APMA E4990 2006 Lectures Summary

Lectures 1-3. Chapter 1 in [HN].

- Definition of a **metric space** with **distance function** (X, d).
- Definition of a **vector space** (or **linear space**). Definition of a norm $\|\cdot\|$ and of a normed vector space $(V, \|\cdot\|)$.
- Definition of closed and open unit ball in normed vector spaces. Definition of convex subsets.
- Notion of equivalence between norms on a vector space.
- Notion of **convergence** $x_n \to x$

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \text{ s.t. } d(x_n, x) \leq \varepsilon \, \forall n \geq N.$$

- Notion of **Cauchy sequence**

$$(x_n) \text{ Cauchy if } \forall \varepsilon, \exists N \text{ s.t. } d(x_n, x_m) \leq \varepsilon \, \forall n, m \geq N.$$

Definition 0.1 *(X, d) is* **complete** *if for each x_n Cauchy, $x_n \to x \in X$.*

Definition 0.2 *A complete normed vector space is called a* **Banach space***.*

- A theorem says that every metric space has a completion. Completion is obtained by equivalence classes of Cauchy sequences as in the construction or $\mathbb{R}$ from $\mathbb{Q}$.
- Notion of **continuity** and **uniform continuity**.
- Notion of **sequential continuity**.

Proposition 0.3 *A function f is continuous at $x \in X$ iif it is sequentially continuous at x.*

- Notion of upper/lower semicontinuity. Notion of limsup and liminf.
- Notion of closed and open sets (topological notion). Open and closed balls $B_r(a)$ and $\overline{B_r(a)}$.

Definition 0.4 *$G \subset X$ is* **open** *if $\forall x \in G$, $\exists r > 0$ such that $B_r(x) \subset G$.*
$G \subset X$ is **closed** *if its complement G^c is open.*

Note that the notion of openness and closedness is related to the choice of the metric.

Proposition 0.5 *A metric space X is a topological space with the family of open sets generated by the above open balls.*

- Very convenient characterization of closed sets:

Proposition 0.6 *$F \subset X$ is closed iif $x_n \to x$ and $x_n \in F$ implies that $x \in F$.*

- Notion of closure $\bar{A} = \{x \in X | x_n \in A \text{ such that } x_n \to x\}$.
  Note that $\bar{\mathbb{Q}} = \mathbb{R}$.

  Definition 0.7 *$A \subset X$ is dense in X if $\bar{A} = X$.*

  Definition 0.8 (Separability) *A metric space is* **separable** *if it has a* **countable** *dense subset.*

- Characterization of continuous functions:

  Proposition 0.9 *$f : (X, d) \to (Y, d')$ is continuous iif $f^{-1}(G)$ is open in X for every open G in Y.*

- Notion of **compactness** and **sequential compactness**.

  Definition 0.10 *$K \subset X$ metric is* **sequentially compact** *if every sequence in K has a converging subsequence in K.*

  Theorem 0.11 (Heine-Borel) *A subset in $\mathbb{R}^d$ is sequentially compact iif it is closed and bounded.*

  Theorem 0.12 (Bolzano-Weierstrass) *Every bounded sequence in $\mathbb{R}^d$ has a convergent subsequence.*

- Notion of cover, finite cover, ε-net, and finite ε-net.

  Definition 0.13 *$A \subset X$ is* **totally bounded** *if it has a finite ε-net for all $\varepsilon > 0$.*

  We have the following characterization (hard to use in practice):

  Theorem 0.14 *$A \subset X$ is sequentially compact iif it is complete and totally bounded.*

- Definition of **compactness**:

  Definition 0.15 *$K \subset X$ is* **compact** *if every open cover of K has a finite subcover.*

  Theorem 0.16 *A subset of a metric space is compact iif it is sequentially compact.*

  So in the future, we refer to compact subsets for the notion of sequential compactness. Sequential compactness is often simpler to use than compactness in metric spaces. Compactness is a more general notion (defined on topological spaces) that may be easier to use when the metric is not involved in the proof.

- A subset A of a metric soace X is called **precompact** (or **relatively compact**) if its closure in X is compact.

- Interplay between compactness and continuous functions.

  Theorem 0.17 *$f : K \to Y$ continuous with K compact metric and Y metric implies that $f(K)$ is compact.*

  Theorem 0.18 *$f : K \to Y$ continuous with K compact metric and Y metric implies that f is uniformly continuous.*

  Theorem 0.19 *$f : K \to \mathbb{R}$ continuous with K compact metric implies that f is bounded and that it attains its minimum and its maximum.*

  Theorem 0.20 *$f : K \to \mathbb{R}$ upper semi-continuous implies that f is bounded from above and attains its maximum. $f : K \to \mathbb{R}$ lower semi-continuous implies that f is bounded from below and attains its minimum.*

Lectures 4-5. Chapter 2 in [HN].

- Notion of convergence of continuous functions $f : X \to \mathbb{R}$.
- Notion of pointwise convergence: $f_n(x) \to f(x)$ for all $x \in X$. This convergence does not preserve continuous functions.
- Notion of uniform continuity. Define the **uniform norm**

  $$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

  Definition 0.21 *f_n converges* **uniformly** *to f if $\lim_{n\to\infty} \|f_n - f\| = 0$.*

  Uniform convergence is much stronger than pointwise convergence.

  Theorem 0.22 *Let f_n bounded and continuous on (X, d) and $f_n \to f$ uniformly. Then f is continuous.*

  So the notion of uniform convergence *preserves* continuity.

- Space of continuous functions.
  Note that continuous functions on arbitrary metric spaces need not be bounded. However we know they are bounded on compact domains.

  Definition 0.23 *We call $\mathcal{C}(K)$ the vector space of continuous functions on K compact.*

  Proposition 0.24 *Let K be a compact metric space. $\|\cdot\|_\infty$ is a norm on $\mathcal{C}(K)$ so that $(\mathcal{C}(K), \|\cdot\|_\infty)$ is a normed vector space.*

  Theorem 0.25 *Let K be a compact metric space. Then $(\mathcal{C}(K), \|\cdot\|_\infty)$ is* **complete** *so that it is a* **Banach** *space.*

- Notion of support of a function and of compact support.
- **Definition 0.26** *$\mathcal{C}_c(X)$ for X metric is the space of continuous function on X with compact support. $\mathcal{C}_b(X)$ for X metric is the space of bounded continuous function on X. $\mathcal{C}_0(X) = \overline{\mathcal{C}}_c(X)$ for the uniform norm.*

  The latter definition is well-defined because $\mathcal{C}_b(X)$ equipped with the uniform norm is a **Banach space** (same proof as for $\mathcal{C}(K)$) and because $\mathcal{C}_c(X) \subset \mathcal{C}_b(X)$. This implies that $\mathcal{C}_0(X)$ is also a Banach space as a closed subset of a Banach space. We thus have

  $$\mathcal{C}_c(X) \subset \mathcal{C}_0(X) \subset \mathcal{C}_b(X) \subset \mathcal{C}(X).$$

  The intermediate two spaces are Banach. $\mathcal{C}_b(X)$ is not even a normed space since $\|\cdot\|_\infty$ is not necessarily defined (it is when X is compact of course).

- We can show that $\mathcal{C}_0(\mathbb{R}^d)$ is the space of continuous functions that vanish at infinity.
- Notion of approximation by polynomials $p(x) = \sum_{k=0}^\infty a_k x^k$ on (a, b).

  Theorem 0.27 (Weierstrass) *Polynomials are dense in $\mathcal{C}([a, b])$ equipped with sup norm.*

On $[a,b]=[0,1]$, this uses the **Bernstein** polynomials

$$B_n(x,f)=\sum_{k=0}^{\infty} f(\frac{k}{n})\binom{n}{k}x^k(1-x)^{n-k}.$$

Calculations show that

$$\|B_n(\cdot,f)-f\|_\infty \le \varepsilon + \frac{\|f\|_\infty}{2n\delta^2},$$

where $\delta(\varepsilon)$ is the constant coming from the uniform continuity of f on $[0,1]$).

- Notion of compact subsets of the space of continuous functions. Notion of **equicontinuity**.

Definition 0.28 *$\mathcal{F}$ family of functions from X to Y metric spaces is* **equicontinuous** *if*

$$\forall x \in X, \forall \varepsilon > 0, \exists \delta > 0 \ s.t.\ d(x,y) < \delta \implies d(f(x),f(y)) < \varepsilon \, \forall f \in \mathcal{F}.$$

Note that δ is independent of $f \in \mathcal{F}$ although it may depend on x. Yet:

Theorem 0.29 *$\mathcal{F}$ equicontinuous on compact metric spaces implies that $\mathcal{F}$ is uniformly equicontinuous (i.e., δ independent of x in above definition).*

Here is a very useful characterization of compact subsets in $\mathcal{C}(K)$.

Theorem 0.30 (Arzelá-Ascoli) *A subset in $\mathcal{C}(K)$ equipped with uniform norm is* **compact** *iif it is* **closed**, **bounded**, *and* **equicontinuous**.

- Notion of **Lipschitz** functions.

Definition 0.31 *$f : X \to \mathbb{R}$ is* **Lipschitz** *continuous on X if $\exists C > 0$ s.t.*

$$|f(x)-f(y)| \le C d(x,y), \quad \forall x,y \in X.$$

Definition 0.32 $Lip(f) = \sup_{x \neq y} \dfrac{|f(x)-f(y)|}{d(x,y)}$.

Proposition 0.33 *$\mathcal{F}_M = \{f | f$ Lipschitz continuous on K and $Lip(f) \le M\}$ is equicontinuous and closed.*

Corollary 0.34 *Let*

$$\mathcal{G} = \{f \in \mathcal{C}([0,1]) | \|f\|_\infty \le M, \quad Lip(f) \ge N\}.$$

Then $\mathcal{G}$ is compact.

- Application to the solution of ODEs and the **Peano** existence theorem for ODEs.
- (1) Using $u(t+h)-u(t) \approx hf(t,u(t))$, construct $u_h(t)$ by finite difference approximation.
- (2) Show that $u_h(t)$ is Lipschitz uniformly in h.
- (3) Use above theorem to obtain convergence of a subsequence to $u(t) \in \mathcal{C}(I)$.
- (4) Show that $u(t)$ satisfies the ODE.

- This proof is typical of the use of compactness theorems in the analysis of differential and integral equations.
- Notion of uniqueness theorem. The above construction provides ones solution. It says nothing about uniqueness. Uniqueness is actually not guaranteed unless $f(t,x)$ is Lipschitz with respect to its second variable. How does one show uniqueness?
- (1) Assume u and v solutions and construct $w = u - v$.
- (2) Show that $w(t)$ satisfies an estimate, here

$$|w(t)| \leq C \int_0^t |w(s)| ds,$$

because f is Lipschitz.
- (3) Show that the estimate implies that $w \equiv 0$. This is done by using **Gronwall's lemma**, which is extremely useful in the analysis of many evolution equations. This shows uniqueness.

Lectures 6-7. Chapters 3-4 in [HN].

- Notion of **Contraction Mapping**.

Definition 0.35 *On (X,d), $T : X \to X$ is a contraction mapping if there exists $0 \leq c < 1$ such that*

$$d(Tx, Ty) \leq cd(x,y), \qquad \textit{for all } x, y \in X.$$

- Note that $TB_r(x) \subset B_{cr}(tx)$, so that contraction maps send balls into subsets of smaller balls. Note also that T is Lipschitz with $Lip(T) =\leq c < 1$. Here we say that $f : X \to Y$ is Lipschitz if $d(f(x), f(y)) \leq Cd(x,y)$ for some constant C uniformly in $x, y \in X$.
- **Contraction mapping**

Theorem 0.36 *Let $T : X \to X$ be a contraction mapping on a* complete *metric space. Then the* **fixed point** *equation $Tx = x$ admits a unique solution.*

The proof is constructive: $x_{n+1} = Tx_n$ from $x_0 \in X$ given. We show that x_n is Cauchy, hence converges to x. Then $x = Tx$ and $y = Ty$ implies $x = y$.

- Application to **Integral equations**. Fredholm integral equations of the second kind have the form

$$f(x) = \int_a^b k(x,y) f(y) dy + g(x).$$

Define $Tf = \displaystyle\int_a^b k(x,y)f(y)dy + g(x)$. Then show that T is a contraction on $(\mathcal{C}([a,b]), \|\cdot\|_\infty)$ provided that $k(x,y)$ is continuous in the x variable and the integral of $|k(x,y)|$ in the y variable is sufficiently small uniformly in x.

- Using $Tf = g + Kf$, show that $(I - K)f = g$ admits for solution the **Neumann series** expansion $f = \sum_{k=0}^\infty K^k g$. Thus we can write $(I-K)^{-1} = \sum_{k=0}^\infty K^k$.
- Application to Differential equation of the form

$$Lu = f + Qu,$$

plus possibly boundary conditions, where L and Q are linear opeators. Sometimes, it is possible to show that L is invertible and to replace the differential equation by an integral equation of the form

$$u = L^{-1}f + L^{-1}Qu.$$

When $L^{-1}Q$ is a contraction in an appropriate Banach space, then the above equation admits a unique solution in that space.

Example:

$$-u'' + \alpha^2 u + q(x)u = f(x), \quad u \in \mathcal{C}_0(\mathbb{R}) \text{ for } f \in \mathcal{C}_0(\mathbb{R}).$$

Here $\mathcal{C}_0(\mathbb{R})$ is equipped with the uniform norm so that it is a Banach space. It turns out that when $q = 0$, the solution to

$$-G''(x; y) + \alpha^2 G(x; y) = \delta(x - y),$$

where the derivatives are in the x variable, is given by

$$G(x) = \frac{1}{2\alpha} e^{-\alpha|x|},$$

as can be verified once one knows what the delta function is. In any event, one verifies that the solution to

$$Lu \equiv -u'' + \alpha^2 u = f \quad u \in \mathcal{C}_0(\mathbb{R}) \text{ for } f \in \mathcal{C}_0(\mathbb{R}),$$

is given by

$$u(x) = \int_{\mathbb{R}} \frac{1}{2\alpha} e^{-\alpha|x-y|} f(y) dy.$$

It is clear that $u \in \mathcal{C}_0(\mathbb{R})$. So the original problem is equivalent to

$$u(x) = \int_{\mathbb{R}} \frac{1}{2\alpha} e^{-\alpha|x-y|} (f(y) - q(y)u(y)) dy.$$

Thus provided that

$$\sup_{x \in \mathbb{R}} \int_{\mathbb{R}} \frac{1}{2\alpha} e^{-\alpha|x-y|} |q(y)| dy < 1,$$

we find that $L^{-1}Q$ with $Qu = q(\cdot)u$ is a contraction on $\mathcal{C}_0(\mathbb{R})$.

Note that $q(x)$ need not be continuous in order for $L^{-1}Q$ to be a contraction on $\mathcal{C}_0(\mathbb{R})$. Piecewise continuous for instance is enough. Note also that the constraint on q shows that it has to go to 0 as $\alpha \to 0$. Note also that there is no solution in $\mathcal{C}_0(\mathbb{R})$ to the problem $-u'' = f$ when $\alpha = 0$ in general.

- Application to ODEs. A function $f(t, u) : I \times \mathbb{R}^n \to \mathbb{R}^n$ is globally Lipschitz in u uniformly in t if there exists C such that

$$\|f(t, u) - f(t, v)\| \le C\|u - v\|, \qquad \forall u, v \in \mathbb{R}^n, t \in I.$$

Theorem 0.37 *When $f(t, u)$ is uniformly Lipschitz, there exists a unique solution to the ODE*

$$\dot{u} = f(t, u(t)), \qquad u(t_0) = u_0,$$

on the interval $I \subset \mathbb{R}$, where $t_0 \in I$.

Proof: show that

$$Tu(t) = u_0 + \int_{t_0}^{t} f(s, u(s)) ds$$

on an interval $(t_0 - \delta, t_0 + \delta)$ for δ such that $C\delta < 1$, where C is the constant of Lipschitzness of f. Then cover I with a countable number of intervals of length less than 2δ and apply the result on each inteval to conclude the proof.

- We can also get a local existence result if f is only locally Lipschitz. This is the Picard construction to ODEs. It is much more straightforward than the Peano construction. However the Peano construction works with less strong hypotheses on f.

- **Topological spaces.** $(X, \mathcal{T})$.

- **Definition 0.38** *A* **topology** *is a collection $\mathcal{T}$ of* **open sets** *in X such that $\emptyset \in \mathcal{T}$, $X \in \mathcal{T}$; arbitrary unions of sets in $\mathcal{T}$ are in $\mathcal{T}$; and finite intersections of sets in $\mathcal{T}$ are in $\mathcal{T}$.*

- A set A is a **closed** set if its compementary $A^c = X \backslash A$ is open.

- A **topological space** $(X, \mathcal{T})$ is a set X equipped with a tolopogy $\mathcal{T}$.

- Examples include the metric spaces where the topology is given by the arbitrary unions and finite intersections of open balls.

- $V \subset X$ is a **neighborhood** of $x \in X$ if $x \in G \subset V$ for $G \in \mathcal{T}$.

- $(X, \mathcal{T})$ is Hausdorff if for all $x \neq y$ in X, we can find two neighborhoods V_x and V_y of x and y, respectively, such that $V_x \cap V_y = \emptyset$.

- Notion of **convergence**. We say that $x_n \to x$ in X if for every neighborhood V of x, there is N such that for all $n \geq N$, we have $x_n \in V$.

- Notion of **Continuity of a map**. A function from $(X, \mathcal{T})$ to $(Y, \mathcal{S})$ is continuous if for all nbhd W of $f(x)$, there is a nbhd V of x such that $f(V) \subset W$.

  Theorem 0.39 *A function from $(X, \mathcal{T})$ to $(Y, \mathcal{S})$ is continuous iif for all $G \in \mathcal{S}$, we have $f^{-1}(G) \in \mathcal{T}$.*

- Notion of **homeomorphism**. $f : X \to Y$ is a homeomorphism if it is one-to-one, onto, and f and f^{-1} are continuous. When such a map exists, then X and Y are homeomorphic topological spaces. I.e., as far as topology is concerned, they are the same spaces.

- Notion of **compactness**. $K \subset X$ is compact if every open cover contains a finite subcover.

- Notion of **comparison of topologies**. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. We say that $\mathcal{T}_1$ is **stronger**, or **finer**, than $\mathcal{T}_2$ if every open set in $\mathcal{T}_2$ is an open set in $\mathcal{T}_1$, i.e., if $\mathcal{T}_2 \subset \mathcal{T}_1$. We then say that $\mathcal{T}_2$ is **weaker** or **coarser** than $\mathcal{T}_1$.

  Weaker topologies are interesting because they have more *compact* sets. Indeed, the fewer open sets there is, the easier (more economical) it is to cover them.

  The topology coming from the distance in a metric space is called the strong topology. GBounded and closed subsets are compact iif the metric space is finite dimensional as in the Heine Borel theorem. It turns out that many metric spaces can also be endowed with a weaker topology, called the weak topology or the weak-* topology in the literature. These weaker topologies have a lot more compact subsets than the strong topology, whence their interest in analysis of PDEs.

- Let f be a continuous map from $(X, \mathcal{T}_2)$ to $(Y, \mathcal{S}_1)$. Let $\mathcal{T}_1$ be finer than $\mathcal{T}_2$ and $\mathcal{S}_2$ be coarser than $\mathcal{S}_1$. Then f is continuous from $(X, \mathcal{T}_1)$ to $(Y, \mathcal{S}_2)$.

  This is an obvious consequence from the definition and the characterisation of continuous functions. So a function from X to $\mathbb{R}$, say, is continuous for the strong topology if it is continous for the weak topology.

- In the same vein, the identity map $I : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is continuous iif $\mathcal{T}_1$ is finer than $\mathcal{T}_2$.

  Note that the above result can be rephrased a $I_Y \circ f \circ I_X$ is continuous when f is continuous and I_X and I_Y are.

Lectures 8-13. Chapter 5 in [HN] and some extras.

- **Properties of Banach spaces.**
- Definition: normed vector space that is a complete metric space.
- Example: l^p of sequences $(x_n)_{n\geq 0}$ such that $\sum |x_n|^p < \infty$ and equipped with norm $\|x\|_p = (\sum |x_n|^p)^{\frac{1}{p}}$.
- Example: Space of continuous functions $\mathcal{C}(K)$ equipped with uniform norm $\|\cdot\|_\infty$, where K is a compact metric space; space of continuous functions vanishing at infinity $\mathcal{C}_0(\mathbb{R}^d)$ equipped with uniform norm; space of continuous functions with k continuous derivatives $\mathcal{C}^k([a,b])$ equipped with uniform norm for all derivatives of order up to k: $\|f\|_{\mathcal{C}^k} = \|f\|_\infty + \ldots + \|f^{(k)}\|_\infty$.
- Example: $L^p(\mathbb{R})$ spaces of functions equipped with norm

$$\|f\|_p = \Big(\int_{\mathbb{R}} |f(x)|^p dx\Big)^{\frac{1}{p}}.$$

This space will be defined more carefully later.

- Example: Sobolev spaces $W^{k,p}(\mathbb{R})$ of functions with first k derivatives in L^p and equipped with norm

$$\|f\|_{W^{k,p}} = \sum_{j=0}^{p} \|f^{(j)}\|_p.$$

Again, this space requires careful definition and will be defined later.

- Proof that l^p is Banach.

(1) Young's inequality

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Proof: log is concave and increasing.

(2) Hölder inequality

$$\sum_{n=1}^{\infty} a_n b_n \leq \Big(\sum_{n=1}^{\infty} a_n^p\Big)^{\frac{1}{p}} \Big(\sum_{n=1}^{\infty} b_n^{p'}\Big)^{\frac{1}{p'}}, \qquad a_n, b_n \geq 0.$$

Proof Using Young's inequality, we have for each $\lambda > 0$ that

$$\sum_{n=1}^{\infty} a_n b_n \leq \sum_{n=1}^{\infty} \frac{\lambda a_n^p}{p} + \sum_{n=1}^{\infty} \frac{\lambda^{-\frac{p'}{p}} b_n^{p'}}{p'}.$$

Choose

$$\lambda = \frac{\|b\|_{p'}}{\|a\|_p^{p-1}} \text{ so that } \lambda^{-\frac{p'}{p}} = \frac{\|a\|_p}{\|b\|_{p'}^{p'-1}}.$$

(3) Minkowski's inequality (triangle inequality).

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p.$$

Proof:

$$\|x+y\|_p^p \leq \textstyle\sum_{n=1}^{\infty} |x_n + y_n|^{p-1}(|x_n| + |y_n|)$$
$$\leq (\|x\|_p + \|y\|_p)\Big(\textstyle\sum_{n=1}^{\infty} |x_n + y_n|^{p'(p-1)}\Big)^{\frac{1}{p'}} = (\|x\|_p + \|y\|_p)\|x+y\|_p^{p-1},$$

using Hölder's inequality. This shows the result. So $(l^p, \|.\|_p)$ is a normed vector space.

(4) Completeness. Take a Cauchy sequence $x^{(n)}$ in l^p. Then $x_k^{(n)}$ is Cauchy in $\mathbb{R}$ and converges to $x_k^{(\infty)}$. Now

$$\sum_{k=1}^{M} |x_k^{(\infty)}|^p = \lim_{m\to\infty} \sum_{k=1}^{M} |x_k^{(m)}|^p \le \liminf_{m\to\infty} \sum_{k=1}^{\infty} |x_k^{(m)}|^p \le C \text{ independent of M.}$$

This implies that $x^{(\infty)} \in l^p$. Whence $x^{(\infty)} - x^{(n)} \in l^p$. Now let $\varepsilon > 0$. Calculate

$$\|x^{(\infty)}-x^{(n)}\|_p^p = \lim_{M\to\infty} \sum_{k=1}^{M} \lim_{m\to\infty} |x_k^{(m)}-x_k^{(n)}|^p \le \frac{\varepsilon}{2} + \sum_{k=1}^{M(\varepsilon)} \lim_{m\to\infty} |x_k^{(m)}-x_k^{(n)}|^p \le \frac{\varepsilon}{2} + \liminf_{m\to\infty} \|x^{(m)}-x^{(n)}\|_p^p.$$

This is less than ε for all n sufficiently large so $x^{(n)}$ converges to $x^{(\infty)}$ in l^p, whence l^p is complete.

- A closed subset of a Banach space is a Banach space (follows from definition of closedness and of convergence of Cauchy sequences).
  Note however that many subspaces of infinite dimensional spaces are not closed. For instance $\mathcal{C}^k([a, b])$ is a subset of $\mathcal{C}([a, b])$ that is not closed for the uniform norm (though it is dense so its closure is $\mathcal{C}([a, b])$).
- Notion of **bounded linear maps.** X and Y are vector spaces.
- $T : X \to Y$ is linear if $T(\lambda x + \mu y) = \lambda Tx + \mu Ty \ \forall \lambda, \mu \in \mathbb{R}$ and $\forall x, y \in X$.
- $T : X \to X$ is called a linear operator on X.
- $T : X \to Y$ that is *one-to-one* (injective) and *onto* (surjective) is called *invertible*. Then $T^{-1} : Y \to X$ is linear and defined by the equivalence: $T^{-1}y = x \iff y = Tx$.
- Notion of bounded operators and of **uniform norm** (also called **operator norm**).
- $T : X \to Y$ linear operator is **bounded** if

  $$\exists M > 0 \text{ s.t. } \|Tx\|_Y \le M\|x\|_X \quad \forall x \in X.$$

  Otherwise we say that T is **unbounded**.
- If T is bounded, then the **operator norm** or **uniform norm** is defined as

  $$\|T\| = \inf\{M, \|Tx\| \le M\|x\| \, \forall x \in X\} = \sup_{\mathbf{x}\neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\|.$$

- Note that $f \mapsto \int_0^x f(t)dt$ is bounded on $\mathcal{C}([0, 1])$ with uniform norm. On the space $\mathcal{C}^\infty([0, 1])$ of infinitely differentiable functions equipped with the uniform norm, $f \mapsto f'$ is *not* bounded.
- Representation of linear operators in finite dimensional spaces. Let $A : X \to Y$ with $dimX = n$ and $dimY = m$. Then in given bases for X and Y, A can be represented by the matrix $m \times n$ matrix $(a_{ij})_{1\le i\neq m;1\le j\le n}$. The matrix obviously depends on the choice of the bases.
- Theorem 5.18. T is bounded $\iff$ T is continuous $\iff$ T is continuous at 0.

  Proof. => Since T bounded, $\|Tx - Ty\| = \|T(x - y)\| \le \|T\|\|x - y\|$ so T Lipschitz.
  <=. T continuous at 0 and $T(0) = 0$. So $\exists\delta > 0$ such that $\|x\| \le \delta \implies \|Tx\| \le 1$. Let $z \in X$. Then $\delta z/\|z\|$ has norm less than δ so that $T\delta z/\|z\|$ has norm less than 1. This is by linearity $\|Tz\| \le \delta^{-1}\|z\|$ so T is bounded.

- Theorem 5.19 **Bounded linear transformation**. Let $T : M \subset X \to Y$ linear with M dense in X n.v.s. and Y Banach. Let T be bounded.
  Then there exists a unique extension $\bar{T} : X \to Y$ such that $\bar{T}x = Tx$ on M and $\|\bar{T}\| = \|T\|$ in operator norm.

  Proof. $\bar{T}$ is defined by continuity. Let $x \in X$ and $x_n \in M$ converging to x for $\|\cdot\|_X$. Then $\bar{T}x = \lim\limits_{n\to\infty} Tx_n$. Limit exists because x_n Cauchy and T bounded so Tx_n Cauchy and Y Banach. Also

  $$\|\bar{T}x\| \leq \lim_{n\to\infty} \|T\|\|x_n\| = \|T\|\|x\|.$$

  So $\bar{T}$ bounded and $\|\bar{T}\| \leq \|T\|$. Since $Tx = \bar{T}x$ on M, $\|\bar{T}\| = \|T\|$. Uniqueness of $\bar{T}$ comes from fact that $\bar{T}$ is sequentially continuous and so is defined uniquely by $\lim \bar{T}x_n = \lim Tx_n$ at $x = \lim x_n$.

  Typical application of the result: Definition of Adjoint operator of unbounded operators with dense domain of definition (to be seen later).

- **Open mapping** theorem and **Closed Graph** theorem.

- **Open mapping** theorem. Let X, Y Banach and $T : X \to Y$ surjective continuous linear map. Then $\exists C > 0$ such that $T(B_X(0,1)) \supset B_Y(0,c)$.

  i.e.: Open sets are mapped into open sets by T.

- Proof is involved and is based on Baire's lemma, which says that in a complete metric space, if $(X_n)_{n\geq 1}$ is a sequence of dense open sets in X, then $\cap_{n=1}^{\infty} X_n$ is still dense (though not necessarily open).

- Corollary (also called **Open mapping** theorem). Let T be as before and be a bijection. Then T^{-1} exists and is bounded.

  Proof: $\|Tx\| < 1$ implies that $\|x\| < 1$ from theorem. So $\|x\| \leq \frac{1}{c}\|Tx\|$ so that T^{-1} is bounded.

- Corollary [Equivalence of norms]. Assume that $(X, \|\cdot\|_1)$ **and** $(X, \|\cdot\|_2)$ are Banach spaces. Then $\|\cdot\|_1 \sim \|\cdot\|_2$, i.e., the norms are equivalent.

  Proof: From the hypotheses, $(X, \|\cdot\|_1 + \|\cdot\|_2)$ is a Banach space since this is clearly a norm and Cauchy sequences converge in *both* norms. Now the identity map from $(X, \|\cdot\|_1)$ to $(X, \|\cdot\|_1 + \|\cdot\|_2)$ is a continuous bijection. So its inverse is also continuous. This implies the existence of a constant such that

  $$\|\cdot\|_1 + \|\cdot\|_2 \leq C\|\cdot\|_1.$$

  Thus $\|\cdot\|_2$ is dominated by $\|\cdot\|_1$. Similarly, $\|\cdot\|_1$ is dominated by $\|\cdot\|_2$ and the proof is complete.

- Notion of Graph of an operator for X and Y n.v.s. $\text{Graph}(T) = \cup_{x\in X}[x, Tx] \subset X \times Y$.

- Notion of closed operator: an operator is closed if its graph is closed.

- T closed means that $(x_n, Tx_n) \to (x, y)$ implies that $y = Tx$ for $x \in X$. Note that $x \in X$ is not necessarily trivial to establish when X is a n.v.s and not a Banach space. Obviously, when T is bounded from a Banach space to a Banach space, it is closed. The converse is also true on Banach spaces:

- **Closed Graph** theorem. Let X, Y Banach spaces. Let $T : X \to Y$ linear and closed. Then T is **bounded**.

  Proof. We show that $(X, \|\cdot\|_X + \|T\cdot\|_Y)$ is a Banach space. Indeed x_n be such that $\|x_n - x_m\|_X + \|Tx_n - Tx_m\|_Y \to 0$. Then $x_n \to x$ in X and $Tx_n \to Tx$ since T is closed. This implies that $\|x_n - x\|_X + \|Tx_n - Tx\|_Y \to 0$ so that $x_n \to x$ for the norm $\|\cdot\|_X + \|T\cdot\|_Y$ in X.
  The corollary above says that $\|\cdot\|_X$ and $\|\cdot\|_X + \|T\cdot\|_Y$ are equivalent so that $\|Tx\|_Y \leq C\|x\|_X$ for some C.

- For an operator $T : X \to Y$ linear from X to Y both Banach spaces, then [T closed $\iff$ T bounded].
- Most operators one encounters in analysis are closed. However they are not necessarily bounded. This does not contradict the above theorem because these operators will then be defined on a domain of definition $D(T)$ dense in X but different from X.
- (Counter-)Example: Take $X = (\mathcal{C}([0,1]), \|\cdot\|_\infty)$ and $D(T) = \mathcal{C}^1([0,1])$. Then we can define $T : f \mapsto Tf = f'$ as an unbounded operator from $X \supset D(T) \to X$. We easily verify that T is closed (graph is closed in $D(T) \times X$ for the $\|\cdot\|_\infty$ norm: for $f_n \in D(T)$ such that $f_n \to f$ for $\|\cdot\|$ and $Tf_n \to g$ for $\|\cdot\|$, we verify that $f \in D(T)$ and that $Tf = g$). Yet it is not bounded.

- Notion of **Range** and **Kernel** of linear maps.
- Let $T : X \to Y$ be a linear map. Its kernel or nullspace is the space $Ker(T) = N(T) = \{x \in X, Tx = 0\} \subset X$. Its range $R(T) = \cup_{x\in X} Tx = \{y \in Y, \exists x \in X, y = Tx\} \subset Y$.
- Theorem 5.25. $Ker(T)$ and $R(T)$ are linear subspaces of X and Y, respectively. When X and Y are n.v.s. and T is bounded, then $Ker(T)$ is closed.
- $T : X \to Y$ is one-to-one (injective) if $Ker(T) = \{0\}$ and is onto (surjective) if $R(T) = Y$.
- Nullity of T=dim($Ker(T)$); rank of T=dim($R(T)$).
- Prop. 5.30. $T : X \to Y$ bounded linear map; X, Y Banach. Then the two following statements are equivalent.

  (a) $\exists c > 0$ s.t. $c\|x\| \le \|Tx\|, \forall x \in X$.

  (b) $R(T)$ is closed and $Ker(T) = \{0\}$.

  The proof uses the open mapping theorem. The result is useful e.g. to show that the adjoint T^* to an opeartor T is surjective.

- Notion of Finite-dimensional Banach spaces
- Lemma 5.32. Let $(X, \|\cdot\|)$ be a finite dimensional Banach space. Then $\|x\| \sim \sum_i |x_i|$, where $\{x_i\}$ are the coordinates of x.
- Theorem 5.36. Let $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ be a finite dimensional Banach spaces. Then $\|\cdot\|_1 \sim \|\cdot\|_2$.
- Theorem 5.33. Every finite dimensional n.v.s. is a Banach space.
- Corollary 5.34. Every finite dimensional subspace (of a possibly infinite dimensional v.s.) is closed (since Banach).

  This corollary is useful in several applications.
- Theorem 5.35. Every lineral operator on a finite dimensional space is bounded.

  Proof: use action of operator on basis elements.

- Notion of **convergence(s)** of bounded operators. There are two important notions: **uniform convergence** and **strong convergence**.
- Recall that $\mathcal{B}(X,Y) = \mathcal{L}(X,Y)$ is the v.s. of linear continuous (equivalently bounded) operators from X to Y, n.v.s. $(\mathcal{B}(X,Y), \|\cdot\|)$ is a n.v.s. with the $\|\cdot\|$ the operator norm.
- Theorem. $T \in \mathcal{B}(X,Y)$ and $S \in \mathcal{B}(Y,Z)$, then $ST \in \mathcal{B}(X,Z)$ and $\|ST\| \le \|S\|\|T\|$. (trivial)
- Commutator $[S,T] = ST - TS$. S and T commute when $[S,T] = 0$.

- Definition of **uniform convergence** (or convergence in operator norm topology). T_n CV uniformly to T if $\lim_{n\to\infty} \|T - T_n\| = 0$.
- Theorem 5.41. Let X n.v.s. and Y Banach. Then $(\mathcal{B}(X,Y), \|\cdot\|)$ is Banach.

  Proof is standard: Assume T_n Cauchy for that norm. Then $T_n f$ converges in Y since it is Cauchy and Y is Banach. This exhibits Tf. Then show T is linear (easy) and then that T is bounded (also not too hard) because T_n is Cauchy, hence uniformly bounded.
- Definition: T is **compact** if $T(B)$ is precompact (i.e. its closure is compact) in Y for all bounded B in X. Equivalently, T is compact iif for each sequence x_n bounded in X, we can extract a subsequence $x_{\phi(n)}$ such that $Tx_{\phi(n)}$ converges in Y.
- We denote by $\mathcal{K}(X,Y)$ the space of compact operators in $\mathcal{B}(X,Y)$.
- Proposition 5.43 (I). Let X n.v.s. and Y Banach. Then $\mathcal{K}(X,Y)$ is a closed subset in $\mathcal{B}(X,Y)$.
- Proof is not so simple. Proof of linearity of $\mathcal{K}(X,Y)$ is trivial. Proof that $T_n \to T$ uniformly implies that T is compact is a bit more involved. We first construct subsequences $x_{\phi_k\circ\ldots\circ\phi_1(n)}$ so that $T_k x_{\phi_k\circ\ldots\circ\phi_1(n)}$ converges to y_k because T_k is compact. Because Y is Banach, y_k converges to y. We then show that $x_{\phi_m\circ\ldots\phi_1(m)}$ is a subsequence of x_m such that $Tx_{\phi_m\circ\ldots\phi_1(m)} \to y$.
- Proposition 5.43 (II). When $\dim R(T) < \infty$, then T is compact. When S and T are compact, then ST (whenever defined) is compact.
- Proof of these items is trivial.
- Definition of **strong convergence**. We say that the sequence $T_n \in \mathcal{B}(X,Y)$ converges to T strongly if

$$\lim_{n\to\infty} T_n x = Tx, \qquad \forall x \in X.$$

- Theorem. $T_n \to T$ uniformly implies $T_n \to T$ strongly.

  Proof trivial.
- **Uniform Boundedness Theorem** (aka Banach Steinhaus Theorem). Let X, Y Banach and $\{T_i\}_{i\in I} \in \mathcal{B}(X,Y)$ such that $\sup_{i\in I} \|T_i x\| < \infty$ for all $x \in X$. Then there exists $c > 0$ such that $\|T_i x\| \le c\|x\|$ for all $x \in X$ and all $i \in I$. That is, $\sup_{i\in I} \|T_i\| \le c$.

  Proof is involved and uses Baire's lemma.
- Corollary. Let X, Y Banach and $T_n \in \mathcal{B}(X,Y)$ such that $T_n \to T$ strongly. Then $\sum_n \|T_n\| < \infty$, $T \in \mathcal{B}(X,Y)$, and $\|T\|_{\mathcal{L}(X,Y)} \le \liminf_{n\to\infty} \|T_n\|_{\mathcal{L}(X,Y)}$.

  Proof from Theorem relatively easy.
- Application to the convergence of approximation schemes.
- Consider a problem $Au = f$ and a sequence of approximations $A_\varepsilon u_\varepsilon = f_\varepsilon$, where A and A_ε are linear operators from X to Y Banach.
- Definition. The approximation scheme is **convergent** if for all $f_\varepsilon \to f$, we have $u_\varepsilon \to u$ for the solutions to the above problems.
- Definition. The approximation scheme is **consistent** if for all $v \in X$, $A_\varepsilon v \to Av$ as $\varepsilon \to 0$.
- Definition. The approximation scheme is **stable** if $\|A_\varepsilon^{-1}\| \le M$ independent of ε.
- Theorem (Lax equivalence). A consistent scheme is convergent iif it is stable.
- proof <=: Write $u - u_\varepsilon = A_\varepsilon^{-1}((A_\varepsilon - A)u + f - f_\varepsilon)$.

- proof =>: Choose $f_\varepsilon = f$ so that $u_\varepsilon = A_\varepsilon^{-1} f \to u$ since scheme is convergent. This implies that $\sup_\varepsilon \|A_\varepsilon^{-1} f\|$ is bounded since it is convergent. Using the above corollary, we deduce that A_ε^{-1} is uniformly bounded.

- Note that strong convergence does not imply uniform convergence. For instance, we can show as in class that on the Banach space $\mathcal{C}_0(\mathbb{R})$ with the uniform norm, the shift operator $\tau_h : f(x) \mapsto f(x+h)$ converges strongly to the identity operator but does not converge uniformly.

- Notion of **Dual Space.**

- In finite dimensions, we use the coordinate functions $x_i : \mathbb{R}^{\to}\mathbb{R}$, which to $\mathbf{x}$ associates $x_i(\mathbf{x}) = \mathbf{x} \cdot \mathbf{e}_i$. In infinite vector spaces, linear forms $\varphi : X \to \mathbb{R}$ play a similar role.

- Definition[Dual space]. Let X be a v.s. The space of continuous linear functionals from X to $\mathbb{R}$ is called the (topological) dual space to X.

- It is often denoted by $X^* = \mathcal{L}(X, \mathbb{R}) = \mathcal{B}(X, \mathbb{R})$. On X^*, we have $|\varphi(x)| \le M_\varphi \|x\|$ for all $x \in X$ and define
$$\|\varphi\|_{X^*} = \sup_{x \neq 0} \frac{|\varphi(x)|}{\|x\|_X}.$$

- We often use the notation $\langle \varphi, x \rangle_{X^*,X} = \varphi(x)$, which is the duality product.

- Since $\mathbb{R}$ is Banach, then so is $(X^*, \|\cdot\|_{X^*})$.

- We have $(\mathbb{R}^n)^* \sim \mathbb{R}^n$ by showing that $\{x_i\}$ forms a basis of $(\mathbb{R}^n)^*$.

- The duals of Hilbert spaces are also isomorphic to the Hilbert spaces.

- This is not true in general, and not true for L^p spaces for $p \neq 2$.

- For $X = \mathcal{C}([a,b])$ with the uniform norm, the dual space is the space $X^* = \mathcal{M}_n([a,b])$ of Radon measures on $[a,b]$. This is a much larger space than X. Note that by the Radon-Riesz representation theorem, for each $\varphi \in X^*$, there is a Radon measure in $\mathcal{M}_n([a,b])$ which can be represented as $\mu = \mu^+ - \mu^-$, where $\mu^\pm$ are Borel measures. So we have the representation
$$\varphi(f) = \langle \varphi, f \rangle = \int_a^b f d\mu^+ - \int_a^b f d\mu^-.$$
The duality product may therefore be "seen" as an integral.

- Linear functionals are often constructed on finite dimensional spaces and need to be extended to the whole space X. The following theorem says that an extension (not necessarily unique) exists that does not increase the norm.

- Theorem 5.58 [**Hahn Banach**]. Let $Y \subset X$ be a vectorial subspace. Let $\psi : Y \to \mathbb{R}$ be a bounded linear functional with norm $\|\psi\|_{Y^*} = M$. Then there exists ϕ from X to $\mathbb{R}$ such that $\varphi = \psi$ on Y and $\|\varphi\|_{X^*} = M$.

  Proof is quite involved and is based on Zorn's lemma or on the axiom of choice.

- Corollary. $\forall x_0 \in X$, $\exists f_0 \in X^*$ such that $\|f_0\|_{X^*} = \|x_0\|_X$ and $\langle f_0, x_0 \rangle_{X^*,X} = \|x_0\|^2$.

  Proof. Choose $Y = \mathbb{R}x_0$ and $\psi(t, x_0) = t\|x_0\|^2$.

- Corollary. Let $x \in X$. Then $\|x\| = \sup_{\|f\| \le 1\}} | < f, x > | = \max_{\|f\| \le 1\}} | < f, x > |$.

  Proof. Choose $f_1 = \|x\|^{-1} f_0$ where $\|f_0\| = \|x\|$.

- Corollary. Assume that $\varphi(x) = \varphi(y)$ for all $\varphi \in X^*$. Then $x = y$. (X^* separates X).

  Proof. If not, $\exists f_0$ such that $\|f_0\| = \|x - y\|$ and $< f_0, x - y >= \|x - y\|^2$.

- $X^{**} = (X^*)^*$ is a Banach space. We have $X \subset X^{**}$.
- When $X \sim X^{**}$ we say that X is **reflexive**.
- Many spaces are reflexive; for instance L^p for $1 < p < \infty$. However L^1 and L^∞ are not reflexive.

- Notion of **weak convergence**.
- Definition. $x_n \in X$ converges weakly to x in X if $\varphi(x_n) \to \varphi(x)$ in $\mathbb{R}$ as $n \to \infty$ for all $\varphi \in X^*$.
- Strong convergence (for $\|\cdot\|$ in X) implies weak convergence since φ is continuous. The reverse is *not* true. For instance $f_n(x) = \cos nx$ converges to 0 weakly in L^2 but does not converge strongly (to anything).
- $\varphi_n \in X^*$ converges weak * to $\varphi \in X^*$ if $\varphi_n(x) \to \varphi(x)$ in $\mathbb{R}$ as $n \to \infty$ for all $x \in X$.
- On X we can generate the weak topology $(X, \mathcal{T}_w)$ by enforcing that $x \to \varphi(x)$ is continuous for all $\varphi \in X^*$ (and $(X, \mathcal{T}_w)$ is the coarsest topology (ie the most economical in terms of number of open sets) to do that). Then the above definition is equivalent to (topological) convergence for the weak topology.
- Similarly, on X^* we can generate the weak topology $(X^*, \mathcal{T}_{w*})$ by enforcing that $\varphi \to \varphi(x)$ is continuous for all $x \in X$ (and $(X^*, \mathcal{T}_{w*})$ is the coarsest topology (ie the most economical in terms of number of open sets) to do that).
- On X^*, $(X^*, \mathcal{T}_{w*})$ has less open sets than $(X^*, \mathcal{T}_w)$, which itself has less open sets than $(X^*, \|\cdot\|)$. So strong convergence implies weak convergence implies weak star convergence. On reflexive spaces, the first two are equivalent since the weak and weak star topologies are the same.
- The real advantage of having less open sets is that it is then easier to be compact since it is easier to find finite subcoverings. The main reason for introducing the weak star topology is the following extremely important result.
- Theorem 5.61 [**Banach Alaoglu**]. The closed unit ball in X^* is weak-star compact. (ie compact for the weak star topology). This implies that for each bounded sequence in X^*, there is a subsequence that converges in X^* for the weak star topology.
- The proof is fairly involved and uses results on the product topology of a coutable product of spaces.
- Theorem [**Kakutani**]. Let X be Banach. Then X is reflexive if and only if $B_X = \{x \in X, \|x\| = 1\}$ is compact for the weak topology.
- So we see that the weak topology is not making bounded domains into compact domains for spaces that are not reflexive. In some sense, only the weak star topology is of interest in many applications. The weak topology is only interesting when it is the same thing as the weak star topology, because then bounded sequences automatically possess converging subsequences.