MATH 185: COMPLEX ANALYSIS
FALL 2008/09
PROBLEM SET 5 SOLUTIONS

1. Consider the nth Taylor polynomial approximant to $\exp(z)$,

$$f_n(z) := 1 + z + \frac{1}{2!}z^2 + \cdots + \frac{1}{n!}z^n.$$

Show that for all $z \in \mathbb{C}$ with $\operatorname{Re}(z) < 0$,

$$|\exp(z) - f_n(z)| \le |z|^{n+1}$$

for all $n \in \mathbb{N}$.

SOLUTION. Let $\Omega := \{z \in \mathbb{C} \mid \operatorname{Re}(z) < 0\}$. Let $z \in \Omega$ and let Γ_z be the line segment from 0 to z. Then by Proposition **3.9**,

$$\int_{\Gamma_z} \exp(w)\,dw = \exp(z) - \exp(0) = e^z - 1.$$

By Proposition **3.5**,

$$\begin{aligned}\left|\int_{\Gamma_z} \exp(w)\,dw\right| &\le \left(\sup_{w\in\Gamma_z} |\exp(w)|\right) \times \left(\int_{\Gamma_z} |dw|\right)\\ &= |z| \sup_{w\in\Gamma_z} e^{\operatorname{Re}(w)}\\ &\le |z|\end{aligned}$$

where the last step follows since $\Gamma_z \subset \Omega$ and thus $\operatorname{Re}(w) \le 1$. Hence

$$|e^z - 1| \le |z|.$$

Suppose the statement is true for $n-1$, ie.

$$|\exp(z) - f_{n-1}(z)| \le |z|^n$$

for all $z \in \Omega$. Now observe that

$$\begin{aligned}\int_{\Gamma_z} \exp(w) - f_{n-1}(w)\,dw &= \exp(z) - \left[z + \frac{1}{2!}z^2 + \cdots + \frac{1}{n!}z^n\right] - \exp(0)\\ &= \exp(z) - f_n(z).\end{aligned}$$

By Proposition **3.5** and the induction hypothesis,

$$\begin{aligned}\left|\int_{\Gamma_z} \exp(w) - f_{n-1}(w)\,dw\right| &\le \left(\sup_{w\in\Gamma_z} |\exp(w) - f_{n-1}(w)|\right) \times \left(\int_{\Gamma_z} |dw|\right)\\ &\le |z| \sup_{w\in\Gamma_z} |w|^n\\ &= |z|^{n+1}.\end{aligned}$$

Hence

$$|\exp(z) - f_n(z)| \le |z|^{n+1}$$

as required. By mathematical induction, this holds for all n.

2. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. As usual, we write $f(z) = u(x,y) + iv(x,y)$ for $z = x + iy$.
(a) Show that if u is positive valued, ie. $u(x,y) > 0$ for all $x, y \in \mathbb{R}$, then f is constant.

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(b) Show that if $|u(x,y)| < |v(x,y)|$ for all $x, y \in \mathbb{R}$, then f is constant.

(c) Can we still draw the same conclusions if '$<$' is replaced by '$>$' in (a) and (b)?

SOLUTION. Note that e^x is a monotone increasing function on $\mathbb{R}$.

- For (a), we choose $g(z) = e^z$ and note that
$$|e^{f(z)}| = |e^u e^{iv}| = e^u \leq e^0 = 1.$$
- For (b), we choose $g(z) = e^{z^2}$ and note that
$$|e^{f(z)^2}| = |e^{u^2 - v^2} e^{2iuv}| = e^{u^2-v^2} \leq e^0 = 1.$$

Applying Liouville's theorem then implies that $e^{f(z)}$ and $e^{f(z)^2}$ are constant functions. It then follows that f must also be a constant function. Note that $-f$ is also an entire function. So by applying (a) and (b) to $-f$ we can draw the same conclusion if '$<$' is replaced by '$>$'.

3. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function.

(a) Show that if f satisfies the following conditions
$$f(z+1) = f(z), \quad f(z+i) = f(z)$$
for all $z \in \mathbb{C}$, then f is constant.

SOLUTION. Given any real number $x \in \mathbb{R}$, we will write $[x]$ for the integral part of x and $\langle x \rangle$ for the fractional part of x. For example $[-5.12] = -5$ and $\langle -5.12 \rangle = 0.12$. Note that $[x] \in \mathbb{Z}$, $\langle x \rangle \in [0,1)$, and $x = [x] + \langle x \rangle$ for all $x \in \mathbb{R}$. Observe that two conditions given may be applied recursively to obtain
$$f(z + m + ni) = f(z)$$
for any $m, n \in \mathbb{Z}$. In particular, if $z = x + iy$, then
$$f(z) = f(\langle x \rangle + i\langle y \rangle + [x] + i[y]) = f(\langle x \rangle + i\langle y \rangle).$$
Note that for any $z = x + iy \in \mathbb{C}$, $\langle x \rangle + i\langle y \rangle \in [0,1) \times [0,i) \subseteq [0,1] \times [0,i]$ (closed unit square in $\mathbb{C}$), and so
$$\sup_{z \in \mathbb{C}} |f(z)| = \sup_{z \in [0,1) \times [0,i)} |f(z)| \leq \max_{z \in [0,1] \times [0,i]} |f(z)|.$$
The last term is finite by the Extreme Value Theorem in Math **104** (since $[0,1] \times [0,i]$ is compact and f is analytic, therefore continuous) and so f is bounded. Liouville's theorem then implies that f is constant.

(b) Let $\alpha, \beta \in \mathbb{C}^\times$ be such that $\alpha/\beta \notin \mathbb{R}$. Show that if f satisfies the following conditions
$$f(z+\alpha) = f(z), \quad f(z+\beta) = f(z)$$
for all $z \in \mathbb{C}$, then f is constant.

SOLUTION. The condition $\alpha/\beta \notin \mathbb{R}$ implies that α, β span $\mathbb{C}$ as a real vector space of dimension 2. In other words, any $z \in \mathbb{R}$ may be written as $z = x\alpha + y\beta$ where $x, y \in \mathbb{R}$. Now the same argument as in (a) yields
$$\begin{aligned} f(z) &= f(x\alpha + y\beta) \\ &= f(\alpha\langle x \rangle + \beta\langle y \rangle + \alpha[x] + \beta[y]) \\ &= f(\alpha\langle x \rangle + \beta\langle y \rangle). \end{aligned}$$
Since $\alpha\langle x \rangle + \beta\langle y \rangle \in [0,\alpha) \times [0,\beta) \subseteq [0,\alpha] \times [0,\beta]$, ie. the closed parallelogram bounded by the line segments from 0 to α and 0 to β and this is compact, the same argument in (a) implies that f is constant.

(c) Show that if f satisfies the following conditions
$$f(z+1) = f(z), \quad f(z+\sqrt{2}) = f(z)$$
for all $z \in \mathbb{C}$, then f is constant.

SOLUTION. By the given condition

$$f(z+m+n\sqrt{2}) = f(z)$$

for all $m, n \in \mathbb{Z}$. In particular, set $z = 0$ and consider the function

$$g(z) = f(z) - f(0).$$

Hence

$$g(m+n\sqrt{2}) = f(0+m+n\sqrt{2}) - f(0) = 0.$$

Note that the set $\{m+n\sqrt{2} \mid m, n \in \mathbb{Z}\}$ has accumulation points. For instance $\sqrt{2}$ is an accumulation point: given any $\varepsilon > 0$, pick $n \in \mathbb{N}$ with $1/n < \varepsilon$, since there exists $m \in \mathbb{N}$ such that

$$\left|\frac{m}{n} - \sqrt{2}\right| \leq \frac{1}{n^2},$$

we get that

$$|m - n\sqrt{2}| \leq \frac{1}{n} < \varepsilon.$$

Hence g vanishes on a set with accumulation point and so

$$g(z) = 0$$

for all $z \in \mathbb{C}$ by the Uniqueness Theorem for analytic fuctions (Theorem **6.9** in the text) and so

$$f(z) = f(0)$$

for all $z \in \mathbb{C}$. Hence f is constant.

4. Let $S = \{x+iy \in \mathbb{C} \mid x, y \in [0,1]\}$ be the unit square in $\mathbb{C}$. Let f be analytic on a region Ω that contains S. Suppose the following is true:

(i) for all z with $\mathrm{Re}(z) = 0$, $0 \leq \mathrm{Im}(z) \leq 1$,

$$f(z+1) - f(z) \geq 0;$$

(ii) for all z with $0 \leq \mathrm{Re}(z) \leq 1$, $\mathrm{Im}(z) = 0$,

$$f(z+i) - f(z) \geq 0.$$

Show that f is constant.

SOLUTION. Since f is anlytic on Ω and $\Gamma = \partial S$ is a rectangular path contained in Ω, we may apply Cauchy's theorem to get

$$\begin{aligned} 0 &= \int_\Gamma f(z)\,dz \\ &= \int_0^1 f(x)\,dx + i\int_0^1 f(1+yi)\,dy - \int_0^1 f(x+i)\,dx - i\int_0^1 f(yi)\,dy \\ &= \int_0^1 [f(x) - f(x+i)]\,dx + i\int_0^1 [f(1+iy) - f(iy)]\,dy. \end{aligned}$$

Hence we have

$$\int_0^1 [f(x) - f(x+i)]\,dx = 0 \quad \text{and} \quad \int_0^1 [f(1+iy) - f(iy)]\,dy = 0. \tag{4.1}$$

By condition (ii),

$$f(x) - f(x+i) \leq 0$$

for all $0 \leq x \leq 1$; and by condition (i),

$$f(1+iy) - f(iy) \geq 0$$

for all $0 \leq y \leq 1$. Furthermore, since f is analytic and thus continuous in Ω, the integrands in (4.1) must be identically zero for $x, y \in [0,1]$. So

$$f(x) = f(x+i) \quad \text{and} \quad f(iy) = f(iy+1)$$

for $x, y \in [0,1]$. So

$$f(z) = f(z+i) \quad \text{and} \quad f(z) = f(z+1) \tag{4.2}$$

for all $z \in [0,1]$ and $z \in [0,i]$ respectively. Since these are subsets of Ω with limit points, (4.2) must hold for all $z \in \Omega$ by the Uniqueness Theorem. Now we may define a function $F : \mathbb{C} \to \mathbb{C}$ as follows:

$$F(x+iy) = f(\langle x \rangle + i\langle y \rangle).$$

Hence,

$$F(z) = F(z+i) \quad \text{and} \quad F(z) = F(z+1)$$

for all $z \in \mathbb{C}$. Note that F is analytic for all $z \in \mathbb{C}$ (why?) and so by Problem **3**(a), F is constant on $\mathbb{C}$ and so f is constant on Ω.

5. (a) Let f and g be entire functions that satisfy

$$|f(z)| < |g(z)|$$

for all $z \in \mathbb{C}$. Show that there exists a constant $\lambda \in \mathbb{C}$ such that

$$f(z) = \lambda g(z)$$

for all $z \in \mathbb{C}$.

SOLUTION. Note that the given condition implies that $f(z) \neq 0$ for all $z \in \mathbb{C}$ (since $|g(z)| > |f(z)| \geq 0$) and so the function $h : \mathbb{C} \to \mathbb{C}$ defined by

$$h(z) = \frac{f(z)}{g(z)}$$

is also an entire function. The given condition then implies that

$$|h(z)| = \left| \frac{f(z)}{g(z)} \right| < 1$$

and so by Liouville's theorem $h(z) = \lambda$ for some $\lambda \in \mathbb{C}$. Hence

$$f(z) = \lambda g(z)$$

for all $z \in \mathbb{C}$.

(b) Determine all entire functions f that satisfies

$$|f'(z)| < |f(z)|$$

for all $z \in \mathbb{C}$.

SOLUTION. By (a), we have that

$$f'(z) = \lambda f(z)$$

for all $z \in \mathbb{C}$. Multiplying the equation by $\exp(\lambda z)$, we get

$$\exp(-\lambda z) f'(z) - \lambda \exp(-\lambda z) f(z) = 0.$$

By the product rule in Proposition **2.7**, the LHS may be written as a derivative of a product and we get

$$\frac{d}{dz}[\exp(-\lambda z) f(z)] = 0.$$

Hence the product $\exp(-\lambda z) f(z)$ is the constant function. Let $\alpha \in \mathbb{C}$ be its value (in fact, we can see that $\alpha = f(0)$). Hence

$$f(z) = \alpha \exp(-\lambda z).$$