The probing source is emitted at the boundary of the domain and takes the form

$$u(x, \vartheta) = \delta(x - x_0)\delta(\vartheta - \vartheta_0), \quad (2.2)$$

where  $x_0 \in \mathbb{R}^2$ , say  $x_0 \in \partial X$  and  $\vartheta_0$  is *entering* the domain  $X$ , i.e.,  $\vartheta_0 \cdot n(x_0) < 0$  where  $n$  is the outward normal to  $X$  at  $x_0 \in \partial X_0$ . Above the "delta" functions mean that a unit amount of photons is created at  $(x_0, \vartheta_0)$  in the sense that for any domain  $(x_0, \vartheta_0) \in Y \subset \mathbb{R}^2 \times \mathbb{S}^1$ , we have

$$\int_Y u(x, \vartheta) dx d\vartheta = 1.$$

The solution to (2.1)-(2.2), which is a first-order ODE in the appropriate variables, is given by

$$u(x + t\vartheta, \vartheta) = u(x, \vartheta) \exp\left(-\int_0^t a(x + s\vartheta) ds\right), \quad x \in \mathbb{R}^2, \quad \vartheta \in \mathbb{S}^1. \quad (2.3)$$

Indeed, write  $v(t) = u(x + t\vartheta, \vartheta)$  and  $b(t) = a(x + t\vartheta)$  so that  $\dot{v} + bv = 0$  and integrate to obtain the above result. For our specific choice at  $(x_0, \vartheta_0)$ , we thus obtain that

$$u(x, \vartheta) = \delta(x - t\vartheta - x_0)\delta(\vartheta - \vartheta_0) \exp\left(-\int_0^t a(x - s\vartheta) ds\right).$$

In other words, on the half line  $x_0 + t\vartheta_0$  for  $t \geq 0$ , there is a positive density of photons. Away from that line, the density of photons vanishes.

For  $x_1 = x_0 + \tau\vartheta_0 \in \partial X$  different from  $x_0$ , if a detector collects the amount of photons reaching  $x_1$  (without being absorbed), it will measure

$$\exp\left(-\int_0^\tau a(x_1 - s\vartheta) ds\right) = \exp\left(-\int_0^\tau a(x_0 + s\vartheta) ds\right).$$

The travel time (for a particle with rescaled speed equal to 1, so it is also a "distance") from one side of  $\partial X$  to the other side depends on  $(x_0, \vartheta_0)$  and is denoted by  $\tau(x_0, \vartheta_0) > 0$ . By taking the logarithm of the measurements, we have thus access to

$$\int_0^{\tau(x_0, \vartheta_0)} a(x_0 + t\vartheta_0) dt.$$

This is the integral of  $a$  over the line segment  $(x_0, x_1)$ . By varying the point  $x_0$  and the direction of the incidence  $\vartheta_0$ , we can have access to integrals of  $a(x)$  over all possible segments (and since  $a$  can be extended by 0 outside  $X$  without changing the measurements, in fact over all possible lines) crossing the domain  $X$ .

The main question in transmission tomography is thus how one can recover a function  $a(x)$  from its integration over all possible lines in the plane  $\mathbb{R}^2$ . This will be the object of subsequent sections. In practice, we need to consider integrations over a finite number of lines. How these lines are chosen is crucial to obtain a rapid and practical inversion algorithm. We do not consider the problems of discretization here and refer the reader to [47, 48].

## 2.2 Two dimensional Radon transform

We have seen that the problem of transmission tomography consisted of reconstructing a function from its integration along lines. We have considered the two-dimensional problem so far. Since X-rays do not scatter, the three dimensional problem can be treated by using the two-dimensional theory: it suffices to image the object slice by slice using the two dimensional reconstruction, as we did in MRI (Transmission Tomography is indeed a tomographic method since tomos means slice in Greek as we already know).

We need to represent (parameterize) lines in the plane in a more convenient way than by describing them as the line joining  $x_0$  and  $x_1$ . This is done by defining an origin  $\mathbf{0} = (0, 0)$ , a direction  $(\cos \theta, \sin \theta) = \vartheta \in S^1$ , and a scalar  $s$  indicating the (signed) distance of the line to  $\mathbf{0}$ . The line is defined by the set of points  $x$  such that  $x \cdot \vartheta^\perp = s$ , where  $\vartheta^\perp$  is the rotation of  $\vartheta$  by  $\frac{\pi}{2}$ , i.e., the vector given by  $\vartheta^\perp = (-\sin \theta, \cos \theta)$ . More precisely, for a smooth function  $f(x)$  on  $\mathbb{R}^2$ , we define the Radon transform  $Rf(s, \theta)$  for  $(s, \theta) \in Z = \mathbb{R} \times (0, 2\pi)$  as

$$Rf(s, \theta) = \int_{\mathbb{R}} f(s\vartheta^\perp + t\vartheta) dt = \int_{\mathbb{R}^2} f(x) \delta(x \cdot \vartheta^\perp - s) dx. \quad (2.4)$$

Notice that the cylinder  $Z$  is a double covering of the space of lines in the real plane

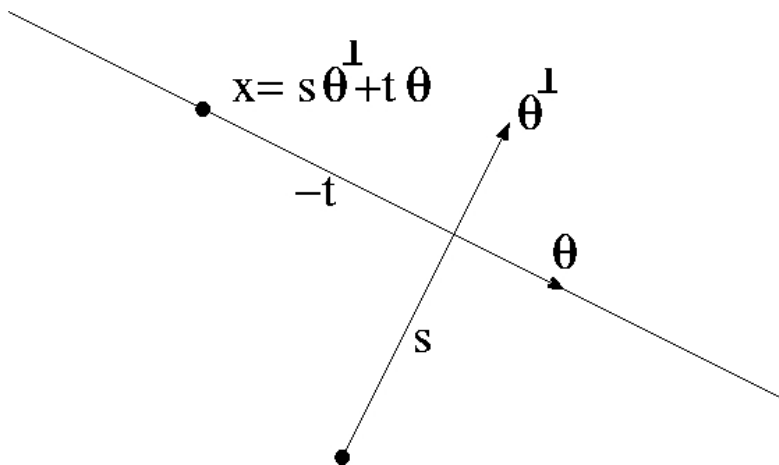


Figure 2.1: Geometry of the Radon transform.

$\mathbb{R}^2$ . Indeed one easily verifies that

$$Rf(s, \theta) = Rf(-s, \theta + \pi), \quad \text{as} \quad \{x \cdot \vartheta^\perp = s\} = \{x \cdot (-\vartheta^\perp) = (-s)\}.$$

Thus there is a redundancy of order two in the parameterization of lines in the Radon transform.

**Radon transform versus X-ray transform.** We pause here to make a remark regarding notation. The notation introduced here for the Radon transform is not the same as the notation introduced from the Radon transform in Chapter 1. Let us explain the reason for this change of notation. A line in a two dimensional space is a hyperplane, as is a plane in a three dimensional space, a notion that is used in the definition of

the three dimensional Radon transform that we consider below. So it is natural to parameterize hyperplanes by their uniquely defined (up to a sign) normal vector.

Now, we have just seen that the reason why line integrals appear in Computerized Tomography is because particles propagate along straight lines. This is true independent of dimension. And it is natural to parameterize lines by their main direction  $\vartheta$ . This is the point of view of the *X-ray* transform (since X-rays roughly follow straight lines in medical imaging applications).

In three dimensions, the three dimensional Radon transform (along hyperplanes) and the three dimensional X-ray transform (along rays) are not the same object (if only because there are many more lines than there are hyperplanes). In two dimensions, however, the X-ray and Radon transforms correspond to the same object. In Chapter 1, we emphasized the parameterization of hyperplanes and thus modeled lines by their normal vector  $\vartheta$ . In this chapter, we emphasize the X-ray transform and parameterize lines by their main direction  $\vartheta$ . One can go from one notation to the other by replacing  $\vartheta$  by its 90 degree rotation  $\vartheta^\perp$ . The rest of this chapter uses the line parameterizations of the X-ray transform. However, in two dimensions, we still refer to the integration of a function along lines, independent of the chosen parameterization, as the Radon transform.

**Some properties of the Radon transform.** Let us derive some important properties of the Radon transform. We first define the operator

$$R_\theta f(s) = Rf(s, \theta). \quad (2.5)$$

This notation will often be useful in the sequel. The first important result on the Radon transform is the *Fourier slice theorem*:

**Theorem 2.2.1** *Let  $f(x)$  be a smooth function. Then for all  $\vartheta \in S^1$ , we have*

$$[\mathcal{F}_{s \rightarrow \varsigma} R_\theta f](\varsigma) = \widehat{R_\theta f}(\varsigma) = \hat{f}(\varsigma \vartheta^\perp), \quad \varsigma \in \mathbb{R}. \quad (2.6)$$

*Proof.* We have that

$$\widehat{R_\theta f}(\varsigma) = \int_{\mathbb{R}} e^{-is\varsigma} \int_{\mathbb{R}^2} f(x) \delta(x \cdot \vartheta^\perp - s) dx ds = \int_{\mathbb{R}^2} e^{-ix \cdot \vartheta^\perp \varsigma} f(x) dx.$$

This concludes the proof.  $\square$

This result should not be surprising. For a given value of  $\vartheta$ , the Radon transform gives the integration of  $f$  over all lines parallel to  $\vartheta$ . So obviously the oscillations in the direction  $\vartheta$  are lost, but not the oscillations in the orthogonal direction  $\vartheta^\perp$ . The oscillations of  $f$  in the direction  $\vartheta^\perp$  are precisely of the form  $\hat{f}(\alpha \vartheta^\perp)$  for  $\alpha \in \mathbb{R}$ . It is therefore not surprising that the latter can be retrieved from the Radon transform  $R_\theta f$ .

Notice that this result also gives us a *reconstruction* procedure. Indeed, all we have to do is to take the Fourier transform of  $R_\theta f$  in the variable  $s$  to get the Fourier transform  $\hat{f}(\sigma \vartheta^\perp)$ . It remains then to obtain the latter Fourier transform for all directions  $\vartheta^\perp$  to end up with the full  $\hat{f}(k)$  for all  $k \in \mathbb{R}^2$ . Then the object is simply reconstructed by using the fact that  $f(x) = (\mathcal{F}_{k \rightarrow x}^{-1} \hat{f})(x)$ . We will consider other (equivalent) reconstruction methods and explicit formulas later on.

Before doing so, we derive additional properties satisfied by  $Rf$ . From the Fourier slice theorem, we deduce that

$$R_\theta \left[ \frac{\partial f}{\partial x_i} \right](s) = \theta_i^\perp \frac{d}{ds} (R_\theta f)(s). \quad (2.7)$$

**Exercise 2.2.1** Verify (2.7).

This is the equivalent for Radon transforms of the property (1.30) for the Fourier transform.

**Smoothing properties of the Radon and X-ray transforms.** Let us now look at the regularizing properties of the Radon transform. To do so, we introduce a function  $\chi(x)$  of class  $C_0^\infty(\mathbb{R}^2)$  (i.e.,  $\chi$  is infinitely many times differentiable) and with compact support (i.e. there is a radius  $R$  such that  $\chi(x) = 0$  for  $|x| > R$ ). When we are interested in an object defined on a bounded domain  $X$ , we can choose  $\chi(x) = 1$  for  $x \in X$ .

As we did for  $\mathbb{R}^n$  in the previous chapter, let us now define the Hilbert scale for the cylinder  $Z$  as follows. We say that  $g(s, \theta)$  belongs to  $H^s(Z)$  provided that

$$\|g\|_{H^s(Z)}^2 = \int_0^{2\pi} \int_{\mathbb{R}} (1 + \sigma^2)^s |\mathcal{F}_{s \rightarrow \sigma} g(\sigma)|^2 d\sigma d\theta < \infty. \quad (2.8)$$

This is a constraint stipulating that the Fourier transform in the  $s$  variable decays sufficiently fast at infinity. No constraint is imposed on the directional variable other than being a square-integrable function. We have then the following result:

**Theorem 2.2.2** *Let  $f(x)$  be a distribution in  $H^s(\mathbb{R}^2)$  for some  $s \in \mathbb{R}$ . Then we have the following bounds*

$$\begin{aligned} \sqrt{2} \|f\|_{H^s(\mathbb{R}^2)} &\leq \|Rf\|_{H^{s+\frac{1}{2}}(Z)} \\ \|R(\chi f)\|_{H^{s+\frac{1}{2}}(Z)} &\leq C_\chi \|\chi f\|_{H^s(\mathbb{R}^2)}. \end{aligned} \quad (2.9)$$

The constant  $C_\chi$  depends on the function  $\chi(x)$ , and in particular on the size of its support.

*Proof.* From the Fourier slice theorem  $\widehat{R_\theta w}(\sigma) = \widehat{w}(\sigma\vartheta^\perp)$ , we deduce that

$$\begin{aligned} \int_Z |\widehat{Rw}(\sigma, \theta)|^2 (1 + \sigma^2)^{s+1/2} d\sigma d\theta &= \int_Z |\widehat{w}(\sigma\vartheta^\perp)|^2 (1 + \sigma^2)^{s+1/2} d\sigma d\theta \\ &= 2 \int_{\mathbb{R}^2} |\widehat{w}(k)|^2 \frac{(1 + |k|^2)^{s+1/2}}{|k|} dk, \end{aligned}$$

using the change of variables from polar to Cartesian coordinates so that  $dk = \sigma d\sigma d\theta$  and recalling that  $\widehat{f}(\sigma\vartheta) = \widehat{f}((-\sigma)(-\vartheta))$ . The first inequality in (2.9) then follows from the fact that  $|k|^{-1} \geq (1 + |k|^2)^{-1/2}$  using  $w(x) = f(x)$ . The second inequality is slightly more difficult because of the presence of  $|k|^{-1}$ . We now choose  $w(x) = f(x)\chi(x)$ . Let

us split the integral into  $I_1 + I_2$ , where  $I_1$  accounts for the integration over  $|k| > 1$  and  $I_2$  for the integration over  $|k| < 1$ . Since  $(1 + |k|^2)^{1/2} \leq \sqrt{2}|k|$  for  $|k| > 1$ , we have that

$$I_1 \leq 2\sqrt{2} \int_{\mathbb{R}^2} |\widehat{\chi f}(k)|^2 (1 + |k|^2)^s dk \leq 2\sqrt{2} \|\chi f\|_{H^s}^2.$$

It remains to deal with  $I_2$ . We deduce from (1.31) that

$$I_2 \leq C \|\widehat{\chi f}\|_{L^\infty(\mathbb{R}^2)}^2.$$

Let  $\psi \in C_0^\infty(\mathbb{R}^2)$  be such that  $\psi = 1$  on the support of  $\chi$  so that  $\psi \chi f = \chi f$ . We define  $\psi_k(x) = e^{-ix \cdot k} \psi(x)$ . Upon using the definition (1.25), the Parseval relation (1.27) and the Cauchy Schwarz inequality  $(f, g) \leq \|f\| \|g\|$ , we deduce that

$$\begin{aligned} |\widehat{\chi f}(k)| &= |\widehat{\psi \chi f}(k)| = \left| \int_{\mathbb{R}^2} \psi_k(x) (\chi f)(x) dx \right| = (2\pi)^{-2} \left| \int_{\mathbb{R}^2} \widehat{\psi_k}(\xi) \widehat{\chi f}(\xi) d\xi \right| \\ &= (2\pi)^{-2} \left| \int_{\mathbb{R}^2} \frac{\widehat{\psi_k}(\xi)}{(1 + |\xi|^2)^{s/2}} (1 + |\xi|^2)^{s/2} \widehat{\chi f}(\xi) d\xi \right| \leq \|\psi_k\|_{H^{-s}(\mathbb{R}^2)} \|\chi f\|_{H^s(\mathbb{R}^2)}. \end{aligned}$$

Since  $\psi(x)$  is smooth, then so is  $\psi_k$  uniformly in  $|k| < 1$  so that  $\psi_k$  belongs to  $H^{-s}(\mathbb{R}^2)$  for all  $s \in \mathbb{R}$  uniformly in  $|k| < 1$ . This implies that

$$I_2 \leq C \|\widehat{\chi f}\|_{L^\infty(\mathbb{R}^2)}^2 \leq C_\chi \|\chi f\|_{H^s(\mathbb{R}^2)}^2,$$

where the function depends on  $\psi$ , which depends on the support of  $\chi$ . This concludes the proof.  $\square$

The theorem should be interpreted as follows. Assume that the function (or more generally a distribution)  $f(x)$  has compact support. Then we can find a function  $\chi$  which is of class  $C^\infty$ , with compact support, and which is equal to 1 on the support of  $f$ . In that case, we can use the above theorem with  $\chi f = f$ . The constant  $C_\chi$  depends then implicitly on the size of the support of  $f(x)$ .

The above inequalities show that  $R$  is a *smoothing* operator. This is not really surprising as an integration over lines is involved in the definition of the Radon transform. However, the result tells us that the Radon transform is smoothing by exactly *one half of a derivative*. The second inequality in (2.9) tells us that the factor  $\frac{1}{2}$  is optimal, in the sense that the Radon transform does not regularize by more than one half of a derivative. Moreover this corresponds to (1.43) with  $\alpha = \frac{1}{2}$ , which shows that the inversion of the Radon transform in two dimensions is a mildly ill-posed problem of order  $\alpha = \frac{1}{2}$ : when we reconstruct  $f$  from  $Rf$ , a differentiation of order one half is involved.

At the same time, seen as an operator from  $H^s(\mathbb{R}^2)$  to  $H^{s+\frac{1}{2}}(Z)$ , the Radon transform is a well posed problem. For  $s = -\frac{1}{2}$ , small noise in the  $L^2(Z)$  sense will generate small reconstruction errors in the  $H^{-\frac{1}{2}}(\mathbb{R}^2)$  sense.

**Filtered-backprojection inversions.** Let us now consider such *explicit* reconstruction formulas. In order to do so, we need to introduce two new operators, the adjoint operator  $R^*$  and the Hilbert transform  $H$ . The adjoint operator  $R^*$  to  $R$  (with respect

to the usual inner products  $(\cdot, \cdot)_{\mathbb{R}^2}$  and  $(\cdot, \cdot)_Z$  on  $L^2(\mathbb{R})$  and  $L^2(Z)$ , respectively) is given for every smooth function  $g(s, \theta)$  on  $Z$  by

$$(R^*g)(x) = \int_0^{2\pi} g(x \cdot \vartheta^\perp, \theta) d\theta, \quad x \in \mathbb{R}^2. \quad (2.10)$$

That  $R^*$  is indeed the adjoint operator to  $R$  is verified as follows

$$\begin{aligned} (R^*g, f)_{\mathbb{R}^2} &= \int_{\mathbb{R}^2} f(x) \int_0^{2\pi} g(x \cdot \vartheta^\perp, \theta) d\theta dx \\ &= \int_{\mathbb{R}^2} f(x) \int_0^{2\pi} \int_{\mathbb{R}} \delta(s - x \cdot \vartheta^\perp) g(s, \theta) ds d\theta dx \\ &= \int_0^{2\pi} \int_{\mathbb{R}} g(s, \theta) \int_{\mathbb{R}^2} f(x) \delta(s - x \cdot \vartheta^\perp) dx ds d\theta \\ &= \int_0^{2\pi} \int_{\mathbb{R}} g(s, \theta) (Rf)(s, \theta) ds d\theta = (g, Rf)_Z. \end{aligned}$$

We also introduce the Hilbert transform defined for smooth functions  $f(t)$  on  $\mathbb{R}$  by

$$Hf(t) = \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{f(s)}{t - s} ds. \quad (2.11)$$

Here p.v. means that the integral is understood in the principal value sense, which in this context is equivalent to

$$Hf(t) = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R} \setminus (t-\varepsilon, t+\varepsilon)} \frac{f(s)}{t - s} ds.$$

Both operators turn out to be *local* in the Fourier domain in the sense that they are multiplications in the Fourier domain. More precisely we have the following lemma.

**Lemma 2.2.3** *We have in the Fourier domain the following relations:*

$$\begin{aligned} (\mathcal{F}_{x \rightarrow \xi} R^*g)(\xi) &= \frac{2\pi}{|\xi|} \left( (\mathcal{F}_{s \rightarrow \sigma} g)(-|\xi|, \hat{\xi}^\perp) + (\mathcal{F}_{s \rightarrow \sigma} g)(|\xi|, -\hat{\xi}^\perp) \right) \\ (\mathcal{F}_{t \rightarrow \sigma} Hf)(\sigma) &= -i \text{sign}(\sigma) \mathcal{F}_{t \rightarrow \sigma} f(\sigma). \end{aligned} \quad (2.12)$$

We have used the notation  $\hat{\xi} = \xi/|\xi|$ . For  $\vartheta = (\cos \theta, \sin \theta)$  with  $\vartheta \in S^1$  and  $\theta \in (0, 2\pi)$ , we also identify the functions  $g(\theta) = g(\vartheta)$ . Assuming that  $g(s, \theta) = g(-s, \theta + \pi)$ , which is the case in the image of the Radon transform (i.e., when there exists  $f$  such that  $g = Rf$ ), and which implies that  $\hat{g}(\sigma, \vartheta) = \hat{g}(-\sigma, -\vartheta)$  we have using shorter notation the equivalent statement:

$$\begin{aligned} \widehat{R^*g}(\xi) &= \frac{4\pi}{|\xi|} \hat{g}(|\xi|, -\hat{\xi}^\perp) \\ \widehat{Hf}(\sigma) &= -i \text{sign}(\sigma) \hat{f}(\sigma). \end{aligned} \quad (2.13)$$

*Proof.* Let us begin with  $R^*g$ . We compute

$$\begin{aligned}\widehat{R^*g}(\xi) &= \int e^{-ix \cdot \xi} g(x \cdot \vartheta^\perp, \theta) d\theta dx = \int e^{-is\xi \cdot \vartheta^\perp} g(s, \theta) ds d\theta e^{-it\xi \cdot \vartheta} dt \\ &= \int 2\pi \delta(|\xi| \hat{\xi} \cdot \vartheta) \hat{g}(\xi \cdot \vartheta^\perp, \theta) d\theta = \int \frac{2\pi}{|\xi|} \delta(\hat{\xi} \cdot \vartheta) \hat{g}(\xi \cdot \vartheta^\perp, \theta) d\theta \\ &= \frac{2\pi}{|\xi|} \left( \hat{g}(-|\xi|, \hat{\xi}^\perp) + \hat{g}(|\xi|, -\hat{\xi}^\perp) \right).\end{aligned}$$

In the proof we have used that  $\delta(\alpha x) = \alpha^{-1} \delta(x)$  and the fact that there are two directions, namely  $\hat{\xi}$  and  $-\hat{\xi}$  on the unit circle, which are orthogonal to  $\hat{\xi}^\perp$ . When  $g$  is in the form  $g = Rf$ , we have  $\hat{g}(-|\xi|, \hat{\xi}^\perp) = \hat{g}(|\xi|, -\hat{\xi}^\perp)$ , which explains the shorter notation (2.13).

The computation of the second operator goes as follows. We verify that

$$Hf(t) = \frac{1}{\pi} \left( \frac{1}{x} * f(x) \right) (t).$$

So in the Fourier domain we have

$$\widehat{Hf}(\sigma) = \frac{1}{\pi} \frac{\widehat{1}}{x}(\sigma) \hat{f}(\sigma) = -i \text{sign}(\sigma) \hat{f}(\sigma).$$

The latter is a result of the following calculation

$$\frac{1}{2} \text{sign}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{ix\xi}}{i\xi} d\xi.$$

This concludes the proof of the lemma.  $\square$

The above calculations also show that  $H^2 = H \circ H = -Id$ , where  $Id$  is the identity operator, as can easily be seen in from its expression in the Fourier domain. This property is referred to as saying that the Hilbert transform is an *anti-involution*. We are now ready to introduce some reconstruction formulas.

**Theorem 2.2.4** *Let  $f(x)$  be a smooth function and let  $g(s, \theta) = Rf(s, \theta)$  be its Radon transform. Then,  $f$  can explicitly be reconstructed from its Radon transform as follows:*

$$f(x) = \frac{1}{4\pi} R^* \left( \frac{\partial}{\partial s} Hg(s, \theta) \right) (x). \quad (2.14)$$

*In the above formula, the Hilbert transform  $H$  acts on the  $s$  variable.*

*Proof.* The simplest way to verify the inversion formula is to do it in the Fourier domain. Let us denote by

$$w(s, \theta) = \frac{\partial}{\partial s} Hg(s, \theta).$$

Since  $g(-s, \theta + \pi) = g(s, \theta)$ , we verify that the same property holds for  $w$  in the sense that  $w(-s, \theta + \pi) = w(s, \theta)$ . Therefore (2.14) is equivalent to the statement:

$$\hat{f}(\xi) = \frac{1}{|\xi|} \hat{w}(|\xi|, -\hat{\xi}^\perp), \quad (2.15)$$

according to (2.13). Notice that  $\hat{w}$  is the Fourier transform of  $w$  in the *first variable* only.

Since in the Fourier domain, the derivation with respect to  $s$  is given by multiplication by  $i\sigma$  and the Hilbert transform  $H$  is given by multiplication by  $-i\text{sign}(\sigma)$ , we obtain that

$$\mathcal{F}_{\sigma \rightarrow s}^{-1} \frac{\partial}{\partial s} H \mathcal{F}_{s \rightarrow \sigma} = |\sigma|.$$

In other words, we have

$$\hat{w}(\sigma, \theta) = |\sigma| \hat{g}(\sigma, \theta).$$

Thus (2.15) is equivalent to

$$\hat{f}(\xi) = \hat{g}(|\xi|, -\hat{\xi}^\perp).$$

This, however, is nothing but the Fourier slice theorem stated in Theorem 2.2.1 since  $(-\hat{\xi}^\perp)^\perp = \hat{\xi}$  and  $\xi = |\xi| \hat{\xi}$ . This concludes the proof of the reconstruction.  $\square$

The theorem can equivalently be stated as

$$Id = \frac{1}{4\pi} R^* \frac{\partial}{\partial s} H R = \frac{1}{4\pi} R^* H \frac{\partial}{\partial s} R. \quad (2.16)$$

The latter equality comes from the fact that  $H$  and  $\partial_s$  commute as can easily be observed in the Fourier domain (where they are both multiplications). Here,  $Id$  is the identity operator, which maps a function  $f(x)$  to itself  $Id(f) = f$ .

Here is some additional useful notation in the manipulation of the Radon transform. Recall that  $R_\theta f(s)$  is defined as in (2.5) by

$$R_\theta f(s) = Rf(s, \theta).$$

The adjoint  $R_\theta$  (with respect to the inner products in  $L_s^2(\mathbb{R})$  and  $L_x^2(\mathbb{R}^2)$ ) is given by

$$(R_\theta^* g)(x) = g(x \cdot \vartheta^\perp). \quad (2.17)$$

Indeed (since  $\vartheta$  is *frozen* here) we have

$$\int_{\mathbb{R}} (R_\theta f)(s) g(s) ds = \int_{\mathbb{R}^2} \int_{\mathbb{R}} f(x) \delta(s - x \cdot \vartheta^\perp) g(s) dx ds = \int_{\mathbb{R}^2} g(x \cdot \vartheta^\perp) f(x) dx,$$

showing that  $(R_\theta f, g)_{L^2(\mathbb{R})} = (f, R_\theta^* g)_{L^2(\mathbb{R}^2)}$ . We can then recast the inversion formula as

$$Id = \frac{1}{4\pi} \int_0^{2\pi} \vartheta^\perp \cdot \nabla R_\theta^* H R_\theta d\theta. \quad (2.18)$$

The only new item to prove here compared to previous formulas is that  $R_\theta^*$  and the derivation commute, i.e., for any function  $g(s)$  for  $s \in \mathbb{R}$ , we have

$$\vartheta^\perp \cdot \nabla (R_\theta^* g)(x) = (R_\theta^* \frac{\partial}{\partial s} g)(x).$$

This results from the fact that both terms are equal to  $g'(x \cdot \vartheta^\perp)$ .

One remark on the smoothing properties of the Radon transform. We have seen that the Radon transform is a smoothing operator in the sense that the Radon transform is half of a derivative smoother than the original function. The adjoint operator  $R^*$  enjoys

exactly the same property: it regularizes by half of a derivative. It is not surprising that these two half derivatives are exactly canceled by the appearance of a full derivation in the reconstruction formula. Notice that the Hilbert transform (which corresponds to multiplication by the smooth function  $i\text{sign}(\sigma)$  in the Fourier domain) is a bounded operator with bounded inverse in  $L^2(\mathbb{R})$  (since  $H^{-1} = -H$ ).

**Exercise 2.2.2** Show that

$$f(x) = \frac{1}{4\pi} \int_0^{2\pi} (Hg')(x \cdot \vartheta^\perp, \theta) d\theta.$$

Here  $g'$  means first derivative of  $g$  with respect to the  $s$  variable only.

**Exercise 2.2.3** Show that

$$f(x) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_{\mathbb{R}} \frac{\frac{d}{ds}g(s, \theta)}{x \cdot \vartheta^\perp - s} ds d\theta.$$

This is Radon's original inversion formula [52].

**Exercise 2.2.4** Starting from the definition

$$f(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ik \cdot x} \hat{f}(k) dk,$$

and writing it in polar coordinates (with change of measure  $dk = |k|d|k|d\hat{k}$ ), deduce the above reconstruction formulas by using the Fourier slice theorem.

## 2.3 Three dimensional Radon transform

Let us briefly mention the case of the Radon transform in three dimensions (generalizations to higher dimensions being also possible). The Radon transform in three dimensions consists of integrating a function  $f(x)$  over all possible planes. As we mentioned earlier, the Radon transform is therefore a distinct object from the X-ray transform, which integrates a function along all possible lines.

A plane  $\mathcal{P}(s, \vartheta)$  in  $\mathbb{R}^3$  is characterized by its direction  $\vartheta \in S^2$ , where  $S^2$  is the unit sphere, and by its signed distance to the origin  $s$ . Notice again the double covering in the sense that  $\mathcal{P}(s, \vartheta) = \mathcal{P}(-s, -\vartheta)$ . The Radon transform is then defined as

$$Rf(s, \vartheta) = \int_{\mathbb{R}^3} f(x) \delta(x \cdot \vartheta - s) dx = \int_{\mathcal{P}(s, \vartheta)} f d\sigma. \quad (2.19)$$

Notice the change of notation compared to the two-dimensional case. The Fourier slice theorem still holds

$$\widehat{Rf}(\sigma, \vartheta) = \hat{f}(\sigma\vartheta), \quad (2.20)$$

as can be easily verified. We check that  $Rf(s, \vartheta) = Rf(-s, -\vartheta)$ . The reconstruction formula is then given by

$$f(x) = \frac{-1}{8\pi^2} \int_{S^2} g''(x \cdot \vartheta, \vartheta) d\vartheta. \quad (2.21)$$

Here  $d\vartheta$  is the usual (Lebesgue) surface measure on the unit sphere.

The result can be obtained as follows. We denote by  $S^2/2$  half of the unit sphere (for instance the vectors  $\vartheta$  such that  $\vartheta \cdot \mathbf{e}_z > 0$ ).

$$\begin{aligned} f(x) &= \frac{1}{(2\pi)^3} \int_{\frac{S^2}{2}} \int_{\mathbb{R}} \hat{f}(r\vartheta) e^{ir\vartheta \cdot x} |r|^2 dr d\vartheta = \frac{1}{(2\pi)^3} \int_{\frac{S^2}{2}} \int_{\mathbb{R}} \widehat{Rf}(r, \vartheta) e^{ir\vartheta \cdot x} |r|^2 dr d\vartheta \\ &= \frac{1}{(2\pi)^2} \int_{\frac{S^2}{2}} (-g)''(\vartheta \cdot x, \vartheta) d\vartheta = \frac{1}{2} \frac{-1}{(2\pi)^2} \int_{S^2} g''(\vartheta \cdot x, \vartheta) d\vartheta. \end{aligned}$$

Here we have used the fact that the inverse Fourier transform of  $r^2 \hat{f}$  is  $-f''$ .

**Exercise 2.3.1** Generalize Theorem 2.2.2 and prove the following result:

**Theorem 2.3.1** *There exists a constant  $C_\chi$  independent of  $f(x)$  such that*

$$\begin{aligned} \sqrt{2} \|f\|_{H^s(\mathbb{R}^3)} &\leq \|Rf\|_{H^{s+1}(Z)} \\ \|R(\chi f)\|_{H^{s+1}(Z)} &\leq C_\chi \|\chi f\|_{H^s(\mathbb{R}^3)}, \end{aligned} \tag{2.22}$$

where  $Z = \mathbb{R} \times S^2$  and  $H^s(Z)$  is defined in the spirit of (2.8).

The above result shows that the Radon transform is more smoothing in three dimensions than it is in two dimensions. In three dimensions, the Radon transform smoothes by a full derivative rather than a half derivative.

Notice however that the inversion of the three dimensional Radon transform (2.21) is *local*, whereas this is not the case in two dimensions. What is meant by local is the following: the reconstruction of  $f(x)$  depends on  $g(s, \vartheta)$  only for the planes  $\mathcal{P}(s, \vartheta)$  that pass through  $x$  (and an infinitely small neighborhood so that the second derivative can be calculated). Indeed, we verify that  $x \in \mathcal{P}(x \cdot \vartheta, \vartheta)$  and that all the planes passing by  $x$  are of the form  $\mathcal{P}(x \cdot \vartheta, \vartheta)$ . The two dimensional transform involves the Hilbert transform, which unlike differentiations, is a *non-local* operation. Thus the reconstruction of  $f$  at a point  $x$  requires knowledge of *all* line integrals  $g(s, \theta)$ , and not only for those lines passing through  $x$ .

**Exercise 2.3.2** Calculate  $R^*$ , the adjoint operator to  $R$  (with respect to the usual  $L^2$  inner products). Generalize the formula (2.16) to the three dimensional case.

## 2.4 Attenuated Radon Transform

In the previous sections, the integration over lines for the Radon transform was not weighted. We could more generally ask whether integrals of the form

$$R_\alpha f(s, \vartheta) := \int_{\mathbb{R}} f(s\vartheta^\perp + t\vartheta) \alpha(s\vartheta^\perp + t\vartheta, \vartheta) dt,$$

over all possible lines parameterized by  $s \in \mathbb{R}$  and  $\vartheta \in S^1$  and assuming that the (positive) weight  $\alpha(x, \vartheta)$  is known, uniquely determine  $f(x)$ . This is a much more delicate question for which only partial answers are known.

The techniques developed in the next chapter allow one to prove that  $R_\alpha$  is invertible up to a finite dimensional kernel by an application of the Fredholm theory of compact operators. Moreover, Jan Boman [23] constructed an example of a function  $f \in C_0^\infty(\mathbb{R}^2)$  with compact support and a uniformly positive weight  $\alpha(x, \vartheta)$  also of class  $C^\infty(\mathbb{R}^2)$  such that  $R_\alpha f(s, \vartheta) = 0$ . The operator  $R_\alpha f$  is therefore not always injective.

Proving that  $R_\alpha f$  is injective, when it is indeed injective, is a hard problem. In the next chapter, we will see a methodology to prove injectivity based on energy estimates. The alternative to energy estimates is to use what we will refer to as analytic/unique continuation techniques. For the rest of this chapter, we focus on the inversion of one example of such weighted Radon transforms, namely the Attenuated Radon transform (AtRT). The invertibility of the AtRT has been obtained recently by two different methods in [6] and [50]. Both rely on the continuation properties of analytic functions. We focus here on the second method, which recasts the inverse Radon transform and the inverse Attenuated Radon transform as a Riemann-Hilbert problem [1]. The rest of this section is significantly more technical than the inversion of the Radon transform. It is presented here as an example of the complications that often appear in the proof of injectivity of many transforms of integral geometry when they are indeed injective.

### 2.4.1 Single Photon Emission Computed Tomography

An important application for the attenuated Radon transform is SPECT, single photon emission computed tomography. The principle is the following: radioactive particles are injected in a domain. These particles emit then some radiation. The radiation propagates through the tissues and gets partially absorbed. The amount of radiation reaching the boundary of the domain can be measured. The imaging technique consists then of reconstructing the location of the radioactive particles from the boundary measurements.

We model the density of radiated photons by  $u(x, \theta)$  and the source of radiated photons by  $f(x)$ . The absorption of photons (by the human tissues in the medical imaging application) is modeled by  $a(x)$ . We assume that  $\sigma(x)$  is *known* here. The absorption can be obtained, for instance, by transmission tomography as we saw in earlier sections. The density  $u(x, \theta)$  satisfies then the following transport equation

$$\vartheta \cdot \nabla u(x, \theta) + \sigma(x)u(x, \theta) = f(x), \quad x \in \mathbb{R}^2, \vartheta \in S^1. \quad (2.23)$$

We assume that  $f(x)$  is compactly supported and impose that no radiation comes from infinity:

$$\lim_{s \rightarrow \infty} u(x - s\vartheta, \theta) = 0. \quad (2.24)$$

The transport equation (2.23) with conditions (2.24) admits a unique solution that can be obtained by the method of characteristics. Let us define the following *symmetrized* beam transform

$$D_\theta \sigma(x) = \frac{1}{2} \int_0^\infty [\sigma(x - t\vartheta) - \sigma(x + t\vartheta)] dt = \frac{1}{2} \int_{\mathbb{R}} \text{sign}(t) \sigma(x - t\vartheta) dt. \quad (2.25)$$

We verify that  $\vartheta \cdot \nabla D_\theta \sigma(x) = \sigma(x)$  so that  $e^{D_\theta \sigma(x)}$  is an integrating factor for (2.23) in the sense that

$$\vartheta \cdot \nabla (e^{D_\theta \sigma(x)} u(x, \theta)) = (e^{D_\theta \sigma} f)(x, \theta).$$

Therefore the solution  $u(x, \theta)$  is given by

$$e^{D_\theta \sigma(x)} u(x, \theta) = \int_0^\infty (e^{D_\theta \sigma} f)(x - t\vartheta, \theta) dt. \quad (2.26)$$

We recall that  $\vartheta = (\cos \theta, \sin \theta)$  and that  $\vartheta^\perp = (-\sin \theta, \cos \theta)$  and decompose  $x = s\vartheta^\perp + t\vartheta$ . We deduce from (2.26) that

$$\lim_{t \rightarrow +\infty} e^{D_\theta \sigma(s\vartheta^\perp + t\vartheta)} u(s\vartheta^\perp + t\vartheta, \theta) = \int_{\mathbb{R}} (e^{D_\theta \sigma} f)(s\vartheta^\perp + t\vartheta, \theta) dt.$$

In the above expression the left hand side is known from the measurements. Indeed  $u(s\vartheta^\perp + t\vartheta, \theta)$  is the radiation outside of the domain to image and is thus measured and  $e^{D_\theta \sigma(s\vartheta^\perp + t\vartheta)}$  involves the attenuation coefficient  $\sigma(x)$  which we have assumed is known. The objective is thus to reconstruct  $f(x)$  from the right hand side of the above relation, which we recast as

$$(R_\sigma f)(s, \theta) = (R_{\sigma, \theta} f)(s) = (R_\theta(e^{D_\theta \sigma} f))(s), \quad (2.27)$$

where  $R_\theta$  is the Radon transform defined for a function of  $f(x, \theta)$  as

$$R_\theta f(s) = \int_{\mathbb{R}} f(s\vartheta^\perp + t\vartheta, \theta) dt = \int_{\mathbb{R}^2} f(x, \theta) \delta(x \cdot \vartheta^\perp - s) dx.$$

When  $\sigma \equiv 0$ , we recover that the measurements involve the Radon transform of  $f(x)$  as defined in (2.4). Thus in the absence of absorption, SPECT can be handled by inverting the Radon transform as we saw in earlier sections. When absorption is constant, an inversion formula has been known for quite some time [64]. The inversion formula for non-constant absorption is more recent and was obtained independently by two different techniques [6, 50]. We do not consider here the method of  $A$ -analytic functions developed in [6]. We will present the method developed in [50] based on the extension of the transport equation in the complex domain and on the solution of a Riemann Hilbert problem.

### 2.4.2 Riemann Hilbert problem

Riemann Hilbert problems find many applications in complex analysis. We consider here the simplest of Riemann Hilbert problems and refer the reader to [1] for more general cases and applications.

Let  $T$  be a smooth closed curve in the complex plane, which in our application will be the unit circle, i.e., the complex numbers  $\lambda$  such that  $|\lambda| = 1$ . The reason why we choose the notation  $\lambda$  to represent complex numbers will appear more clearly in the next section. We denote by  $D^+$  the open bounded domain inside the curve  $T$ , i.e., in our application the unit disk  $\{\lambda \in \mathbb{C}, |\lambda| < 1\}$ , and by  $D^-$  the open unbounded domain outside of the curve  $T$ , i.e., in our application  $\{\lambda \in \mathbb{C}, |\lambda| > 1\}$ . The orientation of the curve  $T$  is chosen so that  $D^+$  is on the “left” of the curve  $T$ .

For a smooth function  $\phi(\lambda)$  defined on  $D^+ \cup D^-$ , we denote by  $\phi^+(t)$  and  $\phi^-(t)$  the traces of  $\phi$  on  $T$  from  $D^+$  and  $D^-$ , respectively. So in the case where  $T$  is the unit circle, we have

$$\phi^+(t) = \lim_{0 < \varepsilon \rightarrow 0} \phi((1 - \varepsilon)t), \quad \phi^-(t) = \lim_{0 < \varepsilon \rightarrow 0} \phi((1 + \varepsilon)t).$$

We define  $\varphi(t)$  on  $T$  as the jump of  $\phi$ , i.e.,

$$\varphi(t) = \phi^+(t) - \phi^-(t). \quad (2.28)$$

Let  $\varphi(t)$  be a smooth function defined on  $T$ . The Riemann Hilbert problem is stated as follows. Find a function  $\phi(\lambda)$  on  $D^+ \cup D^-$  such that

1.  $\phi(\lambda)$  is analytic on  $D^+$  and analytic on  $D^-$
2.  $\lambda\phi(\lambda)$  is bounded as  $|\lambda| \rightarrow \infty$  on  $D^-$
3. the jump of  $\phi$  is given by  $\varphi(t) = \phi^+(t) - \phi^-(t)$ .

The solution to the above Riemann Hilbert problem is *unique* and is given by the Cauchy formula

$$\phi(\lambda) = \frac{1}{2\pi i} \int_T \frac{\varphi(t)}{t - \lambda} dt, \quad \lambda \in \mathbb{C} \setminus T = D^+ \cup D^-. \quad (2.29)$$

This is the form of the Riemann Hilbert problem we will use in the sequel. We refer the reader to [1] for the theory.

### 2.4.3 Inversion of the Attenuated Radon Transform

We now want to apply the theory of Riemann Hilbert problems to invert the attenuated Radon transform (AtRT). The first step is to extend the transport equation to the complex domain as follows. We parameterize the unit circle in the complex plane as

$$\lambda = e^{i\theta}, \quad \theta \in (0, 2\pi). \quad (2.30)$$

The new parameter takes values on the unit circle  $T$  for  $\theta \in (0, 2\pi)$ . It can also be seen more generally as an arbitrary complex number  $\lambda \in \mathbb{C}$ . With the notation  $x = (x_1, x_2)$ , the transport equation (2.23) can be recast as

$$\left( \frac{\lambda + \lambda^{-1}}{2} \frac{\partial}{\partial x_1} + \frac{\lambda - \lambda^{-1}}{2i} \frac{\partial}{\partial x_2} + \sigma(x) \right) u(x, \lambda) = f(x), \quad x \in \mathbb{R}^2, \quad \lambda \in T. \quad (2.31)$$

We can simplify the above equation by identifying  $x$  with  $z = x + iy$  and by defining

$$\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right). \quad (2.32)$$

The transport equation (2.31) is then equivalent to

$$\left( \lambda \frac{\partial}{\partial z} + \lambda^{-1} \frac{\partial}{\partial \bar{z}} + \sigma(z) \right) u(z, \lambda) = f(z), \quad z \in \mathbb{C}, \quad \lambda \in T. \quad (2.33)$$

The same boundary conditions (2.24) that no information comes from infinity need to be added in the new variables as well.

The above equation can also be generalized to  $\lambda \in \mathbb{C}$  instead of  $T$ . It is in this framework that the Riemann Hilbert problem theory is used to invert the attenuated Radon transform. This will be done in three steps

- (i) We show that  $u(z, \lambda)$  is *analytic* in  $D^+ \cup D^- = \mathbb{C} \setminus T$  and that  $\lambda u(z, \lambda)$  is bounded as  $\lambda \rightarrow \infty$ .
- (ii) We verify that  $\varphi(x, \theta) = u^+(x, \theta) - u^-(x, \theta)$ , the jump of  $u$  at  $\lambda = e^{i\theta}$  can be written as a function of the measured data  $R_a f(s, \theta)$ .
- (iii) We solve the Riemann Hilbert problem using (2.29) and evaluate (2.33) at  $\lambda = 0$  to obtain a reconstruction formula for  $f(z) = f(x)$ .

#### 2.4.4 Step (i): The $\bar{\partial}$ problem, an elliptic equation

Let us now analyze (2.33). In the absence of absorption the fundamental solution of (2.33) solves the following equation

$$\left( \lambda \frac{\partial}{\partial z} + \lambda^{-1} \frac{\partial}{\partial \bar{z}} \right) G(z, \lambda) = \delta(z), \quad |G(z, \lambda)| \rightarrow 0 \text{ as } |z| \rightarrow \infty, \quad (2.34)$$

for  $\lambda \in \mathbb{C} \setminus (T \cup \{0\})$ .

**Lemma 2.4.1** *The unique solution to (2.34) is given by*

$$G(z, \lambda) = \frac{\text{sign}(|\lambda| - 1)}{\pi(\lambda \bar{z} - \lambda^{-1} z)}, \quad \lambda \notin (T \cup \{0\}). \quad (2.35)$$

*Proof.* The formula can be verified by inspection. A more constructive derivation is the following. Let us define the change of variables

$$\zeta = \lambda^{-1} z - \lambda \bar{z}, \quad \bar{\zeta} = \overline{\lambda^{-1} z} - \bar{\lambda} z. \quad (2.36)$$

Let us assume that  $|\lambda| > 1$ . The Jacobian of the transformation is  $|\lambda|^2 - |\lambda|^{-2}$ . We verify that

$$\lambda \frac{\partial}{\partial z} + \lambda^{-1} \frac{\partial}{\partial \bar{z}} = (|\lambda|^{-2} - |\lambda|^2) \frac{\partial}{\partial \bar{\zeta}}.$$

The change of variables (2.36) has been precisely tailored so that the above holds. Denoting  $\tilde{G}(\zeta) = G(z)$ , we thus obtain

$$\frac{\partial}{\partial \bar{\zeta}} \tilde{G}(\zeta) = \frac{1}{|\lambda|^{-2} - |\lambda|^2} \delta(z(\zeta)) = -\delta(\zeta).$$

So  $-\tilde{G}(\zeta)$  is the fundamental solution of the  $\bar{\partial}$  operator  $\frac{\partial}{\partial \bar{\zeta}}$ . We verify that

$$\frac{\partial}{\partial \bar{\zeta}} \frac{1}{\zeta} = \pi \delta(\zeta). \quad (2.37)$$

Indeed let  $\psi(z)$  be a smooth test function in  $C_0^\infty(\mathbb{R}^2)$  and let  $d\mu(\zeta)$  be the Lebesgue measure  $dx dy$  in  $\mathbb{C} \sim \mathbb{R}^2$ . Then

$$\begin{aligned} \int_{\mathbb{C}} \psi(\zeta) \frac{\partial}{\partial \bar{\zeta}} \frac{1}{\zeta} d\mu(\zeta) &= - \int_{\mathbb{C}} \frac{\partial \psi}{\partial \bar{\zeta}} \frac{1}{\zeta} d\mu(\zeta) = - \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{C} \setminus \{|\zeta| < \varepsilon\}} \frac{\partial \psi}{\partial \bar{\zeta}} \frac{1}{\zeta} d\mu(\zeta) \\ &= - \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{C} \setminus \{|\zeta| < \varepsilon\}} \frac{\partial \zeta^{-1} \psi}{\partial \bar{\zeta}} d\mu(\zeta) = \frac{1}{2i} \int_{|\zeta| = \varepsilon} \frac{\psi}{\zeta} d\mu(\zeta), \end{aligned}$$

by the Green formula with complex variables:

$$\int_{\partial X} u dz = \int_{\partial X} (u dx + i u dy) = \int_X \left( i \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy = 2i \int_X \frac{\partial u}{\partial \bar{z}} d\mu(z).$$

Sending  $\varepsilon$  to 0, we find in the limit

$$\int_{\mathbb{C}} \psi(\zeta) \frac{\partial}{\partial \bar{\zeta}} \frac{1}{\zeta} d\mu(\zeta) = \frac{1}{2i} 2\pi i \psi(0) = \pi \int_{\mathbb{R}^2} \psi(\zeta) \delta(\zeta) d\mu(\zeta).$$

This implies that  $\tilde{G}(\zeta) = (-\pi\zeta)^{-1}$ , hence  $G(z) = (-\pi\zeta)^{-1} = (\pi(\lambda\bar{z} - \lambda^{-1}z))^{-1}$ . This is (2.35) for  $|\lambda| > 1$ . For  $|\lambda| < 1$ , we verify that the Jacobian of the transformation  $z \rightarrow \zeta(z)$  becomes  $|\lambda|^{-2} - |\lambda|^2$  so that

$$\frac{\partial}{\partial \bar{\zeta}} \tilde{G}(\zeta) = \frac{1}{|\lambda|^{-2} - |\lambda|^2} \delta(z(\zeta)) = \delta(\zeta).$$

This yields (2.35) for  $|\lambda| < 1$ .  $\square$

The above proof shows that  $(\pi z)^{-1}$  is the fundamental solution of the  $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$  operator. This implies that the solution to the following  $\bar{\partial}$  problem

$$\frac{\partial}{\partial \bar{z}} f(z) = g(z), \quad z \in \mathbb{C}, \quad (2.38)$$

such that  $f(z)$  vanishes at infinity is given by convolution by the fundamental solution, i.e.,

$$f(z) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{g(\zeta)}{z - \zeta} d\mu(\zeta) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g(\zeta)}{z - \zeta} d\zeta \wedge d\bar{\zeta}. \quad (2.39)$$

Here we have used that  $dz \wedge d\bar{z} = (dx + idy) \wedge (dx - idy) = 2idx \wedge dy = 2id\mu(z)$ , whence the change of 2-form in the above integrations.

Notice that the Green function  $G(z, \lambda)$  tends to 0 as  $z \rightarrow \infty$  for  $\lambda \notin T$ . This is clearly not true when  $\lambda \in T$ , where  $G(z, \lambda) = \delta(l_\theta(z))$ , where  $l_\theta(z)$  is the segment  $\{t\vartheta, t > 0\}$ . The reason is that for  $\lambda \notin (T \cup \{0\})$ ,

$$\lambda \frac{\partial}{\partial z} + \lambda^{-1} \frac{\partial}{\partial \bar{z}} \quad \text{and} \quad \frac{\partial}{\partial \bar{z}},$$

are elliptic operators, in the sense that in the Fourier domain, their symbol given by  $\lambda k_z + \lambda^{-1} \bar{k}_z$  and  $\bar{k}_z$ , respectively, are positive provided that  $k_z$  is not 0. Indeed we verify that

$$\lambda k_z + \lambda^{-1} \bar{k}_z = 0 \quad \text{implies} \quad |\lambda|^2 = 1,$$

when  $k_z \neq 0$  since  $|k_z| = |\bar{k}_z| \neq 0$ .

Let us now define  $h(z, \lambda)$  as the solution to

$$\left( \lambda \frac{\partial}{\partial z} + \lambda^{-1} \frac{\partial}{\partial \bar{z}} \right) h(z, \lambda) = \sigma(z), \quad |h(z, \lambda)| \rightarrow 0 \text{ as } |z| \rightarrow \infty, \quad (2.40)$$

for  $\lambda \notin (T \cup \{0\})$ . The solution is given by

$$h(z, \lambda) = \int_{\mathbb{R}^2} G(z - \zeta, \lambda) \sigma(\zeta) d\mu(\zeta). \quad (2.41)$$

We now verify that

$$\left(\lambda \frac{\partial}{\partial z} + \lambda^{-1} \frac{\partial}{\partial \bar{z}}\right)(e^{h(z,\lambda)}u(z,\lambda)) = e^{h(z,\lambda)}f(z),$$

so that for  $\lambda \notin (T \cup \{0\})$ , the solution of (2.33) is given by

$$u(z,\lambda) = e^{-h(z,\lambda)} \int_{\mathbb{R}^2} G(z-\zeta,\lambda) e^{h(\zeta,\lambda)} f(\zeta) d\mu(\zeta). \quad (2.42)$$

We verify that  $G(z,\lambda)$ ,  $h(z,\lambda)$  and  $u(z,\lambda)$  are defined by continuity at  $z=0$  since  $G(z,\lambda)=0$  by continuity. We now verify that  $G(z,\lambda)$  is analytic in  $D^+$  (including at  $z=0$ ) and in  $D^-$ . Assuming that  $a(z)$  and  $f(z)$  are smooth functions, this is also therefore the case for  $h(z,\lambda)$  and  $u(z,\lambda)$ . Moreover we easily deduce from (2.42) that  $\lambda u(z,\lambda)$  is bounded on  $D^-$ . The solution  $u(z,\lambda)$  of the transport equation extended to the complex plane is therefore a good candidate to apply the Riemann Hilbert theorem.

### 2.4.5 Step (ii): jump conditions

We now want to find the limit of  $u(z,\lambda)$  as  $\lambda$  approaches  $T$  from above (in  $D^-$ ) and below (in  $D^+$ ). Let us write  $\lambda = re^{i\theta}$  and let us send  $r-1$  to  $\mp 0$  on  $D^\pm$ . The Green function behaves according to the following result

**Lemma 2.4.2** *As  $r-1 \rightarrow \mp 0$ , the Green function  $G(x,\lambda)$  tends to*

$$G_\pm(x,\theta) = \frac{\pm 1}{2\pi i(\vartheta^\perp \cdot x \mp i0 \operatorname{sign}(\vartheta \cdot x))}. \quad (2.43)$$

*Proof.* Let us assume that  $|\lambda| > 1$ , i.e.,  $r = 1 + \varepsilon$  with  $\varepsilon > 0$ . We then find

$$\begin{aligned} G(z, re^{i\theta}) &= \frac{1}{\pi} \frac{1}{re^{i\theta}\bar{z} - \frac{e^{-i\theta}}{r}z} = \frac{1}{\pi} \frac{1}{(1+\varepsilon)e^{i\theta}\bar{z} - e^{-i\theta}(1-\varepsilon)z + o(\varepsilon)} \\ &= \frac{1}{2\pi} \frac{1}{-i\mathcal{I}(e^{-i\theta}z) + \varepsilon\mathcal{R}(e^{-i\theta}z) + o(\varepsilon)} = \frac{1}{2i\pi} \frac{-1}{\vartheta^\perp \cdot x + i\varepsilon(\vartheta \cdot x) + o(\varepsilon)}. \end{aligned}$$

Passing to the limit  $\varepsilon \rightarrow 0$ , we obtain

$$G_-(x,\theta) = \frac{1}{2i\pi} \frac{-1}{\vartheta^\perp \cdot x + i0 \operatorname{sign}(\vartheta \cdot x)}.$$

Here by convention  $\pm 0$  is the limit of  $\pm\varepsilon$  as  $0 < \varepsilon \rightarrow 0$ . The limit on  $D^+$  is treated similarly.  $\square$

We have chosen to define  $G_\pm(x,\theta)$  as functions of  $\theta \in (0, 2\pi)$  instead of functions of  $e^{i\theta}$ . We have also identified  $x = (x, y)$  with  $z = x + iy$ . The above lemma gives us a convergence in the sense of distributions. We can equivalently say that for all smooth function  $\psi(x)$ , we have

$$\int_{\mathbb{R}^2} G_\pm(x-y,\theta)\psi(y)dy = \pm \frac{1}{2i} (HR_\theta\psi)(x \cdot \vartheta^\perp) + (D_\theta\psi)(x). \quad (2.44)$$

We recall that the Hilbert transform  $H$  is defined in (2.11) and the Radon transform in (2.4)-(2.5).

*Proof.* The derivation is based on the following result. For any  $f(x) \in C_0^\infty(\mathbb{R})$ , we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} \frac{f(x)}{ix + \varepsilon} dx = -i \text{p.v.} \int_{\mathbb{R}} \frac{f(x)}{x} dx + \text{sign}(\varepsilon) \pi f(0). \quad (2.45)$$

**Exercise 2.4.1** Prove the above limit called Plemelj's formula.

Let us denote  $x = \varsigma \vartheta^\perp + \tau \vartheta$  and  $y = s \vartheta^\perp + t \vartheta$ . We have

$$\begin{aligned} \int_{\mathbb{R}^2} G_+(y) \psi(x - y) dy &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\psi((\varsigma - s)\vartheta^\perp + (\tau - t)\vartheta)}{is + 0\text{sign}(t)} ds dt \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \text{p.v.} \int_{\mathbb{R}} \frac{\psi((\varsigma - s)\vartheta^\perp + (\tau - t)\vartheta)}{is} ds dt + \frac{1}{2} \int_{\mathbb{R}} \text{sign}(t) \psi(\varsigma \vartheta^\perp + (\tau - t)\vartheta) dt \\ &= \frac{1}{2\pi} \text{p.v.} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\psi((\varsigma - s)\vartheta^\perp + (\tau - t)\vartheta)}{is} dt ds + \frac{1}{2} \int_{\mathbb{R}} \text{sign}(t) \psi(x - t\vartheta) dt \\ &= \frac{1}{2i} (HR_\theta \psi)(x \cdot \vartheta^\perp) + (D_\theta \psi)(x). \end{aligned}$$

A similar derivation yields the limit on  $D^-$ .  $\square$

We deduce that the function  $h(z, \lambda)$  defined in (2.40) admits the limits

$$h_\pm(x, \theta) = \pm \frac{1}{2i} (HR_\theta \sigma)(x \cdot \vartheta^\perp) + (D_\theta \sigma)(x). \quad (2.46)$$

Notice that  $R_\theta$  and  $D_\theta$  involve integrations in the direction  $\vartheta$  only so that

$$R_\theta[u(x)v(x \cdot \vartheta^\perp)](s) = v(s)R_\theta[u](s), \quad D_\theta[u(x)v(x \cdot \vartheta^\perp)](x) = v(x \cdot \vartheta^\perp)D_\theta[u](x).$$

Using this result and (2.44), we deduce that the limits of the solution  $u(z, \lambda)$  to (2.31) are given by

$$\begin{aligned} u^\pm(x, \theta) &= e^{-D_\theta \sigma} e^{\frac{\pm 1}{2i} (HR_\theta \sigma)(x \cdot \vartheta^\perp)} \frac{\pm 1}{2i} H \left( e^{\frac{\pm 1}{2i} (HR_\theta \sigma)(s)} R_\theta(e^{D_\theta \sigma} f) \right) (x \cdot \vartheta^\perp) \\ &\quad + e^{-D_\theta \sigma} D_\theta(e^{D_\theta \sigma} f)(x). \end{aligned} \quad (2.47)$$

We recall that  $R_\theta(e^{D_\theta \sigma} f) = R_{\sigma, \theta} f(s)$  are our measurements. So whereas  $u^+$  and  $u^-$  do not depend only on the measurements (they depend on  $D_\theta(e^{D_\theta \sigma} f)(x)$  which is not measured), the difference  $u^+ - u^-$  depends *only* on the measurements. This is the property that allows us to invert the attenuated Radon transform. More precisely, let us define

$$\varphi(x, \theta) = (u^+ - u^-)(x, \theta). \quad (2.48)$$

Using (2.47), we deduce that

$$i\varphi(x, \theta) = R_{-\sigma, \theta}^* H_\sigma R_{\sigma, \theta} f(x), \quad (2.49)$$

where we have defined the following operators

$$\begin{aligned} R_{\sigma, \theta}^* g(x) &= e^{D_\theta \sigma(x)} g(x \cdot \vartheta^\perp), \quad H_\sigma = C_c H C_c + C_s H C_s \\ C_c g(s, \theta) &= g(s, \theta) \cos\left(\frac{HR_\theta \sigma(s)}{2}\right), \quad C_s g(s, \theta) = g(s, \theta) \sin\left(\frac{HR_\theta \sigma(s)}{2}\right). \end{aligned} \quad (2.50)$$

Here  $R_{\sigma, \theta}^*$  is the formal adjoint operator to  $R_{\sigma, \theta}$ .

The above derivation shows that  $i\varphi(x, \theta)$  is real-valued and of the form  $e^{-D_\theta a(x)} M(x \cdot \vartheta^\perp, \theta)$  for some function  $M$ . We deduce therefore that

$$\vartheta \cdot \nabla \varphi(x, \theta) + \sigma \varphi(x, \theta) = 0. \quad (2.51)$$

### 2.4.6 Step (iii): reconstruction formulas

We have seen that  $u(z, \lambda)$  is analytic in  $\lambda$  on  $D^+ \cup D^-$  and is of order  $O(z^{-1})$  at infinity. Moreover the jump of  $u(z, \lambda)$  across  $T$  is given by  $\varphi(x, \theta)$  for  $0 \leq \theta < 2\pi$ . We thus deduce from the Cauchy formula (2.29) that

$$u(x, \lambda) = \frac{1}{2\pi i} \int_T \frac{\varphi(x, t)}{t - \lambda} dt, \quad \lambda \in D^+ \cup D^-, \quad (2.52)$$

where we identify  $\varphi(x, t)$  with  $\varphi(x, \theta)$  for  $t = e^{i\theta}$ . We now deduce from (2.33) in the vicinity of  $\lambda = 0$  that

$$f(x) = \lim_{\lambda \rightarrow 0} \lambda^{-1} \frac{\partial}{\partial \bar{z}} u(x, \lambda). \quad (2.53)$$

Indeed we verify that  $u(z, \lambda) = O(\lambda)$  on  $D^+$  so that  $\sigma(x)u(x, \lambda) \rightarrow 0$  as  $\lambda \rightarrow 0$ . Since  $u(x, \lambda)$  is known thanks to (2.52) in terms of the boundary measurements  $R_{\sigma, \theta} f(s)$ , this is our reconstruction formula. Let us be more specific. We verify that

$$u(x, \lambda) = \frac{1}{2\pi i} \int_T \frac{\varphi(x, t)}{t} dt + \lambda \frac{1}{2\pi i} \int_T \frac{\varphi(x, t)}{t^2} dt + O(\lambda^2). \quad (2.54)$$

We thus deduce from (2.53) and the fact that  $u(x, \lambda) = O(\lambda)$  on  $D^+$  that

$$0 = \frac{1}{2\pi i} \int_T \frac{\varphi(x, t)}{t} dt \quad \text{and} \quad f(x) = \frac{1}{2\pi i} \int_T \frac{\partial \varphi}{\partial \bar{z}}(x, t) \frac{1}{t^2} dt. \quad (2.55)$$

The second equality is the reconstruction formula we were looking for since  $\varphi$  is defined in (2.49) in terms of the measurements  $R_{a, \theta} f(s)$ . The first equality is a *compatibility* conditions that  $i\varphi$  must satisfy in order for the data to be the attenuated Radon transform of a function  $f(x)$ . This compatibility condition is similar to the condition  $g(s, \theta) = g(-s, \theta + \pi)$  satisfied by the Radon transform in the absence of absorption. These compatibility conditions are much more difficult to visualize when absorption does not vanish because the integrals along the line  $\{s\vartheta^\perp + t\vartheta; t \in \mathbb{R}\}$  differ depending on the direction of integration.

Let us recast the reconstruction formula so that it only involves real-valued quantities.

**Exercise 2.4.2** Using  $t = e^{i\theta}$  and  $dt = ie^{i\theta}$ , deduce that

$$\begin{aligned} \frac{1}{2\pi i} \int_T \frac{\varphi(x, t)}{t} dt &= \frac{1}{2\pi} \int_0^{2\pi} \varphi(x, \theta) d\theta \\ \frac{1}{2\pi i} \int_T \frac{\partial \varphi}{\partial \bar{z}}(x, t) \frac{1}{t^2} dt &= \frac{1}{4\pi} \int_0^{2\pi} \vartheta^\perp \cdot \nabla(i\varphi)(x, \theta) d\theta + \frac{1}{4\pi} \int_0^{2\pi} \vartheta \cdot \nabla \varphi(x, \theta) d\theta. \end{aligned}$$

Use (2.51) and (2.55) to show that

$$f(x) = \frac{1}{4\pi} \int_0^{2\pi} \vartheta^\perp \cdot \nabla(i\varphi)(x, \theta) d\theta. \quad (2.56)$$

Let us denote by  $g_\sigma(s, \theta) = R_\sigma f(s, \theta)$  the SPECT measurements. From the above results we recast (2.56) as

$$f(x) = [\mathcal{N}g](x) \equiv \frac{1}{4\pi} \int_0^{2\pi} \vartheta^\perp \cdot \nabla (R_{-\sigma, \theta}^* H_\sigma g)(x, \theta) d\theta. \quad (2.57)$$

**Exercise 2.4.3** Show that (2.57) simplifies to (2.14) when  $\sigma \equiv 0$ .

**Exercise 2.4.4 (Reconstruction with constant absorption.)** We assume that  $f(x) = 0$  for  $|x| \geq 1$  and that  $\sigma(x) = \mu$  for  $|x| < 1$ . This corresponds thus to the case of a constant absorption coefficient on the support of the source term  $f(x)$ .

(i) Show that

$$e^{D_\theta \sigma(x)} = e^{\mu x \cdot \vartheta}, \quad |x| < 1.$$

Deduce that

$$\vartheta^\perp \cdot \nabla (e^{D_\theta \sigma(x)} g(x \cdot \vartheta^\perp, \theta)) = e^{\mu x \cdot \vartheta} \frac{\partial g}{\partial s}(x \cdot \vartheta^\perp).$$

(ii) Verify that the operator  $H_\mu$  defined by  $H_\mu = H_a$  for  $a$  constant is diagonal in the Fourier domain and that

$$\widehat{H_\mu u}(\sigma) = -i \text{sign}_\mu(\sigma) \hat{u}(\sigma), \quad \text{sign}_\mu(\sigma) = \begin{cases} \text{sign}(\sigma) & |\sigma| \geq \mu, \\ 0 & |\sigma| < \mu. \end{cases}$$

(iii) Show that

$$\begin{aligned} g_\mu(s, \theta) &= R_\theta(e^{\mu x \cdot \vartheta} f)(s), \\ f(x) &= \frac{1}{4\pi} \int_0^{2\pi} e^{-\mu x \cdot \vartheta} (H_\mu \frac{\partial}{\partial s} g_\mu)(x \cdot \vartheta^\perp, \theta) d\theta. \end{aligned} \quad (2.58)$$

Verify that in the limit  $\mu \rightarrow 0$ , we recover the inversion for the Radon transform.

## Chapter 3

# Integral Geometry. Generalized Ray Transform

The preceding chapter was devoted to the Radon and Attenuated Radon transforms for which explicit inversion formulas are available. This fairly technical chapter presents some mathematical techniques that have been developed to analyze transforms in integral geometry for which no explicit inversion formula is known. Although similar techniques may be applied to more general inverse problems, we focus here on the integration of functions along a family of two dimensional curves. This integral geometry problem is described in section 3.1.

Proving the injectivity of such integral transforms is a difficult problem. We saw in Chapter 2 how the method of unique/analytic continuation was used to obtain the injectivity of the Attenuated Radon transform. In that chapter, we mentioned Jan Boman's example of a weighted Radon transform that was *not* injective. In section 3.4 of this chapter, we present another method based on energy estimates and due to Mukhometov [45] (see also [58]) that shows that the integral transform is injective for a wide class of families of curves.

The second main objective of this chapter is to present explicit inversion procedures for such integral transforms. These procedures are based on constructing parametrices for the transform that allow us to recast the inverse transform as the inversion of operators of the form  $I - T$  with  $T$  compact and  $I$  the identity operator. Most of the material introduced in the chapter is devoted to recasting the inversion in this framework. It requires the introduction micro-local analytic tools such as oscillatory integrals and Fourier integral operators. These tools are presented without any required prior knowledge of micro-local analysis in sections 3.2 and 3.3.

More generally, the representation of integral geometry transforms as Fourier integrals displays one of the fundamental properties of inverse problems, namely that of *propagation of singularities*. Heuristically, an inverse problem is *well-posed* when singularities (such as discontinuities) in the parameters give rise to singularities in the available measurements. The inversion procedure therefore first and foremost must back-propagate the singularities in the data space to those in the parameter space. A well-suited language of propagation of singularities is briefly introduced in section 3.5.

### 3.1 Generalized Ray Transform: Setting in two dimensions.

#### 3.1.1 Family of curves.

We consider the following family of curves

$$\mathbb{R} \ni t \mapsto x = F(t, s, \vartheta) \in \mathbb{R}^2, \quad (s, \vartheta) \in \mathbb{R} \times \mathbb{S}^1. \quad (3.1)$$

We assume that curves are traveled along with unit speed so that  $|\frac{dF}{dt}| = 1$ . We will make several assumptions on the curves as we proceed.

Define  $y = (t, s)$  so that  $F = F(y, \vartheta)$ . We *assume* that the map  $y \mapsto F(y, \vartheta)$  is globally invertible for all values of  $\vartheta$  with inverse  $\tilde{F}$  so that

$$F(\tilde{F}(x, \vartheta), \vartheta) = x, \quad \tilde{F}(F(y, \vartheta), \vartheta) = y.$$

We denote the inverse function

$$\tilde{F}(x, \vartheta) = (t(x, \vartheta), s(x, \vartheta)). \quad (3.2)$$

The function  $s(x, \vartheta)$  provides the unique curve to which  $x$  belongs for a fixed  $\vartheta$ , i.e.,  $x \in F(\mathbb{R}, s(x, \vartheta), \vartheta)$ .

For the Radon transform corresponding to integration along straight lines, we have

$$F(t, s, \vartheta) = s\vartheta^\perp + t\vartheta, \quad s(x, \vartheta) = x \cdot \vartheta^\perp.$$

This corresponds to a parameterization of the set of lines where  $\vartheta$  is the vector tangent to the line and  $\vartheta^\perp = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vartheta$  is the vector  $\vartheta$  rotated by  $\frac{\pi}{2}$ . This notation generalizes to the multi-dimensional cases, where  $\vartheta$  still parameterizes the direction of the lines.

#### 3.1.2 Generalized Ray Transform.

The generalized Ray transform (GRT) is then the integral of functions over curves  $(s, \vartheta) \mapsto F(t, s, \vartheta)$ :

$$\begin{aligned} Rf(s, \vartheta) &= \int_{\mathbb{R}} f(F(t, s, \vartheta)) dt = \int_{\mathbb{R}^2} f(F(t, s_0, \vartheta)) \delta(s - s_0) ds_0 dt \\ &= \int_{\mathbb{R}^2} f(x) \delta(s - s(x, \vartheta)) J(x, \vartheta) dx, \end{aligned} \quad (3.3)$$

where  $J(x, \vartheta)$  is the (uniformly positive) Jacobian of the transformation  $x \rightarrow \tilde{F}(x, \vartheta)$  at fixed  $\vartheta$ :

$$J(x, \vartheta) := \left| \frac{d\tilde{F}}{dx} \right| (x, \vartheta) = \left| \frac{ds_0 dt}{dx} \right| (x, \vartheta). \quad (3.4)$$

**Exercise 3.1.1** *Check the change of variables in detail.*

More generally, we considered weighted GRT of the form

$$R_w f(s, \vartheta) = \int_{\mathbb{R}} f(F(t, s, \vartheta)) w(t, s, \vartheta) dt, \quad (3.5)$$

where  $w(y, \vartheta)$  is a given, positive, weight. Such integrals are of the same form as before:

$$\begin{aligned} R_w f(s, \vartheta) &= \int_{\mathbb{R}} f(F(t, s, \vartheta)) w(t, s, \vartheta) dt = \int_{\mathbb{R}^2} f(F(t, s_0, \vartheta)) \delta(s - s_0) w(t, s_0, \vartheta) ds_0 dt \\ &= \int_{\mathbb{R}^2} f(x) \delta(s - s(x, \vartheta)) J_w(x, \vartheta) dx, \end{aligned}$$

with a different expression for  $J(x, \vartheta)$ :

$$J(x, \vartheta) \equiv J_w(x, \vartheta) := \left| \frac{d\tilde{F}}{dx} \right| (x, \vartheta) w(\tilde{F}(x, \vartheta), \vartheta). \quad (3.6)$$

To simplify, we shall use the notation  $J(x, \vartheta)$  rather than  $J_w(x, \vartheta)$  and  $R_J$  rather than  $R_w$ . Thus generally, we consider an operator of the form

$$R_J f(s, \vartheta) = \int_{\mathbb{R}^2} f(x) \delta(s - s(x, \vartheta)) J(x, \vartheta) dx, \quad (3.7)$$

where  $J(x, \vartheta)$  is a smooth, uniformly positive, and bounded weight.

The objective of this section is to obtain a parametrix for the weighted GRT  $R_J$ . A (left) parametrix is an operator  $P_J$  such that  $P_J R_J = I - T_J$ , where  $T_J$  is a compact operator. Provided that  $+1$  is not an eigenvalue of  $K_J$ , then the Fredholm alternatives guarantees that  $(I - T_J)$  is invertible. This provides the inversion procedure

$$f = (I - T_J)^{-1} P_J (R_J f).$$

A complete characterization of when  $+1$  is not an eigenvalue of  $T_J$  is not known in general, and Boman's counter-example presented in Chapter 2 shows that  $R_J$ , and hence  $I - T_J$ , may not be injective in some situations. Injectivity of  $R_J$  is analyzed in section 3.4. The rest of this section and of sections 3.2 and 3.3 is devoted to the construction of a parametrix for  $R_J$ .

### 3.1.3 Adjoint operator and rescaled Normal operator

When  $J \equiv 1$  and  $s(x, \vartheta) = x \cdot \vartheta^\perp$ , then  $R_J$  is the standard Radon transform in two dimensions. We have obtained in Chapter 2 the inversion formula:

$$I = \frac{1}{4\pi} R_J^* \Lambda R_J, \quad \Lambda = H \frac{\partial}{\partial s}.$$

We thus see the need to introduce the Riesz operator  $\Lambda$  and the adjoint operator  $R_J^*$ . In curved geometries, however, no explicit formula such as the one given above can be obtained in general. We no longer have access to the Fourier slice theorem, which uses the invariance by translation of the geometry in the standard Radon transform.

The adjoint operator (AGRT) for the  $L^2$  inner product on  $\mathbb{R} \times \mathbb{S}^1$  is defined as

$$R_K^* g(x) = \int_{\mathbb{S}^1} g(s(x, \vartheta), \vartheta) K(x, \vartheta) d\vartheta = \int_{\mathbb{R} \times \mathbb{S}^1} g(s, \vartheta) K(x, \vartheta) \delta(s - s(x, \vartheta)) d\vartheta ds. \quad (3.8)$$

The “normal” operator is thus given by

$$R_K^* R_J f(x) = \int_{\mathbb{R}^2 \times \mathbb{S}^1} f(y) K(x, \vartheta) J(y, \vartheta) \delta(s(x, \vartheta) - s(y, \vartheta)) dy d\vartheta. \quad (3.9)$$

**Exercise 3.1.2** *Check this.*

We need to introduce  $H\partial_s$  to make the operator invertible from  $L^2$  to  $L^2$  as in the case of the standard Radon transform. A simple way to do so is to recast the operators as Fourier integral operators (FIOs) as follows. We formally recast the GRT and AGRT as the following oscillatory integrals

$$\begin{aligned} R_J f(s, \vartheta) &= \int_{\mathbb{R}^2 \times \mathbb{R}} f(x) e^{i(s-s(x, \vartheta))\sigma} J(x, \vartheta) \frac{dx d\sigma}{2\pi} \\ R_K^* g(x) &= \int_{\mathbb{R} \times \mathbb{S}^1 \times \mathbb{R}} g(s, \vartheta) e^{-i(s-s(x, \vartheta))\sigma} K(x, \vartheta) \frac{ds d\vartheta d\sigma}{2\pi}. \end{aligned} \quad (3.10)$$

We then introduce the Riesz operator  $\Lambda = H\partial_s$  given by the Fourier multiplier  $|\sigma|$  in the Fourier domain:

$$\Lambda f(s) = H\partial_s f(s) = \mathcal{F}_{\sigma \rightarrow s}^{-1} |\sigma| \mathcal{F}_{s \rightarrow \sigma} f(s).$$

We thus have

$$\Lambda R_J f(s, \vartheta) = \int_{\mathbb{R}^2 \times \mathbb{R}} f(x) |\sigma| e^{i(s-s(x, \vartheta))\sigma} J(x, \vartheta) \frac{dx d\sigma}{2\pi}.$$

**Exercise 3.1.3** *Check this.*

The “normal” operator for the weights  $J$  and  $K$  is then defined as

$$Ff(x) := R_K^* \Lambda R_J f(x) = \int_{\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}} f(y) |\sigma| e^{i(s(y, \vartheta) - s(x, \vartheta))\sigma} K(x, \vartheta) J(y, \vartheta) \frac{dy d\sigma d\vartheta}{2\pi}. \quad (3.11)$$

It remains to analyze such an operator. We shall have two main objectives: (i) find the appropriate value for  $K(x, \vartheta)$  that makes  $F$  an approximation of identity; and (ii) see how  $R_J$  can be inverted in some specific cases.

We first observe that  $Ff(x)$  is real valued if we choose  $J$  and  $K$  to be real-valued. The contribution from  $\sigma > 0$  is the same as that from  $\sigma < 0$  by complex-conjugating (3.11). Thus,

$$Ff(x) = \int_{\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}_+} f(y) e^{i(s(y, \vartheta) - s(x, \vartheta))\sigma} K(x, \vartheta) J(y, \vartheta) \frac{dy \sigma d\sigma d\vartheta}{\pi}. \quad (3.12)$$

The variables  $(\vartheta, \sigma)$  may be recast as  $\zeta = \sigma\vartheta$  in  $\mathbb{R}^2$  so that  $\hat{\zeta} = \vartheta$  and  $|\zeta| = \sigma$ . We then recast the above operator as

$$Ff(x) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(y) e^{i(s(y, \hat{\zeta}) - s(x, \hat{\zeta}))|\zeta|} K(x, \hat{\zeta}) J(y, \hat{\zeta}) \frac{dy d\zeta}{\pi}. \quad (3.13)$$

The phase and the amplitude are then defined as

$$\phi(x, y, \zeta) = (s(x, \hat{\zeta}) - s(y, \hat{\zeta}))|\zeta|, \quad a(x, y, \hat{\zeta}) = \frac{1}{\pi} K(x, \hat{\zeta}) J(y, \hat{\zeta}). \quad (3.14)$$

The phase is homogeneous of degree 1 in  $\zeta$  since  $\phi(x, y, t\zeta) = t\phi(x, y, \zeta)$  for  $t > 0$ . The amplitude is homogeneous of degree 0. We may thus recast the above operator as

$$Ff(x) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{i\phi(x, y, \zeta)} a(x, y, \hat{\zeta}) f(y) dy d\zeta. \quad (3.15)$$

Our first objective is to prove that for an appropriate choice of  $K(x, \vartheta)$ , then  $F$  may be decomposed as  $F = I - T$  where  $T$  is a compact, smoothing, operator. This provides an approximate inversion to the generalized Radon transform and an iterative exact reconstruction procedure in some cases. In general, however, it is difficult to show that  $T$  does not admit 1 as an eigenvalue. This property is expected to hold generically. But it may not hold or it may not be known how to prove it for specific transforms of interest.

A second objective is therefore to look at the operator  $N = R_J^* \Lambda R_J$ , which is a self-adjoint (normal) operator. We shall show that  $N$  is injective in some cases of interest (using the Mukhometov technique) and that  $QN = I - T$  for some operator  $Q$  and compact operator  $T$ . This will allow us to prove that  $N$  is an invertible operator in  $L^2$ . The generalized transform can then be inverted by means of, e.g., a conjugate gradient algorithm. This provides an explicit reconstruction procedure that is guaranteed to provide the correct inverse.

## 3.2 Oscillatory integrals and Fourier Integral Operators

In this section, we collect generic properties on oscillatory integrals and operators of the form (3.15). References are Fourier Integral Operators I by Lars Hörmander and Fourier integrals in classical analysis by Christopher Sogge.

### 3.2.1 Symbols, phases, and oscillatory integrals.

The first main player is the classes of *symbols*:

**Definition 3.2.1** We denote by  $S^m(X \times \mathbb{R}^N)$  the set of  $a \in C^\infty(X \times \mathbb{R}^N)$  s.t. for compact  $K \subset X$  and all multi-orders  $\alpha$  and  $\beta$ , we have

$$|D_x^\beta D_\zeta^\alpha a(x, \zeta)| \leq C_{\alpha, \beta, K} (1 + |\zeta|)^{m - |\alpha|}, \quad (x, \zeta) \in K \times \mathbb{R}^N. \quad (3.16)$$

Throughout the notes,  $X$  itself will be bounded and  $a$  can be chosen of class  $C^\infty(\bar{X} \times \mathbb{R}^N)$  with then  $K = \bar{X}$  above.

The second main player is the *phase*  $\phi(x, \zeta)$  that appears in the following *oscillatory integrals*:

$$I_\phi(au) = \int_{X \times \mathbb{R}^N} e^{i\phi(x, \zeta)} a(x, \zeta) u(x) dx d\zeta, \quad u \in C_0^\infty(X). \quad (3.17)$$

The phase is assumed to be *positively homogeneous of degree 1 with respect to  $\zeta$* , i.e.,  $\phi(x, t\zeta) = t\phi(x, \zeta)$  for  $t > 0$ , and such that  $\phi \in C^\infty$  for  $\zeta \neq 0$ . Typically, the phase is defined for  $|\zeta| = 1$ . It is then extended to all values of  $\zeta$  by homogeneity, except at  $\zeta = 0$  where it is not smooth.

These oscillatory integrals need to be defined carefully since the integrand is not Lebesgue-integrable. When  $\phi$  vanishes on open sets, the above integral may simply not be defined at all. However, a reasonable definition can be given when  $\phi$  has *no critical point in joint variables*  $(x, \zeta)$  for  $\zeta \neq 0$ , i.e.,  $d\phi \neq 0$  for  $\zeta \neq 0$ . Here,

$$d\phi = \phi'_x dx + \phi'_\zeta d\zeta.$$

No critical point means  $|\phi'_x|^2 + |\phi'_\zeta|^2 > 0$  for  $\zeta \neq 0$ . By definition, a *phase* is a smooth function that (i) is positively homogeneous of degree one and (ii) has no critical points in the variables  $(x, \zeta)$  when  $\zeta \neq 0$ .

By hypothesis, the sum

$$\psi := |\zeta|^2 |\phi'_\zeta|^2 + |\phi'_x|^2 > 0$$

for  $\zeta \neq 0$  is homogeneous of degree 2 since  $\phi'_\zeta$  is homogeneous of degree 0 and  $\phi'_x$  is homogeneous of degree 1 (all in the  $\zeta$  variable).

**Exercise 3.2.1** *Check this.*

Let  $\chi(\zeta) \in C_0^\infty(\mathbb{R}^N)$  so that  $\chi = 1$  near  $\zeta = 0$  and define the differential operator

$$M = -\alpha \cdot \nabla_\zeta - \beta \cdot \nabla_x + \chi, \quad \alpha = \frac{i(1-\chi)}{\psi} |\zeta|^2 \nabla_\zeta \phi, \quad \beta = \frac{i(1-\chi)}{\psi} \nabla_x \phi.$$

We verify that  $M$  is a differential operator with smooth coefficients and that

$$M e^{i\phi} = e^{i\phi}$$

**Exercise 3.2.2** *Verify this in detail.*

Let us then define the transpose differential operator

$$L = M^t = \alpha \cdot \nabla_\zeta + \beta \cdot \nabla_x + \chi + \nabla_\zeta \cdot \alpha + \nabla_x \cdot \beta.$$

Assuming that  $a(x, \zeta)$  vanishes for large  $\zeta$ , we can integrate (3.17) by parts and obtain that

$$I_\phi(au) = \int_{X \times \mathbb{R}^N} e^{i\phi(x, \zeta)} L^k(a(x, \zeta)u(x)) dx d\zeta, \quad u \in C_0^\infty(X), \quad k = 0, 1, 2, \dots \quad (3.18)$$

The advantage of these integrations by parts is that  $L^k$  maps  $S^m$  into  $S^{m-k}$ .

**Exercise 3.2.3** Check this latter result as well as the integrations by parts in (3.18).

Once  $m - k < -N$ , then the integral is well defined as a classical Lebesgue integral.

**Exercise 3.2.4** Check this.

We can then pass by continuity to the definition of the integral when  $a(x, \zeta)$  does not vanish for large  $\zeta$ . This is our *definition* of the oscillatory integral (which we can check does not depend on the choice of  $\chi$  or  $k$ ).

Note that  $A : u \mapsto I_\phi(au)$  is then defined as a distribution. This is in fact a distribution of order  $\leq k$  if  $a \in S^m$  and  $m - k < -N$ ; see also (3.20) below.

We also verify using the same integrations by parts that

$$I_\phi(au) = \lim_{\varepsilon \rightarrow 0} \int e^{i\phi(x, \zeta)} a(x, \zeta) \chi(\varepsilon \zeta) u(x) dx d\zeta, \quad u \in C_0^\infty(X), \quad (3.19)$$

for  $\chi \in \mathcal{S}(\mathbb{R}^n)$  and  $\chi(0) = 1$ . If  $a$  and  $\phi$  depend on a *parameter*  $t$  in a continuous manner, then, using the above characterization as a limit when  $\varepsilon \rightarrow 0$ , the integral is also continuous in that parameter:

$$t \mapsto I_\phi(au, t) = \int_{X \times \mathbb{R}^N} e^{i\phi(x, \zeta, t)} a(x, \zeta, t) u(x) dx d\zeta$$

is continuous. We can thus differentiate with respect to these parameters under the integral sign.

### 3.2.2 Parameterized oscillatory integrals

This construction allows us to introduce the following *parameterized oscillatory integrals*. We denote by  $X_\phi$  the open subset of  $X$  such that  $\zeta \mapsto \phi(x, \zeta)$  has no critical point  $\zeta \neq 0$ . This means that

$$|\phi'_\zeta|^2(x, \zeta) > 0, \quad x \in X_\phi, \zeta \neq 0.$$

This is more constraining than the previous constraint in the variable  $(x, \zeta)$  since we have less partial derivatives to be non-vanishing. Seeing  $x$  as a parameter, we can then define the oscillatory integral as

$$I_\phi(a, u) = \int_X A(x) u(x) dx, \quad u \in C_0^\infty(X_\phi), \quad A(x) = \int_{\mathbb{R}^N} e^{i\phi(x, \zeta)} a(x, \zeta) d\zeta, \quad x \in X_\phi. \quad (3.20)$$

Then  $A(x)$  is continuous and even a function in  $C^\infty(X_\phi)$ .

Let us see some consequences on the singularities of the distribution  $A$ . The same integrations by parts as above show that for  $A$  considered as a distribution, we have

$$\text{sing supp } A \subset \{x \in X; \phi'_\zeta(x, \zeta) = 0 \text{ for some } \zeta \neq 0\}.$$

**Exercise 3.2.5** Check this.

More precisely, we have that for  $a$  a symbol vanishing in some conic neighborhood of the set

$$C = \{(x, \zeta), x \in X, \zeta \in \mathbb{R}^N \setminus \{0\}, \phi'_\zeta(x, \zeta) = 0\},$$

then the distribution  $A(x)$  defined by  $u \mapsto I_\phi(au)$  is a  $C^\infty$  function.

By conic neighborhood of  $\zeta$ , we mean a set of vectors such that  $\xi$  belongs to the neighborhood when  $|\hat{\xi} - \hat{\zeta}| < \delta$  for some  $\delta > 0$ . Here and below, we denote by  $\hat{\xi} = \frac{\xi}{|\xi|}$  the direction of  $\xi$ .

**Exercise 3.2.6** *Check this by integrations by parts.*

This result on parameterized oscillatory integrals is useful but it cannot be applied directly to operators. For this, we need to split the variables  $x$  into the set of variables  $(x, y)$ , where  $x$  denotes the parameters in which the operator is defined and  $y$  denotes the variables of integration of the function on which the operator acts. This yields the notion of Fourier Integral Operators we now consider.

### 3.2.3 Definition of Fourier Integral Operators.

Consider  $X \times Y \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$  and  $\zeta \in \mathbb{R}^N$ . Then  $\phi(x, y, \zeta)$  is positively homogeneous of degree 1 in  $\zeta$ . For a symbol  $a \in S^m$ , we consider the operator

$$Au(x) = \int_{Y \times \mathbb{R}^N} e^{i\phi(x, y, \zeta)} a(x, y, \zeta) u(y) dy d\zeta, \quad u \in C_0^\infty(Y), x \in X. \quad (3.21)$$

If  $\phi$  does not have any critical point as a function of all variables  $(x, y, \zeta)$ , then we have seen that the integral was well defined as a distribution. This can be verified by multiplying the above equation by  $v(x) \in C_0^\infty(X)$  and integrating over  $X$ .

If for each fixed  $x$ ,  $\phi(x, y, \zeta)$  does not have critical points as a function of  $(y, \zeta)$ , then (3.21) is well defined. Moreover,  $A$  is a continuous map from  $C_0^k(Y)$  to  $C^j(X)$  if

$$m + N + j < k,$$

using the terminology of the previous sections.

**Exercise 3.2.7** *Check this.*

The same occurs for the adjoint operator  $A^*$  by exchanging the roles of  $x$  and  $y$ . Let  $R_\phi$  the open set of points  $(x, y)$  such that  $\phi'_\zeta \neq 0$  for  $\zeta \neq 0$ . Then

$$K_A(x, y) = \int_{\mathbb{R}^N} e^{i\phi(x, y, \zeta)} a(x, y, \zeta) d\zeta, \quad (x, y) \in R_\phi,$$

is a Schwartz kernel for  $A$  and defines a function in  $C^\infty(R_\phi)$ . If  $R_\phi = X \times Y$ , then  $K_A$  defines a map  $A$  continuous from  $\mathcal{E}'(Y)$  to  $C^\infty(X)$ .

**Example 1.** Let us consider the phase function  $\phi(x, y, \zeta) = (x - y) \cdot \zeta$ . We obtain that  $R_\phi$  is the complement of the diagonal  $x = y$ .

**Example 2.** For the GRT with  $\phi(s, \vartheta, x, \sigma) = \sigma(s - s(x, \vartheta))$ , then  $R_\phi$  is the complement of the set  $s = s(x, \vartheta)$ . In other words, a singularity at a point  $x$  can a priori generate singularities along all the points of the curve in the  $(s, \vartheta)$  plane given by

$s = s(x, \vartheta)$ . (Note that, here “ $(s, \vartheta)$ ” plays the role of “ $x$ ”, “ $x$ ” plays the role of “ $y$ ”, and “ $\sigma$ ” plays the role of “ $\zeta$ ”.) For the RT a singularity at  $(0, x_2)$  for instance generates the curve  $s = x_2 \cos \vartheta$ . Similarly, for the adjoint operator, a singularity at  $(s, \vartheta)$  becomes a potential singularity along the line  $s = x \cdot \vartheta^\perp$  (a curve for the GRT). It looks as if a singularity at one point  $x$  propagates to singularities everywhere after application of  $R^*R$ . That this is not the case shows that the singular support is not sufficient to fully characterize the propagation of singularities: the propagation of singularities for the Radon transform requires the notion of singularities in the phase space, in other words, requires the introduction of the Wave Front Set, which will be done in a later section.

Nonetheless, we have the following result. We call  $\phi$  an operator phase function if for each  $x$  (or  $y$ ) it has no critical point in  $(y, \zeta)$  (or  $(x, \zeta)$ ). Let  $C_\phi$  be the complement of  $R_\phi$ , i.e., the projection on  $X \times Y$  of

$$C = \{(x, y, \zeta) \in X \times Y \times \mathbb{R}^N \setminus \{0\}, \phi'_\zeta(x, y, \zeta) = 0\}.$$

In other words,  $(x, y) \in C_\phi$  if (and only if) there exists  $\zeta \neq 0$  such that  $(x, y, \zeta) \in C$ . Then we have:

$$\text{sing supp } Au \subset C_\phi \text{supp } u, \quad u \in \mathcal{E}'(Y).$$

Here, we have defined

$$C_\phi K = \{x, (x, y) \in C_\phi \text{ for some } y \in K\}.$$

We recall that the support  $\text{supp } u$  of a function  $u$  is the closure of the set of points where the function does not vanish. This notion generalizes to distributions. A point is in the open complement of the support of a distribution if the distribution acting on any function supported in a sufficiently small vicinity of the point vanishes. The singular support  $\text{sing supp}$  is the subset of the support of a distribution or function where the latter is not a function of class  $C^\infty$  in the vicinity of a point in that subset.

In fact we can write  $u = v + w$  with  $v$  supported in the vicinity of the singular support of  $u$  and  $w$  smooth. This allows us to obtain the more refined result:

$$\text{sing supp } Au \subset C_\phi \text{sing supp } u, \quad u \in \mathcal{E}'(Y). \quad (3.22)$$

The latter result is sometimes satisfactory, sometimes not. For the phase function  $\phi = (x-y) \cdot \zeta$ , we obtain that the singularities of  $Au$  are included in the set of singularities of  $u$ . In some sense, this is satisfactory as it states that singularities cannot propagate in the  $x$  variable.

For the phase  $\phi(s, \vartheta, x, \sigma) = \sigma(s - s(x, \vartheta))$ , however, the results are not satisfactory as they imply that singularities at a point  $x$  can propagate to singularities at any point  $(s, \vartheta)$  and it therefore looks like singularities are spread very wildly by the Radon transform. The notion of wave front sets will refine this statement and show that there is in fact a one-to-one (in fact one-to-two) correspondence between properly defined singularities before and after application of the Radon transform.

### 3.3 Pseudo-differential operators and GRT

We now come back to the operator  $Ff(x)$  appearing in the analysis of the GRT. Although the phase appears to be complicated, it is in fact essentially of the form  $\phi(x, y, \xi) = (x - y) \cdot \xi$  after an appropriate change of variables as we now show.

### 3.3.1 Absence of singularities away from the diagonal $x = y$

We start with a first step showing that singularities of  $f(x)$  at a point  $x_0$  do not propagate to singularities in  $Ff(x)$  at any other point than  $x_0$  under reasonable assumptions on the phase.

Let us *assume* that

$$\phi'_\zeta(x, y, \zeta) = 0 \quad \text{implies that} \quad x = y. \quad (3.23)$$

**Exercise 3.3.1** *Show that the above constraint is satisfied for the Radon transform. Show that this constraint is still satisfied for curves  $F$  sufficiently close to the straight lines and for weights  $w$  sufficiently close to 1.*

Let then  $\chi_0(x, y) = 1$  when  $x = y$  and supported in the vicinity of  $x = y$  in the sense that  $\chi_0(x, y) = 0$  when  $|x - y| > \delta$  for  $\delta$  to be chosen arbitrarily small. Also we assume that  $\chi_0 \in C^\infty(X \times Y)$ . Let then

$$Ff(x) = F_0f(x) + F_1f(x), \quad F_0f(x) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(y) e^{i\phi(x, y, \zeta)} a(x, y, \zeta) \chi_0(x, y) dy d\zeta. \quad (3.24)$$

We wish to show that  $F_1 := F - F_0$  is a regularizing operator. This is a repeat of what we saw in section 3.2.3 but we shall do the derivation in detail nonetheless. We recall that  $\chi(\zeta)$  is a smooth function with compact support in  $\mathbb{R}^2$ . Since  $\phi'_\zeta(x, y, \zeta) \neq 0$  is homogeneous of degree 0, it is bounded from below by a positive constant uniformly in  $\zeta$  and uniformly in  $x$  and  $y$  for compact domains  $X$  and  $Y$  on the support of  $1 - \chi_0(x, y)$ . We then define

$$b_j(x, y, \zeta) = \frac{-i(1 - \chi)\partial_{\zeta_j}\phi}{|\phi'_\zeta|^2},$$

which is homogeneous of degree 0 and bounded. We verify that

$$L^t e^{i\phi} := (b_j \partial_{\zeta_j} + \chi) e^{i\phi} = e^{i\phi}$$

where  $L^t$  is a first-order differential operator adjoint to the first-order differential operator  $L$  (as we have done before) so that

$$F_1f(x) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(y) e^{i\phi(x, y, \zeta)} L^k(a(1 - \chi_0))(x, y, \zeta) dy d\zeta, \quad k \in \mathbb{N}.$$

Since  $a$  is bounded and smooth in  $\zeta$  (in fact homogeneous of degree 0),  $\chi$  is compactly supported, and  $b_j$  are homogeneous of degree 0, we verify that  $L^k a$  is bounded by  $|\zeta|^{-k}$ . Thus for  $k \geq 3$ , the above integral is absolutely continuous in dimension  $n = 2$  since  $|\zeta|^{-3} |\zeta| d|\zeta|$  is integrable on  $(1, \infty)$ . Now, we can differentiate the above expression with respect to  $x$ . Each differentiation brings a contribution  $\phi'_x$  homogeneous of degree 1 in  $\zeta$ . If we differentiate  $j$  times and choose  $k = j + 3$ , then the integral is absolutely convergent in  $\zeta$  again. With  $j = m$ , we observe that  $F_1$  maps  $L^2(Y)$  to  $H^m(X)$  for all  $m \in \mathbb{N}$  and hence is certainly compact by Sobolev imbedding. These are very similar calculations to those of Exercise 3.2.7.

This calculation shows that the singular support of  $Ff(x)$  is included in the support of  $f$ , and more precisely in the singular support of  $f$  by writing  $f = f_1 + f_2$  with  $f_2$  smooth and  $f_1$  supported in an arbitrary small neighborhood of the singular support of  $f$ . Indeed, by sending  $j \rightarrow \infty$  and  $\delta \rightarrow 0$ , we observe that  $Ff(x)$  is of class  $C^\infty$  away from the support of  $f$ . Thus, as in (3.22),

$$\text{sing supp } Ff \subset \text{sing supp } f. \quad (3.25)$$

This is a satisfactory result. It states that the singularities of  $Ff$  are at the same place as those of  $f$ . As we shall see in a later section, the application of  $R_J$  to  $f$  maps the singularities of  $f$  to singularities in the variables  $(s, \theta)$ . The above results states that applying  $R_K^*$  to  $R_J f$  maps the singularities of  $R_J f$  back to the location of the singularities of  $f$ . Note that the above result describes an inclusion, not an equality. When applying  $F$  to  $f$ , some singularities may be “lost”. It thus remains to show that all the singularities of  $f$  are captured by  $F$  and that  $F$  is invertible in some sense.

### 3.3.2 Change of variables and phase $(x - y) \cdot \xi$

Let us now return to the analysis of  $F_0$ . We want to show that the latter operator is a pseudo-differential operator, namely that after an appropriate change of variables, the phase  $\phi(x, y, \zeta)$  can in fact be recast as  $(x - y) \cdot \xi$ . Let us consider the identity

$$s(x, \hat{\zeta}) - s(y, \hat{\zeta}) = (x - y) \cdot \nabla_x s(z(x, y, \hat{\zeta}), \hat{\zeta}) = s'_{x_j}(z(x, y, \hat{\zeta}), \hat{\zeta})(x_j - y_j),$$

for some smooth function  $z(x, y, \hat{\zeta})$ . Then we define the change of variables  $\xi = \xi(\zeta)$  at fixed  $y$  and  $x$ :

$$\xi = s'_x(z(x, y, \hat{\zeta}), \hat{\zeta})|\zeta| := h(\hat{\zeta}; x, y)|\zeta| := \varphi(\zeta). \quad (3.26)$$

We need this change of variables to be well defined in the vicinity of  $x = y$  since the operator  $F$  has been replaced by  $F_0$  whose kernel is concentrated in the vicinity of  $x = y$ . We verify that

$$L(x, y, \zeta) := \left| \frac{d\xi}{d\zeta} \right| (x, y, \zeta) = \det(h, h')(x, y, \hat{\zeta}). \quad (3.27)$$

Here,  $h'$  is the derivative of the vector  $h$  in the variable  $\phi$  where  $\hat{\zeta} = (\cos \phi, \sin \phi)$ .

**Exercise 3.3.2** *Check these calculations in detail and show that the determinant is indeed homogeneous of degree 0 in  $\zeta$ .*

We *assume* that  $|L(x, y, \zeta)| \geq c_0 > 0$  is uniformly positive for  $(x, y) \in X \times Y$  compact domains and  $|x - y| < \delta$ . For  $\delta$  sufficiently small, it is thus sufficient to *assume* that

$$\left| \frac{d\xi}{d\zeta} \right| (x, x, \hat{\zeta}) = \det(h, h')(x, x, \hat{\zeta}) \geq 2c_0 > 0. \quad (3.28)$$

This is a local property of invertibility. However, we need to *assume* that  $\xi = \varphi(\zeta)$  is a *global* change of variables. In other words, we *assume* that  $\varphi$  is a global diffeomorphism from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  with inverse  $\varphi^{-1}$  for fixed values of  $x$  and  $y$ . Note that since  $\varphi$  is

homogeneous of degree 1 so that  $\lambda\xi = \varphi(\lambda\zeta)$ , then  $\varphi^{-1}$  is also homogeneous of degree 1. As a consequence,  $\zeta = |\xi|\varphi^{-1}(\hat{\xi})$  and

$$\hat{\zeta} = \frac{\varphi^{-1}(\hat{\xi})}{|\varphi^{-1}(\hat{\xi})|} := \hat{\zeta}(x, y, \hat{\xi}).$$

**Exercise 3.3.3** Check that  $\xi = \zeta^\perp$  and that  $L(x, y, \zeta) = 1$  when  $F(t, x, \vartheta) = s\vartheta^\perp + t\vartheta$ .

With this we recast the operator  $F_0$  as

$$F_0f(x) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(y)e^{i(x-y)\cdot\xi} M(x, y, \hat{\xi}) dy d\xi, \quad (3.29)$$

with

$$M(x, y, \hat{\xi}) = K(x, \hat{\zeta}(x, y, \hat{\xi}))J(y, \hat{\zeta}(x, y, \hat{\xi}))L(x, y, \hat{\zeta}(\hat{\xi}))\chi_0(x, y).$$

Note that  $M(x, y, \hat{\xi})$  is a symbol of class  $S^0(X \times Y \times \mathbb{R}^N)$  for  $N = n = 2$ .

Operators of the form

$$Pf(x) = \int_{Y \times \mathbb{R}^n} f(y)e^{i(x-y)\cdot\xi} a(x, y, \xi) dy d\xi, \quad (3.30)$$

with  $a \in S^0(X \times Y \times \mathbb{R}^n)$  are called *pseudo-differential operators* ( $\Psi$ DOs) of order zero. When  $a = a(\xi)$  is a polynomial, then these are *differential operators*. We have therefore obtained that  $F$  could be written as the sum of a  $\Psi$ DO and a smoothing (compact) operator.

### 3.3.3 Choice of a parametrrix.

In the above construction,  $J$  is given by the problem of interest while  $K = K(x, \vartheta)$  is a kernel that we can choose. One way to address the inversion of  $R_J$  is to choose  $K(x, \vartheta)$  so that the above operator is as close to the identity operator as possible. We have already shown that all singularities of the kernel of the above oscillatory integral were located on the diagonal  $x = y$ . We thus define

$$K(x, \hat{\zeta}) = \frac{1}{(2\pi)^2 J(x, \hat{\zeta}) L(x, x, \hat{\zeta})}. \quad (3.31)$$

With this, we find that  $M(x, x, \hat{\xi}) = (2\pi)^{-2}$ . In other words, we have

$$\begin{aligned} F_0f(x) &= f(x) - T_0f(x), \\ T_0f(x) &:= \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(y)e^{i(x-y)\cdot\xi} (M(x, x, \hat{\xi}) - M(x, y, \hat{\xi})) dy d\xi. \end{aligned} \quad (3.32)$$

We will prove in detail that  $T_0$  is a compact operator. This will show that  $R_K^* \Lambda$  is an inverse of  $R_J$  up to a remainder that is a compact operator, in other words that  $R_K^* \Lambda R_J = I - T$  where  $T$  is a compact operator since we already know that  $F_1$  is a compact operator. In other words,  $R_K^* \Lambda$  is a left-parametrix for the generalized Radon transform  $R_J$ .

### 3.3.4 Proof of smoothing by one derivative

As an operator from functions on a bounded domain  $Y$  to functions on a bounded domain  $X$ ,  $T_0$  is a compact operator in  $L^2$ , and in fact an operator mapping  $L^2(Y)$  to  $H^1(X)$  of these respective domains. This will be proved by showing that  $T_0$  and  $\partial_{x_j}T_0$  for  $j = 1, 2$  are bounded operators from  $L^2(Y)$  to  $L^2(X)$ .

We consider the operators  $\partial_{x_j}T_0$  first and calculate:

$$\partial_{x_j}T_0f(x) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(y)e^{i(x-y)\cdot\xi} i\xi_j(x-y) \cdot \nabla_x M(x, \tau(x, y, \hat{\xi}), \hat{\xi}) dy d\xi + T_{00}f(x),$$

for some operator  $T_{00}f(x)$  with a symbol in  $S^0(X \times Y \times \mathbb{R}^2)$  (which will then be bounded from  $L^2$  to  $L^2$  as we shall see in the next section; check the details as an exercise) and for some smooth function  $\tau(x, y, \hat{\xi})$  so that the components of  $\nabla_x M(x, \tau(x, y, \hat{\xi}), \hat{\xi})$  belong to  $S^0(X \times Y \times \mathbb{R}^N)$ .

Let  $\chi(\xi)$  still be the compactly supported, smooth function in  $\mathbb{R}^2$  such that  $\chi(0) = 1$ . The above integrand is multiplied by  $\chi + (1 - \chi)$ . The contribution  $\chi$  clearly generates a smooth, bounded, contribution. The contribution  $(1 - \chi)$  may be recast as a sum of terms of the form

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2} f(y)e^{i(x-y)\cdot\xi} \xi_j(x-y)_k M_k(x, y, \xi) dy d\xi = - \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(y)e^{i(x-y)\cdot\xi} \xi_j \partial_{\xi_k} M_k(x, y, \xi) dy d\xi,$$

for some smooth and bounded functions  $M_k(x, y, \xi)$ , which are in fact homogeneous of degree 0 for  $\xi$  outside the support of  $\chi(\xi)$ . Now we observe that  $\xi_j \partial_{\xi_k} M_k(x, y, \xi) = a_{jk}(x, y, \xi)$  is a symbol in  $S^0(X \times Y \times \mathbb{R}^2)$  and is in fact homogeneous of degree 0 for  $\xi$  outside the support of  $\chi(\xi)$ . Indeed, we have in polar coordinates in two dimensions the following explicit expression:

$$\nabla_{\xi} \psi(\hat{\xi}) = \frac{\partial}{\partial|\xi|} \psi(\hat{\xi}) e_{|\xi|} + \frac{1}{|\xi|} \frac{\partial}{\partial \hat{\xi}} \psi(\hat{\xi}) e_{\xi} = \frac{1}{|\xi|} \frac{\partial}{\partial \hat{\xi}} \psi(\hat{\xi}) e_{\xi}.$$

So we are faced with showing that an oscillatory operator of the form (3.30) with a bounded amplitude  $a(x, y, \xi)$ , homogeneous of degree 0 for large  $\xi$ , is bounded from  $L^2(Y)$  to  $L^2(X)$ .

### 3.3.5 Boundedness of $\Psi$ DOs of order 0 in the $L^2$ sense

Consider the operator

$$Pf(x) = \int_{Y \times \mathbb{R}^n} f(y)e^{i(x-y)\cdot\xi} a(x, y, \hat{\xi}) dy d\xi, \quad (3.33)$$

with  $a(x, y, \hat{\xi}) \in S^0(X \times Y \times \mathbb{R}^n)$ . Here, we are in the simplified setting where  $a$  is a symbol of order 0 that also turns out to be homogeneous of degree 0. Since the integral over any ball in  $\xi$  generates a clearly bounded contribution, it is in fact sufficient that  $a(x, y, \hat{\xi})$  be homogeneous of degree 0 for  $|\xi|$  sufficiently large.

**Lemma 3.3.1** *Let  $X$  and  $Y$  be compact domains in  $\mathbb{R}^n$ . Then the operator  $P$  in (3.33) is a bounded operator from  $L^2(Y)$  to  $L^2(X)$  with a constant  $C$  independent of  $f \in L^2(Y)$  such that*

$$\|Pf\|_{L^2(X)} \leq C\|f\|_{L^2(Y)}. \quad (3.34)$$

*Proof.* We first need to separate low frequencies from high frequencies to handle the lack of regularity of the amplitude at  $\xi = 0$ . Let  $\chi(\xi)$  be smooth, compactly supported and so that  $\chi(0) = 1$  and define

$$P = P_0 + P_1, \quad P_0 f(x) = \int_{Y \times \mathbb{R}^n} f(y) e^{i(x-y) \cdot \xi} (1 - \chi(\xi)) a(x, y, \hat{\xi}) dy d\xi,$$

with  $P_1$  clearly bounded from  $L^2(Y)$  to  $L^2(X)$ . Define  $a_\chi(x, y, \xi) = (1 - \chi(\xi)) a(x, y, \hat{\xi})$ , noting that for  $|\xi|$  sufficiently large, then  $a_\chi$  is a function of  $\hat{\xi}$  since  $\chi = 1$ . For  $\xi$  sufficiently small, then  $a_\chi$  vanishes.

By Taylor expansion, we write

$$a_\chi(x, y, \xi) = \sum_{|\alpha| < k} (y - x)^\alpha a_\alpha(x, x, \xi) + \sum_{|\alpha| = k} (y - x)^\alpha a_\alpha(x, y, \xi).$$

Note that again all functions  $a_\alpha$  are functions of  $\hat{\xi}$  for  $|\xi|$  sufficiently large and vanish for  $\xi$  sufficiently small. Now from

$$\partial_\xi^\alpha e^{i(x-y) \cdot \xi} = i^{|\alpha|} (x - y)^\alpha e^{i(x-y) \cdot \xi},$$

we deduce that

$$\int_{Y \times \mathbb{R}^n} f(y) e^{i(x-y) \cdot \xi} (y - x)^\alpha a_\alpha(x, y, \xi) dy d\xi = \int_{Y \times \mathbb{R}^n} f(y) e^{i(x-y) \cdot \xi} i^{|\alpha|} \partial_\xi^\alpha a_\alpha(x, y, \xi) dy d\xi.$$

For  $|\alpha| < k$ , the above amplitude is a function of  $(x, x)$ . With these expressions, we find that

$$P_0 f(x) = \int_{Y \times \mathbb{R}^n} f(y) e^{i(x-y) \cdot \xi} P_k(x, \xi) d\xi dy + \int_{Y \times \mathbb{R}^n} f(y) e^{i(x-y) \cdot \xi} R_k(x, y, \xi) d\xi dy,$$

where  $P_k(x, \xi)$  is a smooth bounded function in both variables, and where  $R_k(x, y, \xi)$  is a smooth function bounded by  $|\xi|^{-k}$  and vanishing for  $\xi$  small.

**Exercise 3.3.4** *Check this. Hint: differentiate  $k$  times functions that depend only on  $\hat{\xi}$ .*

Upon choosing  $k = n + 1$ , we find that the integral in  $\xi$  involving  $R_k$  is absolutely convergent. That expression therefore clearly has  $L^2$  norm bounded by that of  $f$  by an application of the Cauchy-Schwarz inequality. It remains to address the middle term involving  $P_k(x, \xi)$ . That this operator is bounded in  $\mathcal{L}(L^2)$  is not completely straightforward.

After extending  $f$  by 0 on  $\mathbb{R}^n \setminus Y$ , we recast that term as

$$P_k f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix \cdot \xi} P_k(x, \xi) d\xi.$$

We write for the two-dimensional case  $n = 2$ :

$$P_k(x, \xi) = P_k(0, \xi) + \int_0^{x_1} \partial_1 P_k(t_1, 0, \xi) dt_1 + \int_0^{x_2} \partial_2 P_k(0, t_2, \xi) dt_2 + \int_0^{x_1} \int_0^{x_2} \partial_{12}^2 P_k(t_1, t_2, \xi) dt_1 dt_2.$$

**Exercise 3.3.5** Write the corresponding expansion for arbitrary dimension  $n \geq 2$ .

Let us define

$$P_{k1}f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix \cdot \xi} \int_0^{x_1} \partial_1 P_k(t_1, 0, \xi) dt_1 d\xi.$$

We observe by Fubini, which holds for oscillatory integrals, that

$$P_{k1}f(x) = \int_0^{x_1} \left( \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix \cdot \xi} \partial_1 P_k(t_1, 0, \xi) d\xi \right) dt_1.$$

But then by Minkowski's integral inequality, we have

$$\|P_{k1}f\|_{L^2(X)} \leq \int_0^{x_1} \left\| \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix \cdot \xi} \partial_1 P_k(t_1, 0, \xi) d\xi \right\|_{L^2(X)} dt_1.$$

By using the Plancherel identity and dealing with the other contributions in a similar manner, we obtain that

$$\|P_k f\|_{L^2(X)} \leq C \left\| \int_{\mathbb{R}^n} e^{ix \cdot \xi} \hat{f}(\xi) Q(\xi) d\xi \right\|_{L^2(X)} \leq C \|\hat{f}(\xi) Q(\xi)\|_{L^2(\mathbb{R}^n)},$$

where  $Q(\xi)$  is the maximum over  $t$  of the operators  $\partial^\alpha P_k(t, \xi)$  for  $|\alpha| \leq n$  and  $t \in X$ . This is a bounded quantity. Moreover,  $Q(\xi)$  is bounded since the symbol  $P_k(x, \xi)$  is bounded in  $\xi$ . This shows the result.  $\square$

Coming back to the generalized Radon transform, we have shown that  $T_0$  was bounded from  $L^2(Y)$  to  $L^2(X)$  and that taking partial derivatives in  $x$  composed with  $T_0$  formed an operator that was bounded from  $L^2(Y)$  to  $L^2(X)$ . This shows that  $T_0$  is a bounded operator from  $L^2(Y)$  to  $H^1(X)$ . This also shows that  $F$  is bounded from  $L^2(Y)$  to  $L^2(X)$ . We summarize these results as follows: We have thus obtained that

$$\|T_0 f(x)\|_{H^1(X)} + \|F f(x)\|_{L^2(X)} \leq C \|f(x)\|_{L^2(Y)}. \quad (3.35)$$

### 3.3.6 Injectivity and implicit inversion formula.

The above result states that  $F = I - T$  with  $T$  compact from  $L^2(X)$  to  $L^2(Y)$  since the injection  $i : H^1(X) \rightarrow L^2(X)$  is compact. The Fredholm alternative thus says that  $F$  is *invertible* if it is *injective*. It is not known how to prove injectivity of  $F$  in general. All we know is that  $F$  is injective if 1 is not an eigenvalue of  $T$ . Also, even if 1 is an eigenvalue of  $T$ , the operator  $F$  can be inverted on the complement of a finite dimensional linear space. But in general, we do not know how to prove that  $F$  is invertible. Of course, we know that  $T = 0$  for the Radon transform. Therefore, when  $F(t, s, \vartheta)$  is close to  $s\vartheta^\perp + t\vartheta$ , then by continuity,  $T$  is of norm less than 1 (we then do not need  $T$  to be compact) and then  $F = \sum_{k=0}^{\infty} T^k$ .

Something else can be done, however, when we can prove (typically by another method) that  $R_J$  is injective. As the counter-example of Jan Boman shows,  $R_J$  is not always injective. As we have mentioned already in the last chapter, it is typically very difficult to prove that a given transform is injective. We were able to do so for the attenuated Radon transform in Chapter 2. In the next section, we an injectivity result for  $R_J$  by using a technique of energy estimates developed by Mukhometov.

Let us therefore *assume* that  $R_J$  is injective and define the normal operator

$$Nf(x) = R_J^* \Lambda R_J f(x) \quad (3.36)$$

as an operator from  $L^2(X)$  to  $L^2(X)$ . Since  $\Lambda^{\frac{1}{2}}$  is invertible and self-adjoint, then  $N$  is also injective and self-adjoint for the usual inner product on  $L^2(X)$ . Indeed if  $A$  is injective then  $A^*Au = 0$  implies that

$$(A^*Au, u) = (Au, Au) = 0$$

so that  $u = 0$ .

We want to show that  $N$  is not only injective but in fact invertible in the  $L^2$  sense. Note that  $N$  is no longer of the form  $I - T$  with  $T$  compact and that we therefore again need to work a bit to get such an invertibility statement. As a first step, using the micro-local techniques of the preceding section, we show that

$$QN = I - T,$$

for  $T$  compact and  $Q$  a parametrix of order 0. The construction of  $Q$  goes as follows. We write

$$N = N_0 + T_1,$$

where  $T_1$  is compact and  $N_0$  is given by

$$N_0 f(x) = \int_{\mathbb{R}^{2n}} e^{i(x-y) \cdot \xi} a(y, \xi) f(y) \frac{1}{(2\pi)^n} dy d\xi.$$

**Exercise 3.3.6** *Check that  $T_1$  is indeed a compact operator. Hint: use the same proof showing that  $T_0$  in (3.32) is compact.*

Here  $a(y, \xi)$  is a symbol of order 0 uniformly bounded from below by a positive constant by construction of  $N$ .

**Exercise 3.3.7** *Give the explicit expression satisfied by  $a(y, \xi)$  and show that  $a(y, \xi)$  is indeed uniformly bounded from below by a positive constant.*

Define  $Q$  as the operator

$$Qf(x) = \int_{\mathbb{R}^{2n}} e^{i(x-y) \cdot \xi} \frac{1}{a(x, \xi)} f(y) \frac{1}{(2\pi)^n} dy d\xi. \quad (3.37)$$

Then we find after some cancellations that

$$QN_0 f(x) = \int_{\mathbb{R}^{2n}} e^{i(x-y) \cdot \xi} \frac{a(y, \xi)}{a(x, \xi)} f(y) \frac{1}{(2\pi)^n} dy d\xi.$$

**Exercise 3.3.8** Check the cancellations leading to the preceding calculation. Note that the amplitude depends on  $y$  in the definition of  $N_0$  and depends on  $x$  in the definition of  $Q$ . It is the only “combination” for which the product  $QN_0$  admits a nice compact expression as given above.

How would you define  $Q$  if you wanted an approximate right inverse instead, i.e., an operator  $Q$  such that  $N_0Q = I - T$  for  $T$  compact ?

This shows that

$$T_2f(x) = (QN_0 - I)f(x) = \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} \frac{a(y, \xi) - a(x, \xi)}{a(x, \xi)} f(y) \frac{1}{(2\pi)^n} dy d\xi,$$

is a compact operator from  $L^2(Y)$  to  $H^1(X)$  for the same reasons that  $T_0$  in (3.32) is compact. This shows the existence of a parametrix  $Q$  such that  $QN = I - T$  with  $T$  compact.

We have seen that such operators  $Q$  were bounded from  $L^2(X)$  to  $L^2(X)$  and therefore deduce that

$$\|f\| \leq \|QNf\| + \|Tf\| \leq C\|Nf\| + \|Tf\|. \quad (3.38)$$

Since  $N$  is injective, we can in fact show that there exists  $C_0 > 0$  such that

$$\|f\| \leq C_0\|Nf\|. \quad (3.39)$$

*Proof.* Indeed, assume that there is no such constant so that we can construct  $f_n$  such that

$$1 = \|f_n\| = n\|Nf_n\|.$$

Then  $f_n$  converges weakly to  $f$  in  $L^2$  with then  $\|Nf_n\| \rightarrow 0$ . But then  $(Nf_n, f) = (f_n, Nf) \rightarrow (f, Nf) = 0$ , which implies that  $f = 0$  since  $N$  is self-adjoint and injective. Since  $T$  is compact, we obtain that  $Tf_n$  converges to  $Tf = 0$ . Now (3.38) implies that

$$1 = \|f_n\| \leq C\|Nf_n\| + \|Tf_n\| \leq \frac{C}{n}\|f_n\| + o(1) = o(1),$$

which is a contradiction. Here,  $o(1)$  means a sequence of real numbers that converges to 0 as  $n \rightarrow \infty$ . This shows the existence of  $C_0 > 0$  such that (3.39) holds.  $\square$

This proves that  $N$  is invertible with inverse  $N^{-1}$  bounded by  $C_0$  (indeed the above estimate proves that  $N$  has closed range and trivial kernel so that its range, which is closed, is all of  $L^2(X)$ ).

The inversion of the generalized ray transform may therefore be done as follows. We first apply  $R_j^*\Lambda$  to the data  $R_j f(s, \vartheta)$  and then apply  $N^{-1}$  the inverse of the normal operator. The latter step can be done iteratively for instance by a conjugate gradient method.

## 3.4 Kinematic Inverse Source Problem

We now present an entirely different technique of energy estimates developed by Mukhometov showing that  $R = R_j$  defined above is injective in the case where the weight  $w \equiv 1$  and the curves are parameterized so that  $|\dot{F}| = 1$ , i.e., curves are traveled along with speed equal to 1.

### 3.4.1 Transport equation

We consider a bounded domain  $X \subset \mathbb{R}^2$  with smooth surface  $\partial X$  parameterized by  $0 \leq \tau \leq T$  and points  $x = S(\tau)$  with  $S(0) = S(T)$  and  $|\dot{S}(\tau)| = 1$ .

For a point  $x$  in  $\bar{X}$  and  $0 \leq \tau \leq T$ , we denote by  $\tilde{F}(x, \tau)$  the unique curve joining  $x$  and  $S(\tau)$ . For a function  $f$  supported in  $X$ , we define the curve integrals

$$g(\tau_1, \tau_2) = \int_{\tilde{F}(S(\tau_1), \tau_2)} f dt, \quad (3.40)$$

where  $dt = \sqrt{dx^2 + dy^2}$  is the Lebesgue distance measure along the curve. We thus travel along the curve with speed equal to 1.

We assume  $g(\tau_1, \tau_2)$  known for all  $0 \leq \tau_1, \tau_2 \leq T$ , which corresponds to the curve integrals of  $f$  for all possible curves in the family passing through  $X$ . We then have the following result:

**Theorem 3.4.1** *Under the above hypotheses for the family of curves  $F$ , a function  $f \in C^2(\bar{X})$  is uniquely determined by its integrals  $g(\tau_1, \tau_2)$  given by (3.40) along the curves of  $F$ . Moreover we have the stability estimate*

$$\|f\|_{L^2(X)} \leq C \left\| \frac{\partial g(\tau_1, \tau_2)}{\partial \tau_1} \right\|_{L^2((0,T) \times (0,T))}. \quad (3.41)$$

The rest of the section is devoted to the proof of this theorem. The proof of injectivity of the reconstruction of  $f$  from knowledge of  $g$  is based on analyzing the following transport equation. We introduce the function

$$u(x, \tau) = \int_{\tilde{F}(x, \tau)} f dt \quad (3.42)$$

for  $x \in \bar{X}$ . We denote by  $\vartheta(x, \tau)$  the unit tangent vector to the curve  $\tilde{F}(x, \tau)$  at  $x$  and orientated such that

$$\vartheta(x, \tau) \cdot \nabla u(x, \tau) = f(x), \quad \vartheta(x, \tau) = \begin{pmatrix} \cos \phi(x, \tau) \\ \sin \phi(x, \tau) \end{pmatrix}. \quad (3.43)$$

The latter relation is obtained by differentiating (3.42) with respect to arc length.

**Exercise 3.4.1** *Check this.*

### 3.4.2 Variational form and energy estimates

We now differentiate the above with respect to  $\tau$  and obtain

$$\frac{\partial}{\partial \tau} (\vartheta \cdot \nabla u) = 0. \quad (3.44)$$

We find that

$$\frac{\partial}{\partial \tau} \vartheta = \phi_\tau J \vartheta, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \phi_\tau := \partial_\tau \phi.$$

We calculate

$$J\vartheta \cdot \nabla u \frac{\partial}{\partial \tau} \vartheta \cdot \nabla u = \phi_\tau (J\vartheta \cdot \nabla u)^2 + J\vartheta \cdot \nabla u \vartheta \cdot \nabla u_\tau$$

with  $u_\tau = \partial_\tau u$ . Similarly, we have

$$-\vartheta \cdot \nabla u \frac{\partial}{\partial \tau} J\vartheta \cdot \nabla u = \phi_\tau (\vartheta \cdot \nabla u)^2 - \vartheta \cdot \nabla u J\vartheta \cdot \nabla u_\tau.$$

Upon adding these two identities and using (3.44), we obtain

$$\begin{aligned} -\frac{\partial}{\partial \tau} \left( J\vartheta \cdot \nabla u \vartheta \cdot \nabla u \right) &= \phi_\tau |\nabla u|^2 + J\vartheta \cdot \nabla u \vartheta \cdot \nabla u_\tau - \vartheta \cdot \nabla u J\vartheta \cdot \nabla u_\tau \\ &= \phi_\tau |\nabla u|^2 + \nabla \cdot (J\nabla u u_\tau). \end{aligned}$$

Indeed, denoting by  $R = (\vartheta | J\vartheta)$  the rotation matrix and  $T = (\nabla u | \nabla u_\tau) := (a | b)$ , we find that

$$\vartheta \cdot a J\vartheta \cdot b - J\vartheta \cdot a \vartheta \cdot b = \det(R^t T) = \det(R) \det T = \det T = J a \cdot b,$$

independent of  $\vartheta$ . This little miracle occurs in dimension  $n = 2$ . For  $a = \nabla u$  and  $b = \nabla u_\tau$ , this gives  $J\nabla u \cdot \nabla u_\tau = \nabla \cdot (J\nabla u u_\tau)$  since  $\nabla \cdot J\nabla = 0$ . It remains to integrate over  $X \times (0, T)$  and use the fact that  $S(0) = S(T)$  on the surface of  $X$  to obtain that

$$\int_0^T \int_X \phi_\tau |\nabla u|^2 dx d\tau = \int_0^T \int_{\partial X} \nabla u \cdot Jn(x, \tau) u_\tau(x, \tau) d\Sigma(x) d\tau, \quad (3.45)$$

where  $n$  is the outward unit normal to  $X$  at  $x \in \partial X$  and  $d\Sigma(x)$  the surface (length) measure on  $\partial X$ . Now,  $S(\tau')$  at the surface has tangent vector  $\dot{S}(\tau') d\tau' = -Jn(\tau') d\Sigma(x)$  assuming the parameterization  $S(\tau)$  counter-clock-wise. Since  $u(S(\tau'), \tau) = g(\tau', \tau)$ , we find that  $\nabla u \cdot \dot{S}(\tau') = \partial_{\tau'} g(\tau', \tau)$  and  $u_\tau(x, \tau) = \partial_\tau g(\tau', \tau)$  so that eventually,

$$\int_0^T \int_X \phi_\tau(x, \tau) |\nabla u|^2(x, \tau) dx d\tau = - \int_0^T \int_0^T \frac{\partial}{\partial \tau_1} g(\tau_1, \tau_2) \frac{\partial}{\partial \tau_2} g(\tau_1, \tau_2) d\tau_1 d\tau_2. \quad (3.46)$$

From the definition of  $\tau$  and  $\phi_\tau$ , we observe that

$$\det(\vartheta | \partial_\tau \vartheta) = \phi_\tau \det(\vartheta | J\vartheta) = \phi_\tau.$$

The *assumption* we make on the family of curves is such that the vector  $\partial_\tau \vartheta$  cannot vanish and cannot be parallel to  $\vartheta$ . In the choice of orientation of  $S$ , we find that

$$\phi_\tau > 0. \quad (3.47)$$

Note that this is a non-local assumption on the curves. It states that the curves passing by a point  $x$  separate sufficiently rapidly in the sense that  $\phi$  increases sufficiently rapidly as the boundary parameter  $\tau$  increases.

### 3.4.3 Injectivity result

Since  $|f(x)| \leq |\nabla u(x, \tau)|$  from the definition of the transport equation and  $\phi_\tau$  integrates to  $2\pi$  in  $\tau$ , we find that

$$2\pi \int_X |f(x)|^2 dx = \int_0^T \int_X \phi_\tau |f(x)|^2 dx d\tau \leq - \int_0^T \int_0^T \frac{\partial}{\partial \tau_1} g(\tau_1, \tau_2) \frac{\partial}{\partial \tau_2} g(\tau_1, \tau_2) d\tau_1 d\tau_2. \quad (3.48)$$

Since  $g(\tau, \tau') = g(\tau', \tau)$ , this shows that

$$\|f\|_{L^2(X)} \leq \frac{1}{\sqrt{2\pi}} \|\partial_\tau g\|_{L^2((0,T) \times (0,T))}. \quad (3.49)$$

This concludes the proof of Theorem 3.4.1.

When two measurements  $g$  are equal so that their difference and hence the difference of their differential vanishes, then the difference of sources  $f$  also vanishes. This provides the injectivity of the transform  $Rf(s, \vartheta)$  for  $f$  supported on a compact domain. Indeed, if  $g = 0$ , the  $\partial_\tau g = 0$  and hence  $f = 0$ . This gives the injectivity. However, the proof of injectivity is not a proof of invertibility as we have for the normal operator  $N$  and is not as constructive as the result obtained for  $F = I - T$ .

### 3.4.4 Summary on GRT.

What have we done so far? We have defined a generalized ray transform  $R_J f(s, \vartheta)$ . We have then quickly brought our functions back into the space of positions by applying a rescaled adjoint operator  $R_K^* \Lambda$ . This lead to the definition of the operators  $F = R_K^* \Lambda R_J$  and  $N = R_J^* \Lambda R_J$ . We have seen that by an appropriate choice of  $K$ , then  $F = I - T$  where  $T$  is a compact operator mapping  $L^2(X)$  to  $H^1(X)$ . However, we do not know that  $F$  is invertible in general although by continuity we know that it is so when the curves are close to the straight lines and the weight  $w$  is close to 1. Since  $T$  is compact, we know that the space of functions such that  $T\psi = \psi$  is finite dimensional. But we do not know whether it is trivial.

We have then changed gears slightly and have looked at the normal operator  $N = R_J^* \Lambda R_J$ . Such an operator, like  $F$ , is a pseudo-differential operator. Moreover, it is invertible up to compact perturbations in the sense that  $QN = I - T$  for  $T$  compact and  $Q$  another pseudo-differential operator of order 0. Here again, we do not know that 1 is not an eigenvalue of  $T$  nor that  $Q$  is invertible. However, we have seen that in some situations,  $R_J$ , and hence  $N$ , was injective by using a transport equation. This allowed us to show that  $N$  was in fact an invertible operator in  $L^2(X)$ . Once properly discretized, the resulting equations may then be solved by the method of, e.g., conjugate gradient.

This provides a reasonable theory for the reconstruction of functions from full data, i.e., from knowledge of  $R_J f(s, \vartheta)$  for all  $(s, \vartheta) \in \mathbb{R} \times \mathbb{S}^1$ . In many practical problems, such data are not available. It then remains a very difficult problem to prove injectivity of the transform. In the case of the Radon transform, the Fourier slice theorem shows that the Radon transform is injective as soon as an open set of values of  $\vartheta$  is available (and all values of  $s$ ). This is because the Fourier transform of a compactly supported function is an analytic function in the Fourier variable and that an analytic function

known on an arbitrarily small open set is known everywhere. For the generalized ray transform, no such results are available. However, it is interesting to understand which singularities of the function  $f(x)$  may be reconstructed from available measurements. This requires that we understand how singularities propagate when we apply the Radon transform and the adjoint of the Radon transform.

## 3.5 Propagation of singularities for the GRT.

### 3.5.1 Wave Front Set and Distributions.

As we have seen, the notion of singular support is not sufficient to describe the propagation of singularities for operators that are not  $\Psi$ DOs, such as for instance the Radon transform or the generalized ray transform. The singular supports have to be extended and refined to a phase space notion (the cotangent bundle). We then need a map from cotangent bundle to cotangent bundle describing how singularities propagate.

For  $u \in \mathcal{D}'(X)$  and  $X \subset \mathbb{R}^n$ , we define the Wave Front Set of  $u$  denoted by  $WF(u)$  as follows. We say that  $(x_0, \xi_0) \notin WF(u)$  iff there exists a function  $\phi \in C_0^\infty(X)$  with  $\phi(x_0) \neq 0$  such that the Fourier transform  $\widehat{\phi u}(\xi)$  is rapidly decreasing in a conic neighborhood of the half ray with direction  $\xi_0$ , i.e., for  $\xi$  such that  $\hat{\xi} - \hat{\xi}_0$  is sufficiently small.

The main result on Wave Front Sets is then as follows:

**Theorem 3.5.1** *Let  $X \in \mathbb{R}^n$ ,  $\Gamma$  an open cone in  $X \times (\mathbb{R}^n \setminus \{0\})$  and  $\phi$  a phase function in  $\Gamma$ . If  $a \in S^m(X \times \mathbb{R}^N)$ , vanishes near the zero section and  $\text{cone supp } a \subset \Gamma$ , then for  $A$  seen as a distribution and defined in (3.20), we have*

$$WF(A) \subset \{(x, \phi'_x); (x, \zeta) \in \text{cone supp } a, \phi'_\zeta(x, \zeta) = 0\}. \quad (3.50)$$

*Proof.* The proof of this result goes as follows. Let  $\varphi(x)$  be a function concentrating near a point  $x_0$ . Then

$$\widehat{A\varphi}(\xi) = \int_{\mathbb{R}^N \times \mathbb{R}^n} e^{i\phi(x, \zeta) - ix \cdot \xi} a(x, \zeta) \varphi(x) d\zeta dx.$$

Let  $\psi(x, \zeta) = \phi(x, \zeta) - \xi \cdot x$ . Then

$$d\psi = \phi'_\zeta d\zeta + (\phi'_x - \xi) dx.$$

This means that for  $\xi$  in a cone away from  $\xi_0 = \phi'_x$ , we can define a smooth differential operator of the form  $L = a \cdot \nabla_x + c$  so that  $L^k e^{i\psi(x, \zeta)} = e^{i\psi(x, \zeta)}$ . The reason is that since  $\xi$  is away from  $\xi_0 = \phi'_x$ , which is homogeneous of degree one in  $\zeta$ , then

$$|\phi'_x(x, \zeta) - \xi| \geq C(|\zeta| + |\xi|).$$

The usual integrations by parts then give us that  $|\widehat{A\varphi}(\xi)| \leq C|\xi|^{-k}$  for all  $k \in \mathbb{N}$ . This proves that  $(x_0, \xi_0) \notin WF(A)$  and concludes the proof of the result.  $\square$

In particular, if  $A$  is a  $\Psi$ DO, then  $\phi(x, y, \zeta) = (x - y) \cdot \zeta$  and we find that

$$WF(K_A) \subset N^* \Delta := \{(x, x, \zeta, -\zeta), x \in X, \zeta \in \mathbb{R}^n - 0\}.$$

For more general operators, rules of propagation of singularities can also be defined. We present them without derivations, which are fairly technical.

To do so, it is instructive to look at the product of distributions and understand when such products are defined. Let  $\Gamma_j$ ,  $j = 1, 2$  be two closed cones in  $X \times (\mathbb{R}^N - 0)$ . (What we mean is that they are cones in the  $\zeta$  variable, not the  $x$  variable.) We *assume* that

$$\Gamma_1 + \Gamma_2 = \{(x, \zeta_1 + \zeta_2), (x, \zeta_j) \in \Gamma_j\} \subset X \times (\mathbb{R}^N - 0). \quad (3.51)$$

The “ $-0$ ” above is the important information. We assume that  $\zeta_1$  and  $\zeta_2$  *cannot* be linearly dependent for an  $x$  as the base point in both cones. Then  $(\Gamma_1 + \Gamma_2) \cup \Gamma_1 \cup \Gamma_2$  is also a closed cone in  $X \times (\mathbb{R}^N - 0)$ . We have the following result:

**Theorem 3.5.2** *Let  $\Gamma_j$  be closed cones as above. Then the product of distributions  $u_j$  such that  $WF(u_j) \subset \Gamma_j$  can be defined in one and only one way so that it is sequentially continuous with values in  $\mathcal{D}'$ . Moreover, we have*

$$WF(u_1 u_2) \subset (\Gamma_1 + \Gamma_2) \cup \Gamma_1 \cup \Gamma_2.$$

A prototypical example is the product of  $\delta_{x_1}$  and  $\delta_{x_2}$  in  $\mathbb{R}^n$  for  $n \geq 2$ . Indeed, for  $n = 2$ ,  $\Gamma_1 = \{(0, x_2, \zeta_1, 0) - 0\}$  and  $\Gamma_2 = \{(x_1, 0, 0, \zeta_2) - 0\}$ . Then  $\Gamma_1 + \Gamma_2 = \{(0, 0, \zeta_1, \zeta_2) - 0\}$  and the above WFS inclusion is clearly satisfied for  $\delta_x := \delta_{x_1} \delta_{x_2}$ .

Note that in fact,  $WF(\delta_x) = \Gamma_1 + \Gamma_2 = \{(0, \zeta) - 0\}$ . The inclusion may thus not be an equality. The reason is that singularities of  $\delta_{x_1}$  at  $x_2 \neq 0$  are no longer present in the product since  $u_2 = \delta_{x_2}$  vanishes there. Inclusions may therefore be strict as such cancellations do occur.

Let us look at the propagation of singularities in a linear transformation. Let a distribution  $K \in \mathcal{D}'(X \times Y)$  for  $X \subset \mathbb{R}^n$  and  $Y \subset \mathbb{R}^m$ . Then  $K$  defines a continuous map from  $C_0^\infty(Y)$  to  $\mathcal{D}'(X)$  via

$$\langle K\varphi, \psi \rangle := K(\psi \otimes \varphi); \quad \varphi \in C_0^\infty(Y), \psi \in C_0^\infty(X).$$

We use  $K$  both for the operator (on the left) and the distribution (on the right).

Let  $u \in C_0^\infty(Y)$ . Then

$$WF(Ku) \subset WF_X(K) := \{(x, \xi); (x, \xi, y, 0) \in WF(K)\}.$$

In other words, if  $K$  has no singularities that are purely in  $(x, \xi)$ , then  $Ku \in C^\infty(X)$ .

The definition of  $Ku$  for  $u$  a distribution is a dual question. Let  $u_1 = K$  the distribution on  $X \times Y$ . Let  $u_2 = 1 \otimes u$  another distribution. The product is well defined if the cones  $\Gamma_1 = WF(K)$  and  $\Gamma_2 = WF(1 \otimes u) = X \times WF(u)$  are such that the sum  $\Gamma_1 + \Gamma_2$  satisfies (3.51), i.e., does not meet  $\zeta_1 + \zeta_2 = 0$ . This means that  $WF(u)$  does not meet

$$WF'_Y(K) = \{(y, \eta); (x, 0, y, -\eta) \in WF(K) \text{ for some } x\}.$$

When  $WF'_Y(K) = \emptyset$ , then  $K$  defines a continuous map from  $\mathcal{E}'(Y)$  to  $\mathcal{D}'(X)$ .

### 3.5.2 Propagation of singularities in FIOs

With this, we arrive at the main result:

**Theorem 3.5.3** *Let  $X \subset \mathbb{R}^n$  and  $Y \subset \mathbb{R}^m$  and  $K \in \mathcal{D}'(X \times Y)$ . If  $WF_X(K) = \emptyset$  and for  $u \in \mathcal{E}'(Y)$ ,  $WF(u)$  does not meet  $WF'_Y(K)$ , then*

$$WF(Ku) \subset WF'(K)WF(u), \quad (3.52)$$

where we have defined

$$WF'(K) = \{(x, \xi, y, -\eta) \in X \times \mathbb{R}^n \times Y \times \mathbb{R}^m; (x, \xi, y, \eta) \in WF(K)\} \quad (3.53)$$

where  $WF'(K)$  is regarded as a relation mapping sets in  $Y \times \mathbb{R}^m - 0$  to sets in  $X \times \mathbb{R}^n - 0$ .

In other words,  $(x, \xi) \in WF'(K)WF(u)$  when there exists  $(y, \eta) \in WF(u)$  such that  $(x, \xi, y, -\eta) \in WF(K)$ , the latter being equivalent to  $(x, \xi, y, \eta) \in WF'(K)$ .

For  $K = \delta(x - y) = c_n \int e^{i(x-y)\cdot\xi} d\xi$ , which corresponds to the phase  $\phi(x, y, \xi) = (x - y) \cdot \xi$ , we have seen that  $WF'(K) = \{(x, x, \xi, \xi), (x, \xi) \in X \times \mathbb{R}^n - 0\}$ . Thus,  $WF(Ku) \subset WF(u)$ . More generally, for any distribution of a  $\Psi$ DO, we have

$$K(x, y) = \int e^{i(x-y)\cdot\xi} a(x, y, \xi) d\xi,$$

with also  $WF'(K) = \{(x, x, \xi, \xi), (x, \xi) \in X \times \mathbb{R}^n - 0\}$ . This shows that  $\Psi$ DOs do not propagate singularities. This is a refined version of the statement we had regarding the singular support of a distribution.

Let us now consider the phase  $\phi(s, \vartheta, x, \sigma) = (s - s(x, \vartheta))\sigma$  and the distribution kernel

$$K(s, \vartheta, x) = \int_{\mathbb{R}} e^{i(s-s(x, \vartheta))\sigma} \frac{J(x, \vartheta)}{2\pi} d\sigma.$$

The phase stations at  $s = s(x, \vartheta)$ , the set where  $\phi'_\sigma = 0$ . Away from that set, the distribution kernel is smooth. So all the action takes place on the set

$$WF(K) = \left\{ (s, \vartheta, x, \sigma, -\sigma \partial_\vartheta s(x, \vartheta), -\sigma \nabla_x s(x, \vartheta)), s = s(x, \vartheta) \right\}.$$

This implies that

$$WF'(K) = \left\{ (s(x, \vartheta), \vartheta, x, \sigma, -\sigma \partial_\vartheta s(x, \vartheta), \sigma \nabla_x s(x, \vartheta)) \right\}.$$

How are singularities propagated by the generalized ray transform then? Let us assume we have a singularity of  $f(x)$  at  $(x, \xi) \in X \times \mathbb{R}^2 - 0$ . Then in order for that singularity to propagate to the Radon domain, it has to be of the form

$$\xi = \sigma \nabla_x s(x, \vartheta).$$

Our hypotheses on  $s$  show that  $(\sigma, \vartheta) \rightarrow \xi$  is in fact a diffeomorphism of  $\mathbb{R}_+ \times \mathbb{S}_1$  to  $\mathbb{R}^2 - 0$  at each  $x$ . Therefore,  $\vartheta$  and  $\sigma$  are uniquely determined by  $\xi$ . This means that the singularity of  $f$  at  $(x, \xi)$  propagates to a singularity of  $R_J f$  at

$$(s(x, \vartheta), \vartheta, \sigma) \in \mathbb{R} \times \mathbb{S}^1 \times \mathbb{R}_+ - 0.$$

Note that the singularity in the latter space is uniquely defined. This should be contrasted to the very different result we obtained for the singular supports, stating that a singularity at  $x$  may propagate into singularities for all  $(s(x, \vartheta), \vartheta)$  with  $\vartheta \in \mathbb{S}^1$ .

We have to worry about a slight additional complication here. The above singularity in the phase space of the Radon domain is uniquely defined for  $\sigma > 0$ . It is also uniquely defined for  $\sigma < 0$ . So it turns out that a singularity at  $(x, \xi)$  actually propagates into two singularities in  $(s, \vartheta, \xi_s, \xi_\vartheta)$ . This is related to the fact that the parameterization  $(s, \vartheta) \in \mathbb{R} \times \mathbb{S}^1$  is a double covering of the space of lines: a line parameterized by  $(s, \vartheta)$  is the same line as the one parameterized by  $(-s, -\vartheta)$ . The two corresponding curves are different in general. But we also see in the setting of integrals along curves that one singularity propagates to two places.

At any rate, we observe that the notion of Wave Front Set is crucial to our understanding of the propagation of singularities in generalized ray transforms.

What happens to the adjoint operator  $R_J^*$ ? Its distribution kernel is

$$K'(x, s, \vartheta) = \int_{\mathbb{R}} e^{i(s-s(x, \vartheta))\sigma} \frac{J(x, \vartheta)}{2\pi} d\sigma.$$

Therefore,

$$WF'(K') = \left\{ (x, s(x, \vartheta), \vartheta, -\sigma \nabla_x s(x, \vartheta), -\sigma, \sigma \partial_\vartheta s(x, \vartheta)) \right\}.$$

Let us now assume that we have a singularity at point  $(s, \vartheta, \zeta_s, \zeta_\vartheta)$ . We have seen that  $(s(x, \vartheta), \partial_\vartheta s(x, \vartheta))$  uniquely characterizes  $x$ . This is a property of the curves. This implies that  $x$  is uniquely determined by knowledge of  $(s, \vartheta, \zeta_s, \zeta_\vartheta)$ . And so is  $\xi = -\sigma \nabla_x s(x, \vartheta)$ . We thus obtain again that singularities in the Radon domain corresponded to uniquely determined singularities in the spatial domain. Moreover, the two singularities generated by a single  $(x, \xi)$  both back-propagate to  $(x, \xi)$  by composition with  $WF'(K')$ .

We can therefore apply the operator  $R_J^* R_J$ , and observe by composition of the previous two propagations that a singularity at  $(x, \xi)$  is mapped into the same singularity at  $(x, \xi)$ . This is consistent with the fact that  $R_J^* R_J$  is a pseudo-differential operator with a phase that can be recast as  $(x - y) \cdot \xi$ .

**Strength of the propagating singularities.** In the above discussion, the phase  $\phi$  was seen as the main player responsible for the propagation of singularities in  $X \times \mathbb{R}^n - 0$  to singularities in  $Y \times \mathbb{R}^m - 0$ . However, does such transfer always occur? This depends on the amplitude  $a(x, y, \zeta)$ . Let us recall the form of the kernel

$$K(x, y) = \int e^{i\phi(x, y, \zeta)} a(x, y, \zeta) d\zeta.$$

$\phi$  is a function in a  $(n + m + N)$ -dimensional manifold. Then  $(x, y, \phi'_x, \phi'_y)$  is defined in a  $(n + m + N)$ -dimensional manifold. However, we restrict ourselves to the submanifold where  $\phi'_\zeta = 0$ , which imposes  $N$  constraints. Therefore,  $\{(x, y, \phi'_x, \phi'_y), \phi'_\zeta = 0\}$  a priori is defined in a  $(n + m)$ -dimensional manifold. This is half the dimension of the product space  $X \times \mathbb{R}^n \times Y \times \mathbb{R}^m$ . This forms what is called a Lagrangian manifold. And if we

want a one-to-one correspondence between singularities (or a one-to-two correspondence as we have seen is the case for the ray transform), then  $m = n$ . For our purpose, this means that once a singularity is known at  $(y_0, \xi = \phi'(y_0))$ , then this also defines  $x_0$  and  $\zeta_0$  on that Lagrangian manifold.

We now want to know whether  $a(x_0, y_0, \zeta_0)$  vanishes or not. If it vanishes, then the singularity does not propagate to the  $x$  variables. If  $a(x_0, y_0, \zeta_0) \neq 0$ , then the singularity propagates to the  $x$  variables. Moreover, the strength of  $a(x_0, y_0, \zeta_0)$  tells us how the singularities are attenuated (or amplified, though this is rarely the case in practical inverse problems) by the transform.

There is no simple explanation for the amount of smoothing obtained by FIOs. We describe some generic rules in a very formal and introductory manner. As a general procedure, let  $n$  be the dimension of  $X$  and  $Y$  and let  $N$  be the dimension of the fiber variable  $\zeta$ . For the GRT, this is  $n = 2$  and  $N = 1$ . Then we define a symbol  $a(x, y, \zeta)$  as an element in  $S^{m+\frac{1}{4}(2n-N)}(X \times Y \times \mathbb{R}^N)$ . This strange normalization means that a symbol that looks non-smoothing with  $m + \frac{1}{4}(2n - 2N) = 0$  as we have in the GRT, in fact is an operator smoothing by  $-m$  derivatives with  $m = \frac{1}{2}(N - n)$ , which is  $m = -\frac{1}{2}$  for the Radon transform. This is the reason why we have to multiply the symbol  $J(x, \vartheta)$  by  $\sqrt{|\sigma|}$  to obtain an operator of order 0 (i.e., bounded from  $L^2$  to  $L^2$ ).

**Summary.** So, to summarize this discussion, we have obtained that the phase  $\phi(x, y, \zeta)$  dictates how singularities propagate from  $(y, \eta)$  to  $(x, \xi)$  via the canonical relation  $(x, \phi'_x, y, -\phi'_y)$  defined on the manifold  $\phi'_\zeta = 0$ . On that manifold, for  $(y, \eta)$  known, then  $\zeta$  is also known so that  $(x, y, \zeta)$  is known and thus so is  $a(x, y, \zeta)$ . When the latter does not vanish, then the singularity does propagate. When  $a(x, y, \zeta)$  is strictly positive for all such points  $C_\phi$ , then we say that the FIO is *elliptic*. We are then guaranteed that all singularities will propagate. These singularities are then typically attenuated in the sense that the amplitude of the singularity is multiplied by a constant times  $|\zeta|^m$  (asymptotically for large  $\zeta$ ). What the value of  $m$  is is dictated by  $a(x, y, \zeta)$ , which is a symbol in  $S^{m+\frac{n}{2}-\frac{N}{4}}$ . Again, for the GRT, this means that  $m = -\frac{1}{2}$ . Singularities are attenuated by  $|\zeta|^{\frac{1}{2}}$  by the GRT and then attenuated by  $|\zeta|^{\frac{1}{2}}$  by the adjoint transform. Therefore,  $R_J^* R_J$  has to be multiplied by  $(-\Delta)^{\frac{1}{2}}$  or replaced by  $R_J^* \Lambda R_J$  in order to define an operator that is bounded from  $L^2$  to  $L^2$  and in which singularities are neither amplified nor attenuated.



# Chapter 4

## Inverse wave problems

In the preceding two chapters, the probing mechanisms were based on particles: propagation of X-rays in CT and propagation of particles in media with varying indices of refraction in the generalized Radon transform. All these inverse problems were seen to display good stability properties. We have seen that reconstructions based on M.R.I. also enjoyed good stability properties. The third major way of probing unknown domains with good stability reconstructions is to use *waves*. In some sense, particles may be seen as high frequency wave packets with a frequency that is much larger than any other scale in the problem. There is also considerable interest in considering waves with lower, but still large, frequencies. Unsurprisingly, the propagation of such waves is modeled by wave equations. Three typical partial differential equations are then used. A scalar wave equation models the propagation of acoustic waves, the system of Maxwell's equations models the propagation of electromagnetic waves, and the system of elasticity models the propagation of elastic waves. In this chapter, we restrict ourselves to simplified scalar models for the three types of waves.

As waves propagate, they interact with the medium of interest because of variations in a spatially varying parameter called the sound speed (or light speed, or speed of elastic waves; but since we restrict ourselves to a scalar model, we call such a “speed” *sound speed*). The objective of inverse wave problems is therefore typically the reconstruction of such a sound speed from available measurements at the boundary of the domain. This reconstruction can be performed from time-dependent measurements or from frequency measurements. The theories of reconstructions of sound speeds, called inverse scattering theories, can be fairly involved mathematically. In this chapter, we consider several relatively simple, but still representative, inverse wave problems.

The first problem is a linearization of the inverse scattering problem in one dimension of space with time-dependent measurements. The second inverse problem is a linearization of the inverse scattering problem with measurements at the domain's boundary for one (sufficiently high) frequency. The third problem is an inverse source problem consisting of reconstructing the initial condition in a wave equation from spatio-temporal measurements of wave fields at the boundary of a domain enclosing the support of the initial condition. This inverse source problem finds applications in the medical imaging modality Photo-acoustic Tomography, which will be further analyzed in a later chapter. The fourth and final problem is a nonlinear inverse coefficient problem in one dimension of space.

This selection of inverse problems aims to demonstrate the following statement: Inverse problems based on (sufficiently high-frequency) waves that propagate in not-too-scattering environments involve measurement operators with good stability properties. Reconstructions for such inverse problems are therefore typically high resolution. Although such features remain true in higher spatial dimensions and for more general non-linear inverse problems, the geometric descriptions and mathematical analyses can become extremely complex. For general references on inverse scattering and related inverse problems, we refer the reader to, e.g., [28, 30, 39].

## 4.1 One dimensional inverse scattering problem

In this section, we consider an example of a well-posed inverse problem, i.e., an inverse problem with a Lipschitz stability in the  $L^2$  sense. Inverse problems related to the wave equation are often of this nature. Indeed, the wave equation “propagates” singularities without dampening them, as does the Fourier transform for instance. Here, we consider a simple one dimensional wave equation and a linearization of the inverse scattering problem.

Let us consider the one dimensional wave equation

$$\frac{1}{c_s^2(x)} \frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} = \delta(t)\delta(x - x_s), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}, \quad (4.1)$$

with delta source term at time  $t = 0$  and position  $x = x_s$ . Here,  $c_s$  is the unknown sound speed, which takes a constant value  $c = c_s$  for  $|x| \geq R > 0$ . We assume causality so that  $p(x, t; x_s) = 0$  for  $t < 0$  and assume that  $p$  is bounded. We measure  $p(x_s, t; x_s)$  as the domain’s boundary as a function of time and want to reconstruct the unknown profile  $c_s(x)$ .

It is convenient to analyze the problem in the frequency domain. Let us define  $u(x, \omega; x_s)$  the *causal Fourier transform* of  $p(x, t; x_s)$  in the time variable

$$u(x, \omega; x_s) = \int_0^\infty p(x, t; x_s) e^{i\omega t} dt. \quad (4.2)$$

This transform can be inverted as follows:

$$p(x, t; x_s) = \frac{1}{2\pi} \int_{-\infty}^\infty u(x, \omega; x_s) e^{-i\omega t} d\omega. \quad (4.3)$$

The equation for  $u(x, \omega; x_s)$  is the well-known Helmholtz equation

$$\frac{d^2 u}{dx^2} + \frac{\omega^2}{c_s^2(x)} u = -\delta(x - x_s), \quad \omega \in \mathbb{R}, \quad x \in \mathbb{R}, \quad (4.4)$$

augmented with the following *radiation conditions*

$$\frac{du}{dx} \mp \frac{i\omega}{c} u \rightarrow 0, \quad \text{as } x \rightarrow \pm \infty. \quad (4.5)$$

Since  $p(x_s, t; x_s)$  is measured, then  $u(x_s, \omega; x_s)$  is known by taking the Fourier transform of the available data.

Let us make a few assumptions. We assume that  $c_s(x)$  is known on  $(-\infty, x_s)$  (in Earth profile reconstructions, one is interested in positive depths only) and that  $c_s(x)$  is close to the background sound speed  $c$  on  $(x_s, \infty)$  in the sense that

$$\frac{1}{c_s^2(x)} = \frac{1}{c^2}(1 - \alpha(x)), \quad (4.6)$$

where  $\alpha(x)$  is small compared to 1. In effect we *linearize* the problem of the reconstruction of  $c_s(x)$  from the scattering measurements  $u(x_s, \omega; x_s)$ .

The advantage is that the resulting problem is linear, relatively straightforward to invert and admits an explicit solution for small  $\alpha(x)$ . Let us define by  $u_i$  ( $i$  for incident) the solution of the unperturbed problem

$$\frac{d^2 u_i}{dx^2} + \frac{\omega^2}{c^2} u_i = \delta(x - x_s), \quad \frac{du_i}{dx} \mp \frac{i\omega}{c} u_i \rightarrow 0, \quad \text{as } x \rightarrow \pm \infty. \quad (4.7)$$

The solution to the above problem is nothing but the *Green's function* of the Helmholtz equation with constant coefficients. It is given explicitly by

$$u_i(x, \omega; x_s) = g(x - x_s, \omega) = \frac{ce^{i\frac{\omega}{c}|x-x_s|}}{2i\omega}. \quad (4.8)$$

**Exercise 4.1.1** *Verify the above formula for the Green's function (and verify that the radiation condition is satisfied).*

Let us now decompose the Helmholtz solution as the superposition of the incident field and the *scattered field*:

$$u(x, \omega; x_s) = u_i(x, \omega; x_s) + u_s(x, \omega; x_s).$$

From the equations for  $u$  and  $u_i$ , we verify that  $u_s$  satisfies the following equation

$$\frac{d^2 u_s}{dx^2} + \frac{\omega^2}{c^2} u_s = \frac{\omega^2}{c^2} \alpha(x) (u_i + u_s), \quad (4.9)$$

with appropriate radiation conditions. By the principle of superposition, this implies that

$$u_s(x, \omega; x_s) = \omega^2 \int_{x_s}^{\infty} \frac{\alpha(y)}{c^2} (u_s + u_i)(y, \omega; x_s) g(x - y, \omega) dy. \quad (4.10)$$

So far, we have not used the assumption that  $\alpha(x)$  was small. This approximation, called the Born approximation, allows us to deduce from the above equation that  $u_s$  is also of order  $\alpha$ . This implies that  $\alpha u_s$  is of order  $\alpha^2$ , hence much smaller than the other contributions in (4.10). So neglecting  $u_s$  on the right hand side of (4.10) and replacing  $u_i$  and  $g$  by their expression in (4.8), we deduce that a good approximation of  $u_s$  is

$$u_s(x_s, \frac{ck}{2}; x_s) = \int_{\mathbb{R}} \frac{\alpha(x)}{4} e^{ikx} dx, \quad k \in \mathbb{R}. \quad (4.11)$$

This implies that the scattering data  $u_s(x_s, \omega; x_s)$  uniquely determines the fluctuation  $\alpha$  and that the reconstruction is *stable*: all we have to do is to take the inverse Fourier transform of  $u_s$  to obtain  $\alpha(x)$ . Namely, we have

$$\alpha(x) = \frac{2}{\pi} \int_{\mathbb{R}} e^{-ikx} u_s(x_s, \frac{ck}{2}; x_s) dk. \quad (4.12)$$

Several assumptions have been made to arrive at this result. However as was the case with the MRI problem, we obtain in the end a very simple reconstruction procedure: all we have to do is to compute an inverse Fourier transform.

We can summarize the above result as follows.

**Theorem 4.1.1** *Let  $u_i$  be given by (4.8) and  $u_s$  be the solution of*

$$\frac{d^2 u_s}{dx^2} + \frac{\omega^2}{c^2} u_s = \frac{\omega^2}{c^2} \alpha(x) u_i, \quad (4.13)$$

for  $\alpha$  supported in  $(x_s, R)$  and such that the radiation conditions (4.5) hold. Consider the measurement operator

$$\alpha \mapsto \mathfrak{M}(\alpha) = u_s(x_s, \cdot; x_s), \quad (4.14)$$

mapping  $\alpha(x) \in L^2(x_s, R)$  to  $\omega \mapsto u_s(x_s, \omega; x_s) \in L^2(\mathbb{R})$ . Then the measurement operator uniquely determines  $\alpha$  by means of the explicit inversion formula (4.12). The inversion is Lipschitz stable in the sense that

$$\|\alpha - \tilde{\alpha}\|_{L^2(x_s, R)} \leq C \|\mathfrak{M}(\alpha) - \mathfrak{M}(\tilde{\alpha})\|_{L^2(\mathbb{R})}, \quad (4.15)$$

for a constant  $C > 0$ .

**Exercise 4.1.2** *Write the proof of the above theorem in detail. State a similar theorem for a linearization of the wave equation in the time domain (4.1).*

## 4.2 Linearized Inverse Scattering problem

In the preceding section, we analyzed the linearization of an inverse scattering problem in one spatial dimension. In this section, we consider extensions to two and three dimensions of space. The model is again a wave equation written in the frequency domain (a Helmholtz equation). Unlike the one dimensional setting, in dimension two or more, plane waves with a given frequency are sufficient to uniquely determine (the low frequency component of) a sound speed. This is the setting that we present here.

### 4.2.1 Setting and linearization

Let us consider the wave equation in dimension  $n = 2, 3$  given by

$$\frac{1}{c_s^2(x)} \frac{\partial^2 p}{\partial t^2} - \Delta p = 0, \quad x \in \mathbb{R}^n, \quad t > 0, \quad (4.16)$$

with appropriate initial conditions. The velocity  $c_s(x)$  is the unknown parameter. Let us assume that  $c_s(x) = c$  for  $|x| > R$  for a given radius  $R > 0$ . We assume that  $p = 0$  for  $t < 0$  and as in the preceding section, pass to the frequency domain by introducing

$$u(x, \omega) = \int_0^\infty e^{i\omega t} p(x, t) dt, \quad p(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\omega t} u(x, \omega) d\omega. \quad (4.17)$$

The equation for  $u$  is then the following *Helmholtz* equation

$$\begin{aligned} (\Delta + \frac{\omega^2}{c_s^2(x)})u(x, \omega) &= 0, \quad x \in \mathbb{R}^n, \quad \omega \in \mathbb{R}, \\ \hat{x} \cdot \nabla u(x, \omega) - i\frac{\omega}{c}u(x, \omega) &= o(|x|^{-(n-1)/2}). \end{aligned} \quad (4.18)$$

As usual  $\hat{x} = \frac{x}{|x|}$  and the second equation is the *radiation* condition, which ensures that no energy comes from infinity (only waves radiating *out* are allowed at infinity). The notation  $o(x)$  means a quantity such that  $o(x)/x \rightarrow 0$  as  $0 < x \rightarrow 0$ . So the decay at infinity should be faster than  $|x|^{-1/2}$  in two dimensions and faster than  $|x|^{-1}$  in three dimensions.

Let us now introduce the following linearization for the velocity and the frequency:

$$\frac{1}{c_s^2(x)} = \frac{1}{c^2}(1 - \alpha(x)), \quad k = \frac{\omega}{c}. \quad (4.19)$$

We recast the Helmholtz equation as

$$\begin{aligned} (\Delta + k^2)u(x, \omega) &= \alpha(x)k^2u(x, \omega), \\ \hat{x} \cdot \nabla u(x, \omega) - ik u(x, \omega) &= o(|x|^{-(n-1)/2}). \end{aligned} \quad (4.20)$$

Let  $\vartheta \in \mathbb{S}^n$  be a unit vector. We verify that

$$(\Delta + k^2)u_i(x, \omega; \vartheta) = 0, \quad \text{where} \quad u_i(x, \omega; \vartheta) = e^{ik\vartheta \cdot x}. \quad (4.21)$$

Thus plane waves with the right wavenumber  $k = |k\vartheta|$  are solutions of the homogeneous Helmholtz equation. Notice however that they *do not* satisfy the radiation conditions (they do radiate out in the direction  $\vartheta$  but certainly not in the direction  $-\vartheta$  since they *come* from infinity in that direction).

The forward problem we are interested in is the following: we have a probing plane wave coming from infinity and want to find a solution  $u_s(x, \omega)$  modeling the response of the system to the probing. Therefore we impose that  $u_s$  does not radiate at infinity (i.e., satisfies the radiation condition) and that the whole field  $u = u_i + u_s$  satisfies the Helmholtz equation. We thus end up with the following *scattering* problem

$$\begin{aligned} (\Delta + k^2)u_s(x, \omega) &= \alpha(x)k^2(u_s + u_i)(x, \omega), \\ \hat{x} \cdot \nabla u_s(x, \omega) - ik u_s(x, \omega) &= o(|x|^{-(n-1)/2}). \end{aligned} \quad (4.22)$$

In the above equation we have used (4.21). Under general assumptions on  $\alpha(x)$ , the above equation admits a unique solution [30]. The *inverse scattering* problem consists

then of reconstructing  $\alpha(x)$  from measurements of  $u_s$  at infinity in all possible directions  $\hat{x}$  for all possible incoming plane waves  $\vartheta \in S^n$ . We will not be concerned with this general problem and refer the reader to [30] for more details.

Instead, we concentrate on the linearization of the inverse scattering problem about the constant velocity profile  $c_s(x) = c$ . Let us assume that  $\alpha$  is small (in some appropriate sense that we do not describe further). As a consequence,  $u_s$  is also small as can be seen from (4.22). We therefore neglect the term  $\alpha u_s$ , which is second-order in  $\alpha$ . This approximation, as in the preceding section, is called the Born approximation. It also has the advantage of linearizing the problem of reconstructing  $\alpha(x)$  from scattering measurements, which will be described in detail below. We are thus now concerned with the linearized problem

$$\begin{aligned} (\Delta + k^2)u_s(x, \omega) &= \alpha(x)k^2u_i(x, \omega), \\ \hat{x} \cdot \nabla u_s(x, \omega) - ik u_s(x, \omega) &= o(|x|^{-(n-1)/2}). \end{aligned} \quad (4.23)$$

This equation can be solved explicitly as

$$u_s(x, \omega) = k^2 \int_{\mathbb{R}^n} \alpha(y)u_i(y, \omega)g_n(x - y)dy, \quad (4.24)$$

where  $g_n$  is the Green function, solution of the following equation

$$\begin{aligned} (\Delta + k^2)g_n(x) &= \delta(x) \\ \hat{x} \cdot \nabla u_s(x, \omega) - ik u_s(x, \omega) &= o(|x|^{-(n-1)/2}), \end{aligned} \quad (4.25)$$

and is given for  $n = 2, 3$  by

$$g_2(x) = \frac{i}{4}H_0(k|x|), \quad g_3(x) = \frac{e^{ik|x|}}{4\pi|x|}. \quad (4.26)$$

Here,  $H_0$  is the 0th order Hankel function of the first kind, given by

$$H_0(k|x|) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{\sqrt{k^2 - p^2}} e^{i(px + \sqrt{k^2 - p^2}y)} dp, \quad (4.27)$$

where we have decomposed  $x = (x, y)$  in Cartesian coordinates.

## 4.2.2 Far field data and reconstruction

The measurements we consider in this section are the far field scattering data. They correspond to the scattered waves propagating outwards at infinity. This simplification amounts to saying that the other component of the radiating field, composed of the *evanescent* waves, is not measured. Mathematically, we consider the asymptotic limit of  $u_s$  as  $|x| \rightarrow \infty$ . Let us consider the three dimensional case first. Since  $x$  goes to infinity, then  $|x - y|$  is equal to  $|x|$  plus a smaller order correction. So we have

$$u_s(x, \omega) = \frac{k^2}{4\pi|x|} \int_{\mathbb{R}^3} \alpha(y)e^{ik\vartheta \cdot y} e^{ik|x-y|} dy + \text{l.o.t.} .$$

Upon using the following approximation

$$|x - y| = |x| \left| \hat{x} - \frac{y}{|x|} \right| = |x| \left( 1 + \frac{|y|^2}{|x|^2} - 2 \frac{\hat{x} \cdot y}{|x|} \right)^{\frac{1}{2}} = |x| - \hat{x} \cdot y + \text{l.o.t.},$$

we obtain

$$u_s(x, \omega) = \frac{k^2 e^{ik|x|}}{4\pi|x|} \int_{\mathbb{R}^3} \alpha(y) e^{ik(\vartheta - \hat{x}) \cdot y} dy + \text{l.o.t.} .$$

We thus observe that

$$\begin{aligned} u_s(x, \omega) &= \frac{k^2 e^{ik|x|}}{4\pi|x|} A(\hat{x}) + o\left(\frac{1}{|x|}\right), \\ A(\hat{x}) &= \hat{\alpha}(k(\hat{x} - \vartheta)) = \int_{\mathbb{R}^3} \alpha(y) e^{ik(\vartheta - \hat{x}) \cdot y} dy. \end{aligned} \tag{4.28}$$

Recall that  $\omega = ck$ . So for a plane wave at a given frequency  $\omega$ , i.e., at a given wavenumber  $k$ , and direction  $\vartheta$ , the far field measurement is  $A(\hat{x}) = A(\hat{x}; k, \vartheta)$  in the direction  $\hat{x}$  (obtained by multiplying the measured signal by  $4\pi|x|e^{-ik|x|}k^{-2}$ ).

In two space dimensions ( $n = 2$ ), the final result is similar in the sense that  $u_s$  is proportional to  $|x|^{-\frac{1}{2}}$  at infinity with a coefficient of proportionality  $A(\hat{x})$  taking the same expression as given in (4.28).

The measurement operator we consider in this section thus takes the following form

$$L^\infty(B(0, R)) \ni \alpha \mapsto \mathfrak{M}[\alpha] = A(\cdot; k, \cdot) \in L^\infty(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}), \tag{4.29}$$

which to a (say bounded) sound speed fluctuation  $\alpha(x)$  associates the far field measurement  $(\hat{x}, \vartheta) \mapsto A(\hat{x}; k, \vartheta)$ , which is also bounded as the Fourier transform of a bounded, compactly supported, function. Note that  $\alpha$  lives in the  $n$ -dimensional space  $B(0, R)$  whereas  $A$  for a fixed frequency  $k$  lives in the  $2(n-1)$ -dimensional space  $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ . The latter dimensions agree when  $n = 2$  whereas in dimensions  $n \geq 3$ , the far field data are *richer* than the unknown object  $\alpha$  since  $2(n-1) > n$  then. Note that within the (Born) approximation of linearization, the measurement operator again provides information about  $\alpha$  in the Fourier domain since

$$\mathfrak{M}[\alpha](\hat{x}, \vartheta) = \hat{\alpha}(k(\vartheta - \hat{x})). \tag{4.30}$$

Each measurement  $(\hat{x}, \vartheta)$  provides different information about the Fourier transform of the velocity fluctuation  $\alpha(x)$ . We distinguish two types of measurements. The first ones correspond to directions of measurements  $x$  such that  $x \cdot \vartheta > 0$ . These measurements are called *transmission* measurements since they correspond to the radiated wave that have passed through the object we wish to image. The second ones correspond to the directions such that  $x \cdot \vartheta < 0$ . They are called *reflection* measurements.

**Transmission inverse scattering.** Let us consider transmission measurements first, with  $\hat{\alpha}(k(\hat{x} - \vartheta))$  known for  $\hat{x} \cdot \vartheta > 0$ . In particular we obtain for  $\hat{x} = \hat{\vartheta}$  the value  $\hat{\alpha}(0)$ , which is the average of the fluctuation  $\alpha(x)$  over the whole domain. More generally as  $\hat{x}$  varies in  $\mathbb{S}^1$  such that  $\hat{x} \cdot \vartheta > 0$ , we obtain  $\hat{\alpha}(\xi)$  over a half-circle passing through 0, of radius  $k$  and symmetric about the axis  $\vartheta$ . As  $\vartheta$  varies on the unit circle, we observe

that  $\hat{\alpha}(k(\hat{x} - \vartheta))$  fills the disk of radius  $\sqrt{2}k$ . At a fixed value of  $k$ , this is therefore all we can get:  $\hat{\alpha}(\xi)$  for  $\xi$  such that  $|\xi| \leq \sqrt{2}k$ .

The picture in three dimensions is very similar: for a given  $\vartheta \in S^2$ , we have access to  $\hat{\alpha}(k)$  for  $k$  on a half-sphere of radius  $\sqrt{2}k$  passing through 0 and invariant by rotation about  $\vartheta$ . As  $\vartheta$  varies over the sphere  $S^2$ , we thus get  $\hat{\alpha}(k)$  for all  $k$  such that  $|k| \leq \sqrt{2}k$ , as in the two-dimensional case.

The diffraction inverse problem is therefore *not injective*. All we can reconstruct from the measured data is a low-pass filter of the object  $\alpha(x)$ . The high frequencies are not measured. The high frequencies of  $\alpha$  are encoded in the radiation field  $u_s$ . However they are the evanescent part of the waves. They decay therefore much more rapidly than  $|x|^{-1}$  (when  $n = 3$ ), actually exponentially, and thus cannot be measured accurately in practice.

Let us now consider reconstruction formulas. Since frequencies above  $\sqrt{2}k$  cannot be reconstructed, we assume that

$$\alpha(x) = (\mathcal{F}_{\xi \rightarrow x}^{-1} \chi_{\sqrt{2}k}(\xi) \mathcal{F}_{x \rightarrow \xi} \alpha)(x), \quad (4.31)$$

where  $\chi_{\sqrt{2}k}(\xi) = 1$  when  $|\xi| < \sqrt{2}k$  and 0 otherwise, i.e.  $\alpha$  does not have high wavenumbers. Note that this assumption is inconsistent with our earlier assumption that  $\alpha$  was supported in  $B(0, R)$ . We do not deal with this minor technical difficulty here. Then the reconstruction is obviously unique according to what we just saw. Let us consider the two-dimensional case. We want to reconstruct  $\alpha(x)$  from  $\hat{\alpha}(k(\hat{x} - \vartheta))$ , where  $\hat{x}$  and  $\vartheta$  run over the unit circle  $S^1$ . The inverse Fourier transform tells us that

$$\alpha(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix \cdot \xi} \hat{\alpha}(\xi) d\xi = \frac{k^2}{(2\pi)^2} \int_0^{2\pi} \int_0^{\sqrt{2}} e^{ik\rho x \cdot \vartheta} \hat{\alpha}(k\rho\vartheta) \rho d\rho d\theta.$$

Observe that as  $\vartheta$  covers the unit circle, all points of the disk  $|\xi| < \sqrt{2}k$  are covered twice as  $\hat{x}$  varies, once for a point such that  $\hat{\xi} \cdot \vartheta^\perp > 0$  and once for a point such that  $\hat{\xi} \cdot \vartheta^\perp < 0$ . Therefore the information corresponding to  $\hat{\xi} \cdot \vartheta^\perp > 0$  is sufficient. This information is parameterized as follows: for a given  $\vartheta$  we write  $\hat{x}$  as

$$\hat{x}(\phi, \vartheta) = \sin \phi \vartheta + \cos \phi \vartheta^\perp, \quad 0 \leq \phi \leq \frac{\pi}{2}. \quad (4.32)$$

We thus obtain that

$$\hat{\alpha}(k(\hat{x} - \vartheta)) = \hat{\alpha}\left(k\rho(\phi)\left(\frac{\sin \phi - 1}{\rho(\phi)}\vartheta + \frac{\cos \phi}{\rho(\phi)}\vartheta^\perp\right)\right) = \hat{\alpha}(k\rho(\phi)\mathcal{R}(\phi)\vartheta),$$

with  $\rho(\phi) = \sqrt{2}\sqrt{1 - \cos \phi}$  and  $\mathcal{R}(\phi)$  an explicitly defined rotation depending on  $\phi$ . Here is the rest of the reconstruction:

$$\begin{aligned} \alpha(x) &= \frac{k^2}{(2\pi)^2} \int_0^{2\pi} \int_0^{\pi/2} e^{ik\rho(\phi)x \cdot \vartheta} \hat{\alpha}(k\rho(\phi)\vartheta) \rho(\phi) \frac{d\rho(\phi)}{d\phi} d\phi d\theta \\ &= \frac{k^2}{(2\pi)^2} \int_0^{2\pi} \int_0^{\pi/2} e^{ik\rho(\phi)x \cdot \mathcal{R}(\phi)\vartheta} \hat{\alpha}(k\rho(\phi)\mathcal{R}(\phi)\vartheta) \frac{1}{2} \frac{d\rho^2(\phi)}{d\phi} d\phi d\theta, \end{aligned}$$

so that finally

$$\alpha(x) = \frac{2k^2}{(2\pi)^2} \int_0^{2\pi} \int_0^{\pi/2} e^{ik\rho(\phi)x \cdot \mathcal{R}(\phi)\vartheta} \hat{\alpha}(k|\hat{x}(\phi, \vartheta) - \vartheta|) \sin \phi d\phi d\theta. \quad (4.33)$$

This is the reconstruction formula we were after.

**Reflection inverse scattering.** Let us conclude the section with a few words about reflection tomography. In that case, we only measure data in directions  $\hat{x}$  such that  $\hat{x} \cdot \vartheta < 0$ . Following the same techniques as above, we see that we can reconstruct wavenumbers of  $\alpha(x)$  in the corona of wavenumbers  $\xi$  such that  $\sqrt{2}k < |\xi| < 2k$ . The reconstruction from reflection data is therefore by no means unique. We cannot reconstruct low-frequency components of  $\alpha$  and cannot reconstruct very-high frequencies either. Assuming that the wavenumber content of  $\alpha(x)$  is in the above corona, then the reconstruction is unique. A reconstruction formula similar to what we just obtained can also be derived. Notice that when both the transmission and reflection data can be measured, we can reconstruct all wavenumbers  $\xi$  of  $\alpha(x)$  such that  $|\xi| < 2k$ .

All these result are in sharp contrast to the one-dimensional example we saw in the preceding section. There, a given wavenumber  $k$  allows us to reconstruct one wavenumber of  $\alpha(x)$ . All wavenumbers are thus required (i.e. measurements for all frequencies  $\omega$ ) to reconstruct  $\alpha(x)$ . Here  $\alpha(x)$  is also uniquely determined by measurements obtained for all values of  $k$  (since each value of  $k$  allows us to reconstruct all wavenumbers  $|\xi| < 2k$ ). However because of the multidimensional nature of the measurements (the variable  $\hat{x}$  is discrete in one-dimension instead of living on the unit sphere  $\mathbb{S}^n$ ), measurements for all values of  $k$  is quite redundant: once we have obtained measurements at a given value of  $k_0$ , all measurements obtained for wavenumbers  $k < k_0$  are redundant.

### 4.2.3 Comparison to X-ray tomography

Let us consider the case of transmission data in two space dimensions. We have seen that wavenumbers of  $\alpha(x)$  up to  $\sqrt{2}k$  could be reconstructed. However as  $k$  tends to  $\infty$ , this essentially means that all wavenumbers of  $\alpha(x)$  can be reconstructed.

Indeed in that limit we observe that the half circle of radius  $k$  becomes the full line orthogonal to  $\vartheta$ . That is, as  $k \rightarrow \infty$ , the measurements tend to

$$\hat{\alpha}(\sigma\vartheta^\perp) = R\alpha\left(\sigma, \theta + \frac{\pi}{2}\right).$$

**Exercise 4.2.1** Show that the reconstruction formula (4.33) indeed converges to the inverse Radon transform as  $k \rightarrow \infty$ .

In the limit of infinite frequency, we therefore obtain that the transmission measurements tend to the Radon transform of  $\alpha$ . We have seen in Chapter 2 that the knowledge of  $R_\alpha(\sigma, \theta + \pi/2)$  for all values of  $\sigma$  and  $\theta$  was sufficient to uniquely reconstruct the fluctuation  $\alpha(x)$ .

So how should we consider the inverse diffraction problem? How ill-posed is it? As we already mentioned, the first problem with diffraction tomography is that for a fixed frequency  $\omega$ , the function  $\alpha(x)$  cannot uniquely be reconstructed. Only the wavenumbers below  $\sqrt{2}k$  (below  $2k$ ) in the case of transmission (transmission and reflection) measurements can be reconstructed. However in the class of functions  $\alpha(x) \in L^2(\mathbb{R}^2)$  such that (4.31) holds, we have uniqueness of the reconstruction. In this class we can perform a similar analysis to what we obtained in Theorem 2.2.2.

Let us consider the measurements  $d(\phi, \vartheta) = \hat{\alpha}(k(\hat{x} - \vartheta))$  for  $\vartheta \in S^1$  and  $0 \leq \phi \leq \pi/2$  using (4.32). We verify that  $1 \leq \rho'(\phi) \leq \sqrt{2}$  for  $0 \leq \phi \leq \pi/2$ .

Let us assume that the error we make is of the same order for every angle  $\phi$  and every angle  $\vartheta$ . An estimate of the total error will thus involve

$$\begin{aligned}
& \int_{S^1} \int_0^{\pi/2} |d(\phi, \vartheta)|^2 d\phi d\vartheta = \int_{S^1} \int_0^{\pi/2} |\hat{\alpha}(k\rho(\phi)\mathcal{R}(\phi)\vartheta)|^2 d\phi d\vartheta \\
& = \int_{S^1} \int_0^{\sqrt{2}} |\hat{\alpha}(k\rho\vartheta)|^2 (\rho')^{-1} d\rho d\vartheta \sim \int_{S^1} \int_0^{\sqrt{2}} |\hat{\alpha}(k\rho\vartheta)|^2 d\rho d\vartheta \\
& \sim \frac{1}{k} \int_{S^1} \int_0^{\sqrt{2}k} |\hat{\alpha}(u\vartheta)|^2 du d\vartheta \sim \frac{1}{k} \int_{S^1} \int_0^{\sqrt{2}k} \frac{|\hat{\alpha}(u\vartheta)|^2 u}{u} du d\vartheta \\
& \geq \frac{1}{k} \|\alpha\|_{H^{-\frac{1}{2}}(\mathbb{R}^2)}^2.
\end{aligned}$$

In some sense, the above formula also shows that the data  $d(\phi, \vartheta)$  are more regular than the function  $\alpha(x)$  by half of a derivative. This is consistent with the Radon transform in the limit as  $k \rightarrow \infty$ . To be more consistent with the Radon transform, notice in the limit  $k \rightarrow \infty$  that  $k \cos \phi \sim \sigma$  so that  $k \sin \phi d\phi \sim d\sigma$  as the half circle converges to the real line. Since  $\sin \phi \sim 1$  for most of the wavenumbers  $\sigma$  as  $k \rightarrow \infty$ , this implies that  $kd\phi \sim d\sigma$ . Therefore a total error in the angular measurements in diffraction tomography consistent with the measurement errors for the Radon transform is given by

$$\int_{S^1} \int_0^{\pi/2} |d(\phi, \vartheta)|^2 kd\phi d\vartheta \geq \|\alpha\|_{H^{-1/2}(\mathbb{R}^2)}^2.$$

We recover in this limit that the measurements in diffraction tomography regularize the function  $\alpha$  by half of a derivative.

Note that we see here again that the ill-posedness of a problem very much depends on the norm in which the error on the data is measured.

### 4.3 Inverse source problem in PAT

Consider the following wave equation

$$\begin{aligned}
\frac{\partial^2 p}{\partial t^2} - \Delta p &= 0, & t > 0, \quad x \in \mathbb{R}^n \\
p(0, x) &= f(x), & x \in \mathbb{R}^n \\
\partial_t p(0, x) &= 0, & x \in \mathbb{R}^n.
\end{aligned} \tag{4.34}$$

The inverse wave source problem consists of reconstructing the initial condition  $f$  supported in a open, convex, bounded domain  $X$  from knowledge of  $p(t, x)$  for  $t > 0$  and  $x \in \partial X$ . This inverse problem finds application in Photo-acoustic tomography (PAT). We will come back to the modeling of this imaging modality in Chapter 8.

#### 4.3.1 An explicit reconstruction formula for $\Sigma$ the unit sphere

Explicit inversion formulas are known when  $\Sigma = \partial X$  is a simple geometry. Explicit reconstruction procedures based on time reversal are also known for general surfaces  $\Sigma$  enclosing the support of the source  $f(x)$ , which we assume is compactly supported.

All explicit inversion formulas are based one way or another on the Fourier transform.

Let us assume that  $\Sigma$  is the unit sphere  $|x| = 1$ . Then an explicit reconstruction formula by L. Kunyanski shows that in dimension  $n = 3$ , we have

$$f(x) = \frac{1}{8\pi^2} \nabla_x \cdot \int_{|y|=1} \nu(y) \left( \frac{1}{t} \frac{d}{dt} \frac{p(y, t)}{t} \right)_{t=|y-x|} dS_y. \quad (4.35)$$

The above formula generalizes to arbitrary dimension. We shall not delve into the details of this inversion formula. Like the formula based on the Radon transform, it uses the symmetries of the Fourier transform. However, it is based on Fourier transforms on the sphere (spherical harmonics) that we do not present here. The above reconstruction shows that the source term  $f(x)$  can uniquely and stably be reconstructed from knowledge of  $u$  on the surface  $\Sigma$ .

### 4.3.2 An explicit reconstruction for detectors on a plane

In this section, we present an inversion procedure for point-wise detectors based on the use of the Fourier transform. We assume that  $f$  is compactly supported in the unit ball  $B(0, 1)$  and assume that the measurements are performed at each point of the hyperplane  $x_n = 1$ . The reconstruction procedure is essentially independent of dimension  $n \geq 2$  and is therefore presented for arbitrary dimensions. We denote by  $x' = (x_1, \dots, x_{n-1})$  and by  $z = x_n$ . We want to reconstruct  $f(x)$  from knowledge of  $p(t, x', z = 1)$  for all  $t \geq 0$  and  $x' \in \mathbb{R}^{n-1}$  knowing that

$$\left( \frac{\partial^2}{\partial t^2} - \Delta \right) p = \delta'_0(t) f(x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n.$$

We denote by  $u(\omega, \xi', z)$  the partial Fourier transform of  $p$  given by

$$u(\omega, \xi', z) = \int_{\mathbb{R}^n} e^{-i(\omega t + \xi' \cdot x')} p(t, x', z) dt dx',$$

and by  $\tilde{f}(\xi', z) = \int_{\mathbb{R}^{n-1}} e^{-i\xi' \cdot x'} f(x', z) dx'$ .

Since differentiation in  $t$  corresponds to multiplication by  $i\omega$  in the Fourier domain, with a similar expression in the  $x'$  variable, we find that

$$\left( \omega^2 - |\xi'|^2 + \frac{\partial^2}{\partial z^2} \right) u = -i\omega \tilde{f} := s(\omega, \xi', z).$$

For the reconstructions, it turns out that we need only the subset  $\omega > |\xi'|$  and define  $\theta = \sqrt{\omega^2 - |\xi'|^2}$ . The above equation is a second-order non-homogeneous equation similar to the forced harmonic oscillator. Its solution is of the form

$$u(\omega, \xi', z) = A(\omega, \xi') e^{i\theta z} + B(\omega, \xi') e^{-i\theta z} + \int_0^z s(z') \frac{\sin \theta(z - z')}{\theta} dz',$$

which we recast as

$$u(\omega, \xi', z) = \left( A + \int_0^z s(z') \frac{e^{-i\theta z'}}{2i\theta} dz' \right) e^{i\theta z} + \left( B - \int_0^z s(z') \frac{e^{i\theta z'}}{2i\theta} dz' \right) e^{-i\theta z}.$$

The support of  $f$  and hence of  $s$  is a subset of  $(-1, 1)$  in the  $z$  variable. As  $z \rightarrow \infty$ , we want only outgoing waves (no incoming waves), which imposes that

$$B(\omega, \xi') = \int_0^1 s(z') \frac{e^{i\theta z'}}{2i\theta} dz'.$$

Similarly, as  $z \rightarrow -\infty$ , the radiation condition imposes that

$$A(\omega, \xi') = \int_{-1}^0 s(z') \frac{e^{-i\theta z'}}{2i\theta} dz'.$$

At  $z = 1$ , where information is available, this means,

$$u(\omega, \xi', 1) = e^{i\theta} \int_{-1}^1 s(\omega, \xi', z) \frac{e^{-i\theta z'}}{2i\theta} dz' = -\frac{\omega}{2\theta} e^{i\theta} \int_{\mathbb{R}} e^{-i\theta z} \tilde{f}(\xi', z) dz.$$

Since  $f$  is real valued, then  $\tilde{f}^*(-\xi', z) = \tilde{f}(\xi', z)$ . Let us therefore define

$$v(\theta, \xi') = \begin{cases} \frac{-2\theta}{\sqrt{\theta^2 + |\xi'|^2}} u(\sqrt{\theta^2 + |\xi'|^2}, \xi', 1) e^{-i\theta} & \theta > 0 \\ \frac{2\theta}{\sqrt{\theta^2 + |\xi'|^2}} u(-\sqrt{\theta^2 + |\xi'|^2}, -\xi', 1) e^{-i\theta} & \theta < 0. \end{cases}$$

Then we find that

$$v(\theta, \xi') = \int_{\mathbb{R}} e^{-i\theta z} \tilde{f}(\xi', z) dz := \hat{f}(\xi)$$

is the Fourier transform of  $f$  in all variables.

Note that only the frequencies  $|\omega| > |\xi'|$  are used in the reconstruction. This corresponds to the propagating modes. The frequencies  $|\omega| < |\xi'|$  are still measured but are not necessary in the reconstruction. These evanescent modes decay exponentially in  $z$  and their use might in fact yield unstable reconstructions.

Using the Parseval relation and the fact that  $|\theta|d\theta = |\omega|d\omega$ , we have

$$\|f\|_{L^2(B(0,1))} = c \|\hat{f}\|_{L^2(\mathbb{R}^n)} = c \|v\|_{L^2(\mathbb{R} \times \mathbb{R}^{n-1})} \leq C \|u|_{z=1}\|_{L^2(\mathbb{R} \times \mathbb{R}^{n-1})} \leq C \|u|_{z=1}\|_{L^2(\mathbb{R} \times \mathbb{R}^{n-1})}$$

since  $|\frac{\theta}{\omega}| \leq 1$ . Consequently, we find that

$$\|f\|_{L^2(B(0,1))} \leq C \|p(t, x', 1)\|_{\mathbb{R} \times \mathbb{R}^{n-1}}.$$

The inverse wave problem is therefore injective and well posed in the  $L^2$  sense. This means that there exists a constant  $C > 0$  such that

$$\|f\|_{L^2(B(0,1))} \leq C \|\mathfrak{M}f\|_{L^2(\mathbb{R}^n)}.$$

The above inverse wave problem is therefore stable. Measurement errors are not amplified too drastically during the reconstruction and the inverse wave source problem is consequently a high resolution inverse problem.

## 4.4 One dimensional inverse coefficient problem

The three inverse problems considered so far in this chapter were linear inverse problems. The first two inverse scattering problems were defined as the linearization of nonlinear inverse scattering problems. The last inverse source problem is linear without any approximation. In this section, we consider a one-dimensional inverse coefficient problem, which is nonlinear and treated as such. This nonlinear inverse problem enjoys the same properties as their linear counterparts: Wave equations propagate singularities and thus many measurement operators based on solutions of wave equations are Lipschitz stable.

Consider the equation

$$\begin{aligned} \frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} - q(x)p &= 0, & t > 0, \quad x > 0 \\ p(t, 0) &= f(t), & t > 0, \\ p(0, x) &= \partial_t p(0, x) = 0 & x > 0. \end{aligned} \tag{4.36}$$

This problem can equivalently be posed for  $t \in \mathbb{R}$  by assuming that the source term  $f(t)$  is supported in  $t \in \mathbb{R}_+$ . The solution  $p$  then vanishes for  $t \leq 0$  by finite speed of propagation. For the same reason, the solution is supported in the domain  $t \geq x \geq 0$ .

Here,  $q(x)$  is an unknown bounded potential. We then assume that  $g(t) = \partial_x p(t, 0) + f'(t)$  is measured at the domain's boundary. This defines the measurement operator

$$L^\infty(0, L) \ni q \mapsto \mathfrak{M}_f(q) = g \in L^\infty(0, 2L). \tag{4.37}$$

The objective is to show that  $g(t)$  for  $t \in (0, 2L)$  uniquely determines  $q(x)$  on  $(0, L)$  under some conditions on  $f(t)$ . Moreover, depending on the structure of  $f(t)$  at  $t = 0$ , we obtain different stability estimates showing that the reconstruction of  $q(x)$  is a well posed problem in appropriate topologies.

**Cauchy problems at  $x = 0$  and  $t = 0$ .** Let us define  $p_1(t, x) = p(t, x) - f(t - x)$ , where  $f(t - x)$  is seen to be the solution of the above problem when  $q \equiv 0$ . Then we find that

$$\begin{aligned} \frac{\partial^2 p_1}{\partial t^2} - \frac{\partial^2 p_1}{\partial x^2} &= q(x)p(t, x) := s(t, x) & t \in \mathbb{R}, \quad x > 0 \\ p_1(t, 0) &= 0, \quad \frac{\partial p_1}{\partial x}(t, 0) = g(t), & t \in \mathbb{R} \\ p_1(0, x) &= \frac{\partial p_1}{\partial t}(0, x) = 0 & x > 0. \end{aligned} \tag{4.38}$$

The above problem may be seen in two ways. It can either be seen as a Cauchy problem for  $x > 0$  and  $t \in \mathbb{R}$  with Cauchy data at  $x = 0$ . The solution is then given by

$$p_1(t, x) = \frac{1}{2} \int_{t-x}^{t+x} g(\tau) d\tau + \frac{1}{2} \int_{\Delta(t,x)} q(y)p(\tau, y) d\tau dy, \tag{4.39}$$

where we have defined the triangular-shaped area:

$$\Delta(t, x) = \{(\tau, y), \quad 0 < y < x, \quad t - (x - y) < \tau < t + (x - y)\}. \tag{4.40}$$

Alternatively, we can write the above problem as a Cauchy problem with data at  $t = 0$ . We need to ensure that the constraint  $p_1(t, 0) = 0$  is satisfied. In order to do so, let us define  $f(t, x) = f(t - x)$  for  $t > 0$  and  $x > 0$  and  $f(t, x) = -f(t + x)$  for  $t > 0$  and  $x < 0$  and let us extend the potential so that  $q(x) := q(-x)$  for  $x < 0$ . We then recast the wave equation for  $p_1$  as

$$\begin{aligned} \frac{\partial^2 p_1}{\partial t^2} - \frac{\partial^2 p_1}{\partial x^2} &= q(x)(f(t, x) + p_1(t, x)) & t > 0, x \in \mathbb{R} \\ p_1(0, x) &= \frac{\partial p_1}{\partial t}(0, x) = 0 & x \in \mathbb{R}. \end{aligned} \quad (4.41)$$

The solution to that equation then satisfies the constraint  $p(t, -x) = -p(t, x)$  and with  $s(t, x) := q(x)(f(t, x) + p_1(t, x))$  is given explicitly by

$$p_1(t, x) = -\frac{1}{2} \int_{\tilde{\Delta}(t, x)} s(\tau, y) dy d\tau, \quad (4.42)$$

where  $\tilde{\Delta}(t, x) = \{(\tau, y), 0 < \tau < t, x - (t - \tau) < y < x + (t - \tau)\}$ .

**Exercise 4.4.1** Check the above two formulas for  $p_1(t, x)$ .

We thus obtain again that

$$p_1(t, x) = -\frac{1}{2} \int_{\tilde{\Delta}(t, x)} q(y) [f(\tau, y) + p_1(\tau, y)] d\tau dy. \quad (4.43)$$

For  $f(t) = \delta(t)$ , we observe that  $p_1(t, x)$  is a bounded function. This implies that  $g(t)$  is also a bounded function. We define here  $f(t) = \delta(0)$  as the limit of  $f_\varepsilon(t)$  with  $f_\varepsilon(t) = \frac{1}{\varepsilon}$  for  $t \in (0, \varepsilon)$  and  $f_\varepsilon(t) = 0$  otherwise. We also write  $p$  and  $p_1$  the corresponding limits as  $\varepsilon \rightarrow 0$ .

**Exercise 4.4.2** Show that  $p_1$  is indeed a bounded function. Find that the right-hand side in (4.43) with  $p_1$  set to 0 is indeed bounded. Then use an argument based on a Gronwall lemma as indicated below to conclude.

**Nonlinear problem for  $q(x)$ .** Let us come back to (4.39) and evaluate it at  $t = 0$  for  $x > 0$ . Then we find

$$p_1(0, x) = \frac{1}{2} \int_{-x}^x g(\tau) d\tau + \frac{1}{2} \int_0^x \int_{-(x-y)}^{x-y} s(\tau, y) d\tau dy.$$

Differentiating with respect to  $x$  yields

$$g(x) + \int_0^x q(y) p(x - y, y) dy = 0. \quad (4.44)$$

This may be seen as a nonlinear equation for  $q$  since  $p$  depends on  $q$ .

The same equation may also be obtained from (4.43). Indeed, differentiating in  $x$  yields

$$\partial_x p_1(t, x) = -\frac{1}{2} \int_0^t (s(\tau, x + (t - \tau)) - s(\tau, x - (t - \tau))) \tau = -\int_0^t s(\tau, x + (t - \tau)) d\tau.$$

Evaluated at  $x = 0$ , this yields

$$\partial_x p_1(t, 0) = g(t) = - \int_0^t q(\tau) p(t - \tau, \tau) d\tau, \quad (4.45)$$

which is equivalent to (4.44). The latter may be recast as

$$g(x) + \int_0^x q(y) f(x - 2y) dy + \int_0^x q(y) p_1(x - y, y) dy = 0.$$

Let us now assume that  $f(t) = \delta(t)$  as the limit indicated above. Then we find that

$$g(x) + \frac{1}{2} q\left(\frac{x}{2}\right) + \int_0^x q(y) p_1(x - y, y) dy = 0.$$

However, by the finite speed of propagation,  $p(t, x)$  and  $p_1(t, x)$  are supported on  $t \geq x$ . This means that  $p_1(x - y, y)$  is supported on  $y \leq \frac{x}{2}$ . As a consequence, changing  $x$  to  $2x$ , we find that for all  $x > 0$ ,

$$q(x) = -2g(2x) - 2 \int_0^x q(y) p_1(2x - y, y) dy. \quad (4.46)$$

This is a nonlinear equation for  $q(x)$  of Fredholm type. We define  $\tilde{g}(x) = -2g(2x)$ .

**Error estimates and Gronwall lemma.** Let us consider  $\tilde{g} = \mathfrak{M}_f(\tilde{q})$  and  $g = \mathfrak{M}_f(q)$ . We define  $\delta q = q - \tilde{q}$  and  $\delta p = p - \tilde{p}$  as well as  $\delta \tilde{g}$  with obvious notation. Then we find that

$$\begin{aligned} \delta q(x) &= \delta \tilde{g}(x) - 2 \int_0^x \delta q(y) p_1(2x - y, y) dy - 2 \int_0^x \tilde{q}(y) \delta p_1(2x - y, y) dy \\ \delta p_1(t, x) &= \frac{1}{2} \int_{t-x}^{t+x} \delta g(\tau) d\tau + \frac{1}{2} \int_{\Delta(t, x)} [\delta q(y) p(\tau, y) + \tilde{q}(y) \delta p_1(\tau, y)] d\tau dy. \end{aligned} \quad (4.47)$$

Here, we used (4.39) for the expression of  $p_1$ .

Let us define

$$\Gamma(T) = \{(x, t) \in \mathbb{R}^2, 0 \leq t \leq T, x = t\} \cup \{(x, t) \in \mathbb{R}^2, 0 \leq t \leq T, x = 2T - t\}, \quad (4.48)$$

the part of the boundary of  $\Delta(T, T)$  that does not include the segment  $(0, 0 < t < T)$ . We next define

$$\delta P(T) = \sup_{\Gamma(T)} |\delta p_1(t, x)|, \quad \delta Q(T) = \sup_{t \in [0, T]} |\delta q|(t). \quad (4.49)$$

Then we deduce from (4.47) that

$$\begin{aligned} \delta Q(T) &\leq \|\delta g\|_{L^\infty(0, 2T)} + C \int_0^T \delta Q(\tau) d\tau + C \int_0^T \delta P(\tau) d\tau \\ \delta P(T) &\leq T \|\delta g\|_{L^\infty(0, 2T)} + C \int_0^T \delta Q(\tau) d\tau + C \int_0^T \delta P(\tau) d\tau. \end{aligned}$$

**Exercise 4.4.3** Verify the above expression. The main difficulty involves the analysis of the term

$$\int_{\Delta(t,x)} \delta q(y) \delta(\tau - y) d\tau dy,$$

which has to be seen as the limit with source term  $f_\varepsilon$  as  $\varepsilon \rightarrow 0$ . It is bounded by  $\int_0^T |\delta q|(\tau) d\tau$  and hence by  $\int_0^T \delta Q(\tau) d\tau$ .

We deduce that

$$(\delta P + \delta Q)(T) \leq (1 + T) \|\delta g\|_{L^\infty(0,2T)} + C \int_0^T (\delta P + \delta Q)(\tau) d\tau$$

with constant  $C$  uniform in  $T$ . As an application of the Gronwall lemma, we deduce that

$$(\delta P + \delta Q)(T) \leq C e^{CT} \|\delta g\|_{L^\infty(0,2T)},$$

for some positive constant  $C$  also independent of  $T$ .

**Exercise 4.4.4** Verify that the Gronwall lemma applies to the above problem and yields the latter inequality.

This proves the following result:

**Theorem 4.4.1** Let  $f(t) = \delta(t)$  and  $q$  and  $\tilde{q}$  be bounded potentials. Then  $\mathfrak{M}_\delta(q) = \mathfrak{M}_\delta(\tilde{q})$  implies that  $q = \tilde{q}$  and we have the following error estimate

$$\|q - \tilde{q}\|_{L^\infty(0,T)} \leq C e^{CT} \|\mathfrak{M}_\delta(q) - \mathfrak{M}_\delta(\tilde{q})\|_{L^\infty(0,2T)}. \quad (4.50)$$

The constant  $C$  is uniform in  $T > 0$ .

Let us now assume that the initial condition is not  $f(t) = \delta(t)$  but rather  $f_n(t) = \frac{1}{n!} t^n$  for  $n \geq 0$ . Then obviously  $f_n^{(n+1)}(t) = \delta(t)$ . Moreover, since  $q$  is independent of  $t$ , then  $\partial_t^{n+1} u$  solves the same equation as  $u$  with  $f_n$  replaced by  $\delta(t)$ . As a consequence the measurements  $g_n(t)$  with  $f = f_n$  are such that  $g_n^{(n+1)}(t) = g(t)$ , the measurements corresponding to  $f = \delta(t)$ . This yields

**Corollary 4.4.2** Let  $f_n(t) = \frac{1}{n!} t^n$  and  $q$  and  $\tilde{q}$  be bounded potentials. Then  $\mathfrak{M}_{f_n}(q) = \mathfrak{M}_{f_n}(\tilde{q})$  implies that  $q = \tilde{q}$  and we have the following error estimate

$$\|q - \tilde{q}\|_{L^\infty(0,T)} \leq C e^{CT} \|\mathfrak{M}_{f_n}(q) - \mathfrak{M}_{f_n}(\tilde{q})\|_{W^{n+1,\infty}(0,2T)}. \quad (4.51)$$

The constant  $C$  is uniform in  $T > 0$ .

This straightforward corollary indicates that the stability estimate strongly depends on the probing mechanism. The smoother the probing mechanism, the worse is the stability estimate.

Intuitively, the above result is clear. A signal emitted at  $t = 0$  propagates to position  $x$  during time  $t = x$  and then back to  $x = 0$  at time  $t = 2x$ . Only that signal provides information about  $q$  at position  $x$ . When multiple signals are emitted, then superpositions of signals create some blurring. The above corollary quantifies such a blurring.

# Chapter 5

## Inverse Kinematic and Inverse Transport Problems

Several problems of Integral Geometry model the propagation of particles interacting with the underlying medium. This is the case for the Radon transform seen as a transmission tomography problem in (2.1) and for the Attenuated Radon transform in (2.23). Another transport equation was briefly encountered in (3.43) in the analysis of the Generalized Ray transform. These source problems belong to a larger class of inverse kinematic and inverse transport problems that we analyze in this chapter.

The propagation of particles essentially follows two dynamics: a Hamiltonian (of classical mechanics) describes the “free” propagation of particles in the absence of interactions with the underlying structure; a scattering operator models how particles are absorbed and scattered when they interact with the underlying medium. The inverse kinematic and inverse transport problems aim to reconstruct the structures of the Hamiltonian and the scattering operator from measurements of particle densities.

The first part of this chapter considers the reconstruction of a simple Hamiltonian of the form  $H(x, k) = c(x)|k|$  from travel time boundary measurements. The second part of the chapter assumes the Hamiltonian known with  $c$  a constant and considers the reconstruction of the absorption and scattering coefficients from boundary measurements of particle densities.

**Scattering: a transition to ill-posedness.** Before addressing these inverse problems in more detail, let us comment on the influence of scattering on the structure of an inverse problem and the stability properties of a measurement operator.

The inverse problems we have encountered in the preceding three chapters, to which we can add the simplified M.R.I. description of the first chapter, were “well-posed” inverse problems in the sense that they involved a stable inversion operator from a Hilbert space to another Hilbert space. Wave propagation (as in diffraction tomography) or particle propagation along straight lines (as in computerized tomography) all generate well-posed inverse problems.

What causes then an inverse problem to be ill-posed? As we mentioned in the introduction, an inverse problem is ill-posed when the measurement operator is a highly smoothing/regularizing operator. We have seen that solving the heat equation forward was a highly regularizing process. The main mechanism explaining its regularizing

property is *scattering*: using a kinematic description of the diffusion equation, particles represented by Brownian motions scatterer infinitely often before they exit the domain of interest.

Scattering is one of the main mechanisms that regularizes solutions to PDEs and hence renders inverse problems ill-posed. The linear transport equation, also known as the linear Boltzmann equation or the radiative transfer equation, offers an ideal example of a transition from non-scattering to scattering environments.

## 5.1 Inverse Kinematic Problem

In the absence of scattering or absorption, which we assume in this section, the propagation of particles is modeled by the following Hamiltonian dynamics

$$\begin{aligned} \frac{dx}{dt} &= \dot{x} = \nabla_k H(x(t), k(t)) & x(0) &= x_0, \\ \frac{dk}{dt} &= \dot{k} = -\nabla_x H(x(t), k(t)) & k(0) &= k_0. \end{aligned} \tag{5.1}$$

As we did in the preceding chapter, we consider the simplest example of a Hamiltonian

$$H(x, k) = c(x)|k|. \tag{5.2}$$

This Hamiltonian models the propagation of high frequency acoustic waves in a medium with a spatially varying sound speed  $c(x)$ . It also models the propagation of light in media with varying indices of refraction when polarization effects are neglected. The index of refraction  $n(x)$  is defined as  $n(x) = \frac{c}{c(x)}$  with  $c$  light speed in a vacuum. The same Hamiltonian may be used to model the propagation of seismic (shear) waves in the Earth when pressure waves and polarization properties of shear waves are neglected.

The equation (5.1) models the propagation of a single particle knowing its information  $(x_0, k_0)$  at time 0. The density  $u(t, x, k)$  of an ensemble of such particles then solves the corresponding Liouville equation

$$\frac{\partial u}{\partial t} + \{H, u\} = 0, \quad t > 0, \quad u(0, x, k) = u_0(x, k), \tag{5.3}$$

where  $u_0(x, k)$  is the density of particles at time  $t = 0$  and where

$$\{H, u\} := \nabla_k H \cdot \nabla_x u - \nabla_x H \cdot \nabla_k u = c(x)\hat{k} \cdot \nabla_x u - |k|\nabla c(x) \cdot \nabla_k u, \tag{5.4}$$

is the Poisson bracket of the Hamiltonian  $H$  and the density  $u$ . The inverse kinematic problem thus concerns the reconstruction of the Hamiltonian  $H$  from measurements of the density  $u$  at the boundary of a domain of interest.

In the following section, we first consider a one-dimensional version of the inverse kinematic problem in a geometry with spherical symmetry. This inverse kinematic problem was solved by Herghlotz in 1905 as a means to understand the inner structure of the Earth from travel time measurements. We then revisit the Mukhometov method to solve the inverse kinematic problem in two dimensions of space.

### 5.1.1 Spherical symmetry

We wish to reconstruct the velocity field  $c(r)$  inside the Earth assuming spherical symmetry of the Hamiltonian:

$$H(x, k) = c(r)|k| \quad \text{with} \quad r = |x|.$$

Our data is the time it takes from a particle following (5.1) and emitted at the surface of a sphere with a given direction to come back to the surface of that sphere. To simplify we assume that the Earth radius is normalized to 1. We also assume that  $c(1)$  is known. We want to reconstruct  $c(r)$  from the time it takes to travel along all possible geodesics (the trajectories of (5.1)). Because the geodesics depend on  $c(r)$ , the travel time is a *nonlinear* functional of the velocity profile  $c(r)$ .

Let us denote  $x = r\hat{x}$  and  $k = |k|\vartheta$ . The Hamiltonian dynamics take the form

$$\dot{x} = c(r)\vartheta, \quad \dot{k} = -c'(r)|k|\hat{x}. \quad (5.5)$$

We are interested in calculating the travel times between points at the boundary of the domain  $r = |x| < 1$ . This implies integrating  $dt$  along particle trajectories. Since we want to reconstruct  $c(r)$ , we perform a change of variables from  $dt$  to  $dr$ . This will allow us to obtain integrals of the velocity  $c(r)$  along curves. The objective will then be to obtain a reconstruction formula for  $c(r)$ .

In order to perform the change of variables from  $dt$  to  $dr$ , we need to know where the particles are. Indeed the change of variables should only involve position  $r$  and no longer time  $t$ . This implies to solve the problem  $t \mapsto r(t)$ . As usual it is useful to find invariants of the dynamical system. The first invariant is as always the Hamiltonian itself:

$$\frac{dH(x(t), k(t))}{dt} = 0,$$

as can be deduced from (5.1). The second invariant is *angular momentum* and is obtained as follows. Let us first introduce the basis  $(\hat{x}, \hat{x}^\perp)$  for two dimensional vectors (this is the usual basis  $(e_r, e_\theta)$  in polar coordinates). We decompose  $k = k_r\hat{x} + k_\theta\hat{x}^\perp$  and  $\vartheta = \hat{k}_r\hat{x} + \hat{k}_\theta\hat{x}^\perp$ . We verify that

$$\dot{r} = c(r)\hat{k}_r \quad \text{since} \quad \dot{x} = \dot{r}\hat{x} + r\dot{\hat{x}} = c(r)\vartheta. \quad (5.6)$$

We also verify that

$$\frac{d(rk_\theta)}{dt} = \frac{dx^\perp \cdot k}{dt} = \dot{x}^\perp \cdot k + x \cdot \dot{k}^\perp = c(r)\vartheta^\perp \cdot k - c'(r)|k|x \cdot \hat{x}^\perp = 0. \quad (5.7)$$

This is conservation of angular momentum, which implies that

$$r(t)k_\theta(t) = k_\theta(0),$$

since  $r(0) = 1$ .

By symmetry, we observe that the travel time is decomposed into two identical components: the time it takes to go down the Earth until  $k_r = 0$ , and the time it takes to go back up. On the way up to the surface,  $k_r$  is non-negative. Let us denote  $p = \hat{k}_\theta(0)$

with  $0 < p < 1$ . The lowest point is reached when  $\hat{k}_\theta = 1$ . This means at a point  $r_p$  such that

$$\frac{r_p}{c(r_p)} = \frac{p}{c(1)}.$$

To make sure that such a point is uniquely defined, we impose that the function  $rc^{-1}(r)$  be *increasing* on  $(0, 1)$  since it cannot be decreasing. This is equivalent to the constraint:

$$c'(r) < \frac{c(r)}{r}, \quad 0 < r < 1. \quad (5.8)$$

This assumption ensures the uniqueness of a point  $r_p$  such that  $pc(r_p) = c(1)r_p$ .

Since the Hamiltonian  $c(r)|k|$  is conserved, we deduce that

$$\dot{r} = c(r)\sqrt{1 - \hat{k}_\theta^2} = c(r)\sqrt{1 - \left(\frac{\hat{k}_\theta(0)c(r)}{rc(1)}\right)^2},$$

so that

$$dt = \frac{dr}{c(r)\sqrt{1 - \left(\frac{\hat{k}_\theta(0)c(r)}{rc(1)}\right)^2}}. \quad (5.9)$$

Notice that the right-hand side depends only on  $r$  and no longer on functions such as  $\hat{k}_r$  that depend on time. The travel time as a function of  $p = \hat{k}_\theta(0)$  is now given by twice the time it takes to go back to the surface:

$$T(p) = 2 \int_{t(r_p)}^1 dt = 2 \int_{r_p}^1 \frac{dr}{c(r)\sqrt{1 - \left(\frac{\hat{k}_\theta(0)c(r)}{rc(1)}\right)^2}}. \quad (5.10)$$

Our measurements are  $T(p)$  for  $0 < p < 1$  and our objective is to reconstruct  $c(r)$  on  $(0, 1)$ . We need a theory to invert this integral transform. Let us define

$$u = \frac{c^2(1)r^2}{c^2(r)} \quad \text{so that} \quad du = \frac{2rc^2(1)}{c^2(r)} \left(1 - \frac{rc'(r)}{c(r)}\right) dr.$$

Upon using this change of variables we deduce that

$$T(p) = 2 \int_{p^2}^1 \left(\frac{dr}{du} \frac{u}{r}\right)(u) \frac{du}{\sqrt{u - p^2}}. \quad (5.11)$$

It turns out that the function in parenthesis in the above expression can be reconstructed from  $T(p)$ . This is an *Abel integral*. Before inverting the integral, we need to ensure that the change of variables  $r \mapsto u(r)$  is a diffeomorphism (a continuous function with continuous inverse). This implies that  $du/dr$  is positive, which in turn is equivalent to (5.8). The constraint (5.8) is therefore useful both to obtain the existence of a minimal point  $r_p$  and to ensure that the above change of variables is admissible. The constraint essentially ensures that no rays are trapped in the dynamics so that energy entering the system will eventually exit it. We can certainly consider velocity profiles such that

the energy is attracted at the origin. In such situation the velocity profile cannot be reconstructed.

Let us denote by  $f = \frac{dr}{du} \frac{u}{r}$ . We will show in the following section that  $f(u)$  can be reconstructed from  $T(p)$  and is given by

$$f(u) = -\frac{2}{\pi} \frac{d}{du} \int_u^1 \frac{T(\sqrt{p})}{\sqrt{p-u}} dp. \quad (5.12)$$

Now we reconstruct  $r(u)$  from the relations

$$\frac{f(u)}{u} du = \frac{dr}{r}, \quad u(1) = 1, \quad \text{so that} \quad r(u) = \exp\left(\int_u^1 \frac{f(v)dv}{v}\right).$$

Upon inverting this diffeomorphism, we obtain  $u(r)$  and  $g(r) = f(u(r))$ . Since

$$g(r) = \frac{1}{2} \frac{1}{1 - rc'/c},$$

we now know  $rc'/c$ , hence  $(\log c)'$ . It suffices to integrate  $\log c$  from 1 to 0 to obtain  $c(r)$  everywhere. This concludes the proof of the reconstruction.

### 5.1.2 Abel integral and Abel transform

For a smooth function  $f(x)$  (continuous will do) defined on the interval  $(0, 1)$ , we define the Abel transform as

$$g(x) = \int_x^1 \frac{f(y)}{(y-x)^{1/2}} dy. \quad (5.13)$$

This transform can be inverted as follows:

**Lemma 5.1.1** *The Abel transform (5.13) admits the following inversion*

$$f(y) = -\frac{1}{\pi} \frac{d}{dy} \int_y^1 \frac{g(x)}{(x-y)^{1/2}} dx. \quad (5.14)$$

*Proof.* Let us calculate

$$\int_z^1 \frac{g(x)}{(x-z)^{1/2}} dx = \int_z^1 \int_x^1 \frac{f(y)}{(x-z)^{1/2}(y-x)^{1/2}} dx dy = \int_z^1 dy f(y) k(z, y) dy.$$

The kernel  $k(z, y)$  is given by

$$k(z, y) = \int_z^y \frac{dx}{(x-z)^{1/2}(y-x)^{1/2}} = \int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = \pi.$$

The latter equality comes from differentiating arccos. Thus we have

$$\int_z^1 \frac{g(x)}{(x-z)^{1/2}} dx = \pi \int_z^1 f(y) dy.$$

Upon differentiating both sides with respect to  $z$ , we obtain the desired result.  $\square$

We can also ask ourselves how well-posed the inversion of the Abel transform is. Since the transforms are defined on bounded intervals, using the Hilbert scale introduced in Chapter 1 would require a few modifications. Instead we will count the number of differentiations. The reconstruction formula shows that the Abel transform is applied once more to  $g$  before the result is differentiated. We can thus conclude that the Abel transform regularizes as one half of an integration (since it takes one differentiation to compensate for two Abel transforms). We therefore deduce that the Abel transform is a smoothing operator of order  $\alpha = 1/2$  using the terminology introduced in Chapter 1.

### 5.1.3 Kinematic velocity Inverse Problem

Let us generalize the construction of the Earth velocity from boundary measurements in the setting of a “two-dimensional” Earth.

As in section 3.4.1, we consider a bounded domain  $X \subset \mathbb{R}^2$  with smooth surface  $\partial X$  parameterized by  $0 \leq \tau \leq T$  and points  $x = S(\tau)$  with  $S(0) = S(T)$  and  $|\dot{S}(\tau)| = 1$ .

Local travel time and distance are related by

$$ds^2 = \frac{1}{c^2(x)}(dx^2 + dy^2) = n^2(x)(dx^2 + dy^2),$$

which defines a Riemannian metric with tensor proportional to the  $2 \times 2$  identity matrix.

The geodesics of that metric generate a family of curves  $F(t, s, \vartheta)$  as in (3.1) in Chapter 3. We assume that the family of geodesics satisfies the hypotheses stated there, namely the existence of an inverse in (3.2). For a point  $x$  in  $\bar{X}$  and  $0 \leq \tau \leq T$ , we recall that  $\tilde{F}(x, \tau)$  is the unique curve joining  $x$  and  $S(\tau)$ .

We are interested in reconstructing  $n(x)$  from knowledge of

$$G(\tau_1, \tau_2) = \int_{\tilde{F}(S(\tau_1), \tau_2)} d\tau = \int_{\tilde{F}(S(\tau_1), \tau_2)} n(x) \sqrt{dx^2 + dy^2}, \quad (5.15)$$

for every possible boundary points  $\tau_1$  and  $\tau_2$ , where  $\gamma(t_1, t_2)$  is an extremal for the above functional, i.e., a geodesic of the Riemannian metric  $d\tau^2$ . Notice that since the extremals (the geodesics) of the Riemannian metric depend on  $n(\mathbf{x})$ , the above problem is *non-linear* as in the one-dimensional case.

Let  $G_k$  for  $k = 1, 2$  correspond to measurements for two slownesses  $n_k$ ,  $k = 1, 2$ . We then have the following result

**Theorem 5.1.2** *Let  $n_k$  be smooth positive functions on  $X$  such that the family of extremals are sufficiently regular. Then  $n_k$  can uniquely be reconstructed from  $G_k(\tau_1, \tau_2)$  and we have the stability estimate*

$$\|n_1 - n_2\|_{L^2(X)} \leq C \left\| \frac{\partial}{\partial \tau_1} (G_1 - G_2) \right\|_{L^2((0,T) \times (0,T))}. \quad (5.16)$$

*Proof.* Even though the inverse kinematic problem is non-linear, the proof is similar to that of Theorem 3.4.1 for the corresponding linear inverse problem. The reason is that the same energy estimate can be used in both cases. The regular family of curves

$F$  is defined as the geodesics of the Riemannian metric  $d\tau^2$ . Let us define the travel time

$$t(x, \tau) = \int_{\tilde{F}(x, \tau)} \text{nds}, \quad (5.17)$$

so that as before  $t(S(\tau_1), \tau_2) = G(\tau_1, \tau_2)$ . We deduce as in (3.43) that

$$\vartheta \cdot \nabla_x t = n(x). \quad (5.18)$$

We recall  $\vartheta(x, \tau) = (\cos \theta(x, \tau), \sin \theta(x, \tau))$  is the unit tangent vector to the curve  $\tilde{F}(x, \tau)$  at  $x$  and orientated such that (5.18) holds.

Because  $t$  is an integration along an extremal curve of the Riemannian metric (this is where we use that the curves are geodesics), we deduce that

$$\vartheta^\perp \cdot \nabla t = 0, \quad \text{so that} \quad \nabla_x t = n(x)\vartheta \quad \text{and} \quad |\nabla_x t|^2(x, \tau) = n^2(x).$$

Upon differentiating the latter equality we obtain

$$\frac{\partial}{\partial \tau} |\nabla_x t|^2 = 0.$$

Let us now define  $u = t_1 - t_2$ , the difference of travel times for the two possible sound speeds  $c_1$  and  $c_2$ , so that  $\nabla u = n_1\vartheta_1 - n_2\vartheta_2$ . We deduce from the above expression that

$$\frac{\partial}{\partial t} (\nabla u \cdot (\vartheta_1 + \frac{n_2}{n_1}\vartheta_2)) = 0.$$

We multiply the above expression by  $2\vartheta_1^\perp \cdot \nabla u$  and express the product in divergence form. We obtain as in the preceding section that

$$\begin{aligned} & 2\vartheta_1^\perp \cdot \nabla u \frac{\partial}{\partial t} \vartheta_1 \cdot \nabla u - \frac{\partial}{\partial t} \left( \vartheta_1^\perp \cdot \nabla u \vartheta_1 \cdot \nabla u \right) \\ &= \dot{\theta}_1 |\nabla u|^2 + \vartheta_1 \cdot \nabla (\vartheta_1^\perp \cdot \nabla u u_t) - \vartheta_1^\perp \cdot \nabla (\vartheta_1 \cdot \nabla u u_t). \end{aligned}$$

We now show that the second contribution can also be put in divergence form. More precisely, we obtain, since  $n_1$  and  $n_2$  are independent of  $t$ , that

$$\begin{aligned} & 2\vartheta_1^\perp \cdot \nabla u \frac{\partial}{\partial \tau} \left( \frac{n_2}{n_1} \vartheta_2 \cdot \nabla u \right) = 2\vartheta_1^\perp \cdot (n_1\vartheta_1 - n_2\vartheta_2) \frac{\partial}{\partial \tau} (n_2\vartheta_1 \cdot \vartheta_2 - \frac{n_2^2}{n_1}) \\ &= -2n_2^2 \vartheta_1^\perp \cdot \vartheta_2 (\vartheta_1 \cdot \vartheta_2) = -2n_2^2 (\vartheta_1^\perp \cdot \vartheta_2)^2 \frac{\partial(\theta_1 - \theta_2)}{\partial \tau} = \\ &= -2n_2^2 \sin^2(\theta_1 - \theta_2) \frac{\partial(\theta_1 - \theta_2)}{\partial \tau} = \frac{\partial}{\partial \tau} \left( n_2^2 \left[ \frac{\sin(2(\theta_1 - \theta_2))}{2} - (\theta_1 - \theta_2) \right] \right). \end{aligned}$$

The integration of the above term over  $X \times (0, T)$  thus yields a vanishing contribution. Following the same derivation as in Chapter 3, we deduce that

$$\int_0^T \int_X \frac{\partial \theta_1}{\partial \tau} |\nabla u|^2 dx d\tau = \int_0^T \int_0^T \frac{\partial \delta G(\tau_1, \tau_2)}{\partial \tau_1} \frac{\partial \delta G(\tau_1, \tau_2)}{\partial \tau_2} d\tau_1 d\tau_2, \quad (5.19)$$

where we have defined  $\delta G = G_1 - G_2$ . To conclude the proof, notice that

$$\begin{aligned} \nabla u \cdot \nabla u &= |n_1\vartheta_1 - n_2\vartheta_2|^2 = n_1^2 + n_2^2 - 2n_1n_2\vartheta_1 \cdot \vartheta_2 \\ &\geq n_1^2 + n_2^2 - 2n_1n_2 = (n_1 - n_2)^2, \end{aligned}$$

since both  $n_1$  and  $n_2$  are non-negative. With (5.19), this implies that  $n_1 = n_2$  when  $G_1 = G_2$  and using again the Cauchy-Schwarz inequality yields the stability estimate (5.16).  $\square$

## 5.2 Forward transport problem

So far, the kinetic model accounted for spatial changes in the speed of propagation but not for scattering or absorption mechanisms. We recall that physically, the transport equation models high frequency waves or particles propagating in scattering environments. With the specific form of the Hamiltonian considered in (5.2), and in the time-independent setting to simplify, the transport equation then takes the form

$$c(x) \frac{v}{|v|} \cdot \nabla_x u - |v| \nabla c(x) \cdot \nabla_v u + \sigma(x)u = \int_{\mathbb{R}^d} k(x, v', v) u(x, v') \delta(c(x)|v| - c(x)|v'|) dv', \quad (5.20)$$

where boundary conditions are imposed at the boundary of a domain of interest. Here,  $u(x, v)$  still denotes the density of particles at position  $x$  with velocity  $v$  and  $c(x)$  is the speed of propagation. Since we are primarily interested in scattering effects in this chapter, we assume that  $c(x) = 1$  is constant and normalized. The equation thus becomes

$$\frac{v}{|v|} \cdot \nabla_x u + \sigma(x)u = \int_{\mathbb{R}^d} k(x, v', v) u(x, v') \delta(\omega - |v'|) dv'. \quad (5.21)$$

Here,  $\omega = c|v|$  is the frequency (e.g. the color for light) of the propagating waves. Scattering is assumed to be elastic so that  $\omega = c|v|$  is preserved by scattering and hence wave packets with different frequencies satisfy uncoupled equations. We also normalize  $\omega = 1$  so that  $|v| = \omega = 1$ . In other words,  $v$  is now the direction of propagation of the wave packets (photons). We thus have an equation of the form

$$v \cdot \nabla_x u + \sigma(x)u = \int_{\mathbb{S}^{d-1}} k(x, v', v) u(x, v') dv', \quad (5.22)$$

where  $x \in \mathbb{R}^n$  and  $v \in \mathbb{S}^{n-1}$  the unit sphere in  $\mathbb{R}^n$ . It remains to describe the parameters participating in scattering:  $\sigma(x)$  the attenuation (aka total absorption) coefficient, and  $k(x, v', v)$  the scattering coefficient.  $\sigma(x)$  models the amount of particles that are either absorbed or scattered per unit distance of propagation. This is the same coefficient already encountered in CT and SPECT. Unlike high-energy CT and SPECT, in lower energies such as for visible light, many “absorbed” photons are re-emitted into another direction, i.e., scattered. Then  $k(x, v', v)$  gives the density of particles scattered into direction  $v$  from a direction  $v'$ . The right-hand side in (5.22) corresponds to a re-emission of scattered particles into the direction  $v$ .

In an **inverse transport problem**,  $\sigma(x)$  and  $k(x, v', v)$  are the unknown coefficients. They have to be reconstructed from knowledge of  $u(x, v)$  measured, say, at the boundary of the domain of interest. When  $k \equiv 0$ , this is Computerized Tomography, where the appropriate logarithm of  $u$  provides line integrals of  $\sigma$ .

To be more specific, we need to introduce ways to probe the domain of interest  $X$  and to model measurements. In the most general setting, photons can enter into  $X$  at any point  $x \in \partial X$  and with any incoming direction. In the most general setting, photon densities can then be measured at any point  $x \in \partial X$  and for any outgoing direction. The sets of incoming conditions  $\Gamma_-$  and outgoing conditions  $\Gamma_+$  are defined by

$$\Gamma_{\pm} = \{(x, v) \in \partial X \times V, \text{ s.t. } \pm v \cdot \nu(x) > 0\}, \quad (5.23)$$

where  $\nu(x)$  is the outgoing normal vector to  $X$  at  $x \in \partial X$  and  $V = \mathbb{S}^{n-1}$ . Denoting by  $g(x, v)$  the incoming boundary conditions, we then obtain the following transport equation

$$\begin{aligned} v \cdot \nabla_x u + \sigma(x)u &= \int_{\mathbb{S}^{d-1}} k(x, v', v)u(x, v')dv' & (x, v) \in X \times V \\ u|_{\Gamma_-}(x, v) &= g(x, v) & (x, v) \in \Gamma_- \end{aligned} \quad (5.24)$$

From the functional analysis point of view, it is natural to consider the  $L^1$  norm of photon densities, which essentially counts numbers of particles (the  $L^1$  norm of the density on a domain is the number of particles inside that domain). Let us introduce the necessary notation.

We say that the optical parameters  $(\sigma, k)$  are admissible when

$$\begin{aligned} 0 &\leq \sigma \in L^\infty(X) \\ 0 &\leq k(x, v', \cdot) \in L^1(V) \text{ a.e. in } X \times V \\ \sigma_s(x, v') &:= \int_V k(x, v', v)dv \in L^\infty(X \times V). \end{aligned} \quad (5.25)$$

Here  $\sigma_s$  is also referred to as the scattering coefficient. In most applications,  $\sigma_s(x)$  is independent of  $v'$ .

We define the times of escape of free-moving particles from  $X$  as

$$\tau_\pm(x, v) = \inf\{s > 0 | x \pm sv \notin X\} \quad (5.26)$$

and  $\tau(x, v) = \tau_+(x, v) + \tau_-(x, v)$ . On the boundary sets  $\Gamma_\pm$ , we introduce the measure  $d\xi(x, v) = |v \cdot \nu(x)|d\mu(x)dv$ , where  $d\mu(x)$  is the surface measure on  $\partial X$ .

We define the following Banach space

$$W := \{u \in L^1(X \times V) | v \cdot \nabla_x u \in L^1(X \times V), \tau^{-1}u \in L^1(X \times V)\}, \quad (5.27)$$

with its natural norm. We recall that  $\tau$  is defined below (5.26). We have the following trace formula [29]

$$\|f|_{\Gamma_\pm}\|_{L^1(\Gamma_\pm, d\xi)} \leq \|f\|_W, \quad f \in W. \quad (5.28)$$

This allows us to introduce the following lifting operator

$$\mathcal{I}g(x, v) = \exp\left(-\int_0^{\tau_-(x, v)} \sigma(x - sv, v)ds\right)g(x - \tau_-(x, v)v, v). \quad (5.29)$$

It is proved in [29] that  $\mathcal{I}$  is a bounded operator from  $L^1(\Gamma_-, d\xi)$  to  $W$ . Note that  $\mathcal{I}g$  is the solution  $u_0$  of

$$v \cdot \nabla_x u_0 + \sigma(x)u_0 = 0 \quad (x, v) \in X \times V, \quad u_0 = g \quad (x, v) \in \Gamma_-.$$

**Exercise 5.2.1** *Prove this. This is the same calculation as for the X-ray transform.*

Let us next define the bounded operators

$$\begin{aligned}\mathcal{K}u(x, v) &= \int_0^{\tau^-(x, v)} \exp\left(-\int_0^t \sigma(x - sv, v) ds\right) \int_V k(x - tv, v', v) u(x - tv, v') dv' dt \\ \mathcal{L}S(x, v) &= \int_0^{\tau^-(x, v)} \exp\left(-\int_0^t \sigma(x - sv, v) ds\right) S(x - tv, v) dt\end{aligned}\quad (5.30)$$

for  $(x, v) \in X \times V$ . Note that  $\mathcal{L}S$  is the solution  $u_S$  of

$$v \cdot \nabla_x u_S + \sigma(x) u_S = S \quad (x, v) \in X \times V, \quad u_S = 0 \quad (x, v) \in \Gamma_-.$$

**Exercise 5.2.2** *Prove this.*

Note that

$$\mathcal{K}u(x, v) = \mathcal{L}\left[\int_V k(x, v', v) u(x, v') dv'\right](x, v),$$

which allows us to handle the right-hand side in (5.24). Looking for solutions in  $W$ , the integro-differential equation (5.24) is thus recast as

$$(I - \mathcal{K})u = \mathcal{I}g. \quad (5.31)$$

**Exercise 5.2.3** *Prove this.*

Then we have the following result [13, 29].

**Theorem 5.2.1** *Assume that*

$$(I - \mathcal{K}) \text{ admits a bounded inverse in } L^1(X \times V, \tau^{-1} dx dv). \quad (5.32)$$

*Then the integral equation (5.31) admits a unique solution  $u \in W$  for  $g \in L^1(\Gamma_-, d\xi)$ . Furthermore, the albedo operator*

$$\mathcal{A} : L^1(\Gamma_-, d\xi) \rightarrow L^1(\Gamma_+, d\xi), \quad g \mapsto \mathcal{A}g = u|_{\Gamma_+}, \quad (5.33)$$

*is a bounded operator.*

*The invertibility condition (5.32) holds under either of the following assumptions*

$$\sigma_a := \sigma - \sigma_s \geq 0 \quad (5.34)$$

$$\|\tau\sigma_s\|_\infty < 1. \quad (5.35)$$

We shall not prove this theorem here. The salient features are that the transport equation is well-posed provided that (5.32) is satisfied, which is not necessarily true for arbitrary admissible coefficients  $(\sigma, k)$ . The conditions (5.34) or (5.35) are sufficient conditions for (5.32) to be satisfied. The first condition is the most natural for us and states that particles that are “created” by scattering into  $v$  by  $k(x, v', v)$  are particles

that are “lost” for direction  $v$ . In other words, the scattering mechanism does not create particles. This is quite natural for photon propagation. In nuclear reactor physics, however, several neutrons may be created by fission for each incoming scattering neutron. There are applications in which (5.34) is therefore not valid. In most medical and geophysical imaging applications, however, (5.34) holds and the transport solution exists. Note that  $\sigma_a = \sigma - \sigma_s$  is the *absorption* coefficient, and corresponds to a measure of the particles that are “lost” for direction  $v'$  and do not reappear in any direction  $v$  (i.e., particles that are absorbed).

Using these operators, we may recast the transport solution as

$$u = \mathcal{I}g + \mathcal{K}\mathcal{I}g + (I - \mathcal{K})^{-1}\mathcal{K}^2\mathcal{I}g, \quad (5.36)$$

where  $u_0 := \mathcal{I}g$  is the ballistic component,  $u_1 := \mathcal{K}\mathcal{I}g$  the single scattering component and  $u_2 := u - u_0 - u_1 = (I - \mathcal{K})^{-1}\mathcal{K}^2\mathcal{I}g$  is the multiple scattering component.

Note that when the problem is subcritical, its solution may be expressed in terms of the following Neumann expansion in  $L^1(X \times V)$

$$u = \sum_{m=0}^{\infty} \mathcal{K}^m \mathcal{I}g. \quad (5.37)$$

The contribution  $m = 0$  is the ballistic part of  $u$ , the contribution  $m = 1$  the single scattering part of  $u$ , and so on. It is essentially this decomposition of the transport solution into orders of scatterings that allows us to stably reconstruct the optical parameters in the following sections. Note that the above Neumann series expansions has an additional benefit. Since the optical parameters are non-negative, each term in the above series is non-negative provided that  $g$  and  $S$  are non-negative so that the transport solution itself is non-negative. A little more work allows us to prove the maximum principle, which states that  $u$  in  $X \times V$  is bounded a.e. by the (essential) supremum of  $g$  in  $\Gamma_-$  when  $S \equiv 0$ .

Finally, the albedo operator  $\mathcal{A}$ , which maps incoming conditions to outgoing densities models our measurements. We control the fluxes of particles on  $\Gamma_-$  and obtain information about  $X$  by measuring the density of particles on  $\Gamma_+$ . This allows us to define the *measurements operator of inverse transport*. Let

$$\mathfrak{X} = \{(\sigma, k) \text{ such that } 0 \leq \sigma \in L^\infty(X), 0 \leq k \in L^\infty(X \times V \times V), \sigma \geq \sigma_s\}, \quad (5.38)$$

and let  $\mathfrak{Y} = \mathcal{L}(L^1(\Gamma_-, d\xi), L^1(\Gamma_+, d\xi))$ . Then we define the measurement operator  $\mathfrak{M}$

$$\mathfrak{M} : \mathfrak{X} \ni (\sigma, k) \mapsto \mathfrak{M}(\sigma, k) = \mathcal{A}[(\sigma, k)] \in \mathfrak{Y}, \quad (5.39)$$

where  $\mathcal{A}[(\sigma, k)]$  is the albedo operator constructed in (5.33) with coefficient  $(\sigma, k)$  in (5.24). Note that the measurement operator, as for the Calderón problem in (1.14) is an operator, which to a set of coefficients maps a coefficient-valued operator, the albedo operator.

The main question of inverse transport consists of knowing what can be reconstructed in  $(\sigma, k)$  from knowledge of the full operator  $\mathfrak{M}$  or knowledge of only parts of the operator  $\mathfrak{M}$ . In these notes, we shall mostly be concerned with the full measurement operator.

### 5.3 Inverse transport problem

One of the main results for inverse transport is the decomposition (5.36). The first term  $u_0 := \mathcal{I}g$  is the ballistic component and corresponds to the setting of vanishing scattering. It is therefore the term used in CT and the X-ray transform. It turns out that this term is more *singular*, in a sense that will be made precise below, than the other contributions. It can therefore be extracted from the measurement operator and provide the X-ray transform of the attenuation coefficient  $\sigma$ .

The second term in (5.36) is  $u_1 := \mathcal{K}\mathcal{I}g$  and is the the single scattering component of the transport solution. Finally,  $u_2 := u - u_0 - u_1 = (I - \mathcal{K})^{-1}\mathcal{K}^2\mathcal{I}g$  is the multiple scattering component, which corresponds to particles that have interacted at least twice with the underlying medium. A fundamental property of the transport equation is that single scattering is also more singular than multiple scattering in dimension three (and higher dimensions), but not in dimension two. We shall describe in more detail below in which sense single scattering is more *singular*. The main conclusion, however, is that the single scattering contribution can also be extracted from the full measurement operator in dimension  $n \geq 3$ . As we shall see, single scattering provides a linear operator that allows us to invert for the scattering coefficient  $k$  once  $\sigma$  is known.

Multiple scattering is then *less singular* than ballistic and single scattering. Intuitively, this means that multiple scattering contributions are smoother functions. In some sense, after multiple scattering, we do not expect the density to depend too much on the exact location of the scattering events. Multiply scattered particles visit a large domain and hence are less specific about the scenery they have visited.

#### 5.3.1 Decomposition of the albedo operator and uniqueness result

Following (5.36), we decompose the albedo operator as

$$\begin{aligned} \mathcal{A}g &= \mathcal{I}g|_{\Gamma_+} + \mathcal{K}\mathcal{I}g|_{\Gamma_+} + \mathcal{K}^2(I - \mathcal{K})^{-1}\mathcal{I}g|_{\Gamma_+} \\ &:= \mathcal{A}_0g + \mathcal{A}_1g + \mathcal{A}_2g. \end{aligned} \quad (5.40)$$

We denote by  $\alpha$  the Schwartz kernel of the albedo operator  $\mathcal{A}$ :

$$\mathcal{A}g(x, v) = \int_{\Gamma_-} \alpha(x, v, y, w)g(y, w)d\mu(y)dw.$$

Any linear operator, such as the albedo operator, admits such a decomposition. Knowledge of the operator  $\mathcal{A}$  is equivalent to knowledge of its kernel  $\alpha$ . The decomposition for  $\mathcal{A}$  then translates into the decomposition for  $\alpha$ :

$$\alpha = \alpha_0 + \alpha_1 + \alpha_2.$$

Here,  $\alpha_0$  corresponds to the ballistic part of the transport solution,  $\alpha_1$  corresponds to the single scattering part of the transport solution, and  $\alpha_2$  corresponds to the rest of the transport solution.

After some algebra, we have the following expression for the first two contributions in the time independent case:

$$\alpha_0(x, v, y, w) = \exp\left(-\int_0^{\tau_-(x,v)} \sigma(x - sv, v) ds\right) \delta_v(w) \delta_{\{x - \tau_-(x,v)v\}}(y). \quad (5.41)$$

$$\begin{aligned} \alpha_1(x, v, y, w) = & \int_0^{\tau_-(x,v)} \exp\left(-\int_0^t \sigma(x - sv, v) ds - \int_0^{\tau_-(x-tv,w)} \sigma(x - tv - sw, w) ds\right) \\ & k(x - tv, w, v) \delta_{\{x - tv - \tau_-(x-tv,w)w\}}(y) dt. \end{aligned} \quad (5.42)$$

Here,  $\delta_v(w)$  denotes a delta function at  $v$  on the unit sphere, i.e., the distribution so that

$$\delta_v(f) = \int_{\mathbb{S}^{n-1}} f(w) \delta_v(w) dw = f(v),$$

for  $f$  a continuous function on  $\mathbb{S}^{n-1}$ . Similarly,  $\delta_{\{x\}}(y)$  denotes the delta function at  $x$  on the boundary  $\partial X$ , i.e., the distribution so that

$$\delta_{\{x\}}(f) = \int_{\partial X} f(y) \delta_{\{x\}}(y) d\mu(y) = f(x),$$

for  $f$  a continuous function in the vicinity of  $x \in \partial X$ .

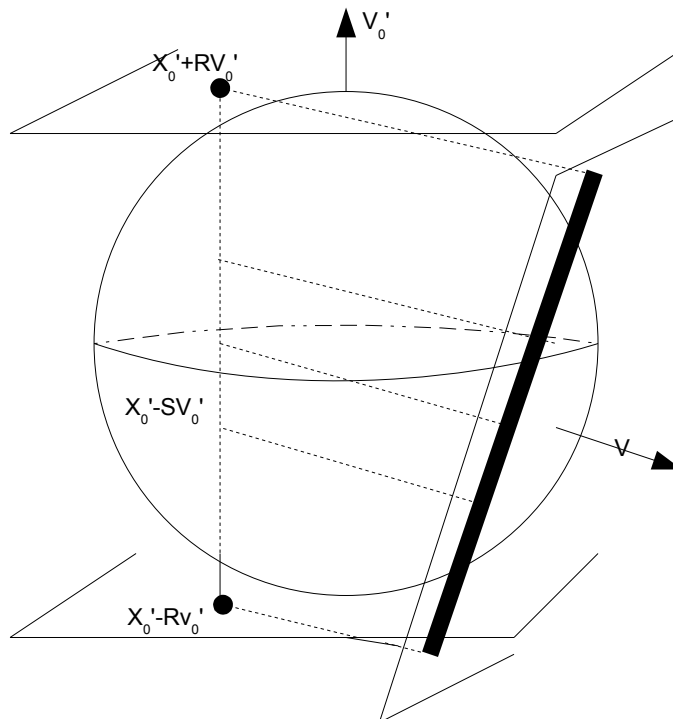
**Exercise 5.3.1** *Prove the above two formulas (5.41) and (5.42). Note that the first formula is nothing but the expression for the X-ray transform.*

Note that  $\alpha_0$  in any dimension and  $\alpha_1$  in dimension  $n \geq 3$  are *distributions*: they involve “delta” functions. In dimension  $n = 2$ , we verify that  $\alpha_1$  is in fact a *function*. A tedious and lengthy calculation shows that the kernel corresponding to multiple scattering  $\alpha_2$  is also a *function*, and when  $k$  is bounded, satisfies that

$$|\nu(y) \cdot w|^{-1} \alpha_2(x, v, y, w) \in L^\infty(\Gamma_-, L^p(\Gamma_+, d\xi)), \quad 1 \leq p < \frac{d+1}{d}. \quad (5.43)$$

We refer the reader to [13] (see also [29]) for the derivation of such a result. The exact regularity of the function  $\alpha_2$  is not very important for us here. For the mathematical derivation of stability estimates, the fact that we can take  $p > 1$  above is important. For a picture of the geometry of the singularities, see Fig. 5.1: particles emitted with direction  $v'_0$  at  $x'_0 - Rv'_0$  scatter along the segment  $x_0 - (R-t)v'_0$  for  $t \geq 0$ . Singly scattered photons reach the plane orthogonal to a given direction  $v$  on the right-hand side of fig. 5.1 along a segment of the plane. We can verify that photons that are at least doubly scattered may reach any point in that plane and in any other plane. This is consistent with the fact that  $\alpha_2$  is a function.

The strategy to recover  $\sigma$  and  $k$  in dimension  $n \geq 3$  thus goes as follows: We send beams of particles into the medium that concentrate in the vicinity of a point  $(y_0, v_0) \in \Gamma_-$ . More precisely, let  $g_\varepsilon$  be a sequence of normalized  $L^1$  functions on  $\Gamma_-$  converging as  $\varepsilon \rightarrow 0$  to  $\delta_{v_0}(v) \delta_{\{y_0\}}(y)$ .

Figure 5.1: Geometry of single scattering for  $n = 3$ .

Since the ballistic term is more singular than the rest, if we place detectors on the support of the ballistic term, then such detectors will overwhelmingly be measuring ballistic particles and very little scattered particles. Since single scattering is more singular than multiple scattering in dimension  $n \geq 3$ , we can use the same strategy and place detectors at the location of the support of single scattering. Such detectors will overwhelmingly be measuring singly scattered particles and very little multiply scattered particles. We now briefly explain the rather tedious mathematical construction.

### Recovery of the attenuation coefficient $\sigma(x)$

Let  $(y_0, v_0) \in \Gamma_-$  be defined as above and  $(x_0, v_0) \in \Gamma_+$  such that  $y_0 = x_0 - \tau_-(x_0, v_0)v_0$ . Let  $\phi_\varepsilon$  be a sequence of bounded functions on  $\Gamma_+$  equal to 1 in the vicinity of  $(x_0, v_0)$  and with vanishing support as  $\varepsilon \rightarrow 0$ . Then we verify that

$$\int_{\Gamma_+ \times \Gamma_-} \alpha_m(x, v, y, w) \phi_\varepsilon(x, v) g_\varepsilon(y, w) d\mu(x) dv d\mu(y) dw \xrightarrow{\varepsilon \rightarrow 0} 0, \quad m = 1, 2,$$

so that

$$\begin{aligned} \langle \phi_\varepsilon, \mathcal{A}g_\varepsilon \rangle &:= \int_{\Gamma_+ \times \Gamma_-} \alpha(x, v, y, w) \phi_\varepsilon(x, v) g_\varepsilon(y, w) d\mu(x) dv d\mu(y) dw \\ &\xrightarrow{\varepsilon \rightarrow 0} \exp\left(-\int_0^{\tau_-(x_0, v_0)} \sigma(x_0 - sv_0, v_0) ds\right). \end{aligned} \quad (5.44)$$

**Exercise 5.3.2** Verify (5.44); see [13] for the details.

In other words, the measurements corresponding to the above choices of functions  $g_\varepsilon$  and  $\phi_\varepsilon$  then converge to the function

$$\langle \phi_\varepsilon, \mathcal{A}g_\varepsilon \rangle \xrightarrow{\varepsilon \rightarrow 0} E(x, y) := \exp \left( - \int_0^{|x-y|} \sigma \left( x - s \frac{x-y}{|x-y|} \right) ds \right). \quad (5.45)$$

This proves that knowledge of the albedo operator  $\mathcal{A}$ , which allows one to construct  $\langle \phi_\varepsilon, \mathcal{A}g_\varepsilon \rangle$ , provides knowledge of  $E(x, y)$ , the exponential of minus the X-ray transform of  $\sigma$  along the segment  $(x, y)$ . This can be obtained from any segment  $(x, y)$  with  $x$  and  $y$  on  $\partial X$ . As a consequence, knowledge of  $\mathcal{A}$  provides knowledge of the X-ray transform of  $\sigma$ . We know that  $\sigma$  is then uniquely and stably reconstructed from such knowledge since the X-ray transform is a smoothing operator by (only) one-half of a derivative.

### Recovery of the scattering coefficient $k(x, v, w)$

We assume that  $\sigma = \sigma(x)$  is now recovered. Let  $z_0 \in X$ ,  $v_0 \in V$ , and  $v_0 \neq w_0 \in V$ . Define  $x_0 = z_0 + \tau_+(z_0, v_0)v_0$  so that  $(x_0, v_0) \in \Gamma_+$  and  $y_0 = z_0 - \tau_-(z_0, w_0)w_0$  so that  $(y_0, w_0) \in \Gamma_-$ . We formally show how the scattering coefficient may be uniquely reconstructed from full knowledge of  $\mathcal{A}$ .

Let us define  $g_{\varepsilon_1}$  as before and  $\phi_\varepsilon$  as a sequence of bounded functions on  $\Gamma_+$  equal to a constant in the vicinity of  $(x_0, v_0)$  and with vanishing support as  $\varepsilon \rightarrow 0$ . Since  $v_0 \neq w_0$ , we find that

$$\int_{\Gamma_+ \times \Gamma_-} \alpha_0(x, v, y, w) \phi_\varepsilon(x, v) g_{\varepsilon_1}(y, w) d\mu(x) dv d\mu(y) dw = 0, \quad 0 \leq \varepsilon, \varepsilon_1 < \varepsilon_0(x_0, v_0, y_0, w_0).$$

i.e., the ballistic contribution vanishes with such measurements. Let us define  $g_{\varepsilon_1}$  such that  $|\nu(y_0) \cdot w_0|^{-1} g_{\varepsilon_1}(y, w)$  converges to a delta function. The factor  $|\nu(y_0) \cdot w_0|^{-1}$  is here to ensure that the number of emitted particles is independent of  $y_0$  and  $w_0$ . The ballistic part of the transport solution is then approximately concentrated on the line passing through  $y_0$  and with direction  $w_0$ . Scattering occurs along this line and particles scattered in direction  $v_0$  are approximately supported on the plane with directions  $v_0$  and  $w_0$  passing through  $x_0$ . The intersection of that plane with the boundary  $\partial X$  is a one-dimensional curve  $F(x_0, v_0, w_0) \subset X$ . In two space dimensions, the curve  $F$  has the same dimension as  $\partial X$ . As a consequence,  $\alpha_1$  is a function and therefore is not more singular than  $\alpha_2$  in the time independent setting when  $n = 2$ .

Let  $\phi_\varepsilon(x, v)$  be a bounded test function supported in the  $\varepsilon$ -vicinity of  $F$ . Because  $F$  is of measure 0 in  $\partial X$  when  $d \geq 3$ , we find using (5.43) that

$$\int_{\Gamma_+ \times \Gamma_-} \alpha_2(x, v, y, w) \phi_\varepsilon(x, v) g_{\varepsilon_1}(y, w) d\mu(x) dv d\mu(y) dw \xrightarrow{\varepsilon, \varepsilon_1 \rightarrow 0} 0,$$

i.e., the multiple scattering contribution is asymptotically negligible with such measurements. Now, choosing  $\phi_\varepsilon(x, v)$  properly normalized and supported in the  $\varepsilon_2$ -vicinity of  $(x_0, v_0)$  (for  $\varepsilon \ll \varepsilon_2 \ll 1$ ), we find that

$$\langle \phi_\varepsilon, \mathcal{A}g_{\varepsilon_1} \rangle \xrightarrow{\varepsilon, \varepsilon_1, \varepsilon_2 \rightarrow 0} E(y_0, z_0) E(z_0, x_0) k(z_0, w_0, v_0),$$

at each point of continuity of  $k(z_0, w_0, v_0)$ , where  $E(x, y)$  is defined in (5.45). Since  $\sigma(x)$  and hence  $E(x, y)$  are known from knowledge of  $\mathcal{A}$ , then so is  $k(z_0, w_0, v_0)$  at each point of continuity in  $X \times V \times V$  thanks to the above formula.

**Exercise 5.3.3** *Verify the above formula; see [13] for the details.*

The above reconstruction of the attenuation coefficient is valid in any dimension  $n \geq 2$ . The reconstruction of the scattering coefficient, however, is valid only in dimension  $n \geq 3$ . The reason again is that in dimension  $n = 2$ , the single scattering contribution is also a function, not a distribution, and thus cannot be separated from the multiple scattering component. What we have obtained so far may be summarized as:

**Theorem 5.3.1** ([29]) *Let  $(\sigma, k)$  and  $(\tilde{\sigma}, \tilde{k})$  be two admissible pairs of optical parameters associated with the same albedo operator  $\mathcal{A}$  and such that  $\sigma$  and  $\tilde{\sigma}$  are independent of the velocity variable. Then  $\sigma = \tilde{\sigma}$  in dimension  $n \geq 2$ . Moreover,  $k = \tilde{k}$  in dimension  $n \geq 3$ .*

### 5.3.2 Stability in inverse transport

Let us assume the existence of two types of measurements  $\mathcal{A}$  and  $\tilde{\mathcal{A}}$ , say, corresponding to the optical parameters  $(\sigma, k)$  and  $(\tilde{\sigma}, \tilde{k})$ , respectively. The question of the stability of the reconstruction is to bound the errors  $\sigma - \tilde{\sigma}$  and  $k - \tilde{k}$  as a function of  $\mathcal{A} - \tilde{\mathcal{A}}$ .

We obtain stability estimates in dimension  $n \geq 3$ . In dimension  $n = 2$ , only the estimate on  $\sigma(x)$  is valid. The construction of the incoming source  $\phi_\varepsilon(x, v)$  is such that  $\phi_\varepsilon \in C^1(\Gamma_-)$  is supported in the  $\varepsilon_1$  vicinity of  $(x_0, v_0)$  and normalized so that  $\int_{\Gamma_-} \phi_\varepsilon d\xi = 1$ . Let  $\psi$  be a compactly support continuous function, which models the array of detectors, on  $\Gamma_+$  such that  $\|\psi\|_\infty \leq 1$ . Then

$$\left| \int_{\Gamma_+} \psi(x, v) ((\mathcal{A} - \tilde{\mathcal{A}})\phi_\varepsilon)(x, v) d\xi(x, v) \right| \leq \|(\mathcal{A} - \tilde{\mathcal{A}})\|_{\mathcal{L}(L^1)}, \quad (5.46)$$

where now  $\|\cdot\|_{\mathcal{L}(L^1)} = \|\cdot\|_{\mathcal{L}(L^1(\Gamma_-, d\xi), L^1(\Gamma_+, d\xi))}$ . We still introduce

$$I_m(\psi, \varepsilon) = \int_{\Gamma_+} \psi(x, v) ((\mathcal{A}_m - \tilde{\mathcal{A}}_m)\phi_\varepsilon)(x, v) d\xi(x, v), \quad m = 0, 1, 2,$$

and obtain that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} I_0(\psi, \varepsilon) &= \psi(y_0, v_0) \left( E(x_0, y_0) - \tilde{E}(x_0, y_0) \right) \\ \lim_{\varepsilon \rightarrow 0^+} I_1(\psi, \varepsilon) &= \int_V \int_0^{\tau_+(x_0, v_0)} \psi(x(s) + \tau_+(x(s), v)v, v) (E_+k - \tilde{E}_+\tilde{k})(x(s), v_0, v) ds dv \end{aligned} \quad (5.47)$$

where we have introduced  $x(s) = x_0 + sv_0$ .

The estimate (5.43) allows us to show that

$$|I_2(\psi, \varepsilon)| \leq C \int_V \left( \int_{\partial X} |\psi(t, x, v)|^{p'} dx \right)^{\frac{1}{p'}} dv, \quad p' > d. \quad (5.48)$$

Multiple scattering is therefore still negligible when the support of  $\psi := \psi_\lambda$  tends to 0 when  $\lambda \rightarrow 0$ .

The first sequence of functions  $\psi_\lambda$  is chosen to have a small support concentrated in the vicinity of  $(y_0, v_0) \in \Gamma_+$ . Then the single scattering contribution  $I_1(\psi_\lambda) \rightarrow 0$  as  $\lambda \rightarrow 0$ . For  $w_0$  fixed in  $V$ , we choose the sequence of functions  $\psi_\lambda$  such that they are concentrated in the vicinity of the curve  $F(s)$  on  $\Gamma_+$  and that they approximately take the value  $\text{sign}(E_+k - \tilde{E}_+\tilde{k})(x_0 + sv_0, v_0, w_0)$  along that curve. Since  $v_0 \neq w_0$ , we verify that  $I_0(\psi_\lambda) \rightarrow 0$  for such sequence of functions; see [13]. Now, the function  $\psi_\lambda$  has a small support only in dimension  $n \geq 3$ . Indeed, in dimension  $n = 2$ , the curve  $F$  has the same dimensionality as the boundary  $\partial X$ . When  $n = 2$ , multiple scattering may no longer be separated from single scattering by using the singular structure of the albedo operator  $\mathcal{A}$ . This allows us to state the result:

**Theorem 5.3.2** ([13]) *Assume that  $\sigma(x)$  and  $k(x, v', v)$  are continuous on  $\bar{X}$  and  $\bar{X} \times V \times V$ , respectively and that  $(\tilde{\sigma}, \tilde{k})$  satisfy the same hypotheses. Let  $(x_0, v_0) \in \Gamma_-$  and  $y_0 = x_0 + \tau_+(x_0, v_0)v_0$ . Then we have for  $d \geq 2$  that*

$$|E(x_0, y_0) - \tilde{E}(x_0, y_0)| \leq \|\mathcal{A} - \tilde{\mathcal{A}}\|_{\mathcal{L}(L^1)}, \quad (5.49)$$

while in dimension  $n \geq 3$ , we have

$$\int_V \int_0^{\tau_+(x_0, v_0)} |E_+k - \tilde{E}_+\tilde{k}|(x_0 + sv_0, v_0, v) ds dv \leq \|\mathcal{A} - \tilde{\mathcal{A}}\|_{\mathcal{L}(L^1)}. \quad (5.50)$$

The stability obtained above for the X-ray transform of the absorption coefficient is not sufficient to obtain any stability of  $\sigma$  itself without a priori regularity assumptions on  $\sigma$ . This results from the well known fact that the X-ray transform is a smoothing (compact) operator so that the inverse X-ray transform is an unbounded operator. Let us assume that  $\sigma$  belongs to some space  $H^s(\mathbb{R}^n)$  for  $s$  sufficiently large and that  $\sigma_p$  defined in (5.25) is bounded. More precisely, define

$$\mathcal{M} = \{(\sigma, k) \in C^0(\bar{X}) \times C^0(\bar{X} \times V \times V) | \sigma \in H^{\frac{n}{2}+r}(X), \|\sigma\|_{H^{\frac{n}{2}+r}(X)} + \|\sigma_p\|_\infty \leq M\}, \quad (5.51)$$

for some  $r > 0$  and  $M > 0$ . Then, we have the following result.

**Theorem 5.3.3** ([13, 14]) *Let  $n \geq 2$  and assume that  $(\sigma, k) \in \mathcal{M}$  and that  $(\tilde{\sigma}, \tilde{k}) \in \mathcal{M}$ . Then the following is valid:*

$$\|\sigma - \tilde{\sigma}\|_{H^s(X)} \leq C \|\mathcal{A} - \tilde{\mathcal{A}}\|_{\mathcal{L}(L^1)}^\kappa, \quad (5.52)$$

where  $-\frac{1}{2} \leq s < \frac{n}{2} + r$  and  $\kappa = \frac{n+2(r-s)}{n+1+2r}$ .

When  $d \geq 3$ , we have

$$\|k - \tilde{k}\|_{L^1(X \times V \times V)} \leq \|\mathcal{A} - \tilde{\mathcal{A}}\|_{\mathcal{L}(L^1)}^{\kappa'} (1 + \|\mathcal{A} - \tilde{\mathcal{A}}\|_{\mathcal{L}(L^1)}^{1-\kappa'}), \quad (5.53)$$

where  $\kappa' = \frac{2(r-r')}{n+1+2r}$  and  $0 < r' < r$ .

Such estimates show that under additional regularization assumptions on  $\sigma$ , we have explicit stability expression of Hölder type on  $\sigma$  and  $k$ . The first stability result (5.52) was first established in [67].

The proof of these theorems is fairly technical and will not be presented here in detail; see [13, 14].



# Chapter 6

## Inverse diffusion and severe ill-posedness

This chapter introduces a classical example of a severely ill-posed problem for which the measurement operator is injective. In the absence of any noise, reconstructions are feasible. However, (in)stability is such that even tiny amounts of noise in moderately high frequency will be so amplified during the reconstruction that the resulting noise in the reconstructed parameters may overwhelm the real parameters. We have seen in earlier chapters that solving the heat equation backward was an example of a severely ill-posed problem. In the preceding chapter, we saw that scattering was responsible for smoothing (in the sense that the multiple scattering contribution of the albedo operator in (5.40) was smoother than the single scattering contribution and that the latter was smoother than the ballistic contribution). In this chapter and in the next chapter, we consider two new examples: the Cauchy problem, which is an inverse source problem, and the Inverse Diffusion problem, also known as the Calderón problem, which is a nonlinear inverse problem. In both problems, the forward modeling, based on a diffusion equation, is highly smoothing because of scattering effects.

This chapter is devoted to the analysis of the Cauchy problem and related inverse problems.

### 6.1 Cauchy Problem and Electrocardiac potential

Let us consider the imaging of the electrical activity of the heart. A probe is sent inside the heart, where the electrical potential is measured. The inverse problem consists of reconstructing the potential on the endocardial surface (the inside of the cardiac wall) from the measurements on the probe.

Mathematically, this corresponds to solving a Cauchy problem for an elliptic equation. The problem is modeled as follows. Let  $\Gamma_0$  be a closed smooth surface in  $\mathbb{R}^3$  representing the endocardial surface and let  $\Gamma_1$  be the closed smooth surface inside the volume enclosed by  $\Gamma_0$  where the measurements are performed. We denote by  $X$  the domain with boundary  $\partial X = \Gamma_0 \cup \Gamma_1$ ; see Fig.6.1. The electric potential solves the

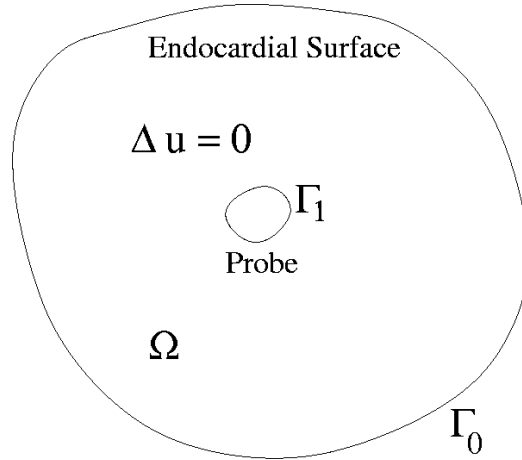


Figure 6.1: Geometry of endocardial measurements

following Laplace equation:

$$\begin{aligned}
 \Delta u &= 0 && \text{in } X, \\
 u &= u_1 && \text{on } \Gamma_1, \\
 \frac{\partial u}{\partial n} &= 0 && \text{on } \Gamma_1.
 \end{aligned} \tag{6.1}$$

The function  $u_1$  models the measurements at the surface of the probe, which is assumed to be insulated so that  $n \cdot \nabla u = 0$  on  $\Gamma_1$ . The objective is then to find  $u = u_0$  on  $\Gamma_0$ .

## 6.2 Half Space Problem

Let us consider the simplified geometry where  $X$  is the slab  $\mathbb{R}^{n-1} \times (0, L)$ . We assume that boundary measurements are given at the boundary  $x_n = 0$  and wish to obtain the harmonic solution in  $X$ . We are thus interested in solving the equation

$$\begin{aligned}
 \Delta u &= 0, && x = (x', x_n) \in X \\
 u(x', 0) &= f(x'), && x' \in \mathbb{R}^{n-1} \\
 \frac{\partial u}{\partial x_n}(x', 0) &= g(x'), && x' \in \mathbb{R}^{n-1}.
 \end{aligned} \tag{6.2}$$

Let us denote by

$$\hat{u}(k', x_n) = (\mathcal{F}_{x' \rightarrow k'} u)(k', x_n). \tag{6.3}$$

Upon Fourier transforming (6.2) in the  $x'$  variable, we obtain

$$\begin{aligned}
 -|k'|^2 \hat{u} + \frac{\partial^2 \hat{u}}{\partial x_n^2} &= 0, && k' \in \mathbb{R}^{n-1}, 0 < x_n < L \\
 \hat{u}(k', 0) &= \hat{f}(k'), && k' \in \mathbb{R}^{n-1}, \\
 \frac{\partial \hat{u}}{\partial x_n}(k', 0) &= \hat{g}(k'), && k' \in \mathbb{R}^{n-1}.
 \end{aligned} \tag{6.4}$$

The solution of the above ODE is given by

$$\hat{u}(k', x_n) = \hat{f}(k') \cosh(|k'|x_n) + \frac{\hat{g}(k')}{|k'|} \sinh(|k'|x_n). \quad (6.5)$$

We thus observe that the solution at  $x_n > 0$  is composed of an exponentially growing component  $e^{|k'|x_n}$  and an exponentially decreasing component  $e^{-|k'|x_n}$ .

### 6.2.1 The well posed problem

Let us assume we are in the favorable case where

$$|k'| \hat{f}(k') + \hat{g}(k') = 0, \quad k' \in \mathbb{R}^{n-1}. \quad (6.6)$$

In the physical domain, this corresponds to satisfying the non-local problem

$$\sqrt{-\Delta} f(x') + g(x') = 0 \quad x' \in \mathbb{R}^{n-1}, \quad (6.7)$$

with  $\sqrt{-\Delta}$  defined as the square root of the Laplacian. It takes the form  $\sqrt{-\Delta} = H \frac{d}{dx}$  in two dimensions  $n = 2$ , where  $H$  is the Hilbert transform.

In the Fourier domain, the solution is thus given by

$$\hat{u}(k', x_n) = \hat{f}(k') e^{-|k'|x_n}. \quad (6.8)$$

**Exercise 6.2.1** (i) In two dimensions  $n = 2$ , prove that

$$u(x_1, x_2) = (f * \frac{1}{\pi} \frac{x_2}{x_1^2 + x_2^2})(x) = \frac{1}{\pi} \int_{\mathbb{R}} f(x_1 - z) \frac{x_2}{z^2 + x_2^2} dz. \quad (6.9)$$

Hint: Use (1.32) and show that

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{-x_2|k|} e^{ix_1 k_1} dk_1 = \frac{1}{\pi} \frac{x_2}{x_1^2 + x_2^2}.$$

(ii) Show that  $\frac{1}{\pi} \frac{y}{x_1^2 + x_2^2}$  is the fundamental solution of (6.2) when  $n = 2$  with  $f(x_1) = \delta(x_1)$ . Calculate the corresponding value of  $g(x_1)$ .

Provided the above compatibility condition is met, the problem (6.2) admits a unique solution and is well-posed, for instance in the sense that

$$\int_{\mathbb{R}^{n-1}} u^2(x', x_n) dx' \leq \int_{\mathbb{R}^{n-1}} f^2(x') dx', \quad \text{for all } x_n > 0.$$

This is an immediate consequence of (6.8) and the Parseval relation. In this situation, the construction of  $u$  for  $x_n > 0$  becomes a well-posed problem.

## 6.2.2 The electrocardiac application

In general, however, (6.6) is not satisfied. For the electro-cardiac application,  $g_0 = 0$  and the harmonic solution is therefore given (for all  $x_n > 0$ ) by

$$\hat{u}(k', x_n) = \hat{f}(k') \cosh |k'|x_n. \quad (6.10)$$

This implies that high frequencies are exponentially amplified. Therefore, the mapping from  $f(x')$  to  $A_{x_n}f(x') := u(x', x_n)$  for a fixed  $x_n > 0$  cannot be bounded from any  $H^s(\mathbb{R}^{n-1})$  (even for  $s$  very large) to any  $H^t(\mathbb{R}^{n-1})$  (even for  $-t$  very large). This does not mean that  $A_{x_n}$  cannot be inverted. Let us define the space

$$\mathfrak{X}_{x_n}(\mathbb{R}^{n-1}) = \{u \in L^2(\mathbb{R}^{n-1}), \cosh(|k'|x_n)\hat{u}(k') \in L^2(\mathbb{R}^{n-1})\}. \quad (6.11)$$

Then,  $A_{x_n}$  is indeed continuous in  $\mathcal{L}(L^2(\mathbb{R}^{n-1}), \mathfrak{X}_{x_n}(\mathbb{R}^{n-1}))$  with an inverse  $A_{x_n}^{-1}$  continuous in  $\mathcal{L}(\mathfrak{X}_{x_n}(\mathbb{R}^{n-1}), L^2(\mathbb{R}^{n-1}))$  and given by

$$A_{x_n}^{-1}u = \mathcal{F}_{k' \rightarrow x'}^{-1} \cosh(|k'|x_n) \mathcal{F}_{x' \rightarrow k'} u. \quad (6.12)$$

This construction is useful when noise is small in  $\mathfrak{X}_{x_n}(\mathbb{R}^{n-1})$ , which means that it is essentially low-frequency. Such a restrictive assumption on noise is often not verified in practice. When noise is small only in some  $L^2$  sense, for instance, then  $A_{x_n}$  above cannot be inverted without assuming prior information about the solution we seek to reconstruct. Chapters 10 and 11 will be devoted to the analysis of such prior information. Here, we look at how stability estimates may still be obtained when simple bounds are available on the harmonic solutions.

## 6.2.3 Prior bounds and stability estimates

Let us consider the solution (6.10) in the setting where  $f$  is small in some sense (modeling the level of noise in the available data since our problem is linear) and where  $u(x)$  the harmonic solution is bounded a priori in some sense. We can then obtain a stability result of the form:

**Theorem 6.2.1** *Let  $u(x)$  be the solution to (6.2) with  $g \equiv 0$ . Let us assume that*

$$\|f\|_{L^2(\mathbb{R}^{n-1})} \leq \eta \leq E, \quad \|u(\cdot, L)\|_{L^2(\mathbb{R}^{n-1})} \leq E. \quad (6.13)$$

*Then we find that*

$$\|u(\cdot, x_n)\|_{L^2(\mathbb{R}^{n-1})} \leq C\eta^{1-\frac{x_n}{L}} E^{\frac{x_n}{L}}, \quad (6.14)$$

*for some universal constant  $C$ .*

*Proof.* The proof is based on the following observation:  $\hat{u}(k', x_n)$  is close to  $\hat{f}(k')$  for small values of  $|k'|$ , where we wish to use the constraint involving  $\eta$ . For large  $|k'|$ , the bound in  $\eta$  on  $\hat{f}(k')$  is no longer sufficient and we need to use the bound at  $L$ , which constrains the growth at high frequencies. We thus calculate

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} |\hat{f}|^2(k') \cosh^2(|k'|x_n) dk' &= \int_{|k'| < k_0} |\hat{f}|^2(k') \cosh^2(|k'|x_n) dk' \\ &+ \int_{|k'| > k_0} |\hat{f}|^2(k') \cosh^2(|k'|x_n) dk' \leq \cosh^2(k_0 x_n) \eta^2 + \frac{\cosh^2(k_0 x_n)}{\cosh^2(k_0 L)} E^2. \end{aligned}$$

We now equate both terms. Since  $k_0$  is “large”, we verify that  $\cosh x \sim \frac{1}{2}e^x$  and choose

$$k_0 = \frac{1}{L} \ln \frac{2E}{\eta}.$$

With this choice, we obtain that

$$\int_{\mathbb{R}^{n-1}} |\hat{f}|^2(k') \cosh^2(|k'|x_n) dk' \leq C\eta^2 \left(\frac{2E}{\eta}\right)^{\frac{x_n}{L}} \leq C \left(\eta^{\frac{x_n}{L}} E^{1-\frac{x_n}{L}}\right)^2.$$

This proves the result.  $\square$

The above result states that even though solving the Cauchy problem is severely ill-posed, if we possess information about the solution at a distance  $L$  from where the measurements are collected, then we can reconstruct the elliptic solution for all  $0 < x_n < L$  with Hölder-type stability, which is a mildly ill-posed problem. However, at  $x_n = L$ , which is where we would like to obtain some stable reconstructions in the electro-cardiac problem, then a mere bound at  $x_n = L$  in the  $L^2$  norm is not sufficient to obtain any meaningful stability. However, a stronger bound would be sufficient as the following result indicates.

**Theorem 6.2.2** *Let  $u(x)$  be the solution to (6.2) with  $g \equiv 0$ . Let us assume that*

$$\|f\|_{L^2(\mathbb{R}^{n-1})} \leq \eta \leq E, \quad \|u(\cdot, L)\|_{H^s(\mathbb{R}^{n-1})} \leq E, \quad s > 0. \quad (6.15)$$

*Then we find that*

$$\|u(\cdot, x_n)\|_{L^2(\mathbb{R}^{n-1})} \leq C \left| \ln \frac{E}{\eta} \right|^{-s \frac{x_n}{L}} \eta^{1-\frac{x_n}{L}} E^{\frac{x_n}{L}}, \quad (6.16)$$

*for some universal constant  $C$ .*

*Proof.* The proof is similar to the preceding one. We calculate

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} |\hat{f}|^2(k') \cosh^2(|k'|x_n) dk' &= \int_{|k'| < k_0} |\hat{f}|^2(k') \cosh^2(|k'|x_n) dk' \\ &+ \int_{|k'| > k_0} |\hat{f}|^2(k') \cosh^2(|k'|x_n) dk' \leq \cosh^2(k_0 x_n) \eta^2 + \frac{\cosh^2(k_0 x_n)}{\cosh^2(k_0 L)} \langle k_0 \rangle^{-2s} E^2. \end{aligned}$$

We recall that  $\langle k_0 \rangle^2 = 1 + k_0^2$ . We then choose

$$k_0 = \frac{1}{L} \ln \frac{2E}{\eta} - \frac{s}{L} \ln \frac{1}{L} \ln \frac{2E}{\eta}.$$

This is a good approximation to the equation  $\cosh^2(k_0 L) \eta^2 = \langle k_0 \rangle^{-2s} E^2$ . We then find

$$\eta e^{k_0 x_n} = \left| \frac{1}{L} \ln \frac{2E}{\eta} \right|^{-s \frac{x_n}{L}} \eta^{1-\frac{x_n}{L}} (2E)^{\frac{x_n}{L}},$$

from which the result follows.  $\square$

For  $x_n < L$ , the latter result is not much more precise than the preceding one. However, when  $x_n = L$ , we obtain that the error in the  $L^2(\mathbb{R}^{n-1})$  norm of  $u(x', x_n)$  is of order  $|\ln \eta|^{-s}$ . This is a *logarithmic* stability result that cannot be improved and indicates the severe ill-posedness of the electro-cardiac problem. Note also that the stability estimate depends on the *strength* of the exponent  $s$  in the prior regularity assumption.

**Exercise 6.2.2** Let us assume that in Theorem 6.2.2, the prior estimate is replaced by an estimate of the form

$$\|u\|_{H^m(\mathbb{R}^{n-1} \times (0, L))} \leq E, \quad m \geq 0 \quad (6.17)$$

- (i) Obtain a result of the form (6.16) in this setting.  
(ii) Show that  $\|u(\cdot, L)\|_{L^2(\mathbb{R}^{n-1})}$  satisfies a logarithmic stability estimate when  $m$  is sufficiently large (how large?). Relate this result to that of Theorem 6.2.2.

### 6.2.4 Analytic continuation

The above results in dimension  $n = 2$  are intimately connected to the analytic continuation of an analytic function given on the real line.

Let  $f(z) = g(z) + ih(z)$  be an analytic function with  $g(z)$  and  $h(z)$  real valued-functions. Let us assume that  $g(z)$  and  $h(z)$  are known on the real line  $\Im(z) = 0$ . The objective is to find them for arbitrary values of  $z$ . We identify  $z = x + iy$  and assume that  $g(x, 0)$  and  $h(x, 0)$  are distributions in  $\mathcal{S}'(\mathbb{R})$  so that their Fourier transform is defined. Since  $f(z)$  is analytic, we have

$$\frac{\partial f}{\partial \bar{z}} = 0,$$

or equivalently that

$$\frac{\partial g}{\partial x} - \frac{\partial h}{\partial y} = 0, \quad \frac{\partial g}{\partial y} + \frac{\partial h}{\partial x} = 0. \quad (6.18)$$

These Cauchy-Riemann relations imply that  $g$  and  $h$  are harmonic, i.e.,  $\Delta g = \Delta h = 0$ . They thus solve the following problems

$$\begin{aligned} \Delta g &= 0, & y &> 0, & \Delta h &= 0, & y &> 0, \\ \frac{\partial g}{\partial y}(x, 0) &= -\frac{\partial h}{\partial x}(x, 0), & & & \frac{\partial h}{\partial y}(x, 0) &= \frac{\partial g}{\partial x}(x, 0), \\ g(x, 0) &\text{ known,} & & & h(x, 0) &\text{ known.} \end{aligned} \quad (6.19)$$

Both problems are of the form (6.2). The solutions in the Fourier domain are given by

$$\begin{aligned} \hat{g}(k_x, y) &= \hat{g}(k_x, 0) \cosh(|k_x|y) - i \operatorname{sign}(k_x) \hat{h}(k_x, 0) \sinh(|k_x|y) \\ \hat{h}(k_x, y) &= \hat{h}(k_x, 0) \cosh(|k_x|y) + i \operatorname{sign}(k_x) \hat{g}(k_x, 0) \sinh(|k_x|y) \end{aligned} \quad (6.20)$$

We verify that the problem is well-posed provided that (6.7) is verified, which in this context means

$$\frac{\partial h}{\partial x} = H \frac{\partial g}{\partial x}, \quad \frac{\partial g}{\partial x} = -H \frac{\partial h}{\partial x}. \quad (6.21)$$

Notice that  $H^2 = -I$  so that both equalities above are equivalent. When the above conditions are met, then the analytic continuation is a stable process. When they are not met, we have seen that high frequencies increase exponentially as  $y$  increases, which renders the analytic continuation process a severely ill-posed problem.

## 6.3 General two dimensional case

We now consider a general two-dimensional geometry as described in section 6.1. We use the Riemann mapping theorem to map such a geometry conformally to an annulus (the region lying between two concentric circles) in the plane. We then solve the problem on the annulus. The Riemann mapping gives us a stable way to transform the original domain to an annulus and back. We will see that solving the problem on the annulus is severely ill-posed similarly to what we saw in the preceding section.

### 6.3.1 Laplace equation on an annulus

Let us begin with a presentation of the problem on the annulus. We assume that the inner circle has radius 1 and the outer circle radius  $\rho > 0$ . By simple dilation, this is equivalent to the more general case of two circles of arbitrary radius  $a$  and  $b$ . In polar coordinates the Laplacian takes the form

$$\Delta u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}. \quad (6.22)$$

The general solution to the above equation periodic in  $\theta$  is decomposed in Fourier modes as

$$u(r, \theta) = a_0 + b_0 \ln r + \sum_{n \in \mathbb{N}^*} \left( \frac{a_n}{2} r^n + \frac{b_n}{2} r^{-n} \right) e^{in\theta}. \quad (6.23)$$

Since  $\nu \cdot \nabla u = 0$  on at  $r = 1$ , we deduce that  $b_0$  and  $b_n - a_n$  vanish. We then find that the solution to (6.1) on the annulus is given by

$$u(r, \theta) = \sum_{n \in \mathbb{N}} \left( \frac{1}{2\pi} \int_0^{2\pi} e^{-in\phi} u_1(\phi) d\phi \right) \frac{r^n + r^{-n}}{2} e^{in\theta}. \quad (6.24)$$

The above solution holds for all  $r > 1$ .

**Exercise 6.3.1** Find the relation that  $u_1(\theta) = u(1, \theta)$  and  $g_1(\theta) = \nu \cdot \nabla u(1, \theta)$  must satisfy so that the problem

$$\begin{aligned} \Delta u &= 0, & |r| &> 1 \\ u(1, \theta) &= u_1(\theta), & n \cdot \nabla u(1, \theta) &= g_1(\theta), & 0 \leq \theta < 2\pi, \end{aligned}$$

is well-posed (in the sense that the energy of  $\theta \rightarrow u(\rho, \theta)$  is bounded by the energy of  $u_1(\theta)$ ).

We verify that an error of size  $\delta$  in the measurement of the coefficient  $a_n$  is amplified into an error of order

$$A_n(\rho) = \frac{\delta}{2} e^{n \ln \rho} \quad \text{at} \quad r = \rho.$$

We verify that  $A_n$  cannot be bounded by any  $Cn^\alpha$  for all  $\alpha > 0$ , which would correspond to differentiating the noise level  $\alpha$  times. This implies that the reconstruction of  $u(\rho, \theta)$  from  $u_1(\theta)$  using (6.24) is a severely ill-posed problem.

### 6.3.2 Riemann mapping theorem

Let  $X$  be an open smooth two dimensional domain with smooth boundary having two smooth connected components. We denote by  $\Gamma_0$  the outer component of  $\partial X$  and  $\Gamma_1$  the inner component; see Fig. 6.1.

For  $z \in X \subset \mathbb{C}$  we construct a holomorphic function  $\Psi(z)$  (i.e., a function such that  $\frac{\partial \Psi}{\partial \bar{z}} = 0$ ) mapping  $X$  to an annulus of the form  $1 < r < \rho$ . The function is constructed as follows. Let first  $v$  be the unique solution to the following Dirichlet problem

$$\Delta v = 0 \quad \text{on } X, \quad v|_{\Gamma_0} = 1, \quad v|_{\Gamma_1} = 0. \quad (6.25)$$

For some  $c$  to be fixed later, let  $G = cv$ . We verify that

$$I = \int_{\Gamma_1} -\frac{\partial G}{\partial y} dx + \frac{\partial G}{\partial x} dy = -c \int_{\Gamma_1} \frac{\partial v}{\partial \nu} ds > 0$$

by the maximum principle ( $v$  is positive inside  $X$  and its normal derivative must be negative on  $\Gamma_1$ ). We fix  $c$  such that  $I = 2\pi$ . In that case we can define a function  $H(z)$ , a conjugate harmonic of  $G(z)$  on  $X$ , by

$$H(z) = \int_p^z -\frac{\partial G}{\partial y} dx + \frac{\partial G}{\partial x} dy, \quad (6.26)$$

where  $p$  is an arbitrary point in  $X$ . Since  $G$  is harmonic, we verify that the definition of  $H$  is independent of the path chosen between  $p$  and  $z$ . Moreover we verify that

$$\frac{\partial H}{\partial x} = -\frac{\partial G}{\partial y}, \quad \frac{\partial H}{\partial y} = \frac{\partial G}{\partial x},$$

so that  $G + iH$  is a holomorphic function on  $X$ . Then so is

$$\Psi(z) = e^{G(z)+iH(z)}. \quad (6.27)$$

We verify that  $\Psi(z)$  maps  $\Gamma_0$  to the circle  $|z| = e^c$  and  $\Gamma_1$  to the circle  $|z| = 1$ . Moreover  $\Psi$  is a diffeomorphism between  $X$  and the annulus  $U_c = \{z \in \mathbb{C}, 1 \leq |z| \leq e^c\}$ . Finally we verify that  $\Delta \Psi(z) = 0$  on  $X$  since  $\Psi$  is holomorphic. Therefore we have replaced the problem (6.1) on  $X$  by solving the Laplace equation on  $U_c$  with boundary conditions  $u_1(\Psi(z))$  on the circle  $r = 1$  and vanishing Neumann boundary conditions (we verify that Neumann boundary conditions are preserved by the map  $\Psi$ ).

**Exercise 6.3.2** *Verify the above statements.*

## 6.4 Backward Heat Equation

Let us now consider the backward heat equation. The forward heat equation is given by

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u &= 0 & t > 0, \quad x \in \mathbb{R}^n \\ u(0, x) &= f(x) & x \in \mathbb{R}^n, \end{aligned} \quad (6.28)$$

with  $f(x)$  an initial condition. Let us assume that  $u(T, x) = \mathfrak{M}f(T, x)$  is measured and that we wish to reconstruct  $f(x)$ . In this setting, we pass to the Fourier domain and obtain as above that

$$\hat{u}(t, k) = e^{-t|k|^2} \hat{f}(k). \quad (6.29)$$

Therefore,  $\hat{f}(k) = e^{T|k|^2} \hat{u}(T, k)$  is the formal inverse to the measurement operator  $\mathfrak{M}$ . Again, this is a severely ill-posed problem. But with prior information about the solution  $u(t, x)$ , we can again obtain Hölder and logarithmic type stability results. For instance, we have

**Theorem 6.4.1** *Let us assume that*

$$\|u(T, \cdot)\|_{L^2(\mathbb{R}^n)} = \eta \leq E, \quad \|f\|_{H^s(\mathbb{R}^n)} \leq E. \quad (6.30)$$

*Then we find that*

$$\|u(t, \cdot)\|_{L^2(\mathbb{R}^n)} \leq C \left| \ln \frac{\eta}{E} \right|^{-\frac{s(T-t)}{2T}} \eta^{\frac{t}{T}} E^{1-\frac{t}{T}}. \quad (6.31)$$

*Proof.* The proof is the same as above. We use the  $\eta$  bound for  $|k| < k_0$  and the other bound for  $|k| > k_0$ . We then choose  $k_0$  so that

$$k_0^2 = \frac{1}{T} \ln \frac{E}{\eta} - \frac{s}{2T} \ln k_0^2 \approx \frac{1}{T} \ln \frac{E}{\eta} - \frac{s}{2T} \ln \frac{1}{T} \ln \frac{E}{\eta}.$$

The rest of the proof proceeds as above. Note that the main difference with respect to the Cauchy problem is that  $s$  is replaced by  $\frac{s}{2}$  in the estimate because frequencies are exponentially damped with a strength proportional to  $|k|^2$  rather than  $|k|$ .  $\square$

Note that for all  $t > 0$ , the error on the reconstruction of  $u(t, x)$  is Hölder in  $\eta$ , which measures the errors in the measurements. At  $t = 0$ , which is the problem of interest in the backward heat equation, we obtain a logarithmic stability: the errors in the reconstruction are proportional to  $|\ln \eta|^{-\frac{s}{2}}$ , which does converge to 0 as  $\eta \rightarrow 0$ , but does so extremely slowly.

The above proof generalizes to a large class of elliptic operators. For instance, consider the problem

$$\begin{aligned} \frac{\partial u}{\partial t} - Lu &= 0, & t > 0, x \in X \\ u(t, x) &= 0 & x \in \partial X \\ u(0, x) &= f(x), & x \in X \end{aligned} \quad (6.32)$$

with  $L$  a symmetric operator such as for instance  $-\nabla \cdot a \nabla + c$  admitting a spectral decomposition

$$-L\phi_n = \lambda_n \phi_n \text{ in } X \quad \phi_n = 0 \text{ on } \partial X, \quad (6.33)$$

with an orthonormal basis of normalized eigenvectors  $\phi_n$  and positive eigenvalues  $0 < \lambda_1 < \lambda_2 \leq \dots$  with  $\lambda_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Then we can decompose

$$f(x) = \sum_{n \geq 1} f_n \phi_n(x), \quad u(t, x) = \sum_{n \geq 1} u_n(t) \phi_n(x).$$

Let us assume that  $u_n(T)$  is measured and that we want to reconstruct  $u_n(t)$  for  $t \geq 0$  and hence also  $f_n$  at  $t = 0$ . Then we find that

$$u_n(t) = e^{-\lambda_n t} f_n = e^{\lambda_n(T-t)} u_n(T).$$

Let us assume that  $f(x)$  admits an a priori bound in  $H^s(X)$  of the form

$$\|f\|_{H^s(X)} := \left( \sum_{n \geq 1} n^{2s} f_n^2 \right)^{\frac{1}{2}} = E. \quad (6.34)$$

Let us assume that we have an error of the form

$$\|u(T, \cdot)\|_{L^2(X)} := \left( \sum_{n \geq 1} u_n^2(T) \right)^{\frac{1}{2}} = \eta \leq E. \quad (6.35)$$

Let us assume that  $\lambda_n \approx n^\alpha$  for some  $\alpha > 0$  asymptotically. Then we calculate

$$\begin{aligned} \sum_{n \geq 1} u_n^2(t) &= \sum_{n \leq n_0-1} (e^{\lambda_n(T-t)} u_n(T))^2 + \sum_{n \geq n_0} (e^{-\lambda_n t} f_n)^2 \\ &\leq e^{2\lambda_{n_0}(T-t)} \eta^2 + e^{-2\lambda_{n_0} t} n_0^{-2s} E^2. \end{aligned}$$

Again, we use the estimate on the noise  $\eta$  for the low frequencies of  $u(t)$  and the regularity estimate  $E$  for the high frequencies of  $u(t)$ . It remains to equate these two terms to find that

$$n_0^\alpha \approx \frac{1}{T} \ln \frac{E}{\eta} - \frac{s}{T\alpha} \ln \frac{1}{T} \ln \frac{E}{\eta}.$$

This yields the error estimate

$$\|u(t, \cdot)\|_{L^2(X)} \leq C \left| \ln \frac{\eta}{E} \right|^{-\frac{(T-t)s}{T\alpha}} \eta^{\frac{t}{T}} E^{1-\frac{t}{T}}. \quad (6.36)$$

We leave it to the reader to fill in the gaps and state theorems. As usual, we obtain a Hölder estimate for all  $0 < t \leq T$ . At  $t = 0$ , the stability estimate becomes again logarithmic and crucially relies on the prior information that  $f$  is bounded in some space  $H^s(X)$  with  $s > 0$ .

# Chapter 7

## Calderón problem

This chapter focuses on the Calderón problem, which is another inverse diffusion problem with a similar behavior to the Cauchy problems considered in the preceding chapter. Because of its central role in inverse problem theory and of the importance of the mathematical tools that are used to solve it, we have devoted a full chapter to its study.

### 7.1 Introduction

The Calderón problem finds applications in two medical imaging methods, *electrical impedance tomography* (EIT) and *optical tomography* (OT). We begin with the setting of EIT, a non-invasive modality that consists of reconstructing the electrical properties (the conductivity) of tissues from current and voltage boundary measurements. The mathematical framework for the Calderón problem and the definition of the measurement operator were presented as Example 5 in section 1.2. For completeness, we repeat this definition here. The Calderón problem is modeled by the following elliptic problem with Dirichlet conditions

$$\begin{aligned} L_\gamma u(x) &\equiv \nabla \cdot \gamma(x) \nabla u(x) = 0, & x \in X \\ u(x) &= f(x), & x \in \partial X, \end{aligned} \tag{7.1}$$

where  $X \subset \mathbb{R}^n$  is a bounded domain with smooth boundary  $\partial X$ . In what follows, we assume  $n \geq 3$ . Here,  $\gamma(x)$  is a conductivity coefficient, which we assume is a smooth function, and  $f(x)$  is a prescribed Dirichlet data for the elliptic problem.

The Dirichlet-to-Neumann or voltage-to-current map is given by

$$\Lambda_\gamma : \begin{aligned} H^{\frac{1}{2}}(\partial X) &\rightarrow H^{-\frac{1}{2}}(\partial X) \\ f(x) &\mapsto \Lambda_\gamma[f](x) = \gamma(x) \frac{\partial u}{\partial \nu}(x). \end{aligned} \tag{7.2}$$

With  $\mathfrak{X} = C^2(\bar{X})$  and  $\mathfrak{Y} = \mathcal{L}(H^{\frac{1}{2}}(\partial X), H^{-\frac{1}{2}}(\partial X))$ , we define the measurement operator

$$\mathfrak{M} : \mathfrak{X} \ni \gamma \mapsto \mathfrak{M}(\gamma) = \Lambda_\gamma \in \mathfrak{Y}. \tag{7.3}$$

The Calderón problem consists of reconstructing  $\gamma$  from knowledge of the Calderón measurement operator  $\mathfrak{M}$ . To slightly simplify the derivation of uniqueness, we also

make the (unnecessary) assumption that  $\gamma$  and  $\nu \cdot \nabla \gamma$  are *known* on  $\partial X$ . The main result of this chapter is the following.

**Theorem 7.1.1** *Define the measurement operator  $\mathfrak{M}$  as in (7.3). Then  $\mathfrak{M}$  is injective in the sense that  $\mathfrak{M}(\gamma) = \mathfrak{M}(\tilde{\gamma})$  implies that  $\gamma = \tilde{\gamma}$ .*

*Moreover, we have the following logarithmic stability estimate:*

$$\|\gamma(x) - \gamma'(x)\|_{L^\infty(X)} \leq C |\log \|\mathfrak{M}(\gamma) - \mathfrak{M}(\tilde{\gamma})\|_{\mathfrak{M}}|^{-\delta}, \quad (7.4)$$

for some  $\delta > 0$  provided that  $\gamma$  and  $\tilde{\gamma}$  are uniformly bounded in  $H^s(X)$  for some  $s > \frac{n}{2}$ .

The proof of the injectivity result was first obtained in [60]. It is based on two main ingredients. The first ingredient consists of recasting the injectivity result as a statement of whether products of functionals of solutions to elliptic equations such as (7.2) are dense in the space of, say, continuous functions. The second step is to construct specific sequences of solutions to (7.2) that positively answer the density question. These specific solutions are Complex Geometric Optics (CGO) solutions. Their construction is detailed in section 7.3. We first start with a section on the uniqueness and stability results.

## 7.2 Uniqueness and Stability

### 7.2.1 Reduction to a Schrödinger equation

We start with the following lemma

**Lemma 7.2.1** *Let  $\Lambda_{\gamma_j}$  for  $j = 1, 2$  be the two operators associated to  $\gamma_j$  and let  $f_j \in H^{\frac{1}{2}}(\partial X)$  for  $j = 1, 2$  be two Dirichlet conditions. Then we find that*

$$\int_{\partial X} (\Lambda_{\gamma_1} - \Lambda_{\gamma_2}) f_1 f_2 d\mu = \int_X (\gamma_1 - \gamma_2) \nabla u_1 \cdot \nabla u_2 dx, \quad (7.5)$$

where  $d\mu$  is the surface measure on  $\partial X$  and where  $u_j$  is the solution to (7.1) with  $\gamma$  replaced by  $\gamma_j$  and  $f$  replaced by  $f_j$ .

Here we use the notation on the left-hand side to mean:

$$\int_{\partial X} (\Lambda_{\gamma_1} - \Lambda_{\gamma_2}) f_1(x) f_2(x) d\mu(x) := \langle (\Lambda_{\gamma_1} - \Lambda_{\gamma_2}) f_1, f_2 \rangle_{H^{-\frac{1}{2}}(\partial X), H^{\frac{1}{2}}(\partial X)}.$$

*Proof.* The proof is a simple integration by parts. Let us consider the equation for  $u_1$ , multiply it by  $u_2$  and integrate the product over  $X$ . Then we find by application of the divergence theorem that

$$\begin{aligned} 0 &= - \int_X u_2 L_{\gamma_1} u_1 dx = \int_X \gamma_1 \nabla u_1 \cdot \nabla u_2 dx - \int_{\partial X} u_2 \gamma_1 \nu \cdot \nabla u_1 d\mu(x). \\ &= \int_X \gamma_1 \nabla u_1 \cdot \nabla u_2 dx - \int_{\partial X} f_2 \Lambda_{\gamma_1} f_1 d\mu(x). \end{aligned}$$

Exchanging the roles of the indices  $j = 1$  and  $j = 2$  and subtracting the result to the above equality yields (7.5).  $\square$

The above lemma shows that when  $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ , the right-hand-side in (7.1) also vanishes for *any* solutions  $u_1$  and  $u_2$  of (7.1) with  $\gamma$  given by  $\gamma_1$  and  $\gamma_2$ , respectively. We are now thus faced with the question of whether products of the form  $\nabla u_1 \cdot \nabla u_2$  are dense in the space of, say, continuous functions. Unfortunately, answering this question affirmatively seems to be a difficult problem. The main difficulty in the analysis of (7.1) is that the unknown coefficient  $\gamma$  appears in the leading order of the differential operator  $L_\gamma$ . The following Liouville change of variables allows us to treat the unknown coefficient as a *perturbation* to a known operator (with constant coefficients):

$$\gamma^{-\frac{1}{2}} L_\gamma \gamma^{-\frac{1}{2}} = \Delta - q, \quad q = \frac{\Delta \gamma^{\frac{1}{2}}}{\gamma^{\frac{1}{2}}}. \quad (7.6)$$

Here  $\Delta$  is the usual Laplacian operator.

**Exercise 7.2.1** Prove (7.6).

Therefore if  $u$  is a solution of  $L_\gamma u = 0$ , then  $v = \gamma^{\frac{1}{2}} u$  is a solution to the Schrödinger equation  $(\Delta - q)v = 0$ . We thus wish to replace the problem of the reconstruction of  $\gamma$  by that of  $q$ .

Consider the Schrödinger equation (still calling the solution “ $u$ ” rather than “ $v$ ”)

$$(\Delta - q)u = 0 \quad \text{in } X, \quad u = f \quad \text{on } \partial X, \quad (7.7)$$

with  $q$  given by (7.6). For  $f \in H^{\frac{1}{2}}(\partial X)$ , we find a unique solution  $u \in H^1(X)$  such that  $\nu \cdot \nabla u \in H^{-\frac{1}{2}}(\partial X)$ . Indeed, the above equation admits a solution since it is equivalent to (7.1) by the change of variables (7.6). We then define the Dirichlet-to-Neumann operator

$$\begin{aligned} \Lambda_q : \quad H^{\frac{1}{2}}(\partial X) &\rightarrow H^{-\frac{1}{2}}(\partial X) \\ f(x) &\mapsto \Lambda_q[f](x) = \frac{\partial u}{\partial \nu}(x), \end{aligned} \quad (7.8)$$

where  $u$  is the solution to (7.7). We then verify that

$$\Lambda_q f = \gamma^{-\frac{1}{2}} \frac{\partial \gamma}{\partial \nu} \Big|_{\partial X} f + \gamma^{-\frac{1}{2}} \Lambda_\gamma (\gamma^{-\frac{1}{2}} \Big|_{\partial X} f), \quad f \in H^{\frac{1}{2}}(\partial X). \quad (7.9)$$

**Exercise 7.2.2** Prove the above result.

We thus observe that knowledge of  $\Lambda_\gamma$ ,  $\gamma|_{\partial X}$  and  $\nu \cdot \nabla \gamma|_{\partial X}$  implies knowledge of  $\Lambda_q$ . It turns out that knowledge of  $\Lambda_\gamma$  implies that of  $\gamma|_{\partial X}$  and  $\nu \cdot \nabla \gamma|_{\partial X}$ :

**Theorem 7.2.2** *Let us assume that  $0 < \gamma_i \in C^m(\bar{X})$  and that  $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ . Then we can show that for all  $|\alpha| < m$ , we have*

$$\partial^\alpha \gamma_1 \Big|_{\partial X} = \partial^\alpha \gamma_2 \Big|_{\partial X}. \quad (7.10)$$

See [42] for a proof of this result. To simplify the analysis of stability, we have assumed here that  $\gamma|_{\partial X}$  and  $\nu \cdot \nabla \gamma|_{\partial X}$  were *known*. Thus,  $\Lambda_\gamma$  uniquely determines  $\Lambda_q$ . Our next step is therefore to reconstruct  $q$  from knowledge of  $\Lambda_q$ . We start with the following lemma:

**Lemma 7.2.3** *Let  $\Lambda_{q_j}$  for  $j = 1, 2$  be the two operators associated to  $q_j$  and let  $f_j \in H^{\frac{1}{2}}(\partial X)$  for  $j = 1, 2$  be two Dirichlet conditions. Then we find that*

$$\int_{\partial X} (\Lambda_{q_1} - \Lambda_{q_2}) f_1 f_2 d\mu = \int_X (q_1 - q_2) u_1 u_2 dx, \quad (7.11)$$

where  $d\mu$  is the surface measure on  $\partial X$  and where  $u_j$  is the solution to (7.7) with  $q$  replaced by  $q_j$  and  $f$  replaced by  $f_j$ .

**Exercise 7.2.3** *Prove this lemma following the derivation in Lemma 7.2.1.*

The above lemma shows that when  $\Lambda_{q_1} = \Lambda_{q_2}$ , then the right-hand-side in (7.11) also vanishes for *any* solutions  $u_1$  and  $u_2$  of (7.7) with  $q$  replaced by  $q_1$  and  $q_2$ , respectively. We are now thus faced with the question of whether products of the form  $u_1 u_2$  are *dense* in the space of, say, continuous functions. This is a question that admits an affirmative answer. The main tool in the proof of this density argument is the construction of complex geometric optics solutions. Such solutions are constructed in section 7.3. The main property that we need at the moment is summarized in the following lemma.

**Lemma 7.2.4** *Let  $\varrho \in \mathbb{C}^n$  be a complex valued vector such that  $\varrho \cdot \varrho = 0$ . Let  $\|q\|_\infty < \infty$  and  $|\varrho|$  be sufficiently large. Then there is a solution  $u$  of  $(\Delta - q)u = 0$  in  $X$  of the form*

$$u(x) = e^{\varrho \cdot x} (1 + \varphi(x)), \quad (7.12)$$

such that

$$|\varrho| \|\varphi\|_{L^2(X)} + \|\varphi\|_{H^1(X)} \leq C. \quad (7.13)$$

The proof of this and more general lemmas can be found in section 7.3. The principle of such solutions is this. When  $q \equiv 0$ , then  $e^{\varrho \cdot x}$  is a (complex-valued) harmonic function, i.e., a solution of  $\Delta u = 0$ . The above result shows that  $q$  may be treated as a perturbation of  $\Delta$ . Solutions of  $(\Delta - q)u = 0$  are fundamentally not that different from solutions of  $\Delta u = 0$ .

Now, coming back to the issue of density of product of elliptic solutions. For  $u_1$  and  $u_2$  solutions of the form (7.12), we find that

$$u_1 u_2 = e^{(\varrho_1 + \varrho_2) \cdot x} (1 + \varphi_1 + \varphi_2 + \varphi_1 \varphi_2). \quad (7.14)$$

If we can choose  $\varrho_1 + \varrho_2 = ik$  for a fixed  $k$  with  $|\varrho_1|$  and  $|\varrho_2|$  growing to infinity so that  $\varphi_1 + \varphi_2 + \varphi_1 \varphi_2$  becomes negligible in the  $L^2$  sense thanks to (7.13), then we observe that in the limit  $u_1 u_2$  equals  $e^{ik \cdot x}$ . The functions  $e^{ik \cdot x}$  for arbitrary  $k \in \mathbb{R}^n$  certainly form a dense family of, say, continuous functions.

## 7.2.2 Proof of injectivity result

Let us make a remark on the nature of the CGO solutions and the measurement operator  $\mathfrak{M}$ . The CGO solutions are complex valued. Since the equations (7.1) and (7.7) are linear, we can assume that the boundary conditions  $f = f_r + i f_i$  are complex valued as a superposition of two real-valued boundary conditions  $f_r$  and  $f_i$ . Moreover, the results (7.5) and (7.11) hold for complex-valued solutions. Our objective is therefore to show

that the product of complex-valued solutions to elliptic equations of the form (7.7) is indeed dense. The construction in dimension  $n \geq 3$  goes as follows.

Let  $k \in \mathbb{R}^n$  be fixed for  $n \geq 3$ . We choose  $\varrho_{1,2}$  as

$$\varrho_1 = \frac{m}{2} + i\frac{k+l}{2}, \quad \varrho_2 = -\frac{m}{2} + i\frac{k-l}{2}, \quad (7.15)$$

where the real-valued vectors  $l$ , and  $m$  are chosen in  $\mathbb{R}^n$  such that

$$m \cdot k = m \cdot l = k \cdot l = 0, \quad |m|^2 = |k|^2 + |l|^2. \quad (7.16)$$

We verify that  $\varrho_i \cdot \varrho_i = 0$  and that  $|\varrho_i|^2 = \frac{1}{2}(|k|^2 + |l|^2)$ . In dimension  $n \geq 3$ , such vectors can always be found. For instance, changing the system of coordinates so that  $k = |k|e_1$ , we can choose  $l = |l|e_2$  with  $|l| > 0$  arbitrary and then  $m = \sqrt{|k|^2 + |l|^2}e_3$ , where  $(e_1, e_2, e_3)$  forms a family of orthonormal vectors in  $\mathbb{R}^n$ . Note that this construction is possible only when  $n \geq 3$ . It is important to notice that while  $k$  is fixed,  $|l|$  can be chosen arbitrarily large so that the norm of  $\varrho_i$  can be arbitrarily large while  $\varrho_1 + \varrho_2 = k$  is fixed.

Upon combining (7.11) and (7.14), we obtain for the choice (7.15) that  $\Lambda_{q_1} = \Lambda_{q_2}$  implies that

$$\left| \int_X e^{ik \cdot x} (q_1 - q_2) dx \right| \leq \left| \int_X e^{ik \cdot x} (q_1 - q_2) (\varphi_1 + \varphi_2 + \varphi_1 \varphi_2) dx \right| \leq \frac{C}{|l|}$$

thanks to (7.13) since  $|l|(\varphi_1 + \varphi_2 + \varphi_1 \varphi_2)$  is bounded in  $L^1(X)$  by an application of the Cauchy-Schwarz inequality and  $e^{ik \cdot x}(q_1 - q_2)$  is bounded in  $L^\infty(X)$ . Since the above inequality holds independent of  $l$ , we deduce that the Fourier transform of  $(q_1 - q_2)$  (extended by 0 outside of  $X$ ) vanishes, and hence that  $q_1 = q_2$ . So far we have thus proved that

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \implies \Lambda_{q_1} = \Lambda_{q_2} \implies q_1 = q_2,$$

where  $q_j$  and  $\gamma_j$  are related by (7.6). From (7.6) still, we deduce that

$$0 = \gamma_1^{\frac{1}{2}} \Delta \gamma_2^{\frac{1}{2}} - \gamma_2^{\frac{1}{2}} \Delta \gamma_1^{\frac{1}{2}} = \nabla \cdot (\gamma_1^{\frac{1}{2}} \nabla \gamma_2^{\frac{1}{2}} - \gamma_2^{\frac{1}{2}} \nabla \gamma_1^{\frac{1}{2}}) = \nabla \cdot \left( \gamma_1 \nabla \left( \frac{\gamma_2}{\gamma_1} \right)^{\frac{1}{2}} \right). \quad (7.17)$$

Since  $\gamma_1 = \gamma_2$  on  $\partial X$ , this is an elliptic equation for  $\left(\frac{\gamma_1}{\gamma_2}\right)^{\frac{1}{2}}$  whose only solution is identically 1. This shows that  $\gamma_1 = \gamma_2$ . This concludes the proof of the uniqueness result

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \implies \gamma_1 = \gamma_2. \quad (7.18)$$

### 7.2.3 Proof of the stability result

Let us return to (7.11) and assume that  $\Lambda_{q_1} - \Lambda_{q_2}$  no longer vanishes but is (arbitrarily) small. We first want to assess how errors in  $\Lambda_q$  translates into errors in  $q$ . For  $u_j$

solutions as stated in Lemma 7.2.3 and of the form (7.12), we find that

$$\begin{aligned}
\left| \int_X e^{ik \cdot x} (q_1 - q_2) dx \right| &\leq \left| \int_X e^{ik \cdot x} (q_1 - q_2) (\varphi_1 + \varphi_2 + \varphi_1 \varphi_2) dx \right| + \left| \int_{\partial X} (\Lambda_{q_1} - \Lambda_{q_2}) f_1 f_2 d\mu \right| \\
&\leq \frac{C}{|l|} + \|\Lambda_{q_1} - \Lambda_{q_2}\|_{\mathfrak{B}} \|f_1\|_{H^{\frac{1}{2}}(\partial X)} \|f_2\|_{H^{\frac{1}{2}}(\partial X)} \\
&\leq \frac{C}{|l|} + C \|\Lambda_{q_1} - \Lambda_{q_2}\|_{\mathfrak{B}} \|u_1\|_{H^1(X)} \|u_2\|_{H^1(X)} \\
&\leq \frac{C}{|l|} + C |l| \|\Lambda_{q_1} - \Lambda_{q_2}\|_{\mathfrak{B}} e^{C|l|}.
\end{aligned}$$

Indeed,  $f_j = u_j|_{\partial X}$  and  $\|u\|_{H^{\frac{1}{2}}(\partial X)} \leq C \|u\|_{H^1(X)}$  is a standard estimate. This step is where the ill-posedness of the Calderón problem is best displayed.

**Exercise 7.2.4** *Verify that for some constant  $C$  independent of  $|l|$  and for  $u$  given by (7.12)-(7.13), we have:*

$$\|u\|_{H^1(X)} \leq C |l| e^{C|l|}.$$

Define  $\delta q = q_1 - q_2$ . So far, we have obtained a control of  $\widehat{\delta q}(k)$  uniform in  $k \in \mathbb{R}^n$ . Upon choosing

$$|l| = \frac{\sigma}{C} \ln \frac{1}{\varepsilon}, \quad 0 < \sigma < 1,$$

so that  $e^{C|l|} = \varepsilon^{-\sigma}$ , we find that for  $\varepsilon := \min(1, \|\Lambda_{q_1} - \Lambda_{q_2}\|_{\mathfrak{B}})$ ,

$$|\widehat{\delta q}(k)| \leq \eta := C |\ln \varepsilon|^{-1}. \quad (7.19)$$

Since  $q$  is assumed to be bounded and compactly supported, it is square integrable in  $\mathbb{R}^n$  so that  $\|\delta q\|_{L^2(\mathbb{R}^n)} := E < \infty$ . This and the control in (7.19) allows one to obtain a control of  $\delta q$  in  $H^{-s}(\mathbb{R}^n)$  for  $s > 0$ . Indeed

$$\begin{aligned}
\|\delta q\|_{H^{-s}(\mathbb{R}^n)}^2 &= \int \langle k \rangle^{-2s} |\widehat{\delta q}|^2 dk \\
&\leq k_0^n \eta^2 + k_0^{-2s} E^2,
\end{aligned}$$

by splitting the integration in  $k$  into  $|k| < k_0$  and  $|k| > k_0$  and choosing  $k_0 \geq 1$ . We then choose

$$k_0 = \left( \frac{E}{\eta} \right)^{\frac{2}{n+2s}}.$$

This implies

$$\|q_1 - q_2\|_{H^{-s}(\mathbb{R}^n)} \leq C E^{\frac{n}{n+2s}} |\ln \varepsilon|^{-\frac{2s}{n+2s}}. \quad (7.20)$$

**Exercise 7.2.5** *Assume that  $\|\delta q\|_{H^s(\mathbb{R}^n)} := E < \infty$ . Show that the estimate (7.20) holds with  $s$  replaced by  $s + \varsigma$  on the right-hand-side.*

It remains to convert the estimate on  $q_1 - q_2$  into an estimate for  $\gamma_1 - \gamma_2$ . We find that (7.17) is replaced by

$$(\gamma_1 \gamma_2)^{\frac{1}{2}}(q_1 - q_2) = \nabla \cdot (\gamma_1 \nabla \left( \frac{\gamma_2}{\gamma_1} - 1 \right)^{\frac{1}{2}}) \quad \text{in } X, \quad \left( \frac{\gamma_2}{\gamma_1} - 1 \right)^{\frac{1}{2}} = 0 \quad \text{on } \partial X. \quad (7.21)$$

Standard elliptic regularity results and the fact that  $\gamma_1$  is of class  $C^2$  therefore show that

$$\|\gamma_1 - \gamma_2\|_{H^1(X)} \leq C \|q_1 - q_2\|_{H^{-1}(X)} \leq C |\ln \varepsilon|^{-\delta}, \quad (7.22)$$

with  $\delta = \frac{2}{2+n}$  if  $q$  is bounded in the  $L^2$  sense and  $\delta = \frac{2(1+\varsigma)}{n+2(1+\varsigma)}$  if  $q$  is bounded as in Exercise 7.2.5. The final result in (7.4) then follows from interpolating the a priori bound in  $H^s$  of  $\gamma_1 - \gamma_2$ , the above smallness bound in  $H^1$  to obtain a small bound in  $H^\tau$  for some  $\frac{n}{2} < \tau < s$ . Then by the Sobolev imbedding of  $L^\infty(X)$  into  $H^\tau(X)$ , we conclude the proof of Theorem 7.1.1.

## 7.3 Complex Geometric Optics Solutions

The major technical ingredient in the proof of Theorem 7.1.1 is the existence of complex geometrical optics (CGO) solutions of the form (7.12). The proof of Lemma 7.2.4 and more general results that will be useful in subsequent chapters is undertaken in this section.

Let us consider the equation  $\Delta u = qu$  in  $X$ . When  $q = 0$ , then a rich family of harmonic solutions is formed by the complex exponentials  $e^{\varrho \cdot x}$  for  $\varrho \in \mathbb{C}^n$  a complex valued vector such that  $\varrho \cdot \varrho = 0$ . Indeed, we verify that

$$\Delta e^{x \cdot \varrho} = \varrho \cdot \varrho e^{x \cdot \varrho} = 0. \quad (7.23)$$

A vector  $\varrho = \varrho_r + i\varrho_i$  is such that  $\varrho \cdot \varrho = 0$  if and only if  $\varrho_r$  and  $\varrho_i$  are orthogonal vectors of the same (Euclidean) length.

When  $q \neq 0$ , it is tempting to try and write solutions of  $\Delta u = qu$  as perturbations of the harmonic solutions  $e^{\varrho \cdot x}$ , for instance in the form

$$u(x) = e^{\varrho \cdot x}(1 + \varphi(x)).$$

This provides an equation for  $\varphi$  of the form

$$(\Delta + 2\varrho \cdot \nabla)\varphi = q(1 + \varphi). \quad (7.24)$$

**Exercise 7.3.1** *Check this formula.*

Treating the right-hand side as a source  $f$ , the first part of the construction consists of solving the problem

$$(\Delta + 2\varrho \cdot \nabla)\varphi = f, \quad (7.25)$$

for  $f$  a source in  $X$  and  $\varphi$  defined on  $X$  as well. Surprisingly, the analysis of (7.25) is the most challenging technical step in the construction of solutions to (7.24). The construction with  $f \in L^2(X)$  is sufficient for the proof of Theorem 7.1.1. In later chapters, we will require more regularity for the solution to (7.25) and thus prove the following result.

**Lemma 7.3.1** *Let  $f \in H^s(X)$  for  $s \geq 0$  and let  $|\varrho| \geq c > 0$ . Then there exists a solution to (7.25) in  $H^{s+1}(X)$  and such that*

$$|\varrho| \|\varphi\|_{H^s(X)} + \|\varphi\|_{H^{s+1}(X)} \leq C \|f\|_{H^s(X)}. \quad (7.26)$$

*Proof.* We first extend  $f$  defined on  $X$  to a function still called  $f$  defined and compactly supported in  $\mathbb{R}^n$  and such that

$$\|f\|_{H^s(\mathbb{R}^n)} \leq C(X) \|f\|_{H^s(X)}.$$

That such an extension exists is proved for instance in [59, Chapter 6, Theorem 5]. We thus wish to solve the problem

$$(\Delta + 2\varrho \cdot \nabla)\varphi = f, \quad \text{in } \mathbb{R}^n. \quad (7.27)$$

The main difficulty is that the operator  $(\Delta + 2\varrho \cdot \nabla)$  has for symbol

$$\mathcal{F}_{x \rightarrow \xi}(\Delta + 2\varrho \cdot \nabla)\mathcal{F}_{\xi \rightarrow x}^{-1} = -|\xi|^2 + 2i\varrho \cdot \xi.$$

Such a symbol vanishes for  $\varrho_r \cdot \xi = 0$  and  $2\varrho_i \cdot \xi + |\xi|^2 = 0$ . The original proof of the injectivity result of Theorem 7.1.1 in [60] shows the existence and uniqueness of a solution to (7.27) in appropriate functional spaces. Since uniqueness is of no concern to us here, we instead follow a path undertaken by [36, 55] and construct a solution that can be seen of the product of a plane wave with a periodic solution with different period. Let us define

$$\varphi = e^{i\varsigma \cdot x} \mathbf{p}, \quad f = e^{i\varsigma \cdot x} \mathbf{f}$$

for some vector  $\varsigma \in \mathbb{R}^n$  to be determined. Then we find

$$\left( \Delta + 2(\varrho + i\varsigma) \cdot \nabla + (2i\varrho \cdot \varsigma - |\varsigma|^2) \right) \mathbf{p} = (\nabla + i\varsigma + 2\varrho) \cdot (\nabla + i\varsigma) \mathbf{p} = \mathbf{f}. \quad (7.28)$$

**Exercise 7.3.2** *Verify this.*

Let us now assume that  $\mathbf{f}$  is supported in a box  $Q$  of size  $(-L, L)^n$  for  $L$  sufficiently large. Then we decompose as Fourier series:

$$\mathbf{p} = \sum_{k \in \mathbb{Z}^n} \mathbf{p}_k e^{i\frac{\pi}{L} k \cdot x}, \quad \mathbf{f} = \sum_{k \in \mathbb{Z}^n} \mathbf{f}_k e^{i\frac{\pi}{L} k \cdot x}. \quad (7.29)$$

We then find that (7.28) is equivalent in the Fourier domain to

$$\mathbf{p}_k = \frac{1}{-|\frac{\pi}{L} k + \varsigma|^2 + 2i\varrho \cdot (\varsigma + \frac{\pi}{L} k)} \mathbf{f}_k \quad (7.30)$$

The imaginary part of the denominator is given by  $2\varrho_r \cdot (\frac{\pi}{L} k + \varsigma)$ . It remains to choose

$$\varsigma = \frac{1}{2} \frac{\pi}{L} \frac{\varrho_r}{|\varrho_r|},$$

to obtain that the above denominator never vanishes since  $k \in \mathbb{Z}^n$ . Moreover, for such a choice, we deduce that

$$\left| -\left|\frac{\pi}{L}k + \varsigma\right|^2 + 2i\varrho \cdot \left(\varsigma + \frac{\pi}{L}k\right) \right| \geq C|\varrho|,$$

for some constant  $C$  independent of  $\varrho$ . This shows that

$$|\mathbf{p}_k| \leq C|\varrho|^{-1}|\mathbf{f}_k|.$$

Since  $\mathbf{f} \in H^s(Q)$ , we deduce that  $\|\mathbf{f}\|_{H^s(Q)}^2 = \sum_{k \in \mathbb{Z}^n} |k|^{2s} |\mathbf{f}_k|^2 < \infty$ , from which we deduce that

$$\|\mathbf{p}\|_{H^s(Q)} \leq C|\varrho|^{-1}\|\mathbf{f}\|_{H^s(Q)}.$$

It remains to restrict the constructed solution to  $X$  (and realize that  $e^{i\varsigma \cdot x}$  is smooth) to obtain that  $|\varrho|\|\varphi\|_{H^s(X)} \leq C\|f\|_{H^s(X)}$  and the first step in (7.26).

The result on  $\|\varphi\|_{H^{s+1}(X)}$  requires that we obtain bounds for  $|k|\mathbf{p}_k$ . For  $|k|$  small, say  $|k| \leq \frac{8L}{\pi}|\varrho|$ , then we use the same result as above to obtain

$$|k|\mathbf{p}_k \leq C|\mathbf{f}_k|, \quad |k| \leq \frac{8L}{\pi}|\varrho|.$$

For the other values of  $|k|$ , we realize that the denominator in (7.30) causes no problem and that

$$|k|\mathbf{p}_k \leq C|k|^{-1}|\mathbf{f}_k|, \quad |k| > \frac{8L}{\pi}|\varrho|.$$

This shows that  $|k|\mathbf{p}_k \leq C|\mathbf{f}_k|$  for some constant  $C$  independent of  $k$  and  $|\varrho|$ . The proof that  $\|\varphi\|_{H^{s+1}(X)} \leq C\|f\|_{H^s(X)}$  then proceeds as above. This concludes the proof of the fundamental lemma of CGO solutions to Schrödinger equations.  $\square$

We now come back to the perturbed problem (7.24). We assume that  $q$  is a complex-valued potential in  $H^s(X)$  for some  $s \geq 0$ . We say that  $q \in L^\infty(X)$  has regularity  $s$  provided that for all  $\varphi \in H^s(X)$ , we have

$$\|q\varphi\|_{H^s(X)} \leq q_s\|\varphi\|_{H^s(X)}, \tag{7.31}$$

for some constant  $q_s$ . For instance, when  $s = 0$ , when  $q_s = \|q\|_{L^\infty(X)}$ . Then we have the following result.

**Theorem 7.3.2** *Let us assume that  $q \in H^s(X)$  is sufficiently smooth so that  $q_s < \infty$ . Then for  $|\varrho|$  sufficiently large, there exists a solution  $\varphi$  to (7.24) that satisfies*

$$|\varrho|\|\varphi\|_{H^s(X)} + \|\varphi\|_{H^{s+1}(X)} \leq C\|q\|_{H^s(X)}. \tag{7.32}$$

Moreover, we have that

$$u(x) = e^{\varrho \cdot x}(1 + \varphi(x)) \tag{7.33}$$

is a Complex Geometrical Optics solution in  $H^{s+1}(X)$  to the equation

$$\Delta u = qu \quad \text{in } X.$$

*Proof.* Let  $T$  be the operator which to  $f \in H^s(X)$  associates  $\varphi \in H^s(X)$  the solution of (7.27) constructed in the proof of Lemma 7.3.1. Then (7.24) may be recast as

$$(I - Tq)\varphi = Tq.$$

We know that  $\|T\|_{\mathcal{L}(H^s(X))} \leq C_s |\varrho|^{-1}$ . Choosing  $|\varrho|$  sufficiently large so that  $|\varrho| > C_s q_s$ , we deduce that  $(I - Tq)^{-1} = \sum_{m=0}^{\infty} (Tq)^m$  exists and is a bounded operator in  $\mathcal{L}(H^s(X))$ . We have therefore constructed a solution so that  $q(1 + \varphi) \in H^s(X)$ . The estimate (7.26) yields (7.32) and concludes the proof of the theorem.  $\square$

Note that the above theorem with  $s = 0$  yields Lemma 7.2.4.

Let us now consider the elliptic equation (7.1). The change of variables in (7.6) shows that  $u = \gamma^{-\frac{1}{2}}v$  with  $v$  a solution of  $\Delta v = qv$ , is a solution of (7.1). We therefore have the

**Corollary 7.3.3** *Let  $\gamma$  be sufficiently smooth so that  $q = \gamma^{-\frac{1}{2}}\Delta\gamma^{\frac{1}{2}}$  verifies the hypotheses of Theorem 7.3.2. Then for  $|\varrho|$  sufficiently large, we can find a solution  $u$  of  $\nabla \cdot \gamma \nabla u = 0$  on  $X$  such that*

$$u(x) = \frac{1}{\gamma^{\frac{1}{2}}(x)} e^{\varrho \cdot x} (1 + \varphi(x)), \quad (7.34)$$

and such that (7.32) holds.

For instance, for  $s \in \mathbb{N}$ , we verify that (7.31) holds provided that  $\gamma$  is of class  $C^{s+2}(X)$ . The case  $s = 0$  with  $\gamma$  of class  $C^2(X)$  is the setting of Theorem 7.1.1.

## 7.4 The Optical Tomography setting

We have seen that optical tomography measurements could be modeled by radiative transfer equations. In the diffusive regime, where photons travel over distances that are much larger than the mean free path, the photon density is accurately modeled by a diffusion equation. Let us assume that the source of photons used to probe the domain of interest is time harmonic with a frequency modulation  $\omega$ . The density of such time harmonic photons then solves the following elliptic model

$$-\nabla \cdot \gamma \nabla u + (\sigma + i\omega)u = 0 \quad \text{in } X, \quad u = f \quad \text{on } \partial X. \quad (7.35)$$

We assume here that  $f$  is the prescribed distribution of photons at the domain's boundary. This is an approximation as only the density of *incoming* photons can be prescribed. More accurate models would not fundamentally change the main conclusions of this section and we therefore assume Dirichlet conditions to simplify.

The coefficients  $(\gamma, \sigma)$  are the diffusion and absorption coefficients, respectively. We assume that light speed is normalized to  $c = 1$  to simplify (otherwise  $\omega$  should be replaced by  $c^{-1}\omega$ ).

The Dirichlet-to-Neumann or density-to-current map is given in this setting by

$$\Lambda_{\gamma, \sigma}^{\omega} : \begin{array}{l} H^{\frac{1}{2}}(\partial X) \rightarrow H^{-\frac{1}{2}}(\partial X) \\ f(x) \mapsto \Lambda_{\gamma, \sigma}^{\omega}[f](x) = \gamma(x) \frac{\partial u}{\partial \nu}(x). \end{array} \quad (7.36)$$

With  $\mathfrak{X} = \mathcal{C}^2(\bar{X}) \times \mathcal{C}^0(\bar{X})$  and  $\mathfrak{Y} = \mathcal{L}(H^{\frac{1}{2}}(\partial X), H^{-\frac{1}{2}}(\partial X))$ , we define the measurement operator

$$\mathfrak{M}^\omega : \mathfrak{X} \ni (\gamma, \sigma) \mapsto \mathfrak{M}^\omega(\gamma, \sigma) = \Lambda_{\gamma, \sigma}^\omega \in \mathfrak{Y}. \quad (7.37)$$

The measurement operator is parameterized by the (known) modulation frequency  $\omega$  and there are now two unknown coefficients  $\gamma$  and  $\sigma$ . Then we have the following result.

**Theorem 7.4.1** *Define the measurement operator  $\mathfrak{M}^\omega$  as in (7.37). Then for  $\omega \neq 0$ , we have that  $\mathfrak{M}^\omega$  is injective in the sense that  $\mathfrak{M}^\omega(\gamma, \sigma) = \mathfrak{M}^\omega(\tilde{\gamma}, \tilde{\sigma})$  implies that  $(\gamma, \sigma) = (\tilde{\gamma}, \tilde{\sigma})$ .*

*Moreover, we have the following logarithmic stability estimate:*

$$\|\gamma(x) - \gamma'(x)\|_{L^\infty(X)} + \|\sigma(x) - \sigma'(x)\|_{L^\infty(X)} \leq C \left| \log \|\mathfrak{M}^\omega(\gamma, \sigma) - \mathfrak{M}^\omega(\tilde{\gamma}, \tilde{\sigma})\|_{\mathfrak{Y}} \right|^{-\delta}, \quad (7.38)$$

for some  $\delta > 0$  provided that  $(\gamma, \sigma)$  and  $(\tilde{\gamma}, \tilde{\sigma})$  are uniformly bounded in  $(H^s(X))^2$  for some  $s > \frac{n}{2}$ . Here,  $C$  depends on  $\omega$ .

Let now  $\omega = 0$ . Then  $\mathfrak{M}(\gamma, \sigma) = \mathfrak{M}(\tilde{\gamma}, \tilde{\sigma})$  implies that

$$q := \frac{\Delta\gamma^{\frac{1}{2}}}{\gamma^{\frac{1}{2}}} + \frac{\sigma}{\gamma} = \tilde{q} := \frac{\Delta\tilde{\gamma}^{\frac{1}{2}}}{\tilde{\gamma}^{\frac{1}{2}}} + \frac{\tilde{\sigma}}{\tilde{\gamma}}. \quad (7.39)$$

and  $(\gamma, \sigma)$  are defined up to an arbitrary change of variables that leaves  $q$  above invariant. Moreover,  $q$  above can be reconstructed in  $H^{-1}(X)$  as shown in the preceding sections and hence in  $L^\infty(X)$  by interpolation.

*Proof.* The proof is very similar to that of Theorem 7.1.1. We mainly highlight the differences. Let  $u$  be a solution of the above equation and  $v = \gamma^{\frac{1}{2}}u$ . We verify that

$$\Delta v = q_\omega v, \quad q_\omega = \frac{\Delta\gamma^{\frac{1}{2}}}{\gamma^{\frac{1}{2}}} + \frac{\sigma + i\omega}{\gamma}.$$

Let us assume that  $\omega \neq 0$ . Then knowledge of  $\Lambda_{\gamma, \sigma}$  yields knowledge of  $\Lambda_{q_\omega}$  as before. This uniquely determines  $q_\omega$  and allows to obtain an error bound for  $q_\omega - \tilde{q}_\omega$  in  $H^{-1}(X)$ . The imaginary part of  $q_\omega$  thus provides a reconstruction of  $\gamma$  in the same space, and hence in  $L^\infty$  by interpolation between  $H^{-1}(X)$  and  $H^s(X)$  for  $s > \frac{n}{2}$ . Since  $n \geq 3$ , we thus obtain an error in  $\Delta\gamma^{\frac{1}{2}}$  also in  $H^{-1}$ , which then yields an error estimate for  $\sigma$ , also in  $H^{-1}(X)$ . Again, by interpolation, we find a value of  $\delta$  so that (7.38) holds. Note that the value of  $\delta$  is a priori worse than the one found for the Calderón problem since we do not take advantage of solving an elliptic equation for  $\gamma^{\frac{1}{2}}$ .

This concludes the proof when  $\omega \neq 0$ . When  $\omega = 0$ , then clearly we can reconstruct  $q$  in (7.39) with the corresponding stability estimate. Now let us consider two couples  $(\gamma, \sigma)$  and  $(\tilde{\gamma}, \tilde{\sigma})$  such that (7.39) holds. From (7.9), we deduce that

$$\Lambda_{\gamma, \sigma}(f) = \gamma^{\frac{1}{2}} \Lambda_q \gamma^{\frac{1}{2}} \Big|_{\partial X} f - \frac{\partial \gamma}{\partial \nu} \Big|_{\partial X} \gamma^{\frac{1}{2}} \Big|_{\partial X} f, \quad f \in H^{\frac{1}{2}}(\partial X). \quad (7.40)$$

Since we assume  $\gamma$  and  $\nu \cdot \nabla \gamma$  known on  $\partial X$ , then we obtain that  $\Lambda_{\gamma, \sigma} = \Lambda_{\tilde{\gamma}, \tilde{\sigma}}$ . This shows that the reconstruction of  $(\gamma, \sigma)$  is obtained up to any change of variables that leaves  $q$  in (7.39) invariant.  $\square$

The above theorem shows that measurements in Optical Tomography with continuous wave (CW) sources corresponding to  $\omega = 0$  do not guaranty uniqueness of the reconstruction of  $(\gamma, \sigma)$ . The injectivity of the measurement operator is restored when  $\omega \neq 0$ . However, as for the Calderón problem, the stability estimates are logarithmic (and cannot fundamentally be improved; only the value of  $\delta$  may not be optimal in the above derivations).

For additional information about the Calderón problem, we refer the reader to [32, 55, 65] and to the historical references [7, 46, 49, 60].

# Chapter 8

## Coupled-physics IP I: Photo-acoustic Tomography and Transient Elastography

Many inverse diffusion problems, including the Calderón problem, are modeled by measurement operators that are injective but not very stable in the sense that the modulus of continuity  $\omega(e)$  in (1.3) is logarithmic. The main reason for such a behavior is that solutions to elliptic equations are very smooth away from where the singularities are generated. As a consequence, singularities in the parameters do not propagate to strong, easily recognizable, signals in the boundary measurements. High frequencies of the parameters are still present in the measurements when the measurement operator is injective. However, they have been exponentially attenuated and the inverse problem is then best described as severely ill-posed and typically displays poor resolution capabilities. Imaging modalities based on inverse diffusion problems are still useful because of the *high contrast* they often display, for instance between healthy and non-healthy tissues in medical imaging. Modalities such as Electrical Impedance Tomography and Optical Tomography may be described as high contrast, low resolution modalities.

In applications where resolution is not paramount, the severe ill-posedness of inverse diffusion problems might not be an issue. In many instances, however, high resolution in the reconstruction is an important objective of the imaging modality. The remedy to such a low resolution is to find methodologies that *combine* the high contrast modality with another well-posed, high-resolution, inversion, such as for instance, those involving inverse wave problems or inverse problems of integral geometry (such as the Radon transform). In order for the combination to exist, we need to be sufficiently fortunate that a measurable physical phenomenon couples the high resolution wave-like mechanism with the high contrast elliptic-like phenomenon. Such examples exist in nature and give rise to *hybrid* imaging techniques that couples the high contrast (but low-resolution) of the diffusion-like equation with the high resolution (but often low contrast) of the wave-like equation.

In this chapter and the next, we consider several such physical couplings. In this chapter, we consider the coupling of optical waves with ultrasound in the so-called photo-acoustic effect. This gives rise to a modality called Photo-acoustic Tomography (PAT). We also consider the coupling of elastic waves with ultrasound in the modality

called Transient Elastography (TE). TE and PAT are relatively similar mathematically and both involve inverse problems with *internal functionals*.

In the next chapter, we consider another coupling of optical waves with ultrasound called ultrasound modulation. Mathematically, this problem also gives rise to an inverse problem with internal functionals, albeit one whose analysis is more complicated than for TE and PAT.

## 8.1 Introduction to PAT and TE

### 8.1.1 Modeling of photoacoustic tomography

Photoacoustic tomography (PAT) is a hybrid medical imaging modality that combines the high resolution of acoustic waves with the high contrast of optical waves. When a body is exposed to short pulse radiation, typically emitted in the near infra-red spectrum in PAT, it absorbs energy and expands thermo-elastically by a very small amount; this is the photoacoustic effect. Such an expansion is sufficient to emit acoustic pulses, which travel back to the boundary of the domain of interest where they are measured by arrays of transducers.

**Radiation propagation.** The propagation of radiation in highly scattering media is modeled by the following diffusion equation

$$\begin{aligned} \frac{1}{c} \frac{\partial}{\partial t} u - \nabla \cdot \gamma(x) \nabla u + \sigma(x) u &= 0, & x \in X \subset \mathbb{R}^n, t > 0 \\ u &= f & x \in \partial X, t > 0 \end{aligned} \quad (8.1)$$

where  $X$  is an open, bounded, connected domain in  $\mathbb{R}^n$  with  $C^1$  boundary  $\partial X$  (embedded in  $\mathbb{R}^n$ ), where  $n$  spatial dimension;  $c$  is light speed in tissues;  $\gamma(x)$  is a (scalar) diffusion coefficient; and  $\sigma(x)$  is an absorption coefficient. Throughout the paper, we assume that  $\gamma(x)$  and  $\sigma(x)$  are bounded from above and below by (strictly) positive constants. The source of incoming radiation is prescribed by  $f(t, x)$  on the boundary  $\partial X$  and is assumed to be a very short pulse supported on an interval of time  $(0, \eta)$  with  $c\eta$  of order  $O(1)$ .

**Photoacoustic effect.** As radiation propagates, a small fraction is absorbed. This absorbed radiation generates a slight temperature increase, which results in a minute mechanical expansion. The latter expansion is responsible for the emission of ultrasound, which are measured at the domain's boundary. The coupling between the optical and ultrasonic waves is called the photo-acoustic effect. The amount of energy deposited and transformed into acoustic energy is given by:

$$H(t, x) = \Gamma(x) \sigma(x) u(t, x),$$

where  $\Gamma(x)$  is the Grüneisen coefficient quantifying the photo-acoustic effect while  $\sigma(x)u(t, x)$  is the density of absorbed radiation.

A thermal expansion proportional to  $H$  results and acoustic waves are emitted. Such waves are modeled by

$$\frac{1}{c_s^2(x)} \frac{\partial^2 p}{\partial t^2} - \Delta p = \frac{\partial}{\partial t} H(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \quad (8.2)$$

with  $c_s$  the sound speed. We assume here to simplify that the wave equation is posed in the whole space  $\mathbb{R}^n$ . This assumption is justified by the fact that waves propagating away from the domain of interest are assumed not to interfere with the measurements and by the fact that the measurement devices themselves do not modify the wave field. In a typical measurement setting, the acoustic pressure  $p(t, x)$  is then measured on  $\partial X$  as a function of time.

Since light travels much faster than sound with  $c_s \ll c$ , we may assume for short light pulses that radiation propagation occurs over a very short time at the scale of the acoustic waves. This justifies the simplification:

$$H(t, x) \sim H(x)\delta_0(t), \quad H(x) = \Gamma(x)\sigma(x) \int_{\mathbb{R}^+} u(t, x)dt.$$

The acoustic signals are therefore modeled by the following wave equation

$$\frac{1}{c_s^2(x)} \frac{\partial^2 p}{\partial t^2} - \Delta p = \frac{\partial \delta_0(t)}{\partial t} H(x). \quad (8.3)$$

We now set the sound speed to  $c_s = 1$  to simplify. The acoustic pressure  $p(t, x)$  is then measured on  $\partial X$  as a function of time. Assuming a system at rest so that  $p(t, x) = 0$  for  $t < 0$ , the wave equation may then be recast as the following equation:

$$\begin{aligned} \frac{\partial^2 p}{\partial t^2} - \Delta p &= 0, & t > 0, \quad x \in \mathbb{R}^n \\ p(0, x) &= H(x), & x \in \mathbb{R}^n \\ \partial_t p(0, x) &= 0, & x \in \mathbb{R}^n. \end{aligned} \quad (8.4)$$

**Exercise 8.1.1** *Show that under the above simplifying assumptions, (8.3) formally leads to the initial value problem (8.4).*

We retrieve the wave equation with unknown source term  $H(x)$  given by

$$H(x) = \Gamma(x)\sigma(x)u(x), \quad u(x) = \int_{\mathbb{R}^+} u(t, x)dt, \quad (8.5)$$

where  $u(x)$  is the total intensity of radiation reaching a point  $x$  integrated over the time span of the radiation pulse, and thus satisfies the equation

$$\begin{aligned} -\nabla \cdot \gamma(x)\nabla u + \sigma(x)u &= 0, & x \in X \subset \mathbb{R}^n, \\ u &= f & x \in \partial X, \end{aligned} \quad (8.6)$$

where  $f(x) = \int_{\mathbb{R}^+} f(t, x)dt$  is the spatial density of photons emitted into the domain.

### 8.1.2 First step: Inverse wave source problem

The first step in PAT consists of reconstructing  $H(x)$  from available ultrasound measurements. Two different settings of ultrasound measurements are considered now.

**Inverse wave problem with boundary measurements.** The reconstruction of  $H(x)$  in (8.4) from a measurement operator given by  $\mathfrak{M}H(t, x) = p(t, x)|_{\partial X}$  was considered in Chapter 4. Several explicit reconstruction formulas were presented there, including (4.35) and the reconstruction procedure in the Fourier domain presented in section 4.3.2.

**Inverse waves problem with planar measurements.** Other measurement settings than point measurements have been considered in practice. One major issue with point measurements such as those considered above is that the acoustic signal is typically rather weak and thus difficult to measure accurately. An alternative to such point measurements is to consider planar detectors, which integrate the pressure field over a larger domain and thus become less sensitive to noise.

Consider the setting of (8.2) with  $\mathfrak{f}(x) = H(x)$  an unknown source term. Let  $P(s, \omega)$  for  $s \in \mathbb{R}$  and  $\omega \in \mathbb{S}^2$  be the plane of points  $x \in \mathbb{R}^3$  such that  $x \cdot \omega = s$ . Then we recall that the three-dimensional Radon transform of a function is defined as

$$R\mathfrak{f}(s, \omega) = \int_{\mathbb{R}^3} \mathfrak{f}(x) \delta(s - x \cdot \omega) dx = \int_{P(s, \omega)} \mathfrak{f}(x) d\mu(x),$$

where  $d\mu(x)$  is the surface measure on the plane  $P(s, \omega)$ . We thus obtain that the measurements are given as a function of time  $t$  and angle  $\omega$  by

$$\mathfrak{f} \mapsto \mathfrak{M}_3 \mathfrak{f}(t, \omega) = Rp(1, \omega, t), \quad t > 0, \quad \omega \in \mathbb{S}^2.$$

Again, we have in the Fourier domain the Fourier slice theorem

$$\widehat{R\mathfrak{f}}(\sigma, \omega) = \hat{\mathfrak{f}}(\sigma\omega),$$

from which combined with the following representation of the Laplace operator

$$-\Delta = \mathcal{F}_{\xi \rightarrow x}^{-1} |\xi|^2 \mathcal{F}_{x \rightarrow \xi},$$

we deduce the intertwining property of the Radon transform and the Laplacian:

$$R\Delta \mathfrak{f}(s, \omega) = \frac{\partial^2}{\partial s^2} R\mathfrak{f}(s, \omega).$$

This property holds for smooth functions that decay sufficiently fast at infinity and by extension to a large class of distributions.

Thus (8.2) can be recast for  $Rp(s, \omega, t)$  as

$$\frac{\partial^2 Rp}{\partial t^2} - \frac{\partial^2 Rp}{\partial s^2} = 0, \quad t > 0, \quad s \in \mathbb{R}, \quad \omega \in \mathbb{S}^2$$

with conditions  $\partial_t Rp(s, \omega, 0) = 0$  and  $Rp(s, \omega, 0) = R\mathfrak{f}(s, \omega)$ .

This is a one-dimensional wave equation, whose unique solution is given by

$$Rp(s, \omega, t) = \frac{1}{2}(Rp(s+t, \omega, 0) + Rp(s-t, \omega, 0)) = \frac{1}{2}(Rf(s+t, \omega) + Rf(s-t, \omega)).$$

For the planes tangent to the unit sphere,  $s = 1$ , while  $t > 0$ . Then  $Rf(t+s, \omega) = 0$  since  $f$  is supported inside the ball of radius 1 so that  $Rf(s, \omega) = 0$  for  $|s| \geq 1$ . Thus for  $0 < t < 2$ , we have

$$Rf(1-t, \omega) = 2Rp(1, \omega, t) = 2\mathfrak{M}_3f(t, \omega).$$

Up to a slight smooth change of variables, the data are therefore the three-dimensional Radon transform of  $f$ , i.e., the integral of  $f$  along all possible hyperplanes (passing through the support of  $f$ ).

We recall the explicit inversion of the Radon transform:

$$f(x) = \frac{-1}{8\pi^2} R^* \frac{\partial^2}{\partial s^2} Rf(x) = \frac{-1}{8\pi^2} \int_{\mathbb{S}^2} \left( \frac{\partial^2}{\partial s^2} Rf \right) (x \cdot \omega, \omega) d\omega.$$

The Radon transform is injective and stable as we saw in Chapter 2. We have the following stability result:

$$\sqrt{2} \|f\|_{H^s(\mathbb{R}^3)} \leq \|\mathfrak{M}_3f\|_{H^{s+1}(Z)}, \quad (8.7)$$

where  $Z = \mathbb{R} \times \mathbb{S}^2$  and  $H^s(Z)$  is defined in Chapter 2.

**Remarks on the first step of PAT.** The inverse wave source problems considered here are therefore well posed problems. When ultrasound is measured with sufficient precision, then the reconstruction of the initial condition  $H(x) = f(x)$  is stable.

### 8.1.3 Second step: Inverse problems with internal functionals

Once the first step of PAT is satisfactorily performed, a second step consists of reconstructing the unknown coefficients  $(\gamma(x), \sigma(x), \Gamma(x))$  from knowledge of internal functionals of the form

$$H_j(x) = \Gamma(x)\sigma(x)u_j(x), \quad 1 \leq j \leq J, \quad (8.8)$$

for  $J \in \mathbb{N}^*$  illumination maps  $f_j(x)$ , where  $u_j$  is the solution to the steady-state equation

$$\begin{aligned} -\nabla \cdot \gamma(x) \nabla u_j + \sigma(x)u_j &= 0, & x \in X \subset \mathbb{R}^n, \\ u_j &= f_j & x \in \partial X. \end{aligned} \quad (8.9)$$

This second step of PAT is sometimes called quantitative photoacoustic tomography (QPAT). Indeed, the reconstruction of  $H_j(x)$  in (8.8) offers important information about the unknown coefficients but depends on the illumination  $f_j$  used to probe the domain of interest and cannot be used to quantify intrinsic properties of the parameters. The second step of PAT aims to provide quantitative statements on the unknown parameters  $(\gamma(x), \sigma(x), \Gamma(x))$ .

Throughout the paper, we assume that the coefficients in (8.9) are known at the domain's boundary  $\partial X$ . Our objective is to reconstruct them in  $X$ . Formally, the measurement operator in QPAT is therefore

$$(\gamma(x), \sigma(x), \Gamma(x)) \mapsto \mathfrak{M}(\gamma(x), \sigma(x), \Gamma(x)) = (H_j(x))_{1 \leq j \leq J}.$$

One of the main theoretical results about QPAT is that, unfortunately, no matter how large  $J$  and the illuminations  $f_j$  may be chosen, we cannot reconstruct all of  $(\gamma(x), \sigma(x), \Gamma(x))$  uniquely from QPAT measurements of the form (8.8). In other words,  $\mathfrak{M}$  is not injective. However, as we shall see, as soon as one of the coefficients is assumed to be known, then the restriction of  $\mathfrak{M}$  in this setting is injective and enjoys good (Lipschitz or Hölder) stability properties.

### 8.1.4 Reconstruction of one coefficient.

We conclude this introductory section by a simpler problem: the reconstruction of one coefficient in  $(\gamma(x), \sigma(x), \Gamma(x))$  when the other two coefficients are known. In practice, it is not uncommon as a first approximation to assume that  $\Gamma$  and  $\gamma$  are known, at least approximatively. Then the important absorption coefficient  $\sigma$  (for instance to reconstruct the oxygenation properties of human tissues) is uniquely and stably determined.

- (i) When only  $\Gamma$  is unknown, then we solve (8.9) for  $u$  and then construct  $\Gamma = \frac{H}{\sigma u}$ .
- (ii) When only  $\sigma$  is unknown, then we solve the following elliptic equation for  $u$

$$\begin{aligned} -\nabla \cdot \gamma \nabla u(x) + \frac{H}{\Gamma} &= 0, & \text{in } X \\ u(x) &= f(x), & \text{on } \partial X \end{aligned}, \quad (8.10)$$

and then evaluate  $\sigma = \frac{H}{\Gamma u}$ .

**Exercise 8.1.2** *Assume that  $f$  is uniformly bounded above and below by positive constants. Show that the reconstruction of  $\sigma$  is Lipschitz-stable in  $L^\infty(X)$ , i.e., that*

$$\|\sigma - \tilde{\sigma}\|_{L^\infty(X)} \leq C \|H - \tilde{H}\|_{L^\infty(X)}, \quad (8.11)$$

where  $\tilde{H}$  is acquired as in (8.8)-(8.9) with  $\sigma$  replaced by  $\tilde{\sigma}$ .

(iii) When only  $\gamma$  is unknown, we obtain  $u = \frac{H}{\sigma \Gamma}$  and then the above elliptic equation in (8.10) with  $\gamma|_{\partial X}$  known is a transport equation for  $\gamma$ . As soon as  $\beta := \nabla u$  is a sufficiently smooth, non-vanishing vector field, then  $\gamma$  is uniquely determined by the linear equation

$$\begin{aligned} \nabla \cdot \gamma \beta &= \beta \cdot \nabla \gamma + (\nabla \cdot \beta) \gamma = \frac{H}{\Gamma}, & \text{in } X \\ \gamma(x) &= \gamma|_{\partial X}(x), & \text{on } \partial X. \end{aligned} \quad (8.12)$$

This transport equation will be analyzed in more detail later in the chapter.

What the above results say is that the reconstruction of the optical coefficients is relatively straightforward when two out of the three are already known. The interest of the QPAT theory resides in the fact that the acquisition of more measurements in fact allows us to (uniquely and stably) reconstruct two coefficients but not three.

### 8.1.5 Introduction to Transient Elastography

Transient elastography images the (slow) propagation of shear waves using ultrasound. For more details, see, e.g., [44] and its extended list of references. As shear waves propagate, the resulting displacements can be imaged by ultra-fast ultrasound. Consider a scalar approximation of the equations of elasticity

$$\begin{aligned} \nabla \cdot \gamma(x) \nabla u(x, t) &= \rho(x) \partial_{tt} u(x, t), & t \in \mathbb{R}, x \in X \\ u(x, t) &= f(x, t), & t \in \mathbb{R}, x \in \partial X, \end{aligned} \quad (8.13)$$

where  $u(x, t)$  is the (say, downward) displacement,  $\gamma(x)$  is one of the Lamé parameters and  $\rho(x)$  is density. Using ultra-fast ultrasound measurements, the displacement  $u(x, t)$  can be imaged. This results in a very simplified model of transient elastography where we aim to reconstruct  $(\gamma, \rho)$  from knowledge of  $u(x, t)$ ; see [44] for more complex models. We may slightly generalize the model as follows. Upon taking Fourier transforms in the time domain and accounting for possible dispersive effects of the tissues, we obtain

$$\begin{aligned} \nabla \cdot \gamma(x; \omega) \nabla u(x; \omega) + \omega^2 \rho(x; \omega) u(x; \omega) &= 0, & \omega \in \mathbb{R}, x \in X \\ u(x; \omega) &= f(x; \omega), & \omega \in \mathbb{R}, x \in \partial X. \end{aligned} \quad (8.14)$$

The inverse transient elastography problem with dispersion effect would then be the reconstruction of  $(\gamma(x; \omega), \rho(x; \omega))$  from knowledge of  $u(x; \omega)$  corresponding to one or several boundary conditions  $f(x; \omega)$  applied at the boundary  $\partial X$ .

This corresponds to the setting of PAT with  $\Gamma\sigma = 1$  so that  $H = u$ . As in PAT,  $H$  is an internal functional of the unknown coefficients  $(\gamma, \rho)$ . The role of quantitative transient elastography (QTE) is therefore to reconstruct  $(\gamma, \rho)$  from knowledge of one or several internal functionals of the form  $H = u$ .

## 8.2 Theory of quantitative PAT and TE

Let us come back to the general theory of photo-acoustic tomography and transient elastic tomography. The two mathematical problems are quite similar. For concreteness, we first focus on the PAT setting and next state how the results should be modified to solve the TE problem.

### 8.2.1 Uniqueness and stability results in QPAT

We have seen above that one coefficient in  $(\gamma, \sigma, \Gamma)$  could be uniquely reconstructed from one internal functional  $H$  provided that the other two coefficients were known. It turns out that two coefficients in  $(\gamma, \sigma, \Gamma)$  can also be uniquely (and stably) be reconstructed from two (well-chosen) internal functionals  $H_j$ ,  $j = 1, 2$  provided that the third coefficient is known. However, these two internal functionals  $(H_1, H_2)$  uniquely determine any internal functional  $H_3$  obtained by using an arbitrary illumination  $f_3$  on  $\partial X$ . Moreover, the two internal functionals  $(H_1, H_2)$  uniquely characterize two explicit functionals of  $(\gamma, \sigma, \Gamma)$  that do not allow us to reconstruct all parameters in  $(\gamma, \sigma, \Gamma)$  uniquely.

**Measurement operator in QPAT.** We have mentioned above the notion of *well-chosen* internal functionals  $H_j$ ,  $j = 1, 2$ , or equivalently *well-chosen* illuminations  $f_j$ ,  $j = 1, 2$  on  $\partial X$  since the different functions  $H_j$  are characterized by the probing  $f_j$  on  $\partial X$ . To make precise statements, we introduce some notation. For  $f \in H^{\frac{1}{2}}(\partial X)$ , we obtain a solution  $u \in H^1(X)$  of (8.9) and we can define the internal functional operator

$$\begin{aligned} \mathfrak{H}_{(\gamma, \sigma, \Gamma)} : H^{\frac{1}{2}}(\partial X) &\rightarrow H^1(X) \\ f &\mapsto \mathfrak{H}_{(\gamma, \sigma, \Gamma)} f = \Gamma(x)\sigma(x)u(x). \end{aligned} \quad (8.15)$$

Let  $I \in \mathbb{N}^*$  and  $f_i \in H^{\frac{1}{2}}(\partial X)$  for  $1 \leq i \leq I$  be a given set of  $I$  boundary conditions. Define  $\mathfrak{f} = (f_1, \dots, f_I)$ . The measurement operator  $\mathfrak{M}_{\mathfrak{f}}$  is then defined as the following collection of internal functionals:

$$\begin{aligned} \mathfrak{M}_{\mathfrak{f}} : \mathfrak{X} &\rightarrow \mathfrak{Y}^I \\ (\gamma, \sigma, \Gamma) &\mapsto \mathfrak{M}_{\mathfrak{f}}(\gamma, \sigma, \Gamma) = (\mathfrak{H}_{(\gamma, \sigma, \Gamma)} f_1, \dots, \mathfrak{H}_{(\gamma, \sigma, \Gamma)} f_I). \end{aligned} \quad (8.16)$$

Here,  $\mathfrak{X}$  is a subset of a Banach space in which the unknown coefficients are defined; see (H1) below). Also  $\mathfrak{Y}$  is a subset of  $H^1(X)$  where the solutions to (8.9) are defined. We also define  $H_j = \mathfrak{H}_{(\gamma, \sigma, \Gamma)} f_j$  for  $1 \leq j \leq I$ .

**Assumptions on the coefficients and the illuminations.** Here are now some mathematical assumptions on the coefficients and a definition of illuminations that we call *well-chosen*. Here and below, we denote by  $W^{m,p}(X)$  the space of functions with derivatives of order less than or equal to  $m$  belonging to  $L^p(X)$ .

(H1). We denote by  $\mathfrak{X}$  the set of coefficients  $(\gamma, \sigma, \Gamma)$  that are of class  $W^{1,\infty}(X)$ , are bounded above and below by fixed positive constants, and such that the traces  $(\gamma, \sigma, \Gamma)|_{\partial X}$  on the boundary  $\partial X$  are fixed (known) functions.

(H2). The illuminations  $f_j$  are positive functions on  $\partial X$  that are the restrictions on  $\partial X$  of functions of class  $C^3(\bar{X})$ .

(H3). We say that  $\mathfrak{f}_2 = (f_1, f_2)$  is a pair of *well-chosen* illuminations with corresponding functionals  $(H_1, H_2) = (\mathfrak{H}_{(\gamma, \sigma, \Gamma)} f_1, \mathfrak{H}_{(\gamma, \sigma, \Gamma)} f_2)$  provided that (H2) is satisfied and the vector field

$$\beta := H_1 \nabla H_2 - H_2 \nabla H_1 = H_1^2 \nabla \frac{H_2}{H_1} = H_1^2 \nabla \frac{u_2}{u_1} = -H_2^2 \nabla \frac{H_1}{H_2} \quad (8.17)$$

is a vector field in  $W^{1,\infty}(X)$  such that

$$|\beta|(x) \geq \alpha_0 > 0, \quad \text{a.e. } x \in X. \quad (8.18)$$

(H3'). We say that  $\mathfrak{f}_2 = (f_1, f_2)$  is a pair of *weakly well-chosen* illuminations with corresponding functionals  $(H_1, H_2) = (\mathfrak{H}_{(\gamma, \sigma, \Gamma)} f_1, \mathfrak{H}_{(\gamma, \sigma, \Gamma)} f_2)$  provided that (H2) is satisfied and the vector field  $\beta$  defined in (8.17) is in  $W^{1,\infty}(X)$  and  $\beta \neq 0$  a.e. in  $X$ .

**Remark 8.2.1** Note that **(H3')** is satisfied as soon as  $\frac{f_1}{f_2} \neq C$  is not a constant. Indeed, if  $\beta = 0$  on a set of positive measure, then  $\nabla \frac{u_2}{u_1} = 0$  on that same set. Yet,  $\frac{u_2}{u_1}$  solves the elliptic equation (8.20) below. It is known that under sufficient smoothness conditions on the coefficients, the critical points of solutions to such elliptic equations are of measure zero unless these solutions are constant [37, 53].

**Uniqueness/Non-uniqueness result.** Hypothesis **(H3)** will be useful in the analysis of the stability of the reconstructions. For the uniqueness result, the weaker hypothesis **(H3')** is sufficient. Note that almost all illumination pairs  $\mathbf{f}_2$  satisfy **(H3')**, which is a mere regularity statement. Beyond the regularity assumptions on  $(\gamma, \sigma, \Gamma)$ , the domain  $X$ , and the boundary conditions  $f_j$ , the only real assumption we impose is thus (8.18). In general, there is no guaranty that the gradient of  $\frac{u_2}{u_1}$  does not vanish. Not all pairs of illuminations  $\mathbf{f}_2 = (f_1, f_2)$  are *well-chosen* although most are *weakly well-chosen*. That the vector field  $\beta$  does not vanish is a *sufficient* condition for the stability estimates presented below to be satisfied. It is not *necessary*. As we shall see, guaranteeing (8.18) is relatively straightforward in dimension  $n = 2$ . It is much complicated in dimension  $n \geq 3$ . The only available methodology to ensure that (8.18) holds for a large class of conductivities is based on the same method of complex geometric optics (CGO) solutions already used to solve the Calderón problem in Chapter 7.

Under these hypotheses, we obtain the following result:

**Theorem 8.2.2** Let  $\mathfrak{X}$  be defined as in **(H1)** and let  $\mathbf{f}_2$  be well chosen illuminations as indicated in **(H2)** and **(H3')**. Let  $I \in \mathbb{N}^*$  and  $\mathbf{f} = (f_1, \dots, f_I)$  be a set of (arbitrary) illuminations satisfying **(H2)**. Then we have the following:

(i). The measurement operator  $\mathfrak{M}_{\mathbf{f}_2}$  uniquely determines  $\mathfrak{M}_{\mathbf{f}}$  (meant in the sense that  $\mathfrak{M}_{\mathbf{f}_2}(\gamma, \sigma, \Gamma) = \mathfrak{M}_{\mathbf{f}_2}(\tilde{\gamma}, \tilde{\sigma}, \tilde{\Gamma})$  implies that  $\mathfrak{M}_{\mathbf{f}}(\gamma, \sigma, \Gamma) = \mathfrak{M}_{\mathbf{f}}(\tilde{\gamma}, \tilde{\sigma}, \tilde{\Gamma})$ ).

(ii). The measurement operator  $\mathfrak{M}_{\mathbf{f}_2}$  uniquely determines the two following functionals of  $(\gamma, \sigma, \Gamma)$  (meant in the same sense as above):

$$\chi(x) := \frac{\sqrt{\gamma}}{\Gamma\sigma}(x), \quad q(x) := \left( \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}} + \frac{\sigma}{\gamma} \right)(x). \quad (8.19)$$

(iii). Knowledge of the two functionals  $\chi$  and  $q$  uniquely determines (in the same sense as above)  $\mathfrak{M}_{\mathbf{f}_2} = (H_1, H_2)$ . In other words, the reconstruction of  $(\gamma, \sigma, \Gamma)$  is unique up to any transformation that leaves  $(\chi, q)$  invariant.

*Proof.* Let us start with (i). Upon multiplying the equation for  $u_1$  by  $u_2$ , the equation for  $u_2$  by  $u_1$ , and subtracting both relations, we obtain

$$\begin{aligned} -\nabla \cdot (\gamma u_1^2) \nabla \frac{H_2}{H_1} &= 0, & \text{in } X \\ \gamma u_1^2 &= \gamma|_{\partial X} f_1^2, & \text{on } \partial X. \end{aligned} \quad (8.20)$$

This is a transport equation in conservative form for  $\gamma u_1^2$ . More precisely, this is a transport equation  $\nabla \cdot \rho \tilde{\beta} = 0$  for  $\rho$  with  $\rho|_{\partial X} = 1$  and

$$\tilde{\beta} = \chi^2 \beta = (\gamma u_1^2) \nabla \frac{H_2}{H_1}.$$

Since  $\tilde{\beta} \in W^{1,\infty}(X)$  and is divergence free, the above equation for  $\rho$  admits the unique solution  $\rho \equiv 1$  since (8.18) holds. Indeed, we find that  $\nabla \cdot (\rho - 1)^2 \tilde{\beta} = 0$  by application of the chain rule with  $\rho|_{\partial X} - 1 = 0$  on  $\partial X$ . Upon multiplying the equation by  $\frac{H_2}{H_1}$  and integrating by parts, we find

$$\int_X (\rho - 1)^2 \chi^2 H_1^2 \left| \nabla \frac{H_2}{H_1} \right|^2 dx = 0.$$

Using **(H3')** and the result of remark 8.2.1, we deduce that  $\rho = 1$  on  $X$  by continuity. This proves that  $\gamma u_1^2$  is uniquely determined. Dividing by  $H_1^2 = (\Gamma\sigma)^2 u_1^2$ , this implies that  $\chi > 0$  defined in (8.19) is uniquely determined. Note that we do not need the full  $W^{1,\infty}(X)$  regularity of  $\beta$  in order to obtain the above result. However, we still need to be able to apply the chain rule to obtain an equation for  $(\rho - 1)^2$  and conclude that the solution to the transport equation is unique.

Let now  $f$  be an arbitrary boundary condition and let  $u$  be the solution to (8.9) and  $H = \mathfrak{H}_{(\gamma,\sigma,\Gamma)} f$  defined by (8.8). Replacing  $H_2$  above by  $H$  yields

$$-\nabla \cdot \chi^2 H_1^2 \nabla \frac{H}{H_1} = 0, \quad \text{in } X \tag{8.21}$$

$$H = \Gamma|_{\partial X} \sigma|_{\partial X} f, \quad \text{on } \partial X.$$

This is a well-defined elliptic equation with a unique solution  $H \in H^1(X)$  for  $f \in H^{\frac{1}{2}}(\partial X)$ . This proves that  $H = \mathfrak{H}_{(\gamma,\sigma,\Gamma)} f$  is uniquely determined by  $(H_1, H_2)$  and concludes the proof of (i).

Let us next prove (ii). We have already seen that  $\chi$  was determined by  $\mathfrak{M}_{f_2} = (H_1, H_2)$ . Define now  $v = \sqrt{\gamma} u_1$ , which is also uniquely determined based on the results in (i). Define

$$q = \frac{\Delta v}{v} = \frac{\Delta(\sqrt{\gamma} u_1)}{\sqrt{D} u_1},$$

which is the Liouville change of variables used in Chapter 7 to solve the Calderón problem. Since  $u_1$  is bounded from below, is sufficiently smooth, and solves (8.9), the following calculations show that  $q$  is given by (8.19). Indeed, we find that

$$\nabla \cdot \gamma \nabla u_1 = \nabla \cdot (\sqrt{\gamma} \nabla v) - \nabla \cdot ((\nabla \sqrt{\gamma}) v) = \sqrt{\gamma} \Delta v - (\Delta \sqrt{\gamma}) v = \sigma u_1 = \frac{\sigma}{\sqrt{\gamma}} v. \tag{8.22}$$

Finally, we prove (iii). Since  $q$  is now known, we can solve

$$(\Delta - q)v_j = 0, \quad X, \quad v_j = \sqrt{\gamma|_{\partial X}} g_j \quad \partial X, \quad j = 1, 2.$$

Because  $q$  is of the specific form (8.19) as a prescribed functional of  $(\gamma, \sigma, \Gamma)$ , it is known that  $(\Delta - q)$  does not admit 0 as a (Dirichlet) eigenvalue, for otherwise, 0 would also be a (Dirichlet) eigenvalue of the elliptic operator

$$(-\nabla \cdot \gamma \nabla + \sigma) \cdot = (-\sqrt{\gamma} (\Delta - q) \sqrt{\gamma}) \cdot. \tag{8.23}$$

The latter calculation follows from (8.22). Thus  $v_j$  is uniquely determined for  $j = 1, 2$ . Now,

$$H_j = \Gamma \sigma u_j = \frac{\Gamma \sigma}{\sqrt{\gamma}} v_j = \frac{v_j}{\chi}, \quad j = 1, 2,$$

and is therefore uniquely determined by  $(\chi, q)$ . This concludes the proof that  $(\chi, q)$  uniquely determines  $\mathfrak{M}_{f_2}$ .  $\square$

**Reconstruction of two coefficients.** The above result shows that the unique reconstruction of  $(\gamma, \sigma, \Gamma)$  is not possible even from knowledge of a measurement operator  $\mathfrak{M}_f$  corresponding to an arbitrary (in fact possibly infinite) number of internal functionals  $I$ . We therefore face this peculiar situation that two well-chosen illuminations uniquely determine the functionals  $(\chi, q)$  but that acquiring additional measurements does not provide any new information.

However, we can prove the following positive result that if one of the coefficients in  $(\gamma, \sigma, \Gamma)$  is known, then the other two coefficients are uniquely determined.

**Corollary 8.2.3** *Under the hypotheses of the previous theorem, let  $(\chi, q)$  in (8.19) be known. Then*

- (a) *If  $\Gamma$  is known, then  $(\gamma, \sigma)$  are uniquely determined.*
- (b) *If  $\gamma$  is known, then  $(\sigma, \Gamma)$  are uniquely determined.*
- (c) *If  $\sigma$  is known, then  $(\gamma, \Gamma)$  are uniquely determined.*

*Proof.* (a) is probably the most practical case as  $\Gamma$  is often assumed to be constant or known. Since  $\Gamma$  is known, then so is  $\Gamma\chi = \sqrt{\gamma}/\sigma$  so that we have the elliptic equation for  $\sqrt{\gamma}$ :

$$(\Delta - q)\sqrt{\gamma} + \frac{1}{\Gamma\chi} = 0, \quad X, \quad \sqrt{\gamma}|_{\partial X} = \sqrt{\gamma|_{\partial X}}, \quad \partial X. \quad (8.24)$$

Again, because of the specific form of  $q$ ,  $(\Delta - q)$  is invertible and the above equation admits a unique solution. Once  $\sqrt{\gamma}$ , hence  $\gamma$ , is known, then so is  $\sigma = \frac{\sqrt{\gamma}}{\Gamma\chi}$ .

If  $\gamma$  is known in (b), then  $\sigma$  is known from  $q$  and  $\Gamma$  is known from  $\chi$ .

Finally in (c), we obtain that from the expression for  $q$  that

$$\sqrt{\gamma}(\Delta - q)\sqrt{\gamma} + \sigma = 0 \quad X, \quad \sqrt{\gamma}|_{\partial X} = \sqrt{\gamma|_{\partial X}}, \quad \partial X. \quad (8.25)$$

We need to prove a uniqueness result for the above nonlinear equation for  $\sqrt{\gamma}$ . Let us assume that  $\sqrt{\gamma}$  and another solution  $\tau\sqrt{\gamma}$  for  $0 < \tau(x)$  satisfy the above equation for  $\sigma$  fixed. We have

$$-\sqrt{\gamma}(\Delta - q)\sqrt{\gamma}\tau - \frac{\sigma}{\tau} = 0 \quad X.$$

Thanks to (8.23), this implies the following equation for  $\tau$ :

$$-\nabla \cdot \gamma \nabla \tau + \sigma \left( \tau - \frac{1}{\tau} \right) = 0, \quad X, \quad \tau = 1, \quad \partial X.$$

Upon multiplying by  $\tau - 1$  and integrating by parts, we find that

$$\int_X \gamma |\nabla(\tau - 1)|^2 dx + \int_X \sigma |\tau - 1|^2 \frac{\tau + 1}{\tau} dx = 0.$$

Since  $\tau > 0$ , we deduce from the above that  $\tau \equiv 1$  and that  $\gamma$  is uniquely determined by  $q$ . We then retrieve  $\Gamma$  from knowledge of  $\chi$ .  $\square$

**Reconstruction formulas.** Note that the above uniqueness results provide a *constructive* reconstruction procedure. In all cases, we first need to solve a transport equation for the functional  $\chi$ :

$$-\nabla \cdot (\chi^2 \beta) = 0 \quad \text{in } X, \quad \chi|_{\partial X} \text{ known on } \partial X, \quad (8.26)$$

with  $\beta$  the vector field defined in (8.17). This uniquely defines  $\chi > 0$ . Then we find that

$$q(x) = \frac{\Delta(H_1 \chi)}{H_1 \chi} = \frac{\Delta(H_2 \chi)}{H_2 \chi}. \quad (8.27)$$

This provides explicit reconstructions for  $(\chi, q)$ . In case (b), no further equation needs to be solved. In cases (a) and (c), we need to solve an elliptic equation for  $\sqrt{\gamma}$ , which is the linear equation (8.24) in (a) and the nonlinear equation (8.25) in (c). These steps have been implemented numerically with very satisfactory results in [18].

**Stability of the solution of the transport equation for  $\chi(x)$ .** We now derive a stability result for the reconstruction of  $\chi$  obtained from analyzing the transport equation (8.20). Similar stability results can be obtained for  $q$  and then for  $(\gamma, \sigma, \Gamma)$  depending on the reconstruction considered.

**Theorem 8.2.4** *Let  $\mathfrak{M}_{\mathfrak{f}_2}(\gamma, \sigma, \Gamma) = (H_1, H_2)$  be the measurements corresponding to the coefficients  $(\gamma, \sigma, \Gamma)$  such that **(H1)**, **(H2)**, **(H3)** hold. Let  $\mathfrak{M}_{\mathfrak{f}_2}(\tilde{\gamma}, \tilde{\sigma}, \tilde{\Gamma}) = (\tilde{H}_1, \tilde{H}_2)$  be the measurements corresponding to the same illuminations  $\mathfrak{f}_2 = (f_1, f_2)$  with another set of coefficients  $(\tilde{\gamma}, \tilde{\sigma}, \tilde{\Gamma})$  such that **(H1)**, **(H2)** hold. Define  $\delta\mathfrak{M}_{\mathfrak{f}_2} = \mathfrak{M}_{\mathfrak{f}_2}(\tilde{\gamma}, \tilde{\sigma}, \tilde{\Gamma}) - \mathfrak{M}_{\mathfrak{f}_2}(\gamma, \sigma, \Gamma)$ . Then we find that*

$$\|\chi - \tilde{\chi}\|_{L^p(X)} \leq C \|\delta\mathfrak{M}_{\mathfrak{f}_2}\|_{(W^{1, \frac{p}{2}}(X))^2}^{\frac{1}{2}}, \quad \text{for all } 2 \leq p < \infty. \quad (8.28)$$

Let us assume, moreover, that  $\gamma(x)$  is of class  $C^3(\bar{X})$ . Then we have the estimate

$$\|\chi - \tilde{\chi}\|_{L^p(X)} \leq C \|\delta\mathfrak{M}_{\mathfrak{f}_2}\|_{(L^{\frac{p}{2}}(X))^2}^{\frac{1}{3}}, \quad \text{for all } 2 \leq p < \infty. \quad (8.29)$$

By interpolation, the latter result implies that

$$\|\chi - \tilde{\chi}\|_{L^\infty(X)} \leq C \|\delta\mathfrak{M}_{\mathfrak{f}_2}\|_{(L^{\frac{p}{2}}(X))^2}^{\frac{p}{3(d+p)}}, \quad \text{for all } 2 \leq p < \infty. \quad (8.30)$$

We may for instance choose  $p = 4$  above to measure the noise level in the measurement  $\mathfrak{M}_{\mathfrak{f}_2}$  in the square integrable norm when noise is described by its power spectrum in the Fourier domain.

*Proof.* Define  $\nu = \chi^2$  and  $\tilde{\nu} = \tilde{\chi}^2$  with  $\chi$  defined in (8.19) and  $\beta$  and  $\tilde{\beta}$  as in (8.17). Then we find that

$$\nabla \cdot \frac{\nu - \tilde{\nu}}{\nu} (\nu \beta) + \nabla \cdot \tilde{\nu} (\beta - \tilde{\beta}) = 0.$$

Note that  $\nu \beta = \chi^2 H_1^2 \nabla \frac{H_2}{H_1}$  is a divergence-free field. Let  $\varphi$  be a twice differentiable, non-negative, function from  $\mathbb{R}$  to  $\mathbb{R}$  with  $\varphi(0) = \varphi'(0) = 0$ . Then we find that

$$\nabla \cdot \varphi\left(\frac{\nu - \tilde{\nu}}{\nu}\right) (\nu \beta) + \varphi'\left(\frac{\nu - \tilde{\nu}}{\nu}\right) \nabla \cdot \tilde{\nu} (\beta - \tilde{\beta}) = 0.$$

Let us multiply this equation by a test function  $\zeta \in H^1(X)$  and integrate by parts. Since  $\nu = \nu'$  on  $\partial X$ , we find

$$\int_X \varphi \left( \frac{\nu - \tilde{\nu}}{\nu} \right) \nu \beta \cdot \nabla \zeta dx + \int_X \tilde{\nu} (\beta - \tilde{\beta}) \nabla \cdot \left[ \zeta \varphi' \left( \frac{\nu - \tilde{\nu}}{\nu} \right) \right] dx = 0.$$

Upon choosing  $\zeta = \frac{H_2}{H_1}$ , we find

$$\int_X \varphi \nu H_1^2 \left| \nabla \frac{H_2}{H_1} \right|^2 dx + \int_X \tilde{\nu} (\beta - \tilde{\beta}) \cdot \nabla \frac{H_2}{H_1} \varphi' dx + \int_X \tilde{\nu} (\beta - \tilde{\beta}) \cdot \nabla \frac{\nu - \tilde{\nu}}{\nu} \frac{H_2}{H_1} \varphi'' dx = 0.$$

Above,  $\varphi$  stands for  $\varphi\left(\frac{\nu - \tilde{\nu}}{\nu}\right)$  in all integrals. By assumption on the coefficients,  $\nabla \frac{\nu - \tilde{\nu}}{\nu}$  is bounded a.e.. This is one of our main motivations for assuming that the optical coefficients are Lipschitz. The middle term is seen to be smaller than the third term and so we focus on the latter one. Upon taking  $\varphi(x) = |x|^p$  for  $p \geq 2$  and using assumption **(H3)**, we find that

$$\|\nu - \tilde{\nu}\|_{L^p(X)}^p \leq C \int_X |\beta - \tilde{\beta}| |\nu - \tilde{\nu}|^{p-2} dx.$$

By an application of the Hölder inequality, we deduce that

$$\|\nu - \tilde{\nu}\|_{L^p(X)} \leq C \|\beta - \tilde{\beta}\|_{L^{\frac{p}{2}}(X)}^{\frac{1}{2}}.$$

We next write  $\beta - \tilde{\beta} = (H_1 - \tilde{H}_1) \nabla H_2 + \tilde{H}_1 (\nabla (H_2 - \tilde{H}_2) - \dots)$  and use the fact that the solutions to (8.9) and the coefficients are in  $W^{1,\infty}(X)$  to conclude that (8.28) holds.

The other results are obtained by regularity theory and interpolation. Indeed from standard regularity results with coefficients in  $W^{1,\infty}(X)$ , we find that the solutions to (8.9) are of class  $W^{3,q}(X)$  for all  $1 \leq q < \infty$ . Since the coefficient  $\gamma$  is of class  $C^3(\bar{X})$ , then the measurements  $H_j$  are of class  $W^{3,q}(X)$  for all  $1 \leq q < \infty$ . Standard Sobolev estimates show that

$$\|H_j - \tilde{H}_j\|_{W^{1,q}(X)} \leq C \|H_j - \tilde{H}_j\|_{L^q(X)}^{\frac{2}{3}} \|H_j - \tilde{H}_j\|_{W^{3,q}(X)}^{\frac{1}{3}}.$$

The last term is bounded by a constant, which gives (8.29) for  $q = \frac{p}{2}$ . Another interpolation result states that

$$\|\varphi\|_{\infty} \leq \|\nabla \varphi\|_{\infty}^{\theta} \|\varphi\|_p^{1-\theta}, \quad \theta = \frac{d}{d+p}.$$

This provides the stability result in the uniform norm (8.30).  $\square$

**Exercise 8.2.1** Find similar stability estimates for  $q$  and  $(\gamma, \sigma, \Gamma)$  in the different settings considered in Corollary 8.2.3 and in section 8.1.4.

## 8.2.2 Application to Quantitative Transient Elastography

We can apply the above results to the time-harmonic reconstruction in a simplified model of transient elastography. Let us assume that  $\gamma$  and  $\rho$  are unknown functions of  $x \in X$  and  $\omega \in \mathbb{R}$ . Recall that the displacement solves (8.14). Assuming that  $u(x; \omega)$  is known after step 1 of the reconstruction using the ultrasound measurements, then we are in the setting of Theorem 8.2.2 with  $\Gamma\sigma = 1$ . Let us then assume that the two illuminations  $f_1(x; \omega)$  and  $f_2(x; \omega)$  are chosen such that for  $u_1$  and  $u_2$  the corresponding solutions of (8.14), we have that **(H3')** holds. Then, (8.19) shows that the reconstructed function  $\chi$  uniquely determines the Lamé parameter  $\gamma(x; \omega)$  and that the reconstructed function  $q$  then uniquely determines  $\omega^2\rho$  and hence the density parameter  $\rho(x; \omega)$ . The reconstructions are performed for each frequency  $\omega$  independently. We may summarize this as follows:

**Corollary 8.2.5** *Under the hypotheses Theorem 8.2.2 and the hypotheses described above, let  $(\chi, q)$  in (8.19) be known. Then  $(\gamma(x; \omega), \rho(x; \omega))$  are uniquely determined by two well-chosen measurements. Moreover, if **(H3)** holds, the stability results in Theorem 8.2.4 hold.*

Alternatively, we may assume that in a given range of frequencies,  $\gamma(x)$  and  $\rho(x)$  are independent of  $\omega$ . In such a setting, we expect that one measurement  $u(x; \omega)$  for two different frequencies will provide sufficient information to reconstruct  $(\gamma(x), \rho(x))$ . Assume that  $u(x; \omega)$  is known for  $\omega = \omega_j$ ,  $j = 1, 2$  and define  $0 < \alpha = \omega_2^2\omega_1^{-2} \neq 1$ . Then straightforward calculations show that

$$\nabla \cdot \gamma\beta_\alpha = 0, \quad \beta_\alpha = (u_1\nabla u_2 - \alpha u_2\nabla u_1). \quad (8.31)$$

This provides a transport equation for  $\gamma$  that can be solved stably provided that  $|\beta_\alpha| \geq c_0 > 0$ , i.e., a hypothesis of the form **(H3)** applies and  $\beta_\alpha$  does not vanish on  $X$ . Then, Theorem 8.2.2 and Theorem 8.2.4 apply in this setting.

## 8.3 Well-chosen illuminations in PAT and TE

The stability results presented for QPAT and QTE involve *well-chosen* illuminations so that **(H3)** and (8.18) hold. Such a constraint is clearly not satisfied for all possible pairs  $(f_1, f_2)$ . In two dimensions of space, i.e., when  $n = 2$ , a very large class of illuminations can be proved to be *well-chosen*. In dimensions  $n \geq 3$ , the proofs are much more complicated and involve the CGO solutions constructed in Chapter 7.

### 8.3.1 The two dimensional case

In dimension  $n = 2$ , we have:

**Lemma 8.3.1** *Assume that  $h = \frac{f_2}{f_1}$  on  $\partial X$  is an almost two-to-one function, i.e., a function that is a two-to-one map except possibly at its minimum and at its maximum. Then (8.18) is satisfied.*

*Proof.* Upon multiplying the equation for  $u_1$  by  $u_2$ , the equation for  $u_2$  by  $u_1$ , and subtracting both relations, we obtain

$$-\nabla \cdot (\gamma u_1^2) \nabla \frac{u_2}{u_1} = 0, \quad \text{in } X \quad \frac{u_2}{u_1} = \frac{f_2}{f_1}, \quad \text{on } \partial X. \quad (8.32)$$

This implies that  $v := \frac{u_2}{u_1}$  satisfies an elliptic equation with a diffusion coefficient  $\tilde{\gamma} = \gamma u_1^2$  bounded from above and below by positive constants. Note that  $\beta = H_1^2 \nabla v$ . Results in, e.g., [2, Theorem 1.2] show that  $\nabla v$  cannot vanish inside  $X$ . In other words,  $v$  does not admit any *critical point*. By the maximum principle and the assumption on  $h$ , no critical point of  $v$  can occur on  $\partial X$  either. This implies that  $|\nabla v| > 0$  and that we can find a constant such that (8.18) holds since  $H_1^2$  is bounded from below by a positive constant and by continuity  $|\nabla v|$  attains its (strictly positive) minimum in  $\bar{X}$ .  $\square$

### 8.3.2 The $n$ dimensional case

In dimension  $n \geq 3$ , the above result on the (absence of) critical points of elliptic solutions no longer holds. However, by continuity, we may verify that (8.18) is satisfied for a large class of illuminations when  $\gamma$  is close to a constant and  $\sigma$  is sufficiently small. For arbitrary coefficients  $(\gamma, \sigma)$  in dimension  $n \geq 3$ , the only available proof that (8.18) is satisfied for an open set of illuminations is the one obtained by means of complex geometrical optics solutions; see [21]. The main result is:

**Theorem 8.3.2** *Let (H1) hold. Then there is an open set in  $C^3(\partial X)$  of illuminations  $\mathfrak{f}_2 = (f_1, f_2)$  such that (H3) holds.*

The open set, unlike the result obtained in two dimensions in Lemma 8.3.1, is not explicitly constructed.

*Proof.* Let us consider the elliptic equation (8.6). In chapter 7, we have constructed solutions of the form

$$u_{\varrho}(x) = \frac{1}{\sqrt{\gamma}} e^{\varrho \cdot x} (1 + \psi_{\varrho}(x)), \quad (8.33)$$

with  $|\varrho| \psi_{\varrho}(x)$  bounded uniformly in  $H^s(X)$  for arbitrary  $s \geq 0$  provided that  $\gamma$  and  $\sigma$  are sufficiently smooth coefficients. Using the construction of Chapter 7, we can prove the following lemma:

**Lemma 8.3.3** *Let  $u_{\varrho_j}$  for  $j = 1, 2$  be CGO solutions with  $(\gamma, \sigma)$  sufficiently smooth for both  $\varrho_j$  and  $k \geq 1$  and with  $c_0^{-1} |\varrho_1| \leq |\varrho_2| \leq c_0 |\varrho_1|$  for some  $c_0 > 0$ . Then we have*

$$\hat{\beta} := \frac{1}{2|\varrho_1|} e^{-(\varrho_1 + \varrho_2) \cdot x} \left( u_{\varrho_1} \nabla u_{\varrho_2} - u_{\varrho_2} \nabla u_{\varrho_1} \right) = \frac{\varrho_1 - \varrho_2}{2|\varrho_1|} + \hat{h}, \quad (8.34)$$

where the vector field  $\hat{h}$  satisfies the constraint

$$\|\hat{h}\|_{C^k(\bar{X})} \leq \frac{C_0}{|\varrho_1|}, \quad (8.35)$$

for some constant  $C_0$  independent of  $\varrho_j$ ,  $j = 1, 2$ .

**Exercise 8.3.1** *Prove the above lemma using the results obtained in Chapter 7.*

With  $\varrho_2 = \overline{\varrho_1}$  so that  $u_{\varrho_2} = \overline{u_{\varrho_1}}$ , the imaginary part of (8.34) is a vector field that does not vanish on  $X$  for  $|\varrho_1|$  sufficiently large. Moreover, let  $u_{\varrho_1} = v + iw$  and  $u_{\varrho_2} = v - iw$  for  $v$  and  $w$  real-valued functions. Then the imaginary and real parts of (8.34) are given by

$$\Im \hat{\beta} = \frac{1}{|\varrho_1|} e^{-2\Re \varrho_1 \cdot x} (w \nabla v - v \nabla w) = \frac{\Im \varrho_1}{|\varrho_1|} + \Im \hat{h}, \quad \Re \hat{\beta} = 0.$$

Let  $u_1$  and  $u_2$  be solutions of the elliptic problem (8.6) on  $X$  such that  $u_1 + iu_2$  on  $\partial X$  is close (in the  $C^3(\partial X)$  (strong) topology) to the trace of  $u_{\varrho_1}$ . The above result shows that

$$|u_1 \nabla u_2 - u_2 \nabla u_1| \geq c_0 > 0 \quad \text{in } X.$$

This yields (8.18) and the proof of the theorem.  $\square$

The set of illuminations  $\mathfrak{f}_2 = (f_1, f_2)$  for which (8.18) is guaranteed is not known explicitly. All we know is that if  $\mathfrak{f}_2$  is chosen sufficiently close to the traces of CGO solutions constructed above, then indeed the vector field  $\beta$  will satisfy (8.18). One major drawback with such a result is that the CGO solutions depend on the unknown coefficient  $(\gamma, \sigma)$ . That said, there does exist an open set of illuminations  $\mathfrak{f}_2$  such that (8.18) holds.

This result should be contrasted with the case in two dimensions, where we were able to prove that (8.18) held for a very large class of (non-oscillatory) illuminations  $\mathfrak{f}_2$ .

# Chapter 9

## Coupled-physics IP II: Ultrasound Modulation Tomography

### 9.1 Ultrasound Modulation Tomography

The preceding chapter analyzed inverse problems related to the photo-acoustic effect. In this chapter, we consider another physical mechanism that couples optical waves with ultrasound, namely the ultrasound modulation effect. To simplify the presentation, and because most results are known in this simplified setting, we assume that the elliptic equations involve an unknown diffusion coefficient but that the absorption coefficient is assumed to vanish. We refer the reader to [20] to generalizations to the practically physical setting of an elliptic equation with unknown diffusion and absorption coefficients.

Consider thus the following elliptic equation

$$-\nabla \cdot \gamma(x)\nabla u = 0 \quad \text{in } X, \quad u = f \quad \text{on } \partial X. \quad (9.1)$$

Here,  $\gamma$  is the unknown diffusion coefficient, which we assume for the moment is a real-valued, scalar, function defined on a domain  $X \subset \mathbb{R}^n$  for  $n = 2$  or  $n = 3$ . We assume that  $\gamma$  is bounded above and below by positive constants so that the above equation admits a unique solution. We also assume that  $\gamma$  is sufficiently smooth so that the solution to the above equation is continuously differentiable on  $\bar{X}$ , the closure of  $X$  [33]. As before, we denote by  $f(x)$  the imposed (sufficiently smooth) Dirichlet boundary conditions.

As we have seen already, the coefficient  $\gamma(x)$  may model the electrical conductivity in the setting of electrical impedance tomography (EIT) or a diffusion coefficient of particles (photons) in the setting of optical tomography (OT). Both EIT and OT are modalities with high contrast, in the sense that  $\gamma(x)$  takes different values in different tissues and allows one to discriminate between healthy and non-healthy tissues. In OT, high contrasts are mostly observed in the absorption coefficient, which we recall is not modeled here; see [20].

A methodology to couple high contrast with high resolution consists of perturbing the diffusion coefficient acoustically. Let an acoustic signal propagate through the domain. We assume here that the sound speed is constant and that the acoustic signal is a plane wave of the form  $p \cos(k \cdot x + \varphi)$  where  $p$  is the amplitude of the acoustic signal,  $k$  its wave-number and  $\varphi$  an additional phase. The acoustic signal modifies the properties of

the diffusion equation. We assume that such an effect is small and that the coefficient in (9.1) is modified as

$$\gamma_\varepsilon(x) = \gamma(x)(1 + \zeta\varepsilon\mathbf{c}), \quad (9.2)$$

where we have defined  $\mathbf{c} = \mathbf{c}(x) = \cos(k \cdot x + \varphi)$  and where  $\varepsilon = p\Gamma$  is the product of the acoustic amplitude  $p \in \mathbb{R}$  and a measure  $\Gamma > 0$  of the coupling between the acoustic signal and the modulations of the constitutive parameter in (9.1). We assume that  $\varepsilon \ll 1$  so that the influence of the acoustic signal on  $\gamma_\varepsilon$  admits an asymptotic expansion that we truncated at the second order as displayed in (9.2). The size of the terms in the expansion are physically characterized by  $\zeta$  and depend on the specific application.

Let  $u$  and  $v$  be solutions of (9.1) with fixed boundary conditions  $g$  and  $h$ , respectively. When the acoustic field is turned on, the coefficients are modified as described in (9.2) and we denote by  $u_\varepsilon$  and  $v_\varepsilon$  the corresponding solutions. Note that  $u_{-\varepsilon}$  is the solution obtained by changing the sign of  $p$  or equivalently by replacing  $\varphi$  by  $\varphi + \pi$ .

By the standard continuity property of the solution to (9.1) with respect to changes in the coefficients and regular perturbation arguments, we find that  $u_\varepsilon = u_0 + \varepsilon u_1 + O(\varepsilon^2)$ . Let us multiply the equation for  $u_\varepsilon$  by  $v_{-\varepsilon}$  and the equation for  $v_{-\varepsilon}$  by  $u_\varepsilon$ , subtract the resulting equalities, and use standard integrations by parts. We obtain that

$$\int_X (\gamma_\varepsilon - \gamma_{-\varepsilon}) \nabla u_\varepsilon \cdot \nabla v_{-\varepsilon} dx = \int_{\partial X} \gamma_{-\varepsilon} \frac{\partial v_{-\varepsilon}}{\partial \nu} u_\varepsilon - \gamma_\varepsilon \frac{\partial u_\varepsilon}{\partial \nu} v_{-\varepsilon} d\mu(x). \quad (9.3)$$

**Exercise 9.1.1** *Verify the above result.*

Here,  $d\mu(x)$  is the standard surface measure on  $\partial X$ . We assume that  $\gamma_\varepsilon \partial_\nu u_\varepsilon$  and  $\gamma_\varepsilon \partial_\nu v_\varepsilon$  are measured on  $\partial X$ , at least on the support of  $v_\varepsilon = h$  and  $u_\varepsilon = g$ , respectively, for all values  $\varepsilon$  of interest. Note that the above equation holds if the Dirichlet boundary conditions are replaced by Neumann boundary conditions. Let us define

$$J_\varepsilon := \frac{1}{2} \int_{\partial X} \gamma_{-\varepsilon} \frac{\partial v_{-\varepsilon}}{\partial \nu} u_\varepsilon - \gamma_\varepsilon \frac{\partial u_\varepsilon}{\partial \nu} v_{-\varepsilon} d\mu(x) = \varepsilon J_1 + \varepsilon^2 J_2 + O(\varepsilon^3). \quad (9.4)$$

We assume that the real valued functions  $J_m = J_m(k, \varphi)$  are known (measured functions). Notice that such knowledge is based on the physical boundary measurement of the Cauchy data of the form  $(u_\varepsilon, \gamma_\varepsilon \partial_\nu u_\varepsilon)$  and  $(v_\varepsilon, \gamma_\varepsilon \partial_\nu v_\varepsilon)$  on  $\partial X$ .

Equating like powers of  $\varepsilon$ , we find at the leading order that

$$\int_X [\zeta \gamma(x) \nabla u_0 \cdot \nabla v_0(x)] \cos(k \cdot x + \varphi) dx = J_1(k, \varphi). \quad (9.5)$$

This may be acquired for all  $k \in \mathbb{R}^n$  and  $\varphi = 0, \frac{\pi}{2}$ , and hence provides the Fourier transform of

$$H[u_0, v_0](x) = \zeta \gamma(x) \nabla u_0 \cdot \nabla v_0(x). \quad (9.6)$$

Note that when  $v_\varepsilon = u_\varepsilon$ , then we find from the expression in (9.3) that  $J_2 = 0$  in (9.4) so that the expression for  $J_1$  may be obtained from available measurements in (9.4) with an accuracy of order  $O(\varepsilon^2)$ . Note also that

$$H[u_0, v_0](x) = \frac{1}{4} (H[u_0 + v_0, u_0 + v_0] - H[u_0 - v_0, u_0 - v_0])$$

by polarization. In other words, the limiting measurements (for small  $\varepsilon$ ) in (9.6) may also be obtained by considering expressions of the form (9.3) with  $u_\varepsilon = v_\varepsilon$ .

In the setting of optical tomography, the coefficient  $\gamma_\varepsilon$  in (9.2) takes the form

$$\gamma_\varepsilon(x) = \frac{\tilde{\gamma}_\varepsilon}{c_\varepsilon^{n-1}}(x),$$

where  $\tilde{\gamma}_\varepsilon$  is the diffusion coefficient,  $c_\varepsilon$  is the light speed, and  $n$  is spatial dimension. When the pressure field is turned on, the location of the scatterers is modified by compression and dilation. Since the diffusion coefficient is inversely proportional to the scattering coefficient, we find that

$$\frac{1}{\gamma_\varepsilon(x)} = \frac{1}{\gamma(x)}(1 + \varepsilon \mathbf{c}(x)).$$

Moreover, the pressure field changes the index of refraction (the speed) of light as follows

$$c_\varepsilon(x) = c(x)(1 + F\varepsilon \mathbf{c}(x)),$$

where  $F$  is a constant (roughly equal to  $\frac{1}{3}$  for water). This shows that

$$\zeta = -(1 + (d-1)F). \quad (9.7)$$

In the setting of electrical impedance tomography, we simply assume that  $\zeta$  models the coupling between the acoustic signal and the change in the electrical conductivity of the underlying material. The value of  $\zeta$  thus depends on the application.

## 9.2 Inverse problems in ultrasound modulation.

Assuming the validity of the above derivation, the objective of ultrasound modulated tomography is to reconstruct the coefficient  $\gamma(x)$  from knowledge of the interior functionals

$$H_{ij}(x) = \gamma(x) \nabla u_i(x) \cdot \nabla u_j(x), \quad 1 \leq i, j \leq m, \quad (9.8)$$

where  $u_j$  is the solution to the equation

$$\nabla \cdot (\gamma \nabla u_j) = 0 \quad \text{in } X, \quad u_j = f_j \quad \text{on } \partial X, \quad 1 \leq j \leq m, \quad (9.9)$$

for appropriate choices of the boundary conditions  $f_j$  on  $\partial X$ . In practice, the ultrasound modulation effect is extremely weak because the coupling coefficient  $\Gamma$  is small. However, its mathematical analysis shows that UMOT and UMEIT are well posed problems unlike the (non ultrasound-modulated) OT and EIT problems.

We need some notation to introduce the main result of this chapter. We first define the UMT measurement operator. We again define  $\mathbf{f} = (f_1, \dots, f_m)$  the vector of illuminations at the domain's boundary. The corresponding solutions to (9.9) are denoted by  $u_j$ . The measurement operator  $\mathfrak{M}_\mathbf{f}$  is then defined as the following collection of internal functionals:

$$\begin{aligned} \mathfrak{M}_\mathbf{f} : \quad \mathfrak{X} &\rightarrow \mathcal{M}^\sigma(\mathbb{R}^m; \mathfrak{Y}) \\ \gamma &\mapsto \mathfrak{M}_\mathbf{f} \gamma = (\gamma \nabla u_i \cdot \nabla u_j)_{1 \leq i, j \leq m}. \end{aligned} \quad (9.10)$$

Here,  $\mathfrak{X}$  is a subset of a Banach space in which the unknown coefficients are defined,  $\mathfrak{Y}$  is a subset of  $H^1(X)$  where the solutions to (9.9) are defined, and  $\mathcal{M}^\sigma(\mathbb{R}^m; \mathfrak{Y})$  is the space of symmetric second-order tensors of order  $m$  with values in  $\mathfrak{Y}$ .

Again, we observe that  $\mathfrak{M}_f$  is parameterized by the illuminations  $f$ .

**Remark 9.2.1** *Note that for  $m = 1$ , the internal functional is of the form  $H = \gamma|\nabla u|^2$ . The unknown coefficient  $\gamma$  can then be eliminated and formally,  $u$  solves the following non-linear equation*

$$\nabla \cdot \frac{H}{|\nabla u|^2} \nabla u = 0 \quad \text{in } X, \quad u = f \quad \text{on } \partial X. \quad (9.11)$$

*This nonlinear problem is in fact hyperbolic in nature rather than elliptic and its solution is therefore far from guaranteed. See [?].*

*In some sense, the UM inverse problem has two unknowns  $(\gamma, u)$ . In the one-functional setting, elimination of  $\gamma$  to get an equation for  $u$  is rather trivial. However, the resulting equation is difficult to analyze and may not have unique solutions. The multi-functional setting when  $m \geq 2$  aims to simplify the solution of the resulting (redundant system of) equation(s). However, the elimination of unknowns becomes significantly more challenging.*

Following [?, 27], we first perform the change of unknown functions  $S_i = \gamma^{\frac{1}{2}} \nabla u_i$  for every  $i$  and define

$$F(x) := \nabla \log \gamma(x). \quad (9.12)$$

Let  $(e_1, \dots, e_n)$  be the canonical basis in  $\mathbb{R}^n$ . For a given vector field  $V = V^i e_i$  defined on  $X$ , we define the corresponding one-form  $V^\flat := V^i dx^i$ , where  $dx^i$  is the dual basis (of 1-forms) to  $e_i$  in the sense that  $dx^i(e_j) = \delta_{ij}$ , i.e., 0 if  $i \neq j$  and 1 if  $i = j$ . With this notation, we obtain that the vector fields  $S_j$  satisfy the system of equations

$$\nabla \cdot S_j = -\frac{1}{2} F \cdot S_j, \quad (9.13)$$

$$dS_j^\flat = \frac{1}{2} F^\flat \wedge S_j^\flat, \quad 1 \leq j \leq m, \quad (9.14)$$

where  $\wedge$  and  $d$  denote the usual exterior product and exterior derivative, respectively. The first equation stems directly from (9.9) whereas the second one states that the one-form  $\gamma^{-\frac{1}{2}} S_j^\flat = du_j$  is exact, therefore closed, and hence  $d(\gamma^{-\frac{1}{2}} S_j^\flat) = 0$ . In dimension  $n = 3$ , this means that  $\nabla u = \gamma^{-\frac{1}{2}} S$  is a gradient so that its curl vanishes. The above equations are generalizations to arbitrary dimensions.

When  $n = 2, 3$ , equation (9.14) is recast as:

$$n = 2 : \quad [\nabla, S_j] - \alpha[F, S_j] = 0, \quad n = 3 : \quad \text{curl } S_j - \alpha F \times S_j = 0,$$

where for  $n = 2$ , we define  $[A, B] := A_x B_y - A_y B_x$  and  $[\nabla, A] := \partial_x A_y - \partial_y A_x$  and for  $n = 3$ ,  $\times$  denotes the standard cross-product.

**Exercise 9.2.1** *Check the above formulas.*

The strategy now is the following: we wish to *eliminate*  $F$  from the above equations. This will result in an overdetermined system of equations for the vector fields  $S$  that we will prove admits a unique solution. Once all vector fields  $S_j$  are known, we obtain an explicit expression for  $F$ , from which the reconstruction of  $\gamma$  is relatively straightforward. The elimination of  $F$  requires that a certain system be invertible. As for condition **(H3)** of **(H3')** in chapter 8, we therefore need to find *well-chosen* illuminations  $\mathfrak{f}$  for which such an invertibility condition holds. We will again see that a large class of illuminations are *well-chosen* in two dimensions  $n = 2$ . In dimensions  $n \geq 3$ , the construction of *well-chosen* illuminations will again be based on CGO solutions.

The invertibility condition is that the  $m$  gradients  $\nabla u_j$  have maximal rank in  $\mathbb{R}^n$  at every point  $x \in X$ . This hypothesis can be formalized by the somewhat stronger statement: there exists a finite open covering  $\mathcal{O} = \{\Omega_k\}_{1 \leq k \leq N}$  of  $X$  (i.e.  $X \subset \cup_{k=1}^N \Omega_k$ ), an indexing function  $\tau : [1, N] \ni i \mapsto \tau(i) = (\tau(i)_1, \dots, \tau(i)_n) \in [1, m]^n$  and a positive constant  $c_0$  such that

$$\min_{1 \leq i \leq N} \inf_{x \in \Omega_i} \det(S_{\tau(i)_1}(x), \dots, S_{\tau(i)_n}(x)) \geq c_0 > 0. \quad (9.15)$$

This assumption is equivalent to imposing the following condition on the data

$$\min_{1 \leq i \leq N} \inf_{x \in \Omega_i} \det H^{\tau(i)}(x) \geq c_0^2 > 0, \quad (9.16)$$

where  $H^{\tau(i)}$  stands for the  $n \times n$  matrix of elements  $H_{kl}^{\tau(i)} = S_{\tau(i)_k} \cdot S_{\tau(i)_l}$ . We state that  $\mathfrak{f}$  is a well-posed set of illuminations when (9.15) or equivalently (9.16) holds.

This complicated expression simply states that at each point  $x \in X$ , we can find  $n$  linearly independent vectors  $\nabla u_{j(x)}$  with determinant bounded from below uniformly in  $x \in X$ . The elimination of  $F$  is then guaranteed and can be done in a stable fashion as the following lemma indicates.

**Lemma 9.2.2** *Let  $\Omega \subset X$  open where, up to renumbering of solutions, we have*

$$\inf_{x \in \Omega} \det(S(x)) \geq c_0 > 0, \quad S(x) := (S_1(x) | \dots | S_n(x)).$$

*Let us define  $H(x) := \{S_i(x) \cdot S_j(x)\}_{1 \leq i, j \leq n}$  and  $D(x) = \sqrt{\det H(x)}$ . Then we have:*

$$F = \frac{c_F}{D} \sum_{i, j=1}^n (\nabla(DH^{ij}) \cdot S_i) S_j = c_F \left( \nabla \log D + \sum_{i, j=1}^n (\nabla H^{ij} \cdot S_i) S_j \right), \quad (9.17)$$

$$c_F := \left( \frac{1}{2}(n-2) + 1 \right)^{-1}.$$

Here  $H^{ij}$  denotes the element  $(i, j)$  of the matrix  $H^{-1}$ .

Once  $F$  is eliminated, we can write a system of equations for the vectors  $S_j$  that admits a unique solution provided that  $\gamma$  and all  $S_j$ 's are known at one point  $x_0 \in \bar{X}$ , for instance at the domain's boundary  $\partial X$ . This leads to well-posed reconstructions as stated in the following:

**Theorem 9.2.3** *Let  $X \subset \mathbb{R}^n, n \geq 2$  be an open convex bounded domain, and let two sets of  $m \geq n$  solutions of (9.9) generate measurements ( $H = \mathfrak{M}_f(\gamma), \tilde{H} = \mathfrak{M}_f(\tilde{\gamma})$ ) whose components belong to  $W^{1,\infty}(X)$ , and who jointly satisfy condition (9.16) with the same triple  $(\mathcal{O}, \tau, c_0)$ . In other words, we assume that  $f$  is well-chosen for both  $\gamma$  and  $\tilde{\gamma}$ .*

*Let  $x_0 \in \overline{\Omega_{i_0}} \subset \overline{X}$  and  $\gamma(x_0), \tilde{\gamma}(x_0)$  and  $\{S_{\tau(i_0)_i}(x_0), \tilde{S}_{\tau(i_0)_i}(x_0)\}_{1 \leq i \leq n}$  be given. Then we have the stability estimate:*

$$\|\log \gamma - \log \tilde{\gamma}\|_{W^{1,\infty}(X)} \leq C \left( \varepsilon_0 + \|H - \tilde{H}\|_{W^{1,\infty}(X)} \right), \quad (9.18)$$

where  $\varepsilon_0$  is the error at  $x_0$ :

$$\varepsilon_0 := |\log \gamma(x_0) - \log \tilde{\gamma}(x_0)| + \sum_{i=1}^n \|S_{\tau(i_0)_i}(x_0) - \tilde{S}_{\tau(i_0)_i}(x_0)\|.$$

## 9.3 Eliminations and redundant systems of ordinary differential equations

The proof of Theorem 9.2.3, which we complete in this section, involves first proving Lemma 9.2.2 and next solving a redundant system of equations for the vectors  $S_j$ .

### 9.3.1 Elimination of $F$

We now prove Lemma 9.2.2 and first recall some notation from tensor calculus. For  $0 \leq k \leq n$ ,  $\Lambda^k$  denotes the space of  $k$ -forms. We recall the definition of the *Hodge star operator*  $\star : \Lambda^k \mapsto \Lambda^{n-k}$  for  $0 \leq k \leq n$ , such that for any elementary  $k$ -form  $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ , we have

$$\star dx^I = \sigma dx^J, \quad \text{where } \sigma = \text{sign}((1 \dots n) \mapsto (I, J)). \quad (9.19)$$

Here,  $J$  is implicitly defined by the fact that  $(1 \dots n) \mapsto (I, J)$  is a permutation. Note that  $\sigma dx^J$  is independent of the ordering of the  $n-k$  indices in  $J$ . We recall the following useful identities:

$$\star \star = (-1)^{k(n-k)} \quad \text{on } \Lambda^k, \quad \star(u^b \wedge \star v^b) = u \cdot v, \quad \star d \star u^b = \nabla \cdot u, \quad u, v \in \Lambda^1.$$

Because  $S_1(x), \dots, S_n(x)$  forms a basis of  $\mathbb{R}^n$ , a vector  $V$  can be represented in this basis by the following representation

$$V = H^{ij}(V \cdot S_i)S_j. \quad (9.20)$$

For  $j = 1, \dots, n$ , let us introduce the following 1-forms:

$$X_j^b := (-1)^{n-1} \sigma_j \star (S_{i_1}^b \wedge \cdots \wedge S_{i_{n-1}}^b), \quad (i_1, \dots, i_{n-1}) = (1, \dots, \hat{j}, \dots, n), \quad (9.21)$$

where the hat indicates an omission and  $\sigma_j = (-1)^{j-1}$ . We now show that the vector fields  $X_j$  satisfy a simple divergence equation. We compute

$$\begin{aligned} \nabla \cdot X_j &= \star d \star X_j^\flat = \sigma_j \star d(S_{i_1}^\flat \wedge \cdots \wedge S_{i_{n-1}}^\flat) = \sigma_j \star \sum_{k=1}^{n-1} (-1)^k S_{i_1}^\flat \wedge \cdots \wedge dS_{i_k}^\flat \wedge \cdots \wedge S_{i_{n-1}}^\flat \\ &= \sigma_j \star \sum_{k=1}^{n-1} (-1)^k S_{i_1}^\flat \wedge \cdots \wedge \frac{1}{2}(F^\flat \wedge S_{i_k}^\flat) \wedge \cdots \wedge S_{i_{n-1}}^\flat \\ &= \frac{1}{2}(n-1) \star (F^\flat \wedge \star X_j^\flat). \end{aligned}$$

Using the identity  $\star(u^\flat \wedge \star v^\flat) = u \cdot v$ , we deduce that

$$\nabla \cdot X_j = \frac{1}{2}(n-1)F \cdot X_j, \quad j = 1 \dots n. \quad (9.22)$$

We now decompose  $X_j$  in the basis  $(S_1, \dots, S_n)$ . For  $k \neq j$ , there is an  $l$  such that  $i_l = k$  and we have

$$X_j \cdot S_k = \det(S_1, \dots, S_{j-1}, S_k, S_{j+1}, \dots, S_n) = 0,$$

by repetition of the term  $S_k$  in the determinant. Now for  $k = j$ , we have

$$X_j \cdot S_j = \det(S_1, \dots, S_n) = \det S = D.$$

Using formula (9.20), we deduce that  $X_j$  admits the expression

$$X_j = DH^{ij}S_i.$$

Plugging this expression into equation (9.22), and using  $\nabla \cdot (\varphi V) = \nabla \varphi \cdot V + \varphi \nabla \cdot V$ , we obtain

$$\begin{aligned} \nabla(DH^{ij}) \cdot S_i + DH^{ij} \nabla \cdot S_i &= \frac{1}{2}(n-1)F \cdot (DH^{ij}S_i) \\ \Leftrightarrow \nabla(DH^{ij}) \cdot S_i - DH^{ij} \frac{1}{2}F \cdot S_i &= \frac{1}{2}(n-1)DH^{ij}F \cdot S_i \\ \Leftrightarrow \nabla(DH^{ij}) \cdot S_i &= c_F^{-1}DH^{ij}F \cdot S_i. \end{aligned}$$

Finally, using the representation (9.20) for  $F$  itself yields

$$F = (H^{ij}F \cdot S_i)S_j = \frac{c_F}{D}(\nabla(DH^{ij}) \cdot S_i)S_j. \quad (9.23)$$

We can also recast the previous expression as

$$F = c_F [\nabla \log D + ((\nabla H^{ij}) \cdot S_i)S_j], \quad (9.24)$$

and the proof of the lemma is complete.

### 9.3.2 System of ODEs for $S_j$

In this section, we obtain a redundant system of ordinary differential equations for the matrix  $S$ . Let  $S$  be the matrix formed with the column vectors  $S_j$ . Then  $H = S^T S$  is known from the measurement operator. Moreover, we have just found an equation of the form  $dS_j^b = F^b(S) \wedge S_j^b$ , with  $F = F(S)$  given above. We thus possess information about the symmetric part of  $S$  and the skew-symmetric part of  $S$ . It remains to know whether this is enough to write an equation of the form  $\nabla \otimes S_j = \mathcal{F}_j(S)$ . The answer is affirmative as we now show.

We first need to introduce other standard geometric notation, without which the derivations become quickly intractable. Let us denote the Euclidean orthonormal frame  $e_i = \partial_{x^i}$  and  $e^i = dx^i$ . We work on a convex set  $\Omega \subset \mathbb{R}^n$  with the Euclidean metric  $g(X, Y) \equiv X \cdot Y = \delta_{ij} X^i Y^j$  on  $\mathbb{R}^n$ . Following [43], we denote by  $\bar{\nabla}$  the Euclidean connection, which here has the expression we can take as a definition:

$$\bar{\nabla}_X f = X \cdot \nabla f = X^i \partial_i f, \quad \text{and} \quad \bar{\nabla}_X Y = (X \cdot \nabla Y^j) e_j = X^i (\partial_i Y^j) e_j,$$

for given vector fields  $X = X^i e_i$  and  $Y = Y^i e_i$ . An important identity for the sequel is the following characterization of the exterior derivative of a one-form  $\omega$ :

$$d\omega(X, Y) = \bar{\nabla}_X(\omega(Y)) - \bar{\nabla}_Y(\omega(X)) - \omega([X, Y]), \quad (9.25)$$

or equivalently in the Euclidean metric, writing  $\omega = S_i^b$ ,  $X = S_j$  and  $Y = S_k$ ,

$$S_i \cdot [S_j, S_k] = S_j \cdot \nabla(S_i \cdot S_k) - S_k \cdot \nabla(S_i \cdot S_j) - dS_i^b(S_j, S_k), \quad (9.26)$$

where the Lie bracket (commutator) of  $X$  and  $Y$  may be “defined” here as  $[X, Y] = \bar{\nabla}_X Y - \bar{\nabla}_Y X = X \cdot \nabla Y - Y \cdot \nabla X$  seen as a vector field.

**Exercise 9.3.1** *Verify (9.26) directly.*

Note that the right-hand side in (9.26) involves no derivatives in the unknown  $S$  since  $S_i \cdot S_k = H_{ik}$  is known and  $dS_i^b$  is a known functional of  $S$  by (9.14) and (9.17).

The following relation between inner products and Lie brackets of a given frame (see e.g. [43, Eq. 5.1 p. 69]) is very useful

$$\begin{aligned} 2(X \cdot \nabla Y) \cdot Z &= X \cdot \nabla(Y \cdot Z) + Y \cdot \nabla(Z \cdot X) - Z \cdot \nabla(X \cdot Y) \\ &\quad - Y \cdot [X, Z] - Z \cdot [Y, X] + X \cdot [Z, Y]. \end{aligned} \quad (9.27)$$

**Exercise 9.3.2** *Check the above expression directly.*

We thus find using (9.14) and (9.26) that

$$2(S_i \cdot \nabla S_j) \cdot S_k = S_i \cdot \nabla H_{jk} - S_j \cdot \nabla H_{ik} + S_k \cdot \nabla H_{ij} - 2F \cdot S_k H_{ij} + 2F \cdot S_j H_{ik}. \quad (9.28)$$

Note that the right-hand-side no longer involves any derivative of  $S_j$ . Moreover,  $S_j$  forms a frame. We can therefore extract the full gradient of  $S_j$  from the terms  $2(S_i \cdot \nabla S_j) \cdot S_k$ . Geometrically, gradients generalize to tensors via the total covariant derivative, which maps a vector field  $X$  to a tensor of type (1, 1) defined by

$$\bar{\nabla} X(\omega, Y) = \omega(\bar{\nabla}_Y X). \quad (9.29)$$

$S$  is a frame provided that the determinant condition  $\inf_{x \in \Omega} \det S \geq c_0 > 0$  holds. In the frame  $S$ , we may express  $\bar{\nabla} S_i$  in the basis  $\{S_j \otimes S_k^b\}_{j,k=1}^n$  of such tensors by writing  $\bar{\nabla} S_i = a_{ijk} S_j \otimes S_k^b$  and identifying the coefficients  $a_{ijk}$  by writing

$$\bar{\nabla} S_i(S_p^b, S_q) = S_p^b(\bar{\nabla}_{S_q} S_i) = S_q \cdot \nabla S_i \cdot S_p,$$

and also

$$\bar{\nabla} S_i(S_p^b, S_q) = a_{ijk} S_j \otimes S_k^b(S_p^b, S_q) = a_{ijk} H_{jp} H_{kq}.$$

Equating the two, we obtain the representation

$$\bar{\nabla} S_i = H^{qk} H^{jp} [(S_q \cdot \nabla S_i) \cdot S_p] S_j \otimes S_k^b, \quad (9.30)$$

where  $H^{ij}$  denote the coefficients of the matrix  $H^{-1}$ .

**Exercise 9.3.3** Seeing  $\nabla \otimes S_j$  as the matrix with components  $(i, k)$  given by  $\partial_{x_i}(S_j)_k$  and  $S_i \otimes S_m$  as the matrix with components  $(j, k)$  given by  $(S_i)_k(S_m)_k$ , show that

$$\nabla \otimes S_j = \sum_{i,k,l,m} H^{ik} (S_k \cdot \nabla S_j) \cdot S_l H^{lm} S_i \otimes S_m = \mathcal{F}_j(S).$$

Now plugging (9.28) into (9.30), and using  $H_{ij} H^{jk} = \delta_{ik}$ , we obtain

$$\begin{aligned} 2\bar{\nabla} S_i &= 2H^{qk} H^{jp} (\bar{\nabla}_{S_q} S_i \cdot S_p) S_j \otimes S_k^b \\ &= H^{qk} H^{jp} (\nabla H_{iq} \cdot S_p + \nabla H_{ip} \cdot S_q - \nabla H_{pq} \cdot S_i + (H_{pq}(F \cdot S_i) - H_{qi}(F \cdot S_p))) S_j \otimes S_k^b \\ &= (H^{jp} U_{ik} \cdot S_p + H^{qk} U_{ij} \cdot S_q + \nabla H^{jk} \cdot S_i + (H^{jk}(F \cdot S_i) - H^{jp} \delta_{ik}(F \cdot S_p))) S_j \otimes S_k^b, \end{aligned}$$

where we have used  $\nabla H^{jk} = -H^{jp}(\nabla H_{pq})H^{qk}$  and have defined

$$U_{jk} := (\nabla H_{jp})H^{pk} = -H_{jp} \nabla H^{pk}, \quad 1 \leq j, k \leq n. \quad (9.31)$$

Using formulas  $H^{jk} S_j \otimes S_k^b = e_i \otimes e^i$  and  $H^{kl}(V \cdot S_k)S_l = V$  for any smooth vector field  $V$ , we obtain for  $1 \leq i \leq n$

$$\bar{\nabla} S_i = \frac{1}{2} (U_{ik} \otimes S_k^b + S_k \otimes U_{ik}^b + (\nabla H^{jk} \cdot S_i) S_j \otimes S_k^b) + \frac{1}{2} (F \cdot S_i) I_n - \frac{1}{2} F \otimes S_i^b. \quad (9.32)$$

Using (9.23), we observe that  $\bar{\nabla} S_i$  is equal to a polynomial of degree at most three in the frame  $S$  with coefficients involving the known inner products  $H_{ij}$ . For each  $1 \leq i, k \leq n$ ,  $\partial_k S_i$  is nothing but  $\bar{\nabla}_{e_k} S_i = \bar{\nabla} S_i(\cdot, e_k)$ , which can be obtained from (9.32). Denoting  $\mathbf{S} := (S_1^T, \dots, S_n^T)^T$ , we are then able to construct the system of equations

$$\partial_k \mathbf{S} = \sum_{|\beta| \leq 3} Q_\beta^k \mathbf{S}^\beta, \quad \mathbf{S}^\beta = \prod_{i=1}^n S_i^{\beta_i}, \quad 1 \leq k \leq n, \quad (9.33)$$

where  $Q_\beta^k$  depends only on the data and  $\beta$  is an  $n^2$ -index. This redundant system can then be integrated along any curve (where it becomes a system of ordinary differential equations with Lipschitz right-hand sides ensuring uniqueness of the solution) in order to solve for the matrix-valued function  $\mathbf{S}$ .

### 9.3.3 ODE solution and stability estimates

Once (9.33) has been obtained, the derivation of Theorem 9.2.3 follows rapidly. We leave this step as an exercise; see [?] for the details.

**Exercise 9.3.4** *Prove the stability estimates in Theorem 9.2.3 from (9.33), (9.24), and the definition (9.12).*

## 9.4 Well-chosen illuminations

It remains to find boundary conditions such that (9.15) holds. As in the preceding chapter, we need to distinguish dimension  $n = 2$  from dimensions  $n \geq 3$ .

### 9.4.1 The case $n = 2$

In dimension  $n = 2$ , the critical points of  $u$  (points  $x$  where  $\nabla u(x) = 0$ ) are necessarily isolated as is shown in, e.g., [2]. From this and techniques of quasiconformal mappings that are also restricted to two dimensions of space, we can show the following results.

**Lemma 9.4.1** ([3]) *Let  $u_1$  and  $u_2$  be the solutions of (9.1) on  $X$  simply connected with boundary conditions  $f_1 = x_1$  and  $f_2 = x_2$  on  $\partial X$ , respectively, where  $x = (x_1, x_2)$  are Cartesian coordinates on  $X$ . Assume that  $\gamma$  is sufficiently smooth. Then  $(x_1, x_2) \mapsto (u_1, u_2)$  from  $X$  to its image is a diffeomorphism. In other words,  $\det(\nabla u_1, \nabla u_2) > 0$  uniformly on  $\bar{X}$ .*

In other words, in two dimensions of space, there are explicit boundary conditions, such as those above with  $f_j$  the trace of  $x_j$  on  $\partial X$  for  $j = 1, 2$  that guarantees that (9.15) holds uniformly on the whole domain  $X$ . It is shown in [25] that the appropriate extension of this result is false in dimension  $n \geq 3$ .

### 9.4.2 The case $n \geq 3$

In dimension  $n \geq 3$ , we have the following result:

**Lemma 9.4.2** *Let  $n \geq 3$  and  $\gamma \in H^{\frac{n}{2}+3+\varepsilon}(X)$  for some  $\varepsilon > 0$  be bounded from below by a positive constant. Then for  $n$  even, there exists a open set  $G$  of illuminations  $\{f_1, \dots, f_n\}$  such that for any  $\mathfrak{f} \in G$ , the condition (9.15) holds with  $\mathcal{O} = \{X\}$  for some constant  $c_0 > 0$ .*

*For  $n$  odd, there exists an open set  $G$  of illuminations  $\{f_1, \dots, f_{n+1}\}$  such that for any  $\mathfrak{f} \in G$  there exists an open cover of  $X$  of the form  $\{\Omega_{2i-1}, \Omega_{2i}\}_{1 \leq i \leq N}$  and a constant  $c_0 > 0$  such that*

$$\inf_{x \in \Omega_{2i-1}} \det(S_1, \dots, S_{n-1}, \epsilon_i S_n) \geq c_0 \quad \text{and} \quad \inf_{x \in \Omega_{2i}} \det(S_1, \dots, S_{n-1}, \tilde{\epsilon}_i S_{n+1}) \geq c_0, \quad (9.34)$$

for  $1 \leq i \leq N$  and with  $\epsilon_i, \tilde{\epsilon}_i = \pm 1$ .

In other words, this lemma indicates that for appropriate boundary conditions  $f_j$ , we can always find  $n$  corresponding solutions whose gradients form a basis of  $\mathbb{R}^n$ .

*Proof.* Consider the problem  $\nabla \cdot \gamma(x)\nabla u = 0$  on  $\mathbb{R}^n$  with  $\gamma(x)$  extended in a continuous manner outside of  $X$  and such that  $\gamma$  equals 1 outside of a large ball. Let  $q(x) = -\frac{\Delta\sqrt{\gamma}}{\gamma}$  on  $\mathbb{R}^n$ . Then  $q \in H^{\frac{n}{2}+1+\varepsilon}(\mathbb{R}^n)$  since  $\gamma - 1 \in H^{\frac{n}{2}+3+\varepsilon}(\mathbb{R}^n)$  for some  $\varepsilon > 0$ . By Sobolev imbedding,  $\gamma$  is of class  $C^3(\overline{X})$  while  $q$  is of class  $C^1(\overline{X})$ . With the above hypotheses, we can apply Corollary 7.3.3.

Let  $v = \sqrt{\gamma}u$  so that  $\Delta v + qv = 0$  on  $\mathbb{R}^n$ . Let  $\varrho \in \mathbb{C}^n$  be of the form  $\varrho = \rho(\vartheta + i\vartheta^\perp)$  with  $\vartheta, \vartheta^\perp \in \mathbb{S}^{n-1}$ ,  $\vartheta \cdot \vartheta^\perp = 0$ , and  $\rho = |\varrho|/\sqrt{2} > 0$ . Now, as we showed in Corollary 7.3.3, we have

$$v_\varrho = \sqrt{\gamma}u_\varrho = e^{\varrho \cdot x}(1 + \psi_\varrho), \quad \rho\psi_{\varrho|_X} = O(1) \text{ in } C^1(\overline{X}),$$

with  $(\Delta + q)v_\varrho = 0$  and hence  $\nabla \cdot \gamma\nabla u_\varrho = 0$  in  $\mathbb{R}^n$ . We have used again the Sobolev imbedding stating that functions in  $H^{\frac{n}{2}+k+\varepsilon}(Y)$  are of class  $C^k(Y)$  for a bounded domain  $Y$ . Taking gradients of the previous equation and rearranging terms, we obtain that

$$\sqrt{\gamma}\nabla u_\varrho = e^{\varrho \cdot x}(\varrho + \varphi_\varrho), \quad \text{with } \varphi_\varrho := \nabla\psi_\varrho + \psi_\varrho\varrho - (1 + \psi_\varrho)\nabla\sqrt{\gamma}.$$

Because  $\nabla\sqrt{\gamma}$  is bounded and  $\rho\psi_{\varrho|_X} = O(1)$  in  $C^1(\overline{X})$ , the  $\mathbb{C}^n$ -valued function  $\varphi_\varrho$  satisfies  $\sup_{\overline{X}}|\varphi_\varrho| \leq C$  independent of  $\varrho$ . Moreover, the constant  $C$  is in fact independent of  $\gamma$  provided that the norm of the latter is bounded by a uniform constant in  $H^{\frac{n}{2}+3+\varepsilon}(X)$  according to Corollary 7.3.3.

Both the real and imaginary parts of  $u_\varrho$ , denoted  $u_\varrho^{\Re}$  and  $u_\varrho^{\Im}$ , are solutions of the free-space conductivity equation. Thus,  $\sqrt{\gamma}\nabla u_\varrho^{\Re}$  and  $\sqrt{\gamma}\nabla u_\varrho^{\Im}$  can serve as vectors  $S_i$ . More precisely, we have

$$\begin{aligned} \sqrt{\gamma}\nabla u_\varrho^{\Re} &= \rho e^{\rho\vartheta \cdot x} \left( (\vartheta + \rho^{-1}\varphi_\varrho^{\Re}) \cos(\rho\vartheta^\perp \cdot x) - (\vartheta^\perp + \rho^{-1}\varphi_\varrho^{\Im}) \sin(\rho\vartheta^\perp \cdot x) \right), \\ \sqrt{\gamma}\nabla u_\varrho^{\Im} &= \rho e^{\rho\vartheta \cdot x} \left( (\vartheta^\perp + \rho^{-1}\varphi_\varrho^{\Im}) \cos(\rho\vartheta^\perp \cdot x) + (\vartheta + \rho^{-1}\varphi_\varrho^{\Re}) \sin(\rho\vartheta^\perp \cdot x) \right). \end{aligned} \quad (9.35)$$

**Case  $n$  even:** Set  $n = 2p$ , define  $\varrho_l = \rho(e_{2l} + ie_{2l-1})$  for  $1 \leq l \leq p$ , and construct

$$S_{2l-1} = \sqrt{\gamma}\nabla u_{\varrho_l}^{\Re} \quad \text{and} \quad S_{2l} = \sqrt{\gamma}\nabla u_{\varrho_l}^{\Im}, \quad 1 \leq l \leq p.$$

Using (9.35), we obtain that

$$\det(S_1, \dots, S_n) = \rho^n e^{2\rho \sum_{i=1}^p x_{2i}} (1 + f(x)),$$

where  $\lim_{\rho \rightarrow \infty} \sup_{\overline{X}} |f| = 0$ . Letting  $\rho$  so large that  $\sup_{\overline{X}} |f| \leq \frac{1}{2}$  and denoting  $\gamma_0 := \min_{x \in \overline{X}} (\rho^n e^{2\rho \sum_{i=1}^p x_{2i}}) > 0$ , we have  $\inf_{x \in \overline{X}} \det(S_1, \dots, S_n) \geq \frac{\gamma_0}{2} > 0$ .

**Case  $n$  odd:** Set  $n = 2p - 1$ , define  $\varrho_l = \rho(e_{2l} + ie_{2l-1})$  for  $1 \leq l \leq p - 1$ , and  $\varrho_p = \rho(e_n + ie_1)$  and construct

$$S_{2l-1} = \sqrt{\gamma}\nabla u_{\varrho_l}^{\Re} \quad \text{and} \quad S_{2l} = \sqrt{\gamma}\nabla u_{\varrho_l}^{\Im}, \quad 1 \leq l \leq p.$$

Using (9.35), we obtain that

$$\begin{aligned}\det(S_1, \dots, S_{n-1}, S_n) &= \rho^n e^{\rho(x_n + 2 \sum_{l=1}^{n-1} x_{2l})} (-\cos(\rho x_1) + f_1(x)), \\ \det(S_1, \dots, S_{n-1}, S_{n+1}) &= \rho^n e^{\rho(x_n + 2 \sum_{l=1}^{n-1} x_{2l})} (-\sin(\rho x_1) + f_2(x)),\end{aligned}$$

where  $\lim_{\rho \rightarrow \infty} \sup_{\bar{X}} |f_1| = \lim_{\rho \rightarrow \infty} \sup_{\bar{X}} |f_2| = 0$ . Letting  $\rho$  so large that  $\sup_{\bar{X}} (|f_1|, |f_2|) \leq \frac{1}{4}$  and denoting  $\gamma_1 := \min_{x \in \bar{X}} (\rho^n e^{\rho(x_n + 2 \sum_{l=1}^{n-1} x_{2l})}) > 0$ , we have that  $|\det(S_1, \dots, S_{n-1}, S_n)| \geq \frac{\gamma_1}{4}$  on sets of the form  $X \cap \{\rho x_1 \in ]\frac{-\pi}{3}, \frac{\pi}{3}[ + m\pi\}$  and  $|\det(S_1, \dots, S_{n-1}, S_{n+1})| \geq \frac{\gamma_1}{4}$  on sets of the form  $X \cap \{\rho x_1 \in ]\frac{\pi}{6}, \frac{5\pi}{6}[ + m\pi\}$ , where  $m$  is a signed integer. Since the previous sets are open and a finite number of them covers  $X$  (because  $X$  is bounded and  $\rho$  is finite), we therefore have fulfilled the desired requirements of the construction. Upon changing the sign of  $S_n$  or  $S_{n+1}$  on each of these sets if necessary, we can assume that the determinants are all positive.  $\square$

Note that as in the preceding paragraph, we have obtained the existence of an open set of illuminations  $\mathfrak{f}$  such that appropriate determinants remain strictly positive throughout the domain  $X$ . However, these illuminations  $\mathfrak{f}$  are not characterized explicitly.

## 9.5 Remarks on hybrid inverse problems

In the past two chapters, we have briefly presented two hybrid inverse problems based on the photo-acoustic effect (with a similar theory for the imaging modality Transient Elastography) and the ultrasound modulation effect. What characterizes these hybrid inverse problems is that after a preliminary step (involving and inverse wave problem in photoacoustics and an inverse Fourier transform in ultrasound modulation) we obtain an inverse problem with *internal functionals* of the unknown parameters.

These internal functionals have an immediate advantage: singularities of the unknown coefficients no longer need to be propagated to the boundary of the domain by an elliptic operator that severely damps high frequencies. The main reason for ill-posedness of invertible operators, namely the smoothing property of such operators, is therefore no longer an issue. However, injectivity of the measurement operator is not guaranteed. We have seen in QPAT that only two out of three coefficients could be reconstructed in QPAT. QPAT data acquired at one frequency are thus not sufficient to reconstruct three coefficients. With appropriate prior information about the dependency of coefficients with respect to a frequency parameter, then injectivity of the measurement operator can be restored [?]. But again, this requires prior information that one may not wish to make. The alternative is then to come up with *other* prior models that restore injectivity or to combine QPAT measurements with additional measurements. Hybrid inverse problems face the same shortcomings as any other inverse problem.

Once injectivity is guaranteed, then stability of the reconstructions are guaranteed in principle by the fact that singularities no longer need to propagate. We have seen a few such stability results. Note that these results typically require a certain degree of smoothness of the unknown coefficients. This is a shortcoming of the theories presented above. The reason why we have recourse to hybrid inverse problems is to obtain high resolution. The reason we typically need high resolution is because coefficients may

vary rapidly and we wish to quantify such variations. It would therefore be useful to understand how stability estimates degrade when the coefficients are not smooth.

That said, numerical experiments conducted in e.g. [?, 18, 19] show that reconstructions based on algorithms similar to those presented above do allow us to obtain very accurate reconstructions even for highly discontinuous coefficients, and this even in the presence of relatively significant noise.

For additional general references to hybrid inverse problems, we refer the reader to [4, 56].



# Chapter 10

## Priors and Regularization

As we have mentioned several times in these notes, the influence of “*noise*” is largely subjective: we either find it *acceptable* or *unacceptable*. Mathematically, this influence is typically acceptable in a given norm while it may be unacceptable in another (more constraining) norm. In which norms that influence is controlled for a given problem is characterized by stability estimates, which we have presented for the problems considered in these notes.

Once we have decided that “*noise*” had too large an effect in a setting of interest, something must be done. That something inescapably requires that we add *prior information*. Several techniques have been developed to do so. The simplest and most developed is the *regularization* methodology. Typically, such a regularization assumes that the object we are interested in reconstructing has a prior smoothness. We may for instance assume that the object belongs to  $H^s(X)$  for some  $s > 0$ . This assumption indicates that the Fourier transform object of interest decreases rapidly as frequency increases. High frequencies, which are not present, thus do not need to be reconstructed with high accuracy. This allows us to mitigate the effect of high frequency noise in the reconstructions.

The main drawback of regularization theory is that objects of interest may not necessarily be smooth. Smooth means that the first coefficients in a Fourier series expansion are big while the other coefficients are small. In several settings of interest, the objects may be represented by a few big coefficients and a large number of small coefficients, but not in the basis of Fourier coefficients. In other words, the object may be *sparse* in a different, known basis. The objective of sparse regularization is to devise methods to find these coefficients.

In some settings, the problems are so ill-posed that looking even for the first coefficients in a given basis may not provide sufficient accuracy. Other sorts of prior information may be necessary, for instance assuming that the objects of interest are small inclusions with specific structures, or that next to a blue pixel in an image, red pixels are extremely unlikely. In such situations, reconstructions are typically not very accurate and it is often important to characterize this inaccuracy. A very versatile setting to do so is the statistical Bayesian framework. In such a setting, objects of interest are modeled by a set of possible outcomes with prior probabilities of happening. This is the *prior probability distribution*. Then data are acquired with a given noise model. The probability of such data happening conditioned on given parameters is called the *like-*

*likelihood probability distribution.* Using the Bayes rule, a *posterior probability distribution* gives the probability density of the parameters based on availability of the data.

We now briefly consider these three settings, the smoothness regularization methodology, the sparsity regularization methodology, and the Bayesian framework and show their relations.

## 10.1 Smoothness Regularization

We have seen that many of the inverse problems we have considered were either mildly ill-posed (with  $\alpha > 0$  in the sense of Chapter 1) or severely ill-posed (as for instance the Calderón problem or the Cauchy problems for elliptic equations). We present here some techniques to regularize the inversion. Such techniques typically work for mildly ill-posed problems but are often not sufficient for severely ill-posed problems. But this the influence of “*noise*” is subjective anyway, these regularization techniques should be applied first as they are the simplest both theoretically and computationally. We refer the reader to [31, 40, 63] for additional information on these regularization techniques.

### 10.1.1 Ill-posed problems and compact operators

Let  $A$  be an injective and compact operator defined on an infinite dimensional (separable) Hilbert space  $H$  with range  $\text{Range}(A)$  in  $H$ :

$$A : H \rightarrow \text{Range}(A) \subset H. \quad (10.1)$$

We recall that compact operators map the unit ball in  $H$  to a subset of  $H$  whose closure (with respect to the usual norm in  $H$ ) is compact, i.e., verifies that every bounded (with respect to the usual norm in  $H$ ) family of points admits a converging (with respect to the usual norm in  $H$ ) subsequence in the compact set.

Since  $A$  is injective (i.e.,  $Ax = 0$  implies  $x = 0$ ), we can define the inverse operator  $A^{-1}$  with domain of definition  $\text{Range}(A)$  and Range  $H$ :

$$A^{-1} : D(A^{-1}) = \text{Range}(A) \rightarrow H. \quad (10.2)$$

The problem is that  $A^{-1}$  is *never* a continuous operator from  $\text{Range}(A)$  to  $H$  when both spaces are equipped with the usual norm in  $H$ :

**Lemma 10.1.1** *For  $A$  as above, there exists a sequence  $x_n$  such that*

$$\|x_n\|_H = 1, \quad \|Ax_n\|_H \rightarrow 0. \quad (10.3)$$

*The same holds true with  $\|x_n\|_H \rightarrow \infty$ .*

*Proof.* The proof holds in more complicated settings than Hilbert spaces. The Hilbert structure gives us a very simple proof and is based on the existence of an orthonormal basis in  $H$ , i.e., vectors  $x_n$  such that  $\|x_n\|_H = 1$  and  $(x_n, x_m)_H = 0$  for  $n \neq m$ . Since these vectors belong to the unit ball, we deduce that  $y_n = Ax_n$  is a converging sequence (up to taking subsequences), say to  $y \in H$ . Take now  $\tilde{x}_n = 2^{-1/2}(x_n - x_{n+1})$ . We verify that  $\tilde{x}_n$  satisfies (10.3). Now define  $z_n = \tilde{x}_n / \|A\tilde{x}_n\|_H^{1/2}$  when the latter denominator does

not vanish and  $z_n = n\tilde{x}_n$  otherwise. Then  $Az_n$  still converges to 0 while  $\|z_n\|_H$  converges to  $\infty$ .  $\square$

This simple lemma shows that inverting a compact operator can never be a well-posed problem in the sense that  $A^{-1}$  is not continuous from  $D(A^{-1})$  to  $H$  with the  $H$  norm. Indeed take the sequence  $y_n = Ax_n/\|Ax_n\|$  in  $D(A^{-1})$ , where  $x_n$  is the sequence in (10.3). Then  $\|y_n\|_H = 1$  while  $\|A^{-1}y_n\|_H$  tends to  $\infty$ .

The implication for inverse problems is the following. If  $\delta y_n$  is the measurement noise for  $n$  large, then  $\delta A^{-1}y_n$  models the error in the reconstruction, which may thus be arbitrarily larger than the norm of the true object we aim to reconstruct. More precisely, if  $Ax = b$  is the real problem and  $A\tilde{x} = \tilde{b}$  is the exact reconstruction from noisy data, then arbitrarily small errors  $\|b - \tilde{b}\|$  in the measurements is still compatible with arbitrary large errors  $\|x - \tilde{x}\|$  in the space of parameters. This shows that the problem needs to be regularized before any inversion is carried out.

### 10.1.2 Regularity assumptions and error bound

The calculations we have carried out in the preceding section show that an ill-posed inverse problem cannot satisfactorily be solved if no other *assumptions* on the problem are added. A sometimes reasonable and practically useful assumption is to impose, before we start the reconstruction process, that the object we want to reconstruct is *sufficiently smooth*. This allows us to filter out high frequencies that may appear in the reconstruction because we know they are part of the noise and not of the object we want to reconstruct. We present two mathematical frameworks for such a regularization.

In the first framework, we introduce the adjoint operator  $A^*$  to  $A$ , defined from  $H$  to  $\text{Range}(A^*)$  by the relation

$$(Ax, y)_H = (x, A^*y)_H, \quad \text{for all } x, y \in H.$$

Since  $A$  is compact and injective, then so is  $A^*$ . We can also define the inverse operator  $A^{-*} = (A^*)^{-1}$  from  $\text{Range}(A^*)$  to  $H$ .

We may now assume that  $x$ , the object we aim at reconstructing, is sufficiently smooth that it belongs to the range of  $A^*$ , i.e., there exists  $y$  such that  $x = A^*y$ . Since  $A$  and  $A^*$  are compact operators, hence smoothing operators, the above hypothesis means that we assume *a priori* that  $x$  is smoother than being merely an element in  $H$ . We then define the *stronger* norm

$$\|x\|_1 = \|A^{-*}x\|_H. \quad (10.4)$$

We may also assume that the object  $x$  is even smoother than being in the range of  $A^*$ . For instance let us assume that  $x$  belongs to the range of  $A^*A$ , i.e., there exists  $y$  such that  $x = A^*Ay$ . Note that since both  $A$  and  $A^*$  are smoothing operators (because they are compact), the assumption on  $x$  is stronger than simply being in the range of  $A^*$ . We define the even stronger norm

$$\|x\|_2 = \|(A^*A)^{-1}x\|_H. \quad (10.5)$$

We want to use these definitions to show that if the solution  $x$  is *a priori* bounded for the  $\|\cdot\|_1$  or the  $\|\cdot\|_2$  norm and “noise” is small, then the error in the reconstruction

is small. For instance, assume that  $y_j = Ax_j$  for  $j = 1, 2$  so that  $y = Ax$  for  $y = y_1 - y_2$  and  $x = x_1 - x_2$ . If both  $x_j$ ,  $j = 1, 2$  are bounded and  $y$  is small, then how small is  $x$ ? For such questions, we have the following result:

**Theorem 10.1.2** *Let  $x \in H$  such that  $\|x\|_1 \leq E$  and  $\|Ax\|_H \leq \delta$ . Then we have:*

$$\|x\|_H \leq \sqrt{E\delta}. \quad (10.6)$$

*If we now assume that  $\|x\|_2 \leq E$  instead, we obtain the better bound*

$$\|x\|_H \leq E^{\frac{1}{3}}\delta^{\frac{2}{3}}. \quad (10.7)$$

*Proof.* Let  $y = A^{-*}x$  so that  $\|y\|_H \leq E$ . We have then

$$\|x\|_H^2 = (x, A^*y) = (Ax, y) \leq \|Ax\|_H \|y\|_H \leq \delta E.$$

This proves (10.6). For the second bound let  $z = (A^*A)^{-1}x$  so that  $\|z\|_H \leq E$  and compute:

$$\|x\|_H^2 = (x, A^*Az) = (Ax, Az) \leq \delta \|Az\| = \delta (Az, Az)^{\frac{1}{2}} \leq \delta (z, x)^{\frac{1}{2}} \leq \delta E^{\frac{1}{2}} \|x\|_H^{\frac{1}{2}}.$$

This proves the second bound (10.7).  $\square$

The theorem should be interpreted as follows. Consider that  $Ax$  is the noise level in the measured data and that  $\|x\|_1 < E$  or  $\|x\|_2 < E$  is *a priori* smoothness information we have on the object we want to reconstruct. Then the worst error we can make on the reconstruction (provided we find an appropriate inversion method; see below) is given by the bounds (10.6) and (10.7). Note that the latter bound is better (since  $\delta^{\frac{2}{3}} \ll \delta^{\frac{1}{2}}$ ). This results from a more stringent assumption on the image  $x$ .

Let us now consider smoothing operators in the (second) framework of the Hilbert scale  $H^s(\mathbb{R})$  we have introduced in Chapter 1. Then we have the following result:

**Theorem 10.1.3** *Let us assume that the operator  $A$  is mildly ill-posed of order  $\alpha > 0$  so that*

$$\|Af\|_{L^2(\mathbb{R})} \geq m \|f\|_{H^{-\alpha}(\mathbb{R})}. \quad (10.8)$$

*Suppose now that the measurement error is small and that the function we want to reconstruct is regular in the sense that*

$$\|Af\|_{L^2(\mathbb{R})} \leq \delta m, \quad \text{and} \quad \|f\|_{H^\beta(\mathbb{R})} \leq E, \quad (10.9)$$

*for some  $\delta > 0$ ,  $\beta > 0$ , and  $E > 0$ . Then we have*

$$\|f\|_{L^2(\mathbb{R})} \leq \delta^{\frac{\beta}{\alpha+\beta}} E^{\frac{\alpha}{\alpha+\beta}}. \quad (10.10)$$

*Proof.* The proof is a simple but interesting exercise in interpolation theory. Notice that the hypotheses are

$$\|f\|_{H^\beta(\mathbb{R})} \leq E, \quad \text{and} \quad \|f\|_{H^{-\alpha}(\mathbb{R})} \leq \delta,$$

and our objective is to find a bound for  $\|f\|_{L^2(\mathbb{R})}$ . Let us denote  $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$ . We have

$$\begin{aligned} (2\pi)^n \|f\|_{L^2(\mathbb{R})}^2 &= \int_{\mathbb{R}^n} |\hat{f}(\xi)|^{2\theta} \langle \xi \rangle^{2\gamma} |\hat{f}(\xi)|^{2(1-\theta)} \langle \xi \rangle^{-2\gamma} d\xi \\ &\leq \left( \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \langle \xi \rangle^{2\gamma/\theta} d\xi \right)^\theta \left( \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \langle \xi \rangle^{-2\gamma/(1-\theta)} d\xi \right)^{1-\theta}, \end{aligned}$$

thanks to Hölder's inequality

$$\left| \int_{\mathbb{R}} f(x)g(x)dx \right| \leq \|f\|_{L^p(\mathbb{R})} \|g\|_{L^q(\mathbb{R})},$$

which holds for all  $p \geq 1$  and  $q \geq 1$  such that  $p^{-1} + q^{-1} = 1$ , where we have defined for all  $p \geq 1$ ,

$$\|f\|_{L^p(\mathbb{R})} = \left( \int_{\mathbb{R}} |f(x)|^p dx \right)^{1/p}. \quad (10.11)$$

Choosing  $\theta = \frac{\alpha}{\alpha+\beta}$  and  $\gamma = \frac{\alpha\beta}{\alpha+\beta}$  gives (10.10).

Let us briefly recall the proof of the Hölder's inequality [62]. We first verify that

$$x^{\frac{1}{p}} \leq \frac{x}{p} + \frac{1}{q}, \quad x > 0,$$

for  $p^{-1} + q^{-1} = 1$  and  $p \geq 1$ , since  $x^{\frac{1}{p}} - \frac{x}{p}$  attains its maximum at  $x = 1$  where it is equal to  $q^{-1}$ . For  $y > 0$  we use the above inequality for  $x/y$  and multiply by  $y$  to obtain

$$x^{\frac{1}{p}} y^{\frac{1}{q}} \leq \frac{x}{p} + \frac{y}{q}, \quad x > 0, y > 0. \quad (10.12)$$

Choosing  $x = |tf(x)|^p$  and  $y = |t^{-1}g(x)|^q$ , we deduce that

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq \frac{1}{p} \|tf\|_{L^p(\mathbb{R})}^p + \frac{1}{q} \|t^{-1}g\|_{L^q(\mathbb{R})}^q = \frac{t^p}{p} \|f\|_{L^p(\mathbb{R})}^p + \frac{t^q}{q} \|g\|_{L^q(\mathbb{R})}^q,$$

for all  $t > 0$ . Maximizing over  $t$  gives the Hölder inequality.  $\square$

The last theorem applies to a less general class of operators than compact operators (although it applies to operators that are not necessarily compact) but it gives us an accurate result. We should still consider  $\delta$  as the noise level and  $E$  as an a priori bound we have on the object we want to reconstruct. Then depending on the a priori smoothness of the object, we obtain different possible accuracies in the reconstructions. What is important is the relative regularity of the object compared to the smoothing effect of the operator  $A$ . When  $\beta = \alpha$ , this corresponds to assuming the same regularity as  $\|x\|_1 \leq E$  in Theorem 10.1.2. We thus obtain an accuracy of order  $\delta^{\frac{1}{2}}$  in the reconstruction. When  $\beta = 2\alpha$ , this corresponds to  $\|x\|_2 \leq E$  since  $f = (A^*A)^{-1}g$  for some  $g \in L^2(\mathbb{R})$  means that  $f$  is twice as regular as  $A$  is smoothing. We thus recover the accuracy of order  $\delta^{\frac{2}{3}}$  as in Theorem 10.1.2. Theorem 10.1.3 allows us to deal with arbitrary values of  $\beta$ . Notice that as  $\beta \rightarrow \infty$ , we recover that the problem is almost well-posed since the error in the reconstruction is asymptotically of the same order  $\delta$  as the noise level.

### 10.1.3 Regularization methods

Now that we know how noise can optimally be controlled in the reconstruction based on the regularity of the object we want to reconstruct, we need to devise algorithms that indeed control noise amplification in the reconstruction.

Since  $A^{-1}$  is an unbounded operator with domain of definition  $\text{Range}(A)$ , a proper subset of  $H$ , we first need to introduce approximations of the inverse operator. We denote by  $R_\gamma$  defined from  $H$  to  $H$  for  $\gamma > 0$  a sequence of regularizations of  $A^{-1}$  such that

$$\lim_{\gamma \rightarrow 0} R_\gamma Ax = x \quad \text{for all } x \in H. \quad (10.13)$$

Under the hypotheses of Lemma 10.1.1, we can show that the sequence of operators  $R_\gamma$  is not uniformly bounded. A uniform bound would indeed imply that  $A^{-1}$  is bounded. Thus,  $R_\gamma A$  converges to identity *strongly* (since (10.13) is the definition of strong convergence of operators) but not uniformly in the sense that  $\|R_\gamma A - I\|$  does not converge to 0.

**Exercise 10.1.1** Prove this.

One of the main objectives of the regularization technique is to handle noise in an optimal fashion. Let us denote by  $y^\delta$  our measurements and assume that  $\|y^\delta - Ax\|_H \leq \delta$ . We then define

$$x^{\gamma, \delta} = R_\gamma y^\delta. \quad (10.14)$$

We want to find sequences  $R_\gamma$  that deal with noise in an optimal fashion. For instance assuming that  $\|x\|_1 \leq E$  and that  $\|y^\delta - Ax\|_H \leq \delta$ , we want to be able to show that

$$\|x - x^{\gamma, \delta}\| \leq C\sqrt{E\delta},$$

at least for some values of  $\gamma$ . We know from Theorem 10.1.3 that such a bound is optimal. We will consider three regularization techniques: singular value decomposition, Tikhonov regularization, and Landweber iterations.

The choice of a parameter  $\gamma$  is then obviously of crucial importance as the above bound will not hold independently of  $\gamma$ . More precisely, the reconstruction error can be decomposed as

$$\|x^{\gamma, \delta} - x\|_H \leq \delta \|R_\gamma\|_H + \|R_\gamma Ax - x\|_H. \quad (10.15)$$

**Exercise 10.1.2** Prove this. The operator norm  $\|R_\gamma\|_H$  is defined as the supremum of  $\|R_\gamma x\|_H$  under the constraint  $\|x\|_H \leq 1$ .

We thus observe that two competing effects enter (10.15). The first effect comes from the ill-posedness: as  $\gamma \rightarrow 0$ , the norm  $\|R_\gamma\|_H$  tends to  $\infty$  so  $\gamma$  should not be chosen too small. The second effect comes from the regularization: as  $\gamma$  increases,  $R_\gamma A$  becomes a less accurate approximation of identity so  $\gamma$  should not be chosen too large. Only intermediate values of  $\gamma$  will provide an optimal reconstruction.

### Singular Value Decomposition

For a compact and injective operator  $A$  defined on an infinite dimensional Hilbert space  $H$ , let us assume that we know its singular value decomposition defined as follows. Let  $A^*$  be the adjoint operator to  $A$  and  $\lambda_j > 0$ ,  $j \in \mathbb{N}$  the eigenvalues of the symmetric operator  $A^*A$ . Then, the sequence  $\mu_j = \sqrt{\lambda_j}$  for  $j \in \mathbb{N}$  are called the *singular values* of  $A$ . Since  $\mu_j \leq \|A\|_H$ , we order the singular values such that

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq \cdots > 0.$$

Multiple eigenvalues are repeated as many times as their multiplicity (which is necessarily finite since the associated eigenspace for  $A^*A$  needs to be compact).

Then there exist two orthonormal systems  $(\mathbf{x}_j)_{j \in \mathbb{N}}$  and  $(\mathbf{y}_j)_{j \in \mathbb{N}}$  in  $H$  such that

$$A\mathbf{x}_j = \mu_j\mathbf{y}_j \quad \text{and} \quad A^*\mathbf{y}_j = \mu_j\mathbf{x}_j, \quad \text{for all } j \in J. \quad (10.16)$$

We call  $(\mu_j, \mathbf{x}_j, \mathbf{y}_j)$  the singular system for  $A$ . Notice that

$$A\mathbf{x} = \sum_{j=1}^{\infty} \mu_j (\mathbf{x}, \mathbf{x}_j) \mathbf{y}_j, \quad A^*\mathbf{y} = \sum_{j=1}^{\infty} \mu_j (\mathbf{y}, \mathbf{y}_j) \mathbf{x}_j.$$

Here  $(\mathbf{x}, \mathbf{x}_j)$  is the inner product in  $H$ ,  $(\mathbf{x}, \mathbf{x}_j)_H$ . We then have the very useful characterization of the Range of the compact and injective operator  $A$ :

**Lemma 10.1.4 (Picard)** *The equation  $A\mathbf{x} = \mathbf{y}$  is solvable in  $H$  if and only if*

$$\sum_{j \in \mathbb{N}} \frac{1}{\mu_j^2} |(y, y_j)|^2 < \infty, \quad (10.17)$$

*in which case the solution is given by*

$$\mathbf{x} = A^{-1}\mathbf{y} = \sum_{j \in \mathbb{N}} \frac{1}{\mu_j} (y, y_j) \mathbf{x}_j. \quad (10.18)$$

The ill-posedness of the inverse problem appears very clearly in the singular value decomposition. As  $j \rightarrow \infty$ , the singular values  $\mu_j$  tend to 0. And they do so all the faster that the inverse problem is ill-posed. We can extend the definition of ill-posed problems in the sense that a compact operator generates a mildly ill-posed inverse problem of order  $\alpha > 0$  when the singular values decay like  $j^{-\alpha}$  and generates a severely ill-posed problem when the singular values decay faster than any  $j^{-m}$  for  $m \in \mathbb{N}$ .

So in order to regularize the problem, all we have to do is to replace too small singular values by larger values. Let us define  $q(\gamma, \mu)$  for  $\gamma > 0$  and  $\mu \in [0, \|A\|]$  such that

$$|q(\gamma, \mu)| < 1, \quad |q(\gamma, \mu)| \leq c(\gamma)\mu, \quad \text{and} \quad q(\gamma, \mu) - 1 \rightarrow 0 \quad \text{as } \gamma \rightarrow 0, \quad (10.19)$$

(not uniformly in  $\mu$  obviously). Then we define the regularizing sequence

$$R_\gamma \mathbf{y} = \sum_{j \in \mathbb{N}} \frac{q(\gamma, \mu_j)}{\mu_j} (y, y_j) \mathbf{x}_j. \quad (10.20)$$

Compare to (10.18). As  $\gamma \rightarrow 0$ ,  $R_\gamma$  converges to  $A^{-1}$  pointwise. We are interested in estimating (10.15) and showing that the error is optimal based on the assumed regularity of  $\mathbf{x}$ . The total error is estimated by using

$$\|R_\gamma\|_H \leq c(\gamma), \quad \|R_\gamma A\mathbf{x} - \mathbf{x}\|_H = \sum_{j=1}^{\infty} (q(\gamma, \mu_j) - 1)^2 |(x, \mathbf{x}_j)|^2. \quad (10.21)$$

**Exercise 10.1.3** Prove these relations.

We can now prove the following results:

**Theorem 10.1.5** (i) Let us assume that  $\mathbf{x} = A^*z$  with  $\|z\|_H \leq E$  and that  $\|y^\delta - A\mathbf{x}\| \leq \delta$ , where  $y^\delta$  is the measurements. Choose  $q(\gamma, \mu)$  and  $\gamma$  such that

$$|q(\gamma, \mu) - 1| \leq C_1 \frac{\sqrt{\gamma}}{\mu}, \quad c(\gamma) \leq \frac{C_2}{\sqrt{\gamma}}, \quad \gamma = \frac{C_3 \delta}{E}. \quad (10.22)$$

Then we have that

$$\|\mathbf{x}^{\gamma, \delta} - \mathbf{x}\|_H \leq \left( \frac{C_2}{\sqrt{C_3}} + C_1 \sqrt{C_3} \right) \sqrt{\delta E}. \quad (10.23)$$

(ii) Let us assume that  $\mathbf{x} = A^*Az$  with  $\|z\|_H \leq E$  and that  $\|y^\delta - A\mathbf{x}\| \leq \delta$ , where  $y^\delta$  is the measurements. Choose  $q(\gamma, \mu)$  and  $\gamma$  such that

$$|q(\gamma, \mu) - 1| \leq C_4 \frac{\gamma}{\mu^2}, \quad c(\gamma) \leq \frac{C_5}{\sqrt{\gamma}}, \quad \gamma = C_6 \left( \frac{\delta}{E} \right)^{\frac{2}{3}}. \quad (10.24)$$

Then we have that

$$\|\mathbf{x}^{\gamma, \delta} - \mathbf{x}\|_H \leq \left( \frac{C_5}{\sqrt{C_6}} + C_4 C_6 \right) \delta^{\frac{2}{3}} E^{\frac{1}{3}}. \quad (10.25)$$

*Proof.* Since  $\mathbf{x} = A^*z$ , we verify that  $(\mathbf{x}, \mathbf{x}_j) = \mu_j (z, y_j)$  so that

$$\|R_\gamma A\mathbf{x} - \mathbf{x}\|_H^2 = \sum_{j=1}^{\infty} (q(\gamma, \mu_j) - 1)^2 |(z, y_j)|^2 \leq C_1^2 \gamma \|z\|_H^2.$$

This implies that

$$\delta \|R_\gamma\|_H + \|R_\gamma A\mathbf{x} - \mathbf{x}\|_H \leq \frac{C_2 \delta}{\sqrt{\gamma}} + C_1 \sqrt{\gamma} E.$$

Using (10.15) and the expression for  $\gamma$  yields (10.23).

**Exercise 10.1.4** Using similar arguments, prove (10.25).

This concludes the proof.  $\square$

We have thus defined an optimal regularization scheme for the inversion of  $A\mathbf{x} = \mathbf{y}$ . Indeed from the theory in Theorem 10.1.2 we know that up to some multiplicative constants, the above estimates are optimal.

It remains to find filters  $q(\gamma, \mu)$  satisfying the above hypotheses. We propose two:

$$q(\gamma, \mu) = \frac{\mu^2}{\gamma + \mu^2}, \quad (10.26)$$

$$q(\gamma, \mu) = \begin{cases} 1, & \mu^2 \geq \gamma, \\ 0, & \mu^2 < \gamma. \end{cases} \quad (10.27)$$

**Exercise 10.1.5** Show that the above choices verify the hypotheses of Theorem 10.1.5.

### Tikhonov Regularization

One of the main drawbacks of the theory presented in the preceding section is that in most cases, the singular value decomposition of the operator is not analytically available (although it is for the Radon transform; see [47, 48]), and is quite expensive to compute numerically once the continuous problem has been discretized. It is therefore useful to consider regularization techniques that do not depend on the SVD. One of the most popular regularization techniques is the Tikhonov-Phillips regularization technique.

Solving  $Ax = y$  corresponds to minimizing  $\|Ax - y\|_H$ . Instead one may want to minimize the regularized Tikhonov functional

$$J_\gamma(x) = \|Ax - y\|_H^2 + \gamma\|x\|_H^2, \quad x \in H. \quad (10.28)$$

For  $\gamma > 0$  and  $A$  a linear bounded operator on  $H$ , we can show that the above functional admits a unique minimizer  $x^\gamma$  solving the following normal equations

$$A^*Ax^\gamma + \gamma x^\gamma = A^*y. \quad (10.29)$$

**Exercise 10.1.6** Prove the above statement.

We can thus define the regularizing sequence

$$R_\gamma = (\gamma + A^*A)^{-1}A^*. \quad (10.30)$$

The operator is bounded in  $H$  by  $\|R_\gamma\|_H \leq C\gamma^{-1/2}$  for all  $\gamma > 0$ . Notice that for a compact operator  $A$  with singular system  $(\mu_i, x_i, y_i)$ , we verify that the singular value decomposition of  $R_\gamma$  is

$$R_\gamma y = \sum_{j=1}^{\infty} \frac{\mu_j}{\gamma + \mu_j^2} (y, y_j) x_j. \quad (10.31)$$

This means that the Tikhonov regularization corresponds to the SVD regularization with filter given by (10.26) and implies that the Tikhonov regularization is optimal to inverse problem with a priori regularity  $\|x\|_1 \leq E$  or  $\|x\|_2 \leq E$ . It is interesting to observe that the Tikhonov regularization is no longer optimal when the a priori regularity of  $x$  is better than  $\|x\|_2 \leq E$  (see [40]).

Let us make this observation more explicit. Let us consider the operator  $A$  given in the Fourier domain by multiplication by  $\langle \xi \rangle^{-\alpha}$  for some  $\alpha > 0$ . We verify that  $A^* = A$  and that  $R_\gamma$  is given in the Fourier domain by

$$R_\gamma = \mathcal{F}_{\xi \rightarrow x}^{-1} \frac{\langle \xi \rangle^{-\alpha}}{\langle \xi \rangle^{-2\alpha} + \gamma} \mathcal{F}_{x \rightarrow \xi}, \quad \text{so that} \quad \|R_\gamma\| \leq \frac{1}{2\sqrt{\gamma}}.$$

Indeed, we check that  $x/(x^2 + \gamma) \leq 1/(2\sqrt{\gamma})$  and attains its maximum at  $x = \sqrt{\gamma}$ . We now verify that

$$I - R_\gamma A = \mathcal{F}_{\xi \rightarrow x}^{-1} \frac{\gamma}{\langle \xi \rangle^{-2\alpha} + \gamma} \mathcal{F}_{x \rightarrow \xi},$$

so that for a function  $f \in H^\beta(\mathbb{R}^n)$ , we have

$$\|f - R_\gamma A f\| \leq \sup_{\langle \xi \rangle \geq 1} \frac{\gamma \langle \xi \rangle^{-\beta}}{\langle \xi \rangle^{-2\alpha} + \gamma} \|f\|_{H^\beta(\mathbb{R}^n)}.$$

Moreover the inequality is sharp in the sense that there exists functions  $f$  such that the reverse inequality holds (up to a multiplicative constant independent of  $\gamma$ ; Check this). For  $\beta > 2\alpha$ , the best estimate we can have for the above multiplier is that it is of order  $O(\gamma)$  (choose for instance  $\langle \xi \rangle = 1$ ).

**Exercise 10.1.7** Using (10.12) show that

$$\frac{\gamma \langle \xi \rangle^{-2\alpha\theta}}{\langle \xi \rangle^{-2\alpha} + \gamma} \leq \gamma^\theta, \quad 0 \leq \theta \leq 1.$$

Show that the above inequality is sharp.

Let  $Af = g$  be the problem we want to solve and  $g^\delta$  the measurements so that  $\|Af - g^\delta\|_{L^2(\mathbb{R}^n)} \leq \delta$ . Let us assume that  $f$  belongs to  $H^\beta(\mathbb{R}^n)$ . We verify using (10.15) that the error of the regularized problem is given by

$$\|f - R_\gamma g^\delta\| \leq \frac{\delta}{2\sqrt{\gamma}} + \gamma^{\frac{\beta}{2\alpha} \wedge 1} \|f\|_{H^\beta(\mathbb{R}^n)}. \quad (10.32)$$

Here,  $a \wedge b = \min(a, b)$ . This implies that

$$\|f - R_\gamma g^\delta\| \leq C \delta^{\frac{\beta}{\alpha+\beta} \wedge \frac{2}{3}} \|f\|_{H^{\frac{\alpha+\beta}{\alpha+\beta} \wedge \frac{1}{3}}(\mathbb{R}^n)}, \quad (10.33)$$

for a universal constant  $C$ . We therefore obtain that the Tikhonov regularization is optimal according to Theorem 10.1.3 when  $0 < \beta \leq 2\alpha$ . However, for all  $\beta > 2\alpha$ , the error between the Tikhonov regularization and the exact solution will be of order  $\delta^{\frac{2}{3}}$  instead of  $\delta^{\frac{\beta}{\beta+\alpha}}$ .

**Exercise 10.1.8** More generally, consider an operator  $A$  with symbol  $a(\xi)$ , i.e.,

$$A = \mathcal{F}_{\xi \rightarrow x}^{-1} a(\xi) \mathcal{F}_{x \rightarrow \xi},$$

such that  $0 < a(\xi) \in C^\infty(\mathbb{R}^n)$  and for some  $\alpha > 0$  and  $a_\infty \neq 0$ ,

$$\frac{a(\xi)}{\langle \xi \rangle^\alpha} \rightarrow a_\infty, \quad \text{as} \quad |\xi| \rightarrow \infty. \quad (10.34)$$

(i) Show that  $A^*$ , the adjoint of  $A$  for the  $L^2(\mathbb{R}^n)$  inner product, satisfies the same hypothesis (10.34).

(ii) Show that  $R_\gamma$  and  $S_\gamma = R_\gamma A - I$  are bounded operator with symbols given by

$$r_\gamma(\xi) = (|a(\xi)|^2 + \gamma)^{-1} \bar{a}(\xi), \quad s_\gamma(\xi) = \gamma(|a(\xi)|^2 + \gamma)^{-1},$$

respectively.

(iii) Assuming that  $f \in H^\beta(\mathbb{R})$ , show that (10.33) holds.

These results show that for the Radon transform, an a priori regularity of the function  $f(x)$  in  $H^1(\mathbb{R}^2)$  is sufficient to obtain an error of order  $\delta^{\frac{2}{3}}$ . When the function is smoother, a different technique from Tikhonov regularization is necessary to get a more accurate reconstruction.

### Landweber iterations

The drawback of the Tikhonov regularization is that it requires to invert the regularization of the normal operator  $\gamma + A^*A$ . This inversion may be computationally expensive in practice. The Landweber iteration method is an iterative technique in which no inversion is necessary. It is defined to solve the equation  $Ax = y$  as follows

$$x_0 = 0, \quad x_{n+1} = (I - rA^*A)x_n + rA^*y, \quad n \geq 0, \quad (10.35)$$

for some  $r > 0$ . By induction, we verify that  $x_n = R_n y$ , where

$$R_n = r \sum_{k=0}^{n-1} (I - rA^*A)^k A^*, \quad n \geq 1. \quad (10.36)$$

Consider a compact operator  $A$  with singular system  $(\mu_j, x_j, y_j)$ . We thus verify that

$$R_n y = \sum_{j=1}^{\infty} \frac{1}{\mu_j} (1 - (1 - r\mu_j^2)^n) (y, y_j) x_j. \quad (10.37)$$

**Exercise 10.1.9** Check (10.37).

This implies that  $R_n$  is of the form  $R_\gamma$  in (10.20) with  $\gamma = n^{-1}$  and

$$q(\gamma, \mu) = 1 - (1 - r\mu^2)^{1/\gamma}.$$

**Exercise 10.1.10** Show that the above filter verifies the hypotheses (10.19) and those of Theorem 10.1.5.

This implies that the Landweber iteration method is an optimal inversion method by Theorem 10.1.5.

**Exercise 10.1.11** Show that the hypotheses of Theorem 10.1.5 are met provided that the number of iterations  $n$  is chosen as

$$n = c \frac{E}{\delta}, \quad n = c \left( \frac{E}{\delta} \right)^{\frac{2}{3}},$$

when  $\|x\|_1 \leq E$  and  $\|x\|_2 \leq E$ , respectively.

The above result shows that the number of iterations should be chosen carefully: when  $n$  is too small, then  $R_\gamma A$  is a poor approximation of  $I$ , and when  $n$  is too large, then  $\|R_\gamma\|_H$  is too large. Unlike the Tikhonov regularization, we can show that the Landweber iteration method is also optimal for stronger regularity assumptions on  $x$  than those given in Theorem 10.1.5 (see [40] for instance).

Let us come back to the operator  $A$  with symbol  $a(\xi) = \langle \xi \rangle^{-\alpha}$ . We verify that  $R_n$  and  $S_n = R_n A - I$  have respective symbols

$$r_n(\xi) = \frac{1 - (1 - r \langle \xi \rangle^{-2\alpha})^n}{\langle \xi \rangle^{-\alpha}}, \quad s_n(\xi) = -(1 - r \langle \xi \rangle^{-2\alpha})^n.$$

**Exercise 10.1.12** (i) Show that  $s_n(\xi) \langle \xi \rangle^{-\beta}$  is bounded by  $C n^{-\beta/(2\alpha)}$  for  $\langle \xi \rangle$  of order  $n^{1/(2\alpha)}$ . Deduce that for  $f \in H^\beta(\mathbb{R}^n)$ , we have

$$\|S_n f\| \leq C n^{-\frac{\beta}{2\alpha}} \|f\|_{H^\beta(\mathbb{R}^n)}.$$

(ii) Show that provided that  $n$  is chosen as

$$n = C \delta^{\frac{-2\alpha}{\alpha+\beta}} \|f\|_{H^\beta(\mathbb{R}^n)}^{\frac{2\alpha}{\alpha+\beta}},$$

we have the estimate

$$\delta \|R_n\| + \|S_n f\| \leq C \delta^{\frac{\beta}{\alpha+\beta}} \|f\|_{H^\beta(\mathbb{R}^n)}^{\frac{\alpha}{\alpha+\beta}}. \quad (10.38)$$

(iii) Deduce that the Landweber iteration method is an optimal regularization technique for all  $\beta > 0$ .

(iv) Generalize the above results for the operators described in Exercise 10.1.8.

We have thus obtained the interesting result that unlike the Tikhonov regularization method described in (10.33), the Landweber iteration regularization can be made optimal (by choosing the appropriate number of iterations  $n$ ) for all choices on the regularity in  $H^\beta(\mathbb{R}^n)$  of the object  $f$ .

Let us conclude this section by the following summarizing remark. The reason why regularization was necessary was because *the user* decided that noise was too amplified during the not-regularized inversions. Smoothness priors were then considered to restore well-posedness. This corresponds to a choice of the factor  $\beta$  and a bound  $E$ . *In addition*, we need to choose a *regularization* parameter,  $\gamma$  in the Tikhonov regularization algorithm and the stopping iteration  $n$  in the Landweber iteration. How these choices are made depends on the bound  $E$  but also on the *estimated error*  $\delta$ . Various techniques have been developed to choose  $\gamma$  or  $n$  a posteriori (Morozov principle, L-curve). All these techniques require that  $\delta$  be known. There is no free lunch. Regularization does require prior information about the solution to mitigate the perceived lack of information in the available data.

## 10.2 Sparsity and other Regularization Priors

The regularization methods considered in the preceding section have a huge advantage: they replace ill-posed linear systems of equations by well-posed, at least better-posed (better-conditioned), *linear* systems as well. For instance, the inversion of  $A$  has been replaced by that of  $A^*A + \gamma I$  in the simplest version of Tikhonov regularization. Their main disadvantage is that they render the regularized solution typically smoother than the “exact” solution. Such a smoothing is unavoidable with such regularizations.

The reason why smooth objects are well reconstructed by the smoothing regularization method is that such objects can be represented by a small number of large coefficients (e.g., the first Fourier modes in a Fourier series expansion). It turns out that some objects are better represented in other bases. For instance, an image tends to have sharp discontinuities, for instance between a bright area and a dark area. Some bases, such as for instance those based on wavelets, will be much better than Fourier bases to represent this type of information.

A general framework to account for such prior information is to recast the inverse problem as seeking the minimum of an appropriate functional. Solving the inverse problem then amounts to solving a *optimization problem*. Let us assume that we have a problem of the form

$$\mathfrak{M}(u) = v,$$

and assume that  $v_d$  are given data. Let us assume that a functional  $u \mapsto \mathcal{R}(u)$  incorporates the prior information about  $u$  in the sense that  $\mathcal{R}(u)$  is small when  $u$  satisfies the constraints. Let us assume also that  $\rho(u, v)$  is a function that we wish to use to quantify the error between the available data  $v_d$  and the forward model  $\mathfrak{M}(u)$ . Then we want to minimize  $\rho(v, v_d)$  and at the same time minimize  $\mathcal{R}(u)$ . Both constraints can be achieved by introducing a *regularization parameter*  $\alpha$  and minimizing the sum

$$\mathcal{F}_\alpha(u) = \rho(\mathfrak{M}(u), v_d) + \alpha \mathcal{R}(u). \quad (10.39)$$

Solving the inverse problem consists of minimizing the above functional to get

$$\hat{u}_\alpha = \operatorname{argmin} \mathcal{F}_\alpha(u). \quad (10.40)$$

These minimization problems often go by the name of Tikhonov regularization and may be seen as generalizations of (10.28).

The main objective of regularization (or sparsity) theory is then to devise functions  $\rho$ ,  $\mathcal{R}$  and a regularization parameter  $\alpha$ , that best fits our prior information about the problem of interest. Once such a problem has been formulated, it remains to devise a method to solve such an optimization problem numerically. There is a vast literature on the subject; for recent references see the two books [56, 57].

### 10.2.1 Smoothness Prior and Minimizations

The simplest class of regularizations consists of choosing  $\rho(u, v) = \frac{1}{2} \|D(u - v)\|_H^2$  in the  $H = L^2$  sense for an operator  $D$  that may be identity or an operator of differentiation if small errors on derivatives of the solution matter in practice, and choosing  $\mathcal{R}(u)$  also as a quadratic functional, for instance  $\mathcal{R}(u) = \frac{1}{2} \|Ru\|_H^2$ , again for an operator  $R$  that may

be identity or a differential operator. Then associated to the linear problem  $Au = v$ , we have the minimization problem:

$$\mathcal{F}_\alpha(u) = \frac{1}{2}\|D(Au - v_d)\|_H^2 + \frac{\alpha}{2}\|Ru\|_H^2. \quad (10.41)$$

The main advantage of the above quadratic expression is that the Euler-Lagrange equations associated to the above minimization problem is the following linear system of equations

$$((DA)^*(DA) + \alpha R^*R)u = (DA)^*Dv_d. \quad (10.42)$$

When  $D = R = I$ , this is nothing but (10.29). Provided that  $R$  is an invertible matrix, then the above problem can be solved for all  $\alpha > 0$ . We have seen in the preceding section how the method converged (at least when  $D = R = I$ ) as the noise in the data and the regularization parameter  $\alpha$  tend to 0.

### 10.2.2 Sparsity Prior and Minimizations

Choosing  $\rho(u, v) = \frac{1}{2}\|D(u - v)\|_H^2$  for the misfit to the data may appear as relatively “natural” as it corresponds to measuring noise in the  $H = L^2$  sense. The quadratic functional  $\mathcal{R}(u) = \frac{1}{2}\|Ru\|_H^2$  is, however, much less justified in many settings.

Sometimes, prior knowledge about the object we wish to reconstruct shows that the latter is sparse in a given representation (a given basis, say). Sparse means here that the object is represented by a small number of large coefficients. For instance, an audio signal may be represented by a finite number of frequencies. Images typically display sharp edges that can be represented in a more economical fashion than pixel by pixel values.

Let us assume that  $u$  is discrete and  $A$  a matrix to simplify the presentation. Let us also assume that  $Bu$  is sparse, where  $B$  is a known matrix. Sparsity will be encoded by the fact that the  $l_1$  norm of  $Bu$  is small. Penalizing the residual and the  $l_1$  norm yields the minimization of

$$\mathcal{F}_\mu(u) = \|Bu\|_{l_1} + \frac{\mu}{2}\|Au - v_d\|_{l_2}^2. \quad (10.43)$$

This and similar minimization problems have been applied very successfully for a large class of imaging problems.

However, minimizing  $\mathcal{F}_\mu$  above is computationally more intensive than solving (10.42). Several algorithms have been developed to solve such minimization problems. We present one strategy, called the split Bregman iteration, that is both efficient and relatively easy to explain. The main idea is that when  $B$  and  $A$  are the identity operators, then the above minimization can be performed for each component of  $u$  separately. In the general case, we introduce

$$d = Bu,$$

and replace the above minimization by

$$\min_{u,d} \|d\|_{l_1} + \frac{\mu}{2}\|Au - v_d\|_{l_2}^2 + \frac{\lambda}{2}\|d - Bu\|_{l_2}^2. \quad (10.44)$$

Choosing  $\lambda$  sufficiently large provides a good approximation of the problem we wish to solve. Alternatively, we can solve a series of problems of the above form and show that

we minimize (10.43) in the limit; we do not present the details here and refer the reader to [35].

Now the minimization of (10.44) can be performed iteratively by successively minimizing for  $u$  and for  $d$ . The minimization for  $u$  becomes a linear problem while the minimization for  $d$  can be performed for each coordinate independently (this is called soft shrinkage). The iterative algorithm then converges [35]. More precisely, the solution of

$$\min_u \frac{\mu}{2} \|Au - v_d\|_{l_2}^2 + \frac{\lambda}{2} \|d - Bu\|_{l_2}^2,$$

is given by

$$(\mu A^* A + \lambda B^* B)u = \mu A^* v_d + \lambda d. \quad (10.45)$$

This is a linear problem that admits a unique solution. Now the solution of

$$\min_d \|d\|_{l_1} + \frac{\lambda}{2} \|d - Bu\|_{l_2}^2 = \min_d \sum_{j=1}^J |d_j| + \frac{\lambda}{2} |d_j - (Bu)_j|^2 = \sum_{j=1}^J \min_{d_j} |d_j| + \frac{\lambda}{2} |d_j - (Bu)_j|^2.$$

Each element in the sum can then be minimized separately. We find that the solution of

$$\min_d |d| + \frac{\lambda}{2} |d - a|^2, \quad (10.46)$$

is given by the soft thresholding

$$d = \text{sgn}(a) \max\left(|a| - \frac{1}{\lambda}, 0\right).$$

We have presented this algorithm to show that replacing a smoothness regularization as in (10.41) by a sparsity regularization as in (10.43) increased the computational complexity of the reconstruction algorithm: instead of solving one linear system, we have to iteratively solve linear systems of the form (10.45) and soft thresholdings given by (10.46). When the sparsity assumptions are valid, however, these methods have shown to be greatly beneficial in many practical settings of medical imaging; see [56, 57].

## 10.3 Bayesian framework and regularization

The penalty regularization framework seen in the preceding two sections is very efficient when the data are sufficiently informative. When the data are very informative and noise relatively small, then no real regularization is necessary. When data are less informative but still quite informative, prior information becomes necessary and smoothness and sparsity type priors allow us to still obtain very accurate reconstructions. When data are even less informative, for instance because noise is very large, or because the problem is severely ill-posed, then sparsity priors are typically no longer sufficient. What one typically obtains as a result is a function that resembles the minimum of the penalization term. In some cases, that may not be desirable. Also, additional prior information may occasionally be known, for instance that next to a black pixel, there is never a blue pixel. Such information is difficult to include in a penalization method.

A fairly versatile methodology to include various prior informations is the Bayesian framework. To simplify the presentation slightly, let us assume that the problem of interest is

$$y = \mathfrak{M}(x) + n, \quad (10.47)$$

where  $\mathfrak{M}$  is the measurement operator,  $x$  the unknown set of coefficients,  $y$  the measurements and  $n$  models additive noise in the data.

The main assumption of Bayesian inversions is to assume that  $x$  belongs to a class of possible models  $\mathfrak{X}$  and that each  $x \in \mathfrak{X}$  is given an *a priori* probability of being the “true” coefficient. The associated probability (density)  $\pi(x)$  is called the *prior* distribution. A second ingredient in the Bayesian framework is the model for the noise  $n$ . We denote by  $\pi_n(n)$  the probability distribution of  $n$ .

Let us now define

$$\pi(y|x) = \frac{\pi(x, y)}{\pi(x)}$$

the conditional probability density of  $y$  *knowing*  $x$  with  $\pi(x, y)$  the probability density of  $x$  and  $y$ . Note that  $\pi(x) = \int \pi(x, y) dy$  as a marginal density so that the above conditional probability density is indeed a probability density (integrating to 1 in  $y$ ). Note that knowledge of  $\pi(y|x)$  is equivalent to knowledge of  $\pi_n$  since

$$\pi(y|x) = \pi_n(y - \mathfrak{M}(x)) \quad \text{for each fixed } x.$$

Bayes’ rule then essentially states that

$$\pi(x|y)\pi(y) = \pi(y|x)\pi(x) = \pi(x, y). \quad (10.48)$$

In our inverse problem where  $y$  is the measured data and  $x$  the unknown coefficients, this means

$$\pi(x|y) = \frac{1}{\pi(y)} \pi(y|x)\pi(x) \propto \pi(y|x)\pi(x), \quad (10.49)$$

where  $\propto$  means proportional to, i.e., up to a normalizing constant (here  $1/\pi(y)$ ). In other words, if we know the prior density  $\pi(x)$  and the likelihood function  $\pi(y|x)$ , then by Bayes’ rule, we know  $\pi(x|y)$ , which is the *posterior* probability (density).

Let us recapitulate the main ingredients of the Bayesian formalism. We assume the *prior* distribution  $\pi(x)$  known as an indication of our prior beliefs about the coefficients before data are acquired. We assume knowledge of the likelihood function  $\pi(y|x)$ , which as we have seen is a statement about the noise model in the experiment. From these two prior assumptions, we use Bayes’ rule to infer the *posterior* distribution  $\pi(x|y)$  for  $x$  knowing the data.

### 10.3.1 Penalization methods and Bayesian framework

Before going into the advantages and drawbacks of the method, we first show that penalization methods can be seen as an application of the Bayesian framework. Let  $\mathcal{R}(x)$  be a given function and assume that the prior is given by the Gibbs distribution:

$$\pi(x) \propto e^{-\mathcal{R}(x)}.$$

Now assume that the likelihood function is of the form

$$\pi(y|x) \propto e^{-\rho(y-\mathfrak{M}(x))},$$

where  $\rho$  is a distance function. Then by Bayes' rule, we find that

$$\pi(x|y) \propto e^{-(\rho(y-\mathfrak{M}(x))+\mathcal{R}(x))}.$$

The Maximum A Posteriori (MAP)  $x_{\text{MAP}}$  is the parameter that maximizes the posterior distribution, or equivalently the minimum of the functional

$$\mathcal{F}(x) = \rho(y - \mathfrak{M}(x)) + \mathcal{R}(x).$$

Therefore, for appropriate choices of the prior and likelihood function, we retrieve the penalization methods seen in the preceding section.

Note that the minimization problem is solved by linear algebra when both  $\rho$  and  $\mathcal{R}$  are quadratic functionals. For instance if  $\pi_n(n) \sim \mathcal{N}(0, \Sigma)$  a multivariate Gaussian with correlation matrix  $\Sigma$ , then we have  $\rho(n) \propto e^{-\frac{1}{2}n^t \Sigma^{-1} n}$ . Similarly, for  $\pi(x) \sim \mathcal{N}(0, \Gamma)$ , then  $\mathcal{R}(x) \propto e^{-\frac{1}{2}x^t \Gamma^{-1} x}$  so that we need to minimize

$$\mathcal{F}(x) = \frac{1}{2}(y - \mathfrak{M}x)^t \Sigma^{-1} (y - \mathfrak{M}x) + \frac{1}{2}x^t \Gamma^{-1} x. \quad (10.50)$$

If  $\mathfrak{M}$  is a linear operator, then the solution to the minimization problem is, as we already saw, solution of

$$(\mathfrak{M}^* \Sigma^{-1} \mathfrak{M} + \Gamma^{-1})x = \mathfrak{M}^* \Sigma^{-1} y. \quad (10.51)$$

The Bayesian framework can then be used to recover the Tikhonov regularization of linear equations. Moreover, it gives an explicit characterization of the correlation matrices  $\Sigma$  and  $\Gamma$  as the co-variance functions of the measurement noise and of the prior assumptions on the coefficients, respectively.

Note that the  $l_1$  minimization corresponds to a choice  $\mathcal{R}(x) = \sum_i |x_i|$ . This corresponds to assuming that each pixel value satisfies independent and identically distributed random variables with a Laplace distribution. We thus also recover the sparsity regularizations using the Bayesian framework. If we expect nearby pixels to be correlated, then more complex prior models or functionals  $\mathcal{R}(x)$  need to be constructed. This is a very active area of research. Although the derivation of the “best” functional is often more an art than grounded in first principles, the Bayesian framework sometimes allows for very “pleasing” reconstructions (we recall that the ill-posedness of an inverse problem is a subjective notion).

### 10.3.2 Computational and psychological costs of the Bayesian framework

We have seen that the Bayesian framework reduced to an optimization problem when the Maximum A Posteriori (MAP)  $x_{\text{MAP}}$  is what we are looking for. The Bayesian framework allows one to obtain much more information, at least in theory, since the output of the procedure is the full posterior distribution  $\pi(x|y)$  and not only its argmax.

In practice, however, we are faced with daunting tasks: first of all, how do we sample what is often a very high dimensional distribution  $\pi(x|y)$ ? And second of all, even if sampling is possible, how does one represent such a huge object practically? These two questions severely limit the applicability of Bayesian frameworks in practice.

A third, and in some sense more fundamental and structural, question pertains to the choice of the prior  $\pi(x)$ . Where should that prior information come from? There is no “good” answer to this fundamental yet ill-formulated question. In some sense, we have already partially answered it: since *the user* decided that adding no prior information was not working in the sense that “noise” had too large an effect on the reconstruction, then *the user* has to come up with *another* model. There is no such a thing as a “non-informative” prior, since *the user* decided that a prior was necessary. (If data alone were sufficient to obtain “good” reconstructions, then the Bayesian framework would not be necessary.) If data are not sufficient, then the Bayesian framework provides a very versatile framework for *the user* to provide information about the problem that helps to compensate for what is not present in the data. Some researchers will not be satisfied with this way of addressing the inverse problem and the notion of “compensating” for the lack of data. This is a perfectly reasonable position. However, the Bayesian framework at least has this very appealing feature: it provides a logical mapping from the prior information, namely the prior distribution and the likelihood function, to the outcome of the procedure, namely the posterior distribution. If nothing else, it can therefore serve as a very valuable tool to guide intuition and to search what types of prior informations are necessary for a given set of constraints on the posterior distribution. Moreover, nothing prevents us from estimating how the posterior distribution is sensitive to variations in the prior information. This strategy, sometimes referred to as Robust Bayesian analysis, allows one understand which features of the reconstructed parameters strongly or weakly depend on the choices of the prior density and likelihood functions.

Now that the psychological cost of the Bayesian framework has been taken into account, let us come back to its computational cost, which still poses enormous challenges. Let us first address the representation of the posterior distribution. Typically, moments of the posterior distribution are what we are interested in. For instance, one may be interested in the first moment (a vector) and the variance (a matrix)

$$x_m = \int x \pi(x|y) dx, \quad \Gamma_x = \int x \otimes x \pi(x|y) dx - x_m \otimes x_m. \quad (10.52)$$

Of interest are also various quantiles of the posterior distribution, for instance the probability that  $x_j$  be larger than a number  $\gamma$ :  $\int \pi(x|y) \chi(x_j > \gamma) dx$ , where  $\chi(X)$  is the indicatrix function of the set  $X$  equal to 1 on  $X$  and to 0 otherwise.

For each of these moments of the posterior distribution, we need to be able to sample  $\pi(x|y) dx$ . In a few cases, the sampling of  $\pi(x|y) dx$  may be straightforward, for instance when  $\pi(y|x)$  has a Gaussian structure. In most cases, however, sampling is a difficult exercise. The most versatile method to perform such a sampling is arguably the Markov Chain Monte Carlo (MCMC) method. The objective of MCMC samplers is to generate a Markov chain  $X^i$  for  $i \in \mathbb{N}$  whose invariant distribution (distribution at convergence when the algorithm converges) is the posterior distribution. There are two main MCMC samplers, the Gibbs sampler and the Metropolis-Hastings sampler. The latter is defined as follows. We assume that we want to sample a distribution  $\pi(x|y)$ .

Let  $q(x, x')$  be a given, positive, transition density from the vector  $x$  to the vector  $x'$  (it thus sums to 1 integrated in all possible vectors  $x'$  for each  $x$ ). Let us then define

$$\alpha(x, x') := \min \left( \frac{q(x, x')\pi(x'|y)}{q(x', x)\pi(x|y)}, 1 \right). \quad (10.53)$$

Note that the above quantity, which is all we need about  $\pi(x|y)$  in the Metropolis-Hastings sampler, depends only on  $\frac{\pi(x'|y)}{\pi(x|y)}$  and thus is independent of the normalizing constant of  $\pi(x|y)$ , which is typically not known in the Bayesian framework, and whose estimation is typically expensive computationally.

Let  $X^i$  the current state of the Markov chain. Let  $\tilde{X}^{i+1}$  be drawn from the transition kernel  $q(X^i, x')$ . Then with probability  $\alpha(X^i, \tilde{X}^{i+1})$ , we accept the transition and set  $X^{i+1} = \tilde{X}^{i+1}$  while with probability  $1 - \alpha(X^i, \tilde{X}^{i+1})$ , we reject the transition and set  $X^{i+1} = X^i$ .

The transition probability of the chain from  $x$  to  $x'$  is thus  $p(x, x') = \alpha(x, x')q(x, x')$  while the probability to stay put at  $x$  is  $1 - \int p(x, x')dx'$ . The construction is such that  $\pi(x|y)p(x, x') = p(x', x)\pi(x'|y)$ , which means that  $\pi(x|y)dy$  is indeed the invariant distribution of the Markov chain. In practice, we want independent samples of  $\pi(x|y)$  so that the following Monte Carlo integration follows from an application of the law of large numbers (ergodicity), for instance:

$$\int f(x)\pi(x|y)dx \sim \frac{1}{|I|} \sum_{i \in I} f(X^i), \quad (10.54)$$

for any reasonable (continuous) functional  $f$ . Such a rule is accurate if the  $X^i$  are sampled according to  $\pi(x|y)dy$  and are sufficiently independent. This is for instance achieved by choosing  $I = \{1 \leq i \leq i_{\max}, \quad i = Nj, \quad j \in \mathbb{N}\}$ . For instance, we can take  $i_{\max} = 10^7$  and  $N = 1000$  so that  $I$  is composed of  $|I| = 10^4$  points chosen every 1000 points in the Metropolis-Hastings Markov chain. For an accuracy equal to  $\sqrt{|I|} = 0.01$  (as an application of the central limit theorem to estimate the error in (10.54)), we thus need  $10^7$  evaluations of  $\pi(x|y)$ . Using Bayes' rule, this is proportional to  $\pi(x)\pi(y|x)$ , where the latter likelihood function requires that we solve a forward problem (for a given  $x$  drawn from the prior  $\pi(x)$ ) to estimate the law of the "data"  $y$ . In other words, the construction of the above statistical moment with an accuracy of order  $10^{-2}$  requires that we solve  $10^7$  forward problems. In many practical situations, this is an unsurmountable computational cost. Moreover, this assumes that the transition  $q(x, x')$  has been chosen in such a way that every 1000 samples  $X^i$  are indeed sufficiently independent. This is very difficult to achieve in practice and is typically obtained by experienced users rather than from sound, physics- or mathematics- based principles.

Note that in practice, it has been observed that  $I$  in (10.54) should be the set of all runs  $1 \leq i \leq i_{\max}$ . In other words, there is no gain in throwing away points  $X^i$  in the evaluation of the integrals. However, the above heuristics are correct: the error in the approximation (10.54) is indeed proportional to the square root of the number of independent components in  $\{X^i\}$  and not  $\sqrt{i_{\max}}$ .

Many methodologies have been developed to improve the efficiency of MCMC algorithms. It is however fair to say that even with nowadays computational capabilities, many problems of interest are totally out of reach using the standard Bayesian framework. That said, it is still a very versatile methodology that goes a long way to address

the main problem of this chapter, which we recall is: *the user* decided that adding no prior information was not working and thus something had to be done.

General references on the use of the Bayesian framework in inverse problems include [38, 61].

# Chapter 11

## Geometric Priors and Parameterizations

In this chapter, we consider a few other methods to include prior information one might possess about the unknown objects of interest. The Bayesian framework introduced in the preceding chapter is quite versatile, but it is computationally extremely expensive because deterministic reconstructions are replaced by a probability measure on possible reconstructions, which is often daunting to sample and visualize from a numerical point of view.

Other physically motivated methods have been developed to incorporate prior information. In one such method, we aim to reconstruct the support of an inclusion rather than its full description and assume that such an inclusion is embedded in a *known* medium. Several techniques have been developed to do this and we focus here on a method called the *factorization* method. The factorization method is a functional analytical tool that allows us to separate the influence of the unknown inclusion from that of the known background. It is considered in a simple setting in section 11.1.

In another method, we give up the hope to perform global reconstructions and replace the unknown object by a low-dimensional parameterization. Any finite dimensional problem that is injective is necessarily well-posed essentially as an application of the Fredholm alternative. One such quite useful parameterization consists of assuming that the inclusion has *small* volume. We then perform asymptotic expansions in its volume to understand the leading influence of such an inclusion on available measurements. The reconstruction then focuses on the first coefficients appearing in the asymptotic expansion assuming the surrounding background to be known. The method is presented in a very simple setting in section 11.2.

### 11.1 Reconstructing the domain of inclusions

The reconstruction of physical parameters in an elliptic equation from boundary measurements, such as the Neumann-to-Dirichlet map, is a severely ill-posed problem. One should therefore not expect to stably reconstruct more than a few coefficients modeling the physical parameters, such as for instance the first Fourier modes in a Fourier series expansion as we saw in the smoothness penalization considered in Chapter 10.

In certain applications, knowing the first few coefficients in a Fourier series expansion

is not what one is interested in. In this chapter, we assume that the physical parameters are given by a background, which is known, and an inclusion, from which we only know that it differs from the background. Moreover, we are not so much interested in the detailed structure of the inclusion as in its location. We thus wish to reconstruct an interface separating the background from the inclusion.

To reconstruct this interface, we use the method of *factorization*. The method provides a constructive method to obtain the support of the inclusion from the Neumann-to-Dirichlet (NtD) boundary measurements. Notice that the NtD measurements allow us a priori to reconstruct much more than the support of the inclusion. However, because we restrict ourselves to this specific reconstruction, we can expect to obtain more accurate results on location of the inclusion than by directly reconstructing the physical parameters on the whole domain.

### 11.1.1 Forward Problem

We consider here the problem in impedance tomography. The theory generalizes to a certain extent to problems in optical tomography.

Let  $\gamma(x)$  be a conductivity tensor in an open bounded domain  $X \in \mathbb{R}^n$  with Lipschitz boundary  $\partial X$ . We define  $\Sigma$ , a smooth surface in  $X$ , as the boundary of the inclusion. We denote by  $D$  the simply connected bounded open domain such that  $\Sigma = \partial D$ . This means that  $D$  is the domain “inside” the surface  $\Sigma$ . We also define  $D^c = X \setminus \overline{D}$ , of boundary  $\partial D^c = \partial X \cup \Sigma$ . We assume that  $\gamma(x)$  is a smooth known background  $\gamma(x) = \gamma_0(x)$  on  $D^c$ , and that  $\gamma$  and  $\gamma_0$  are smooth but different on  $D$ . For  $\gamma_0$  a smooth known tensor on the full domain  $X$ , this means that  $\gamma$  jumps across  $\Sigma$  so that  $\Sigma$  is the surface of discontinuity of  $\gamma$ . More precisely, we assume that the  $n \times n$  symmetric tensor  $\gamma_0(x)$  is of class  $C^2(\overline{X})$  and positive definite such that  $\xi_i \xi_j \gamma_{0ij}(x) \geq \alpha_0 > 0$  uniformly in  $x \in X$  and in  $\{\xi_i\}_{i=1}^n = \xi \in \mathbb{S}^{n-1}$ , the unit sphere in  $\mathbb{R}^n$ . Similarly, the  $n \times n$  symmetric tensor  $\gamma(x)$  is of class  $C^2(\overline{D}) \otimes C^2(\overline{D^c})$  (in the sense that  $\gamma(x)|_D$  can be extended as a function of class  $C^2(D)$  and similarly for  $\gamma(x)|_{D^c}$ ) and positive definite such that  $\xi_i \xi_j \gamma_{ij}(x) \geq \alpha_0 > 0$  uniformly in  $x \in X$  and in  $\{\xi_i\}_{i=1}^n = \xi \in \mathbb{S}^{n-1}$ .

The equation for the electric potential  $u(x)$  is given by

$$\begin{aligned} \nabla \cdot \gamma \nabla u &= 0, & \text{in } X \\ \nu \cdot \gamma \nabla u &= g & \text{on } \partial X \\ \int_{\partial X} u \, d\sigma &= 0. \end{aligned} \tag{11.1}$$

Here,  $\nu(x)$  is the outward unit normal to  $X$  at  $x \in \partial X$ . We also denote by  $\nu(x)$  the outward unit normal to  $D$  at  $x \in \Sigma$ . Finally  $g(x)$  is a mean-zero current, i.e.,  $\int_{\partial X} g \, d\sigma = 0$ , imposed at the boundary of the domain.

The above problem admits a unique solution  $H_0^1(X)$ , the space of functions in  $u \in H^1(X)$  such that  $\int_{\partial X} u \, d\sigma = 0$ . This results from the variational formulation of the above equation

$$b(u, \phi) \equiv \int_X \gamma \nabla u \cdot \nabla \phi \, dx = \int_{\partial X} g \phi \, d\sigma(x) \equiv l(\phi), \tag{11.2}$$

holding for any test function  $\phi \in H_0^1(X)$ . Indeed from a Poincaré-like inequality, we deduce that  $b(u, v)$  is a coercive and bounded bilinear form on  $H_0^1(X)$  and the existence result follows from the Lax-Milgram theorem. Classical trace estimates show that  $u|_{\partial X} \in H_0^{\frac{1}{2}}(\partial X)$ , the space of functions  $v \in H^{\frac{1}{2}}(\partial X)$  such that  $\int_{\partial X} v d\sigma = 0$ .

We define the Neumann-to-Dirichlet operator  $\Lambda_\Sigma$ , depending on the location of the discontinuity  $\Sigma$ , as

$$\Lambda_\Sigma : H_0^{-\frac{1}{2}}(\partial X) \longrightarrow H_0^{\frac{1}{2}}(\partial X), \quad g \longmapsto u|_{\partial X}, \quad (11.3)$$

where  $u(x)$  is the solution to (11.1) with boundary normal current  $g(x)$ . Similarly, we introduce the “background” Neumann-to-Dirichlet operator  $\Lambda_0$  defined as above with  $\gamma$  replaced by the known background  $\gamma_0$ . To model that the inclusion has a different conductivity from the background, we assume that  $\gamma$  satisfies either one of the following hypotheses

$$\gamma(x) - \gamma_0(x) \geq \alpha_1 > 0 \quad \text{on } D, \quad \gamma_0(x) = \gamma(x), \quad \text{on } D^c, \quad (11.4)$$

$$\gamma_0(x) - \gamma(x) \geq \alpha_1 > 0 \quad \text{on } D, \quad \gamma_0(x) = \gamma(x), \quad \text{on } D^c, \quad (11.5)$$

for some constant positive definite tensor  $\alpha_1$ . The tensor inequality  $\gamma_1 \geq \gamma_2$  is meant in the sense that  $\xi_i \xi_j (\gamma_{1,ij} - \gamma_{2,ij}) \geq 0$  for all  $\xi \in \mathbb{R}^n$ .

### 11.1.2 Factorization method

The purpose of the factorization method is to show that

$$\Lambda_0 - \Lambda_\Sigma = L^* F L, \quad (11.6)$$

where  $L$  and  $L^*$  are operators in duality that depend only on  $\gamma|_D = (\gamma_0)|_D$  and  $F$  or  $-F$  is an operator that generates a coercive form on  $H_0^{\frac{1}{2}}(\Sigma)$  when (11.4) or (11.5) are satisfied, respectively. The operators are constructed as follows. Let  $v$  and  $w$  be the solutions of

$$\begin{aligned} \nabla \cdot \gamma \nabla v &= 0, & \text{in } D^c & & \nabla \cdot \gamma \nabla w &= 0, & \text{in } D^c \\ \nu \cdot \gamma \nabla v &= \phi & \text{on } \partial X & & \nu \cdot \gamma \nabla w &= 0 & \text{on } \partial X \\ \nu \cdot \gamma \nabla v &= 0 & \text{on } \Sigma & & \nu \cdot \gamma \nabla w &= -\phi & \text{on } \Sigma \\ \int_\Sigma v d\sigma &= 0, & & & \int_\Sigma w d\sigma &= 0. & \end{aligned} \quad (11.7)$$

These equations are well-posed in the sense that they admit solutions in  $H^1(D^c)$  with traces in  $H^{\frac{1}{2}}(\Sigma)$  and in  $H^{\frac{1}{2}}(\partial X)$  at the boundary of  $D^c$ . We then define the operator  $L$ , which maps  $\phi \in H_0^{-\frac{1}{2}}(\partial X)$  to  $v|_\Sigma \in H_0^{\frac{1}{2}}(\Sigma)$ , where  $v$  is the unique solution to the left equation in (11.7), and the operator  $L^*$ , which maps  $\phi \in H_0^{-\frac{1}{2}}(\Sigma)$  to  $w|_{\partial X}$ , where  $w$  is the unique solution to the right equation in (11.7). We verify that both operators are in duality in the sense that

$$(L\phi, \psi)_\Sigma \equiv \int_\Sigma \psi L\phi \, d\sigma = \int_{\partial X} \phi L^*\psi \, d\sigma \equiv (\phi, L^*\psi)_{\partial X}.$$

Let us now define two operators  $G_\Sigma$  and  $G_\Sigma^*$  as follows. For any quantity  $f$  defined on  $D \cup D^c$ , we denote by  $f^+(x)$  for  $x \in \Sigma$  the limit of  $f(y)$  as  $y \rightarrow x$  and  $y \in D^c$ , and by  $f^-(x)$  the limit of  $f(y)$  as  $y \rightarrow x$  and  $y \in D$ . Let  $v$  and  $w$  be the unique solutions to the following problems

$$\begin{aligned}
\nabla \cdot \gamma \nabla v &= 0, & \text{in } X \setminus \Sigma & & \nabla \cdot \gamma \nabla w &= 0, & \text{in } X \setminus \Sigma \\
[v] &= 0, & \text{on } \Sigma & & [w] &= \phi, & \text{on } \Sigma \\
[\nu \cdot \gamma \nabla v] &= 0 & \text{on } \Sigma & & [\nu \cdot \gamma \nabla w] &= 0 & \text{on } \Sigma \\
\nu \cdot \gamma \nabla v &= g & \text{on } \partial X & & \nu \cdot \gamma \nabla w &= 0 & \text{on } \partial X \\
\int_\Sigma v \, d\sigma &= 0 & & & \int_{\partial X} w \, d\sigma &= 0. & 
\end{aligned} \tag{11.8}$$

We define  $G_\Sigma$  as the operator mapping  $g \in H_0^{-\frac{1}{2}}(\partial X)$  to  $G_\Sigma g = \nu \cdot \gamma \nabla v|_\Sigma^+ \in H_0^{-\frac{1}{2}}(\Sigma)$  and the  $G_\Sigma^*$  as the operator mapping  $\phi \in H_0^{\frac{1}{2}}(\Sigma)$  to  $G_\Sigma^* \phi = w|_{\partial X} \in H_0^{\frac{1}{2}}(\partial X)$ , where  $v$  and  $w$  are the unique solutions to the above equations (11.8).

Except for the normalization  $\int_\Sigma v \, d\sigma = 0$ , the equation for  $v$  is the same as (11.1) and thus admits a unique solution in  $H^1(X)$ , say. Moreover integrations by parts on  $D^c$  imply that

$$\int_\Sigma \nu \cdot \gamma \nabla v^+ \, d\sigma = \int_{\partial X} g \, d\sigma = 0.$$

This justifies the well-posedness of the operator  $G_\Sigma$  as it is described above. The operator  $G_\Sigma^*$  is more delicate. We first obtain that for any smooth test function  $\psi$ ,

$$\begin{aligned}
\int_{D^c} \gamma \nabla w \cdot \nabla \psi \, dx + \int_\Sigma \nu \cdot \gamma \nabla w \psi^+ \, d\sigma &= 0 \\
\int_D \gamma \nabla w \cdot \nabla \psi \, dx - \int_\Sigma \nu \cdot \gamma \nabla w \psi^- \, d\sigma &= 0,
\end{aligned}$$

so that

$$\int_X \gamma \nabla w \cdot \nabla \psi \, dx = \int_\Sigma (-\nu \cdot \gamma \nabla w) [\psi] \, d\sigma. \tag{11.9}$$

It turns out that  $\|\nu \cdot \gamma \nabla w\|_{H_0^{-\frac{1}{2}}(\Sigma)}$  is bounded by the norm of  $\gamma \nabla w$  in  $H(\text{div}, X)$  (see [34]).

This and a Poincaré-type inequality shows that the above right-hand side with  $\psi = w$  is bounded by  $C \|\phi\|_{H_0^{\frac{1}{2}}(\Sigma)}^2$ . Existence and uniqueness of the solution  $w \in H^1(D) \otimes H^1(D^c)$

to (11.8) is then ensured by an application of the Lax-Milgram theorem. This also shows that the operator  $G_\Sigma^*$  as defined above is well-posed.

Integrations by parts in the equation for  $v$  in (11.8) by a test function  $\varphi$  yields

$$\begin{aligned}
\int_D \gamma \nabla v \cdot \nabla \varphi \, dx - \int_\Sigma \nu \cdot \gamma \nabla v \varphi^- \, d\sigma &= 0 \\
\int_{D^c} \gamma \nabla v \cdot \nabla \varphi \, dx + \int_\Sigma \nu \cdot \gamma \nabla v \varphi^+ \, d\sigma &= \int_{\partial X} g \varphi \, d\sigma,
\end{aligned} \tag{11.10}$$

from which we deduce that

$$\int_X \gamma \nabla v \cdot \nabla \varphi = \int_{\partial X} g \varphi - \int_\Sigma (G_\Sigma g)[\varphi]. \tag{11.11}$$

That  $G_\Sigma$  and  $G_\Sigma^*$  are in duality in the sense that

$$\int_\Sigma G_\Sigma g \phi \, d\sigma = \int_{\partial X} g G_\Sigma^* \phi \, d\sigma, \quad (11.12)$$

follows from (11.11) with  $\varphi = w$  and (11.9) with  $\psi = v$  since  $[v] = 0$ .

We finally define  $F_\Sigma$  as the operator mapping  $\phi \in H_0^{\frac{1}{2}}(\Sigma)$  to  $F_\Sigma \phi = -\nu \cdot \gamma \nabla w \in H_0^{-\frac{1}{2}}(\Sigma)$ , where  $w$  is the solution to (11.8). Based on the above results, this is a well-posed operator. Moreover, we deduce from (11.9) that

$$(F_\Sigma[w], [\psi])_\Sigma = \int_X \gamma \nabla w \cdot \nabla \psi \, dx = ([w], F_\Sigma[\psi])_\Sigma, \quad (11.13)$$

so that  $F_\Sigma = F_\Sigma^*$ . Upon choosing  $[w] = [\psi]$ , we find that  $F_\Sigma$  is coercive on  $H_0^{\frac{1}{2}}(\Sigma)$ . This implies among other things that  $F_\Sigma$  is injective.

We now notice that

$$G_\Sigma^* = L^* F_\Sigma.$$

This follows from the uniqueness of the solution to the elliptic problem on  $D^c$  with conditions defined on  $\partial D^c = \Sigma \cup \partial X$ . By duality, this also implies that  $G_\Sigma = F_\Sigma L$ . The operators  $G_0$  and  $F_0$  are defined similarly except that  $\gamma$  is replaced by  $\gamma_0$  in (11.8). Let us finally define the operator  $M$ , which maps  $g \in H_0^{-\frac{1}{2}}(\partial X)$  to  $u|_{\partial X} \in H_0^{\frac{1}{2}}(\partial X)$ , where  $u$  is the solution to

$$\begin{aligned} \nabla \cdot \gamma \nabla u &= 0, & \text{in } D^c \\ \nu \cdot \gamma \nabla u &= 0, & \text{on } \Sigma \\ \nu \cdot \gamma \nabla u &= g, & \text{on } \partial X \\ \int_{\partial X} u \, d\sigma &= 0. \end{aligned} \quad (11.14)$$

Except for the normalization, the operator  $M$  is the same as the operator  $L$  (so that  $L - M$  is proportional to identity) and is thus well-posed. We now verify from the linearity of the elliptic problems that

$$\Lambda_\Sigma = M - L^* G_\Sigma = M - L^* F_\Sigma L, \quad \Lambda_0 = M - L^* G_0 = M - L^* F_0 L. \quad (11.15)$$

We thus deduce the main factorization result of this section, namely that

$$\Lambda_0 - \Lambda_\Sigma = L^* F L, \quad F = F_\Sigma - F_0. \quad (11.16)$$

The above result would not be very useful if  $F$  did not have specific properties. We now show that  $F$  or  $-F$  generates a coercive form on  $H_0^{\frac{1}{2}}(\Sigma)$  and may be written as  $B^* B$  for some *surjective* operator  $B^*$ . Note that  $F^* = F$  since both  $F_\Sigma$  and  $F_0$  are self-adjoint.

We denote by  $w_\Sigma$  the solution  $w$  to (11.8) and by  $w_0$  the solution to the same equation with  $\gamma$  replaced by  $\gamma_0$ . Upon multiplying the equation for  $w_\Sigma$  by  $w_0$  and subtracting the equation for  $w_0$  multiplied by  $w_\Sigma$ , we obtain since  $\gamma = \gamma_0$  on  $D^c$  that

$$\begin{aligned} \int_D (\gamma - \gamma_0) \nabla w_0 \cdot \nabla w_\Sigma \, dx &= \int_\Sigma (\nu \cdot \gamma \nabla w_\Sigma w_0^- - \nu \cdot \gamma \nabla w_0 w_\Sigma^-) \, d\sigma \\ 0 &= \int_\Sigma (\nu \cdot \gamma \nabla w_\Sigma w_0^+ - \nu \cdot \gamma \nabla w_0 w_\Sigma^+) \, d\sigma. \end{aligned}$$

Notice that both  $\gamma$  and  $\nabla w_\Sigma$  jump across  $\Sigma$  but that  $\nu \cdot \gamma \nabla w_\Sigma$  does not. This yields that

$$\int_D (\gamma - \gamma_0) \nabla w_0 \cdot \nabla w_\Sigma dx = \int_\Sigma (F_\Sigma - F_0) \phi \phi d\sigma = \int_\Sigma F \phi \phi d\sigma. \quad (11.17)$$

Let us now introduce  $\delta w = w_0 - w_\Sigma$ . Upon multiplying  $\nabla \cdot \gamma_0 \nabla \delta w + \nabla \cdot (\gamma_0 - \gamma) \nabla w_\Sigma = 0$  by  $\delta w$  and integrating by parts on  $D^c$  and  $D$  we deduce that

$$\int_X \gamma_0 \nabla \delta w \cdot \nabla \delta w dx + \int_D (\gamma - \gamma_0) \nabla w_\Sigma \cdot \nabla w_\Sigma dx = \int_D (\gamma - \gamma_0) \nabla w_0 \cdot \nabla w_\Sigma dx.$$

By exchanging the roles of the indices  $\Sigma$  and 0 we also obtain

$$\int_X \gamma \nabla \delta w \cdot \nabla \delta w dx + \int_D (\gamma_0 - \gamma) \nabla w_0 \cdot \nabla w_0 dx = \int_D (\gamma_0 - \gamma) \nabla w_0 \cdot \nabla w_\Sigma dx.$$

Combining these results with (11.17) we deduce that

$$\begin{aligned} \int_\Sigma F \phi \phi d\sigma &= \int_X \gamma_0 \nabla \delta w \cdot \nabla \delta w dx + \int_D (\gamma - \gamma_0) \nabla w_\Sigma \cdot \nabla w_\Sigma dx \\ \int_\Sigma -F \phi \phi d\sigma &= \int_X \gamma \nabla \delta w \cdot \nabla \delta w dx + \int_D (\gamma_0 - \gamma) \nabla w_0 \cdot \nabla w_0 dx. \end{aligned} \quad (11.18)$$

Let us assume that (11.4) holds. Then  $F$  generates a coercive form on  $H_0^{\frac{1}{2}}(\Sigma)$ . Indeed, let us assume that the right-hand side of the first equality above is bounded. Then by a Poincaré-type inequality, we have  $\delta w \in H^1(X)$  and  $w_\Sigma|_D \in H^1(D)$  thanks to (11.4). This implies that  $(\nu \cdot \gamma \nabla w_\Sigma)|_\Sigma \in H^{-\frac{1}{2}}(\Sigma)$  and thus based on (11.8) that  $w_\Sigma|_{D^c} \in H^1(D^c)$ . This in turn implies that both  $w_\Sigma^+$  and  $w_\Sigma^-$  belong to  $H^{\frac{1}{2}}(\Sigma)$  so that their difference  $\phi \in H_0^{\frac{1}{2}}(\Sigma)$ . Thus, we have shown the existence of a positive constant  $C$  such that

$$\|\phi\|_{H_0^{\frac{1}{2}}(\Sigma)} \leq C(F\phi, \phi)_\Sigma^{\frac{1}{2}}. \quad (11.19)$$

Exchanging the indices  $\Sigma$  and 0 also yields the existence of a constant  $C$  under hypothesis (11.5) such that

$$\|\phi\|_{H_0^{\frac{1}{2}}(\Sigma)} \leq C(-F\phi, \phi)_\Sigma^{\frac{1}{2}}. \quad (11.20)$$

In what follows, we assume that (11.4) and (11.19) hold to fix notation. The final results are not modified when (11.5) and (11.20) hold instead.

The operator  $F$  is defined from  $H_0^{\frac{1}{2}}(\Sigma)$  to  $H_0^{-\frac{1}{2}}(\Sigma)$ , which have not been identified yet. So writing  $F = B^*B$  requires a little bit of work. Let  $\mathcal{I}$  be the canonical isomorphism between  $H_0^{-\frac{1}{2}}(\Sigma)$  and  $H_0^{\frac{1}{2}}(\Sigma)$ . Since it is positive definite, we can decompose it as

$$\mathcal{I} = \mathcal{J}^* \mathcal{J}, \quad \mathcal{J} : H_0^{-\frac{1}{2}}(\Sigma) \rightarrow L_0^2(\Sigma), \quad \mathcal{J}^* : L_0^2(\Sigma) \rightarrow H_0^{\frac{1}{2}}(\Sigma).$$

Both  $\mathcal{J}$  and  $\mathcal{J}^*$  are isometries as defined above. We can thus recast the coercivity of  $F$  as

$$(F\phi, \phi) = (F\mathcal{J}^*u, \mathcal{J}^*u) = (\mathcal{J}F\mathcal{J}^*u, u) \geq \alpha \|\phi\|_{H_0^{\frac{1}{2}}(\Sigma)}^2 = \alpha \|u\|_{L_0^2(\Sigma)}^2.$$

So  $\mathcal{J}F\mathcal{J}^*$  as a self-adjoint positive definite operator on  $L^2(\Sigma)$  can be written as  $C^*C$ , where  $C$  and  $C^*$  are bounded operators from  $L_0^2(\Sigma)$  to  $L_0^2(\Sigma)$ . Since

$$\|Cu\|_{L_0^2(\Sigma)}^2 \geq \alpha \|u\|_{L_0^2(\Sigma)}^2,$$

we deduce that  $C^*$  is surjective. We thus obtain that  $F = B^*B$  where  $B = C(\mathcal{J}^*)^{-1}$  maps  $H_0^{\frac{1}{2}}(\Sigma)$  to  $L_0^2(\Sigma)$  and its adjoint operator  $B^* = \mathcal{J}^{-1}C^*$  maps  $L_0^2(\Sigma)$  to  $H_0^{\frac{1}{2}}(\Sigma)$ . Since  $\mathcal{J}$  is an isomorphism, we deduce that  $B^*$  is surjective.

From the above calculations we obtain that

$$\Lambda_0 - \Lambda_\Sigma = L^*FL = L^*B^*(L^*B^*)^* = A^*A, \quad A = BL.$$

Since the Range of  $(A^*A)^{\frac{1}{2}}$  for  $A$  acting on Hilbert spaces is equal to the Range of  $A^*$ , we deduce that

$$\mathcal{R}((\Lambda_0 - \Lambda_\Sigma)^{\frac{1}{2}}) = \mathcal{R}(L^*B^*) = \mathcal{R}(L^*) \quad (11.21)$$

since  $B^*$  is surjective. The latter is shown as follows. We always have that  $\mathcal{R}(L^*B^*) \subset \mathcal{R}(L^*)$ . Now for  $y \in \mathcal{R}(L^*)$  there is  $x$  such that  $y = L^*x$  and since  $B^*$  is surjective  $u$  such that  $y = L^*B^*x$  so that  $y \in \mathcal{R}(L^*B^*)$ ; whence  $\mathcal{R}(L^*) \subset \mathcal{R}(L^*B^*)$ .

When (11.5) and (11.20) hold instead of (11.4) and (11.19), we deduce that

$$\mathcal{R}((\Lambda_\Sigma - \Lambda_0)^{\frac{1}{2}}) = \mathcal{R}(L^*), \quad (11.22)$$

instead of (11.21). In both cases, we have shown the following result.

**Theorem 11.1.1** *Provided that (11.4) or (11.5) holds, the range of the operator  $L^*$  defined in (11.7) is determined by the Neumann-to-Dirichlet operator  $\Lambda_\Sigma$  and (11.22) holds.*

### 11.1.3 Reconstruction of $\Sigma$

The above theorem gives us a method to reconstruct  $\Sigma$ : we need to find a family of functions that belong to the Range of  $L^*$  when some probe covers  $D$  and do not belong to it when the probe covers  $D^c$ . Notice that the operator  $L^*$  does not depend on the domain  $D$  and thus depends only on the tensor  $\gamma_0$  and the surface  $\Sigma$ . Consequently, the reconstruction of  $\Sigma$  is independent of  $\gamma$  on  $D$ , except for the existence of a positive definite tensor  $\alpha_0$  such that (11.4) or (11.5) holds.

Let us now introduce the family of functions  $N(\cdot; y)$  indexed by the parameter  $y \in X$  solution of

$$\begin{aligned} \nabla \cdot \gamma_0 \nabla N(\cdot; y) &= \delta(\cdot - y), & \text{in } X \\ \nu \cdot \gamma_0 \nabla N(\cdot; y) &= 0 & \text{on } \partial X \\ \int_{\partial X} N(\cdot; y) d\sigma &= 0. \end{aligned} \quad (11.23)$$

We define the family of functions  $g_y(x) = N(x; y)|_{\partial X}$  on  $\partial X$ . Then we have the following result:

**Theorem 11.1.2** *The function  $g_y(x)$  belongs to  $\mathcal{R}(L^*)$  when  $y \in D$  and does not belong to  $\mathcal{R}(L^*)$  when  $y \in D^c$ .*

This theorem provides us with a constructive method to image  $\Sigma = \partial D$ . For each  $y \in X$ , all we have to do is to solve (11.23) and verify whether the trace on  $\partial X$  belongs to the Range of  $(\pm(\Lambda_0 - \Lambda_\Sigma))^{\frac{1}{2}}$ , which can be evaluated from the known boundary measurements. Only when the verification is positive do we deduce that  $y \in D$ .

*Proof.* The proof of the theorem is as follows. When  $y \in D$ , we have that  $\nu \cdot \gamma \nabla N(x; y)|_\Sigma \in H_0^{-\frac{1}{2}}(\Sigma)$  and  $\nabla \cdot \gamma_0 \nabla N(\cdot; y) = 0$  on  $D^c$  so that  $g_y \in \mathcal{R}(L^*)$ . Let us now assume that  $y \in D^c$  and  $g_y(x) \in \mathcal{R}(L^*)$ . Then there exists  $\phi \in H_0^{-\frac{1}{2}}(\Sigma)$  such that  $g_y = L^* \phi = w|_{\partial X}$ , where  $w$  is the solution to (11.7). Let  $B(y; \varepsilon)$  be the ball of radius  $\varepsilon$  centered at  $y$  for  $\varepsilon$  sufficiently small. On  $D^c \setminus \overline{B_\varepsilon}$ , both  $w$  and  $g_y$  satisfy the same equation. By uniqueness of the solution to the Cauchy problem imposing Dirichlet data and vanishing Neumann data on  $\partial X$ , we deduce that  $w = g_y$  on  $D^c \setminus \overline{B_\varepsilon}$ . On  $\omega_\varepsilon = B_{\varepsilon_0} \setminus \overline{B_\varepsilon}$  for some fixed  $\varepsilon_0 > 0$ , we verify that the  $H^1(\omega_\varepsilon)$  norm of  $w$  remains bounded independently of  $\varepsilon$ , which is not the case for the fundamental solution  $g_y$ ; whence the contradiction implying that  $g_y$  is not in the Range of  $L^*$  when  $y \in D^c$ .  $\square$

The factorization method is one example in the class of so-called qualitative reconstruction methodologies. The interested reader is referred to [26, 41, 51] for additional information about such methods.

## 11.2 Reconstructing small inclusions

This second section concerns the reconstruction of small inclusions. We have seen that the reconstruction of diffusion or absorption coefficients in an elliptic equation resulted in a severely ill-posed problem. The previous section dealt with the issue by reconstructing the support of an inclusion instead of its detailed structure. Because the support of the inclusion remains an infinite dimensional object, the stability of the reconstruction is still a severely ill-posed problem. Here we further simplify the problem by assuming that the inclusions have small support. This introduces a small parameter allowing us to perform asymptotic expansions. We can then characterize the influence of the inclusion on the boundary measurements by successive terms in the expansion. The interest of such a procedure is the following. Since high-order terms in the expansion become asymptotically negligible, the procedure tells us which parameters can be reconstruction from a given noise level in the measurements and which parameters cannot possibly be estimated. Moreover, each term in the asymptotic expansion is characterized by a finite number of parameters. This implies that by truncating the expansion, we are parameterizing the reconstruction with a finite number of parameters. Unlike the previous reconstructions, this becomes a well-posed problem since the ill-posedness comes from the infinite dimensionality of the parameters we want to reconstruct, at least as long as the mapping from the object to be reconstructed to the noise-free measurements is one-to one (injective).

We consider in this chapter a mathematically very simple problem, namely the reconstruction of inclusions characterized by a variation in the absorption coefficient. We also restrict ourselves to the reconstruction from the leading term in the aforementioned asymptotic expansion. The interesting problem of variations of the diffusion coefficient is mathematically more difficult, although the main conclusions are in the end very similar. The presentation follows that in [8].

### 11.2.1 First-order effects

Let us consider the problem of optical tomography modeled by a diffusion equation on a domain  $X$  with current density  $g(x)$  prescribed at the boundary  $\partial X$ . We assume here that the diffusion coefficient is known and to simplify, is set to  $\gamma \equiv 1$ . Our main hypothesis on the absorption coefficient is that it is the superposition of a background absorption, to simplify the constant  $\sigma_0$ , and a finite number of fluctuations of arbitrary size  $\sigma_m - \sigma_0$ , with  $\sigma_m$  constant to simplify, but of small volume. Smallness of the volume of the inclusions compared to the volume of the whole domain is characterized by the small parameter  $\varepsilon \ll 1$ . The diffusion equation with small absorption inclusions then takes the form

$$\begin{aligned} -\Delta u_\varepsilon(x) + \sigma_\varepsilon(x)u_\varepsilon(x) &= 0, & X \\ \frac{\partial u_\varepsilon}{\partial \nu} &= g, & \partial X, \end{aligned} \quad (11.24)$$

where absorption is given by

$$\sigma_\varepsilon(x) = \sigma_0 + \sum_{m=1}^M \sigma_m \chi_{z_m + \varepsilon B_m}(x). \quad (11.25)$$

We have introduced here  $\varepsilon B_m$  as the shape of the  $m$ th inclusion centered at  $z_m$ , and  $\chi_{z_m + \varepsilon B_m}(x) = 1$  if  $x - z_m \in \varepsilon B_m$  and 0 otherwise. The inclusions are centered at *zero* in the sense that

$$\int_{B_m} x dx = 0 \quad \text{for all } m, \quad (11.26)$$

and are assumed to be at a distance greater than  $d > 0$ , independent of  $\varepsilon$ , of each-other and of the boundary  $\partial X$ . The parameter  $\varepsilon$  is a measure of the diameter of the inclusions. In the three-dimensional setting, which we assume from now on, this implies that the volume of the inclusions is of order  $\varepsilon^3$ .

We want to derive an asymptotic expansion for  $u_\varepsilon$  in powers of  $\varepsilon$  and observe which information about the inclusions we can deduce from the first terms in the expansion. Let us first define the Green function of the corresponding homogeneous problem

$$\begin{aligned} -\Delta G(x; z) + \sigma_0 G(x; z) &= \delta(x - z), & X \\ \frac{\partial G}{\partial \nu}(x; z) &= 0, & \partial X, \end{aligned} \quad (11.27)$$

and the homogeneous-domain solution  $U(x)$  of

$$\begin{aligned} -\Delta U(x) + \sigma_0 U(x) &= 0, & X \\ \frac{\partial U}{\partial \nu}(x) &= g(x), & \partial X. \end{aligned} \quad (11.28)$$

As  $\varepsilon \rightarrow 0$ , the volume of the inclusions tends to 0 and  $u_\varepsilon$  converges to  $U$ . To show this, we multiply (11.27) by  $u_\varepsilon$  and integrate by parts to obtain

$$u_\varepsilon(z) = \int_{\partial X} g(x)G(x; z)d\sigma(x) - \sum_{m=1}^M \int_{z_m + \varepsilon B_m} \sigma_m G(x; z)u_\varepsilon(x)dx.$$

Using the same procedure for  $U(x)$ , we obtain

$$u_\varepsilon(z) = U(z) - \sum_{m=1}^M \int_{z_m + \varepsilon B_m} \sigma_m G(x; z) u_\varepsilon(x) dx. \quad (11.29)$$

In three space dimensions, the Green function is given by

$$G(x; z) = \frac{e^{-\sqrt{\sigma_0}|z-x|}}{4\pi|z-x|} + w(x; z), \quad (11.30)$$

where  $w(x; z)$  is a smooth function (because it solves (11.28) with smooth boundary conditions) provided that  $\partial X$  is smooth. For  $z$  at a distance greater than  $d > 0$  away from the inclusions  $x_m + \varepsilon B_m$ , we then deduce from the  $L^\infty$  bound on  $u_\varepsilon$  (because  $g$  and  $\partial X$  are assumed to be sufficiently regular) that

$$u_\varepsilon(z) = U(z) + O(\varepsilon^3).$$

In the vicinity of the inclusions, we deduce from the relation

$$\int_{z_m + \varepsilon B_m} G(x; z) dx = O(\varepsilon^2), \quad z - z_m \in \varepsilon B_m,$$

that  $u_\varepsilon(z) - U(z)$  is of order  $\varepsilon^2$  when  $z$  is sufficiently close to an inclusion. This also shows that the operator

$$K_\varepsilon u_\varepsilon(z) = - \sum_{m=1}^M \int_{z_m + \varepsilon B_m} \sigma_m G(x; z) u_\varepsilon(x) dx \quad (11.31)$$

is a bounded linear operator in  $\mathcal{L}(L^\infty(X))$  with a norm of order  $\varepsilon^2$ . This implies that for sufficiently small values of  $\varepsilon$ , we can write

$$u_\varepsilon(z) = \sum_{k=0}^{\infty} K_\varepsilon^k U(z). \quad (11.32)$$

The above series converges fast when  $\varepsilon$  is small. Notice however that the series does not converge as fast as  $\varepsilon^3$ , the volume of the inclusions, because of the singular behavior of the Green function  $G(x; z)$  when  $x$  is close to  $z$ .

Let us now use that

$$\begin{aligned} u_\varepsilon(z) &= U(z) - \sum_{m=1}^M \int_{z_m + \varepsilon B_m} \sigma_m G(x; z) U(x) dx \\ &+ \sum_{m=1}^M \sum_{n=1}^M \int_{z_m + \varepsilon B_m} \int_{z_n + \varepsilon B_n} \sigma_m \sigma_n G(x; z) G(y; x) u_\varepsilon(y) dy dx. \end{aligned} \quad (11.33)$$

For the same reasons as above, the last term is of order  $\varepsilon^5$ , and expanding smooth solutions  $U(x)$  and  $G(x; z)$  inside inclusions of diameter  $\varepsilon$ , we obtain that

$$u_\varepsilon(x) = U(x) - \sum_{m=1}^M G(z; z_m) C_m U(z_m) + O(\varepsilon^5), \quad (11.34)$$

where  $C_m$  is given by

$$C_m = \varepsilon^3 |B_m| \sigma_m. \quad (11.35)$$

The reason why we obtain a correction term of order  $\varepsilon^5$  in (11.34) comes from the fact that (11.26) holds so that the terms of order  $\varepsilon^4$ , proportional to  $x \cdot \nabla U$  or  $x \cdot \nabla G$ , vanish.

## 11.2.2 Stability of the reconstruction

The above analysis tells us the following. Provided that our measurement errors are of order  $O(\varepsilon^5)$ , the only information that can possibly be retrieved on the inclusions is its location  $z_m$  and the product  $C_m = \varepsilon^3 \sigma_m B_m$  of the absorption fluctuation with the volume of the inclusion. More refined information requires data with less noise. Assuming that the inclusions are sufficiently small so that the above asymptotic expansion makes sense, no other information can be obtained in a stable fashion from the data.

Note that the problem we now wish to solve is finite-dimensional. Indeed, each inclusion is represented by four real numbers, namely the three components of the position  $z_m$  and the product  $C_m$ . Assuming that only  $M$  inclusions are present, this leaves us with  $4M$  parameters to reconstruct. The main advantage of reconstructing a finite number of parameters is that it is natural to expect stability of the reconstruction. We can even show stability of the reconstruction from boundary measurements corresponding to one current density  $g(x)$  provided that the homogeneous solution  $U(x)$  is uniformly positive inside the domain. Here is how it can be proved.

Let us assume that the boundary measurements have an accuracy of order  $O(\varepsilon^5)$ , which is consistent with

$$u_\varepsilon(z) = U(z) - \sum_{m=1}^M C_m (G(z_m; z) U(z_m)) + O(\varepsilon^5). \quad (11.36)$$

We denote by  $u_\varepsilon$  and  $u'_\varepsilon$  the solution of two problems with absorption coefficients  $\sigma_\varepsilon$  and  $\sigma'_\varepsilon$  of the form (11.25). Using (11.36), we obtain that

$$u_\varepsilon(z) - u'_\varepsilon(z) = F(z) + O(\varepsilon^5),$$

with

$$F(z) = - \sum_{m=1}^M \left( C_m (G(z_m; z) U(z_m)) - C'_m (G(z'_m; z) U(z'_m)) \right). \quad (11.37)$$

Here we use  $M = \max(M, M')$  with a small abuse of notation; we will see shortly that  $M = M'$ . The function  $F(z)$  satisfies the homogeneous equation  $-\Delta F + \sigma_0 F = 0$  on  $X$  except at the points  $z_m$  and  $z'_m$ . Moreover, we have that  $\frac{\partial F}{\partial \nu} = 0$  at  $\partial X$ . If  $F = 0$  on  $\partial X$ , we deduce from the uniqueness of the Cauchy problem for the operator  $-\Delta + \sigma_0$  that  $F \equiv 0$  in  $X$ . As  $\varepsilon \rightarrow 0$  and  $u_\varepsilon - u'_\varepsilon \rightarrow 0$ , we deduce that  $F(z)$  becomes small not only at  $\partial X$  but also inside  $X$  (the continuation of  $F$  from  $\partial X$  to  $X \setminus \{z_m \cup z'_m\}$  is independent of  $\varepsilon$ ). However, the functions  $G(z_m; z) U(z_m)$  form an independent family. Each term must therefore be compensated by a term from the sum over the *prime* coefficients. We thus obtain that  $M = M'$  and that

$$\left| C_m (G(z_m; z) U(z_m)) - C'_m (G(z'_m; z) U(z'_m)) \right| \leq C \|u_\varepsilon - u'_\varepsilon\|_{L^\infty(\partial X)} + O(\varepsilon^5).$$

The left-hand side can be recast as

$$(C_m - C'_m) G(z_m; z) U(z_m) + C'_m (z_m - z'_m) \partial_{z_m} (G(\bar{z}_m; z) U(\bar{z}_m))$$

where  $\bar{z}_m = \theta z_m + (1 - \theta) z'_m$  for some  $\theta \in (0, 1)$ . Again these two functions are linearly independent and so we deduce that

$$|C_m - C'_m| + |C'_m| |z_m - z'_m| \leq C \|u_\varepsilon - u'_\varepsilon\|_{L^\infty(\partial X)} + O(\varepsilon^5).$$

Using (11.34) and (11.35), we then obtain assuming that  $\|u_\varepsilon - u'_\varepsilon\|_{L^\infty(\partial X)} \approx \varepsilon^5$ , that

$$|B_m \sigma_m - B'_m \sigma'_m| + |z_m - z'_m| \leq C \varepsilon^{-3} \|u_\varepsilon - u'_\varepsilon\|_{L^\infty(\partial X)} \approx \varepsilon^2. \quad (11.38)$$

Assuming that the accuracy of the measured data is compatible with the expansion (11.34), i.e. that the  $u_\varepsilon$  is known on  $\partial X$  up to an error term of order  $\varepsilon^5$ , we can then reconstruct the location  $z_m$  of the heterogeneities up to an error of order  $\varepsilon^2$ . The product of the volume of the inclusion and the absorption fluctuation is also known with the same accuracy.

The analysis of small volume expansions has been analyzed in much more general settings than the one presented above. We refer the reader to [5] for additional details. In the context of the reconstruction of absorbing inclusions as was considered above, the expression of order  $O(\varepsilon^5)$  and the additional information it provides about inclusions may be found in [8].

# Chapter 12

## Inverse Problems and Modeling

We conclude these notes by a chapter devoted to the interplay between modeling and inverse problems. We have described an inverse problem as the triplet (Measurement Operator, Noise Model, Prior information). In such a setting, the MO is given as our best guess for the operator that maps parameters to measurements. The noise structure is also given in the model, and depending on the potential effect of noise on reconstructions. Priors were considered to mitigate its influence.

In some situations, *modeling* can help to come up with a triplet (Measurement Operator, Noise Model, Prior information) that can greatly simplify the analysis and numerical simulation of an inverse problem. We consider two such situations, one in which modeling helps to derive a MO adapted to a particular situation, and one in which asymptotic expansions may help to come up with simple yet efficient noise models. Modeling may also assist in the choice of Priors. For instance, if the parameter of interest is an image, then one might envision situations in which prior probabilities on images are assigned based on how likely a given image is the transformation of a known reference image for a transformation satisfying prescribed, physically motivated, constraints. We do not explore this aspect here.

As an example of modeling of the MO, consider the reconstruction of objects buried in a highly heterogeneous medium, which means a medium with clutter present at very small scales compared to the overall size of the physical domain of interest. We assume that the medium is probed by electromagnetic waves, say, which as usual in this text are modeled by a scalar wave equation. The space of parameters is therefore the buried object *and* the heterogeneous medium. The latter being highly heterogeneous requires many more parameters than we can afford to consider based on available data. Besides, we are not interested in the heterogeneous medium. We want to use modeling to simplify the inverse problem so that the set of (unknown) parameters is restricted to the buried object plus a number of macroscopic parameters describing the heterogeneous medium that we wish to be as limited as possible. How this might be done is considered in section [12.1](#).

As a second example of the interplay between modeling and inverse problems, let us consider a situation with an inverse problem in which the influence of the high frequencies of the parameters on the measurements becomes negligible compared to the amount of noise. Such high frequencies then cannot be reconstructed. Yet, when the inverse problem is nonlinear, they may very well influence the low frequency compo-

ment of the available data and hence influence the reconstruction of the low frequency component of the parameters. Since said high frequencies cannot be reconstructed, we might as well model them as random. How such a randomness propagates to the available measurements is a modeling exercise that is sometimes tractable analytically. The benefit of such a calculation is a physics-based model for the Noise Model associated to a Measurement Operator focusing on the low frequency component of the parameters. We consider such a setting in section 12.2.

## 12.1 Imaging in Highly Heterogeneous Media

### 12.1.1 Wave model

Consider the propagation of acoustic waves in a heterogeneous medium modeled by the following wave equation:

$$\begin{aligned} \frac{\partial^2 p_\varepsilon}{\partial t^2} - c_\varepsilon^2(x) \Delta p_\varepsilon &= 0, & t > 0, x \in \mathbb{R}^n, \\ p_\varepsilon(0, x) &= p_{\varepsilon 0}(x), \quad \varepsilon \frac{\partial p_\varepsilon}{\partial t}(0, x) = j_{\varepsilon 0}(x), & x \in \mathbb{R}^n, \end{aligned} \quad (12.1)$$

where  $p_\varepsilon(t, x)$  is the pressure potential and  $c_\varepsilon(x)$  is the sound speed.

The parameter  $\varepsilon$  is defined as the ratio of the typical wavelength  $\lambda$  of the probing wave field with the overall distance  $L$  of the domain over which propagation is observed. This is modeled by assuming that e.g.  $p_{\varepsilon 0}(x) = \phi_p(x) \psi_p(\frac{x}{\varepsilon})$  and  $j_{\varepsilon 0}(x) = \phi_j(x) \psi_j(\frac{x}{\varepsilon})$  for some smooth functions  $\phi_{p,j}(x)$  and  $\psi_{p,j}(x)$ .

We are interested in the regime where  $\varepsilon \ll 1$ . The sound speed  $c_\varepsilon(x)$  models both the buried inclusion (for instance  $c$  jumps across an interface; or  $c$  becomes constant inside the inclusion) and the heterogeneous surrounding.

Let us assume that the sound speed  $c_\varepsilon(x)$  takes the form

$$c_\varepsilon^2(x) = c_0^2(x) - \sqrt{\varepsilon} V(x, \frac{x}{\varepsilon}), \quad (12.2)$$

where  $c_0(x)$  is the deterministic background speed and  $V(x, \frac{x}{\varepsilon})$  denotes the rapid oscillations, also at the scale  $\varepsilon$ . This is the so-called weak coupling regime of wave propagation.

The reconstruction of the highly oscillatory  $c_\varepsilon(x)$  from boundary measurements is a formidable task. The objective of the modeling effort is to replace this difficult inverse problem by a simpler one in which the parameters would be the deterministic background fluctuations  $c_0(x)$ , which could model the buried inclusions as well, and a few deterministic parameters modeling the statistics of  $V(x, \frac{x}{\varepsilon})$ , which would be modeled as a random field since we no longer aim to reconstruct it.

We thus assume that  $V(x, y)$  is a random field  $V(x, y; \omega)$ , where  $\omega \in \Omega$ , the state space of possible realizations of the heterogeneous medium constructed on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . We assume that for each fixed  $x$ ,  $V(x, y; \omega)$  is a stationary, mean-zero, random field, with a correlation function defined by

$$R(x, y) = \mathbb{E}\{V(x, z)V(x, z + y)\}, \quad (12.3)$$

where  $\mathbb{E}$  is mathematical expectation with respect to  $\mathbb{P}$  and where  $R$  is independent of  $z$  by stationarity of  $V$ . The correlation function  $R(x, y)$  is the only macroscopic,

fluctuation-dependent parameter that will appear in the macroscopic models we consider below.

### 12.1.2 Kinetic Model

The mathematical modeling we wish to perform consists of assuming that  $\varepsilon \rightarrow 0$  and observing whether a macroscopic, hopefully  $\varepsilon$ -independent, model emerges in that limit. It turns out that  $p_\varepsilon$  itself is not the right quantity to consider. As  $\varepsilon \rightarrow 0$ , wave fields oscillate rapidly and the (weak) limit of  $p_\varepsilon$  does not capture how the wave *energy* propagates.

The local energy density of the waves is given by

$$\mathcal{E}_\varepsilon(t, x) = \frac{1}{2} \left( \frac{1}{c_\varepsilon^2(x)} \left( \varepsilon \frac{\partial p_\varepsilon}{\partial t} \right)^2(t, x) + |u_\varepsilon|^2(t, x) \right), \quad (12.4)$$

where  $\varepsilon \frac{\partial p_\varepsilon}{\partial t}$  is the pressure field and  $u_\varepsilon = -\varepsilon \nabla_x p_\varepsilon$  is the acoustic velocity field. Up to multiplication by a constant density  $\rho_0$ , the first term above is the potential energy and the second term the kinetic energy. The total energy is an invariant of the dynamics so that

$$\mathcal{E}(t) = \int_{\mathbb{R}^n} \mathcal{E}_\varepsilon(t, x) dx = \mathcal{E}(0) = \int_{\mathbb{R}^n} \frac{1}{2} \left( \frac{1}{c_\varepsilon^2(x)} |j_{0\varepsilon}|^2(x) + |\varepsilon \nabla_x p_{0\varepsilon}|^2(x) \right) dx. \quad (12.5)$$

All quantities are rescaled so that  $\mathcal{E}(t)$  is of order  $O(1)$  independent of  $\varepsilon$ .

The local energy density  $\mathcal{E}_\varepsilon(t, x)$  does not satisfy a closed-form equation. As in the classical mechanical description of particles, position needs to be augmented with a momentum variable to obtain closed-form equations such as Newton's laws for instance. The same behavior occurs here and the energy density admit a representation in the phase space of positions and momenta:

$$\lim_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(t, x) = \int_{\mathbb{R}^n} a(t, x, k) dk, \quad (12.6)$$

where  $a(t, x, k)$  is a phase-space energy density for each  $t \geq 0$ .

In the absence of random fluctuations, i.e., when  $V \equiv 0$ , then  $a$  satisfies the same Liouville equation as in Chapter 5:

$$\frac{\partial a}{\partial t} + \nabla_k H \cdot \nabla_x a - \nabla_x H \cdot \nabla_k a = 0, \quad (12.7)$$

where  $H(x, k) = c_0(x)|k|$  is the Hamiltonian of the dynamics.

In the presence of random fluctuations,  $V \neq 0$ , the wave packets interact with the underlying structure and scatter. The Liouville equation (12.7) no longer holds and propagation in a homogeneous medium is no longer an accurate model. Rather, we obtain the following radiative transfer model

$$\frac{\partial a}{\partial t} + \nabla_k H \cdot \nabla_x a - \nabla_x H \cdot \nabla_k a = \int_{\mathbb{R}^n} \sigma(x, k, q) (a(t, x, q) - a(t, x, k)) \delta(H(x, q) - H(x, k)) dk, \quad (12.8)$$

where the scattering coefficient is given by:

$$\sigma(x, k, q) = \frac{\pi H^2(x, k)}{2(2\pi)^n} \hat{R}(x, k - q). \quad (12.9)$$

Here,  $\hat{R}(x, k)$  is the Fourier transform of the correlation  $R(x, z)$  with respect to the second variable only.

The derivation of kinetic models such as (12.8) is a lengthy process that is understood only in specific situations. We refer the reader to the review article [15] for an extensive list of works related to such derivations.

### 12.1.3 Statistical Stability

We observe that the above kinetic model is *deterministic* even though the description of the scattering medium through  $V(x, y)$  was random. It remains, however, to understand in which sense we have convergence in (12.6). Does the whole random variable  $\mathcal{E}_\varepsilon(t, x)$  converge to a deterministic limit, or does the ensemble average  $\mathbb{E}\{E_\varepsilon(t, x)\}$  converge to that limit? The latter would not be very useful in the reconstruction of the profile  $c_0(x)$ , which may for instance model a buried inclusion. Indeed, available measurements of  $p_\varepsilon$ , and hence of  $\mathcal{E}_\varepsilon(t, x)$ , are available for only “one” realization of the random medium  $V(x, y; \omega)$ . Taking ensemble averages is often not possible practically.

It turns out that the whole random field  $\mathcal{E}_\varepsilon(t, x)$  converges in an appropriate sense to its deterministic limit  $\int_{\mathbb{R}^n} a(t, x, k) dk$ . To obtain a more precise statement, we need to introduce some notation. We first introduce the Wigner transform of two possibly complex-valued functions  $f$  and  $g$  at the scale  $\varepsilon$ :

$$W[f, g](x, k) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ik \cdot y} f\left(x - \frac{\varepsilon y}{2}\right) g^*\left(x + \frac{\varepsilon y}{2}\right) dy. \quad (12.10)$$

Note that the Wigner transform is nothing but the Fourier transform of the two point function  $f\left(x - \frac{\varepsilon y}{2}\right) g^*\left(x + \frac{\varepsilon y}{2}\right)$ . As a consequence, it may be seen as a decomposition over wave numbers of the correlation  $f(x)g^*(x)$ , since by integrating both sides, we obtain that

$$\int_{\mathbb{R}^n} W[f, g](x, k) dk = f(x)g^*(x). \quad (12.11)$$

Let us define

$$W_\varepsilon(t, x, k) = \frac{1}{2} \left( \frac{1}{c_\varepsilon(x)} W \left[ \varepsilon \frac{\partial p_\varepsilon}{\partial t}, \varepsilon \frac{\partial p_\varepsilon}{\partial t} \right] (x, k) + W \left[ \varepsilon \nabla_x p_\varepsilon, \varepsilon \nabla_x p_\varepsilon \right] (x, k) \right). \quad (12.12)$$

Then some calculations show that the above quantity, which we also call a Wigner transform, is a decomposition over wave numbers of the field-field correlation (12.4):

$$\mathcal{E}_\varepsilon(t, x) = \int_{\mathbb{R}^n} W_\varepsilon(t, x, k) dk.$$

**Exercise 12.1.1** Check this derivation; see [15] for details.

In other words, we are interested in which sense the limit of  $W_\varepsilon(t, x, k)$  is equal to  $a(t, x, k)$ , the solution of (12.8).

The main theoretical advantage of the Wigner transform  $W_\varepsilon(t, x, k)$  is that it satisfies a closed-form equation [9, 54]. Formal asymptotic expansions developed in [54] allow us to pass to the limit in that equation and obtain the transport equation (12.8). One way to show that the full random process  $W_\varepsilon(t, x, k)$  converges to  $a(t, x, k)$  is to look at the following scintillation function (covariance function):

$$J_\varepsilon(t, x, k, y, p) = \mathbb{E}\{W_\varepsilon(t, x, k)W_\varepsilon(t, y, p)\} - \mathbb{E}\{W_\varepsilon(t, x, k)\}\mathbb{E}\{W_\varepsilon(t, y, p)\}. \quad (12.13)$$

It has been shown in several regimes of wave propagation that  $J_\varepsilon(t, x, k, y, p)$  converges weakly to 0 as  $\varepsilon \rightarrow 0$ ; see [15]. Such results allow us to deduce that, for  $\varphi$  sufficiently smooth, we have

$$\int_{\mathbb{R}^{2n}} W_\varepsilon(t, x, k)\varphi(x, k)dxdk \xrightarrow{\text{probability}} \int_{\mathbb{R}^{2n}} a(t, x, k)\varphi(x, k)dxdk. \quad (12.14)$$

**Exercise 12.1.2** Using the Chebyshev inequality, show that for any function  $\varphi(x, k)$  and  $\delta > 0$ , we have the inequality

$$\mathbb{P}\left(|\langle W_\varepsilon(t, x, k), \varphi \rangle - \langle a(t, x, k), \varphi \rangle| \geq \delta\right) \leq \frac{1}{\delta^2} |\langle J_\varepsilon(t), \varphi \otimes \varphi \rangle|. \quad (12.15)$$

The convergence result (12.14) has one positive aspect and one negative one. Let us start with the former. In our search for a model for the buried inclusion described by  $c_0(x)$ , we have replaced an inverse wave problem for  $c_\varepsilon(x)$  with measurements of the form  $p_\varepsilon(t, x)$  by an inverse kinetic problem for  $(c_0(x), \hat{R}(x, k))$  with measurements of the form  $W_\varepsilon(t, x, k)$ , say (let us pretend for the moment that such quantities are accessible by measurements). If the measurements  $W_\varepsilon(t, x, k)$  obtained for one realization of  $V(x, y, \omega)$  were not approximately *deterministic*, then the reconstruction of  $(c_0(x), \hat{R}(x, k))$  would inevitably strongly depend on the unknown realization of  $V(x, y, \omega)$ . The reconstruction would then be statistically *unstable*.

So the positive aspect of (12.14) is that the random quantity  $W_\varepsilon(t, x, k)$  converges to  $a(t, x, k)$  *in probability*. Let  $m_\varepsilon(t)$  be the left-hand-side in (12.14) and  $m(t)$  its right-hand-side. Then we obtain that for all  $\delta > 0$ , the probability that  $|m_\varepsilon(t) - m(t)| \geq \delta$  (uniformly in  $t \in (0, T)$  in fact) goes to 0 as  $\varepsilon \rightarrow 0$ . Here, (12.14) does not describe how fast the convergence occurs.

Some estimates (sometimes optimal, sometimes not) of the rate of convergence exist in the literature [15]. What is much more damaging in practice, and forms the real negative aspect of (12.14) is that convergence holds only *after* phase-space averaging. Averaging over wavenumbers  $k$  may not be an issue since typically only  $\mathcal{E}_\varepsilon(t, x)$  is measured. Averaging over space is more constraining: the wider the spatial extent of  $\varphi$ , the lesser the spatial resolution capabilities of measurements of the form  $m_\varepsilon(t)$ .

In summary, we were successful in replacing an inverse wave problem for  $c_\varepsilon(x)$  with statistically unstable measurements of the form  $p_\varepsilon(t, x)$  by an inverse kinetic problem for  $(c_0(x), \hat{R}(x, k))$  with statistically stable measurements of the form  $W_\varepsilon(t, x, k)$ . However, statistical stability requires that the kinetic measurements be averaged over a sufficiently large area, with negative consequences for resolution.

### 12.1.4 Inverse transport problem

Let us further simplify the inverse transport problem obtained by the modeling presented above. We assume that the propagating waves are monochromatic waves with frequency  $\omega = c_0|k|$ . We also assume that the background sound speed  $c_0$  is constant except at the location of the buried inclusion(s). Furthermore, we assume that  $\hat{R}(x, k)$  is constant in  $x$  and on the sphere of radius  $|k| \leq 2\frac{\omega}{c_0}$  and equal to a constant  $\hat{R}_0$ . This implies that scattering is isotropic for the frequency  $\omega$ . With all these simplifications, and defining  $u(x, \vartheta) = a(x, |k|\vartheta)$ , we find that  $u$  solves the following transport equation

$$\vartheta \cdot \nabla_x u + \sigma_0 u = \sigma_0 \int_{\mathbb{S}^n} u(x, \vartheta') dv(\vartheta'), \quad (12.16)$$

where  $dv$  is normalized so that  $\int_{\mathbb{S}^n} dv(\vartheta') = 1$ , at least at those points  $x$  where no inclusion is present.

**Exercise 12.1.3** *Derive (12.16) from (12.8) and find the value of  $\sigma_0$ .*

We now assume that we are able to measure the energy density  $\mathcal{E}_\varepsilon(x)$  of time harmonic waves with frequency  $\omega$ . Then obviously

$$\Phi(x) := \int_{\mathbb{S}^n} u(x, \vartheta') dv(\vartheta') = \mu \mathcal{E}_\varepsilon(x) + n_\varepsilon(x) \quad (12.17)$$

is measured for  $x \in \Gamma$  for instance  $\Gamma \subset \partial X$  with  $X$  a bounded domain in  $\mathbb{R}^n$  for a scaling constant  $\mu$ .

**Exercise 12.1.4** *Find the scaling constant  $\mu$ .*

Let us assume that the inclusions are small and that their influence can be suitably linearized. Further assuming that only one inclusion of strength  $\nu$  is present at  $x = x_0$  to simplify, we thus have the model

$$\begin{aligned} \vartheta \cdot \nabla_x u + \sigma_0 u &= \sigma_0 \int_{\mathbb{S}^n} u(x, \vartheta') dv(\vartheta') + \nu \delta(x - x_0), & (x, \vartheta) \in X \times \mathbb{S}^{n-1}, \\ u(x, \theta) &= u_0(x, \theta) & (x, \vartheta) \in \Gamma_-(X), \end{aligned} \quad (12.18)$$

and we have for instance a measurement operator for a prescribed incoming boundary condition  $u_0$ :

$$\mathfrak{M} : (\sigma_0, x_0, \nu) \mapsto \mathfrak{M}(\sigma_0, x_0, \nu)(x) = \Phi(x) \in L^1(\partial X). \quad (12.19)$$

The above simplified model of a measurement operator was used to image inclusions buried in highly heterogeneous media from experimental data in [12]. The kinetic model performed remarkably well. Even inclusions  $x_0$  located behind a known blocker located between  $x_0$  and the array of detectors were reconstructed with very reasonable accuracy. Arguably such reconstructions would be very difficult to achieve with an inverse problem based on the wave model for  $p_\varepsilon$ .

The accuracy of the reconstructions of  $(\sigma_0, x_0, \nu)$  in such a model depends on the structure of the noise  $n_\varepsilon(x)$  in (12.17). The derivation of asymptotic models for  $n_\varepsilon$  is a

difficult task and very few theoretical or even formal results exist in this direction; see [15]. Note that the above derivation accurately describes  $n_\varepsilon$  as the difference between the solution of a deterministic transport equation and the energy density of waves propagating in a random medium. The statistics of  $n_\varepsilon$  can therefore a priori be computed, at least numerically, from those of  $V(x, y)$ , at least when the latter are known. This provides an accurate physics-based model for the likelihood function in the Bayesian framework considered in Chapter 10.

### 12.1.5 Random media and Correlations

Let us conclude this section by a useful generalization of the kinetic model considered in (12.8). The salient feature of the above derivation is that the phase-space *energy density*  $W_\varepsilon(t, x, k)$  is *statistically stable*. The energy density may be seen as the correlation of a wave field with itself. More generally, we can consider correlations of two wave fields, which may or may not propagate in the same medium, or may or may not be generated by the same initial conditions. It turns out that such field-field correlations are also statistically stable.

Consider for instance two wave fields  $p_{\varepsilon,k}$  for  $k = 1, 2$  propagating in two different media modeled by

$$c_{\varepsilon,k}^2(x) = c_0^2(x) - \sqrt{\varepsilon} V_k\left(x, \frac{x}{\varepsilon}\right), \quad (12.20)$$

Define then the following field-field correlation function:

$$\mathcal{C}_\varepsilon(t, x) = \frac{1}{2} \left( \frac{1}{c_{\varepsilon,1}(x)c_{\varepsilon,2}(x)} \varepsilon \frac{\partial p_{\varepsilon,1}}{\partial t} \varepsilon \frac{\partial p_{\varepsilon,2}}{\partial t}(t, x) + \varepsilon \nabla_x p_{\varepsilon,1} \cdot \varepsilon \nabla_x p_{\varepsilon,2}(t, x) \right). \quad (12.21)$$

When the two wave fields are identical, the correlation  $\mathcal{C}_\varepsilon = \mathcal{E}_\varepsilon$  satisfies the decomposition (12.6), where  $a$  solves (12.8).

We could consider  $p_{\varepsilon,2}$  as having the same initial condition as  $p_{\varepsilon,1}$  shifted in a direction  $\tau$ . Alternatively, we may consider that  $p_{\varepsilon,k}$  have the same initial condition but that  $p_{\varepsilon,2}$  propagates in a medium that is shifted by  $-\tau$  compared to the medium of  $p_{\varepsilon,1}$ . Since  $\tau \ll 1$ , this could be approximately (exactly when  $c_0$  is constant) modeled by (12.20). Let us define the following cross-correlation functions

$$R_{jk}(x, y) = \mathbb{E}\{V_j(x, z)V_k(x, z + y)\}, \quad (12.22)$$

and let  $\hat{R}_{jk}(x, k)$  be the Fourier transform of  $R_{jk}(x, y)$  with respect to the second variable only. Following [9], we obtain that (12.8) generalizes to

$$\lim_{\varepsilon \rightarrow 0} \mathcal{C}_\varepsilon(t, x) = \int_{\mathbb{R}^n} a_{12}(t, x, k) dk, \quad (12.23)$$

where  $a_{12}(t, x, k)$  solves the following radiative transfer equation

$$\begin{aligned} & \frac{\partial a_{12}}{\partial t} + \nabla_k H \cdot \nabla_x a_{12} - \nabla_x H \cdot \nabla_k a_{12} + (\Sigma(x, k) + i\Pi(x, k))a_{12} \\ & = \int_{\mathbb{R}^n} \sigma(x, k, q) a_{12}(t, x, q) \delta(H(x, q) - H(x, k)) dk, \end{aligned} \quad (12.24)$$

where we have defined the following kinetic parameters:

$$\begin{aligned}
\Sigma(x, k) &= \frac{\pi H^2(x, k)}{2(2\pi)^n} \int_{\mathbb{R}^n} \frac{\hat{R}_{11} + \hat{R}_{22}}{2}(x, k - q) \delta(H(x, k) - H(x, q)) dq \\
i\Pi(x, k) &= \frac{i}{4(2\pi)^n} \text{p.v.} \int_{\mathbb{R}^n} (\hat{R}_{11} - \hat{R}_{22})(x, k - q) \sum_{\pm} \frac{\pm H(x, k) H(x, q)}{H(x, k) - \pm H(x, q)} dq \\
\sigma(x, k, q) &= \frac{\pi H^2(x, k)}{2(2\pi)^n} \hat{R}_{12}(x, k - q).
\end{aligned} \tag{12.25}$$

In other words, the field-field correlations also satisfy deterministic kinetic models asymptotically, which may then be used to infer properties about the underlying medium and the buried inclusions; see [11] for some applications of field-field correlations in imaging.

### 12.1.6 Imaging with waves in random media

The main heuristic conclusion we can draw from the derivation of this section is that the modeling of an inverse problem should aim to (i) find a space of parameters that is of reasonable size when compared to the quality of the available data; and (ii) find functionals of these parameters that are as immune as possible to all other parameters that were neglected in the derivation.

This is more easily said than done of course and often cannot be achieved satisfactorily. In some sense, the factorization methodology presented in Chapter 11 also aimed to find smaller parameter sets (such as the location of inclusions) that could uniquely be determined independent of neglected parameters (such as the quantitative values inside such inclusions).

In wave imaging in highly heterogeneous media, the proper model is to describe clutter as random and then to look for functionals of the wave fields that are independent of the realization of such a clutter. Field-field correlations are such functionals. They have been successfully exploited to stabilize imaging functionals based on field measurements in a technique called Coherent Interferometry [24] as well as to image inclusions in very strong clutter [16].

The main limitation to the simplified model is encoded in the noise structure; see for instance  $n_\varepsilon$  in (12.17). In the next section, we consider the asymptotic analysis of such a noise in a simpler physical context.

## 12.2 Random fluctuations and Noise models

In Chapter 4, we encountered the nonlinear inverse coefficient problem for  $q(x)$  in (4.36), which was recast as the integral (4.46). In this chapter, we consider a simplified version of such an inverse problem of the form

$$y = Ax + Bx \otimes x, \tag{12.26}$$

where  $y$  is the available measurements and  $x$  is the unknown coefficient. We assume that  $B$  is a bi-linear map, for instance of the form

$$Bx \otimes x(x) = \int_0^x \int_0^y b(x, y, z)x(y)x(z)dzdy, \quad (12.27)$$

to mimic the inverse problem in (4.46). As a first approximation to the same inverse problem, we could assume that  $A$  is the identity operator.

Let us now assume that measurements are *not perfect*, as is the case in practice. More precisely, let us assume that measurements are of the form

$$y_\eta = C_\eta y, \quad C_\eta y(x) = \int_0^\infty \varphi_\eta(x - y)y(y)dy. \quad (12.28)$$

In other words, we assume that  $C_\eta$  is a convolution operator that blurs measurements while preserving some causality. We assume that  $\varphi_\eta \geq 0$ , is smooth, integrates to 1, and has a support of size  $\eta$  for instance.

As a consequence, the measurements  $y_\eta$  do not allow us to stably reconstruct frequencies of order  $\eta^{-1}$  or larger. Note that unique reconstructions are still possible when  $\varphi_\eta$  is known since clearly

$$\widehat{C_\eta y}(\omega) = \widehat{\varphi}_\eta(\omega)\widehat{y}(\omega) \quad \text{so that} \quad \widehat{y}(\omega) = \frac{\widehat{C_\eta y}(\omega)}{\widehat{\varphi}_\eta(\omega)}.$$

However, for  $\omega$  of order  $\eta^{-1}$ ,  $\widehat{\varphi}_\eta(\omega)$  is small and inevitable noise in  $y$  is therefore extremely amplified (the inverse denoising problem is a severely ill-posed problem). As a consequence, typically only the low frequency part of  $x$  can be reconstructed. Let us take  $C_\eta x$  as a proxy for such low frequencies. Let us also make the simplifying assumption that  $C_\eta$  and  $A$  commute, in other words, that  $A$  is itself a convolution, which is certainly the case for  $A = Id$ . Then we find that

$$y_\eta = Ax_\eta + C_\eta Bx \otimes x, \quad x_\eta = C_\eta x. \quad (12.29)$$

This can be written as an equation for  $x_\eta$  as follows:

$$y_\eta = Ax_\eta + C_\eta Bx_\eta \otimes x_\eta + n_\eta, \quad n_\eta := (C_\eta Bx \otimes x - C_\eta Bx_\eta \otimes x_\eta). \quad (12.30)$$

**Noise and Modeling.** We have thus slightly changed problems. After realizing that  $x$  could not be reconstructed explicitly, we changed to a problem for  $x_\eta = C_\eta x$ , which we hope can be reconstructed stably. However, by doing so, we do get a nonlinear equation for  $x_\eta$  that involves a noise term  $n_\eta$ . Here is where the *modeling* step becomes useful in the theoretical understanding of the above inverse problem.

We first realize that

$$n_\eta = C_\eta B(x - x_\eta) \otimes x + C_\eta Bx \otimes (x - x_\eta) - C_\eta B(x - x_\eta) \otimes (x - x_\eta). \quad (12.31)$$

Let us assume that some theoretical arguments allow us to show that  $C_\eta B(x - x_\eta) \otimes (x - x_\eta)$  is negligible compared to the other two terms for  $\eta \ll 1$ , the regime of interest for the rest of this section.

Since  $z_\eta := x - x_\eta$  cannot be reconstructed, the only choice that we can model such a term is as the realization of a *random process*. And because  $z_\eta$  is defined at the scale  $\eta$ , we define it as  $z_\eta(x) = z(\frac{x}{\eta})$ . Moreover, in the absence of information about potential bias, we assume that  $\mathbb{E}\{z(x)\} = 0$ , where  $\mathbb{E}\{\cdot\}$  denotes ensemble average over the randomness, and that  $z(x)$  is a *stationary* process (which means that the statistics of  $z(x)$  are not modified by shifts in  $x$ ). Since it slightly simplifies the presentation, let us assume that  $b(x, y, z) = b(x - y)b(y - z)$  in (12.27) so that  $B$  appears as a reiterated convolution:

$$Bx \otimes x = b \star (xb \star x).$$

Then we find that

$$C_\eta B(x - x_\eta) \otimes x = b_\eta \star (z_\eta b \star x), \quad b_\eta = C_\eta b.$$

In other words,

$$C_\eta B(x - x_\eta) \otimes x(x) = \int_0^x b_\eta(x - y)z\left(\frac{y}{\eta}\right)(b \star x)(y)dy. \quad (12.32)$$

The above contribution to the error term has been recast as a stochastic integral with a highly oscillatory integrand. The limits of such integrals as  $\eta \rightarrow 0$  are well-understood. We find for instance that

$$\frac{1}{\sqrt{\eta}} \int_0^x b_\eta(x - y)z\left(\frac{y}{\eta}\right)(b \star x)(y)dy \xrightarrow[\eta \rightarrow 0]{\text{distribution}} N(x) := \int_0^x b(x - y)(b \star x)(y)\sigma dW(y), \quad (12.33)$$

where  $dW(y)$  is a one dimensional Wiener measure (i.e., white noise).

**Exercise 12.2.1** *Prove the above formula. The analysis of such oscillatory integrals is for instance available in [10].*

Here,  $\sigma$  is the strength of the random fluctuations, given explicitly by

$$\sigma^2 = \int_{\mathbb{R}} \mathbb{E}\{z(0)z(x)\}dx, \quad (12.34)$$

which we assume is positive and bounded.

We first observe that  $b_\eta \star (z_\eta b \star x)$  is of size  $\sqrt{\eta}\sigma$ . This means that for fluctuations  $z(x)$  of order  $O(1)$ , the contribution  $b_\eta \star (z_\eta b \star x)$  is asymptotically not of order  $O(1)$  but rather of order  $O(\sqrt{\eta})$ .

Second, we note that the right-hand side  $N(x)$  of (12.33) is a centered Gaussian process. This means that jointly,  $(N(x_1), \dots, N(x_n))$  for  $n \geq 0$  is a multi-dimensional Gaussian random variable. For instance, we obtain the following expression for covariances:

$$\mathbb{E}\{N(x_1)N(x_2)\} = \sigma^2 \int_0^{x_1 \wedge x_2} b^2(x_1 \wedge x_2 - y)(b \star x)^2(y)dy, \quad (12.35)$$

where  $x \wedge y = \min(x, y)$ . In other words, we have an *estimate* of the correlation of this noise contribution at different values of  $x$ .

**Noise Model and Correlations.** The noise in (12.31) has two leading contributions:

$$n_\eta = b_\eta \star (z_\eta b \star x) + b_\eta \star (x b \star z_\eta) + r_\eta.$$

The second contribution may be written as

$$b_\eta \star (x b \star z_\eta)(x) = \int_0^x \left( \int_y^x b_\eta(x-z) x(z) b(z-y) dy \right) z \left( \frac{y}{\eta} \right) dy.$$

Assuming that the contribution  $r_\eta$  is negligible as  $\eta \rightarrow 0$ , we obtain that

$$\begin{aligned} \frac{1}{\sqrt{\eta}} \left( b_\eta \star (z_\eta b \star x) + b_\eta \star (x b \star z_\eta) \right)(x) &\xrightarrow[\eta \rightarrow 0]{\text{distribution}} \mathcal{N}(x)[x] \\ \mathcal{N}[x](x) &= \int_0^x \mathbf{n}[x](x, y) \sigma dW(y) \\ \mathbf{n}[x](x, y) &= b(x-y)(b \star x)(y) + \left( \int_y^x b(x-z) x(z) b(z-y) dz \right) \end{aligned} \quad (12.36)$$

The above convergence result is the main result of this section. We obtain that  $n_\eta(x)$  is of order  $\sqrt{\eta}$ . Moreover, we obtain a limiting description of the statistics of  $n_\eta(x)$ . What is interesting is that the limiting distribution  $\mathcal{N}(x)[x]$  depends on the unknown  $x$ , which is close to  $x_\eta$ .

**Closed form equation with noise model.** To summarize the above derivation, we obtain that the model for  $x$  in the presence of noise in the data is replaced by the following approximate model for  $x_\eta = C_\eta x$ :

$$y_\eta = Ax_\eta + C_\eta Bx_\eta \otimes x_\eta + n_\eta, \quad n_\eta = \sqrt{\eta} \mathcal{N}[x_\eta]. \quad (12.37)$$

By an application of the Gronwall lemma, it is not difficult to obtain that the above nonlinear equation is uniquely solvable

**Exercise 12.2.2** *Verify this in detail.*

What is particularly interesting in the above model is that the noise model  $n_\eta$  is a *physics-based* noise model whose statistics *depend* on the unknown parameter  $x_\eta$ . In the above derivation,  $\mathcal{N}[x_\eta]$  is linear because we assumed that the data were quadratic in the unknown parameters. However, more general forms for  $\mathcal{N}[x_\eta]$  would appear in higher order terms are kept in the definition of  $y = \mathfrak{M}(x)$  as would be required in the treatment of (4.46) for instance.

Here are the main conclusions we can draw from such an analysis. First, the correlation function of the measurements is very far from being diagonal:

$$\mathbb{E}\{n_\eta(x_1)n_\eta(x_2)\} = \eta\sigma^2 \int_0^{x_1 \wedge x_2} (\mathbf{n}[x](x_1 \wedge x_2, y))^2 dy. \quad (12.38)$$

Assuming that the measurements at different points (or times)  $x_j$  are uncorrelated, as is often done in practice when no priori knowledge is available, generates larger errors than when a reasonable model for correlations is available. The above modeling provides

a means to estimate such correlations in a physically justified manner. For instance, in the Bayesian setting with noise and prior modeled as correlated Gaussian fields, the maximum of the posterior distribution (which then coincides with its mean) is given by the solution of (10.51). The solution depends on the correlation matrix  $\Sigma$ , which is precisely what is provided by (12.38).

Second, the statistics of the noise depend on the unknown parameter  $x$ . In most practical settings, the noise statistics are assumed to be independent of the unknown coefficients. We find that physics-based noise models provide examples where the statistics of the noise do depend on the unknown parameter  $x$ . This has to be accounted for in practical implementations of the minimization procedures or Bayesian formalisms presented in Chapter 10, for instance by assuming that  $\Sigma$  in (10.51) depends on  $x$ .

In the setting of a one-dimensional Sturm-Liouville problem, the gains in accuracy that one might expect from such asymptotic model as presented above was quantified numerically in [17].

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