The decomposition of matrices

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Overview

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Motivation

- Solving linear systems
  1. Gaussian elimination
  2. LU-decomposition
  3. QR-decomposition

- Goal: Faster algorithm
  1. Toeplitz decomposition
  2. Tridiagonal decomposition
Set up

- $M_n$: the space of all $n \times n$ matrices
- $r$: natural number
- $V_1, \ldots, V_r$: algebraic varieties in $M_n$
- morphism $\phi : V_1 \times \cdots \times V_r \rightarrow M_n$

$$\phi(A_1, \ldots, A_r) = A_1 \cdots A_r$$
Questions

- What types of $V_j$’s can make $\phi$ surjective?
- For fixed types of $V_j$’s, what is the smallest $r$ such that $\phi$ is surjective?

Weaker version

- What types of $V_j$’s can make $\phi$ dominant?
- For fixed types of $V_j$’s, what is the smallest $r$ such that $\phi$ is dominant?
Connection to matrix decomposition

**Exact case**

The morphism

$$\phi : V_1 \times \cdots \times V_r \rightarrow M_n$$

is surjective if and only if for every matrix $X \in M_n$, we can decompose $X$ into the product of elements in $V_j$’s.

**Generic case**

The morphism

$$\phi : V_1 \times \cdots \times V_r \rightarrow M_n$$

is dominant if and only if for a generic (almost every) matrix $X \in M_n$, we can decompose $X$ into the product of elements in $V_j$’s.
Examples

- $LU$-decomposition: $X = LUP$
- $QR$-decomposition: $X = QR$
- Gaussian elimination: $X = PDQ$
Non-examples

- the set of all upper triangular matrices
- subgroups of $\text{GL}_n$
- one dimensional linear subspaces of $M_n$
- subspaces of the space of matrices of the form

$$
\begin{bmatrix}
0 & * & \cdots & * \\
0 & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & * & \cdots & *
\end{bmatrix}
$$
### Theorem (open mapping theorem)

*If $f : X \hookrightarrow Y$ is a dominant morphism between algebraic varieties, there exists a subset $V$ of $f(X)$ such that*

1. $V$ is open and dense in $Y$ and
2. $\dim f^{-1}(y) = \dim X - \dim Y$ for any $y \in V$. 
easy to verify whether a morphism is dominant

**Lemma (dominant Lemma)**

Let $f : X \to Y$ be a morphism between algebraic varieties. Assume that exists a point $x \in X$ such that the differential $df|_x$ is surjective, then $f$ is dominant.

passing from open sets to the whole group

**Lemma (generating Lemma)**

Let $G$ be an algebraic group and let $U, V$ be open dense subsets of $G$. Then $G = UV$. 
Method

- $\phi_0 : V_1 \times \cdots \times V_{r_0} \to M_n$
- $\tilde{V}_j = V_j \cap GL_n, j = 1, 2, \ldots, r_0$
- $\tilde{\phi}_0 : (\tilde{V}_1 \times \cdots \times \tilde{V}_{r_0}) \times (\tilde{V}_1 \times \cdots \times \tilde{V}_{r_0}) \to GL_n$
- $\phi : (V_1 \times \cdots \times V_{r_0})^{\times d} \to M_n$
  - $d$: to be determined

**step 1.** find an $r_0$ making $\phi_0$ dominant: dominant Lemma + open mapping theorem

**step 2.** $\tilde{\phi}_0$ is surjective: generating Lemma

**step 3.** $\phi$ is surjective: known decompositions
Definition

- Toep$_n$: space of Toeplitz matrices
- $r_0 = \lfloor \frac{n}{2} \rfloor + 1$
- Toep$_n^{\times r_0} = \underbrace{\text{Toep}_n \times \cdots \times \text{Toep}_n}_{r_0 \text{ copies}}$
- $\phi_0 : \text{Toep}_n^{\times r_0} \rightarrow M_n$
- $t_j$: indeterminants $j = 1, 2, \ldots, r$
- $T_0, T_1, T_{-1}, \ldots, T_{n-1}, T_{-n+1}$: standard basis for Toep$_n$
- $A_j = T_0 + t_j(T_{n-j} - T_{-(n-j)}), j = 1, 2, \ldots, r$
Toeplitz decomposition

first express

\[ d\phi_0|_{(A_1,\ldots,A_r)} \]

as a \( r_0(2n - 1) \times n^2 \) matrix \( M \),
then find a nonzero \( n^2 \times n^2 \) minor (in terms of \( t \)'s) of \( M \), this proves

Theorem

\( \phi_0 \) is a dominant morphism.
Motivation and general problems

Method

Toeplitz decomposition and Hankel decomposition

Bidiagonal decomposition and Tridiagonal decomposition

Toeplitz decomposition

- \( \tilde{\phi}_0 : \text{Toep}_n^{\times 2r_0} \to \text{GL}_n \)
- \( \phi : \text{Toep}_n^{\times (4r_0+1)} \to M_n \)

Open mapping theorem + generating Lemma \( \implies \tilde{\phi}_0 \) surjective

Gaussian elimination \( \implies X = PTQ \) for \( P, Q \in \text{GL}_n, T \in \text{Toep}_n \)

Hence

**Theorem**

\( \phi \) is a surjective morphism. Equivalently, every \( n \times n \) matrix is a product of \( 2n + 5 \) Toeplitz matrices.
the decomposition is not unique
no explicit algorithm is known
$2n + 5$ is not sharp: every $2 \times 2$ matrix can be decomposed as a product of two Toeplitz matrices
Important implication of the decomposition

- Gaussian elimination: \( \frac{n^3}{2} + \frac{n^2}{2} \) operations
- \( LU \)-decomposition: \( \frac{n^3}{3} + \frac{n^2}{3} - \frac{n}{3} \) operations
- \( QR \)-decomposition: \( 2n^3 + 3n^2 \) operations
- Bitrneath & Anderson, or Houssam, Bernard & Michelle: \( O(n \log^2 n) \) operations for Toeplitz linear systems
- K. Ye & L.H Lim: \( O(n^2 \log^2 n) \) operations for general linear systems
Definition

$A = (a_{i,j}): n \times n$ matrix

- Rotation: $A^R = (a_{n+1-j,i})$
- Right swap: $A^S = (a_{i,n+1-j})$
- Left swap: $SA = (a_{n+1-i,j})$

three operations are all isomorphisms and

$A$ Toeplitz $\iff A^R$ Hankel
$A$ Toeplitz $\iff A^S$ Hankel
$A$ Toeplitz $\iff SA$ Hankel
Hankel decomposition

- $A, B$: $n \times n$ matrices
  1. $(AB)^R = B^{RS} A^R = B^R(S(A^R))$
  2. $A^{SR} = A^T$
  3. $(S A)^R = A^T$
  4. $(AB)^S = AB^S$
  5. $S(AB) = SAB$

- $A_1, \ldots, A_m$: $n \times n$ matrices

Relations above $\implies (A_1^S \cdots A_m^S)^R = A_m^{SR} \cdot S(A_{m-1}^{SRS})(A_1^S \cdots A_{m-2}^S)^R$
Hankel decomposition

first consider

\[ f : \text{Hank}_n^{\times r} \xrightarrow{S} \text{Toep}_n^{\times r} \xrightarrow{\phi_0} M_n \xrightarrow{R} M_n \]

S: right swap operator
R: rotation operator
then

\[ \text{im}(f) \simeq \phi_0(\text{Toep}_n^{\times r}) \simeq \phi_0(\text{Hank}_n^{\times r}) \]

this proves

**Theorem**

\( \phi_0 \) is dominant for \( r = \lfloor n/2 \rfloor + 1 \).
Hankel decomposition

same argument $\implies$ exact version for Hankel decomposition

Theorem

$\phi : \text{Hank}_n^{(2n+5)} \rightarrow M_n$ is surjective.
• **$U$**: space of upper triangular matrices
• **$L$**: space of lower triangular matrices
• **$D_{1,\geq 0}$**: space of upper bidiagonal matrices
• **$D_{1,\leq 0}$**: space of lower bidiagonal matrices
• **$\phi_U : D_{\geq 0}^{\times n} \mapsto U$**
• **$\phi_L : D_{\leq 0}^{\times n} \mapsto L$**
bidiagonal decomposition

- rank of the differential at a generic point
  \[ \phi_U, \phi_L \text{ dominant} \]
- open mapping theorem + generating Lemma
  \[ \text{element in } U = \text{product of } 2n \text{ elements in } D_{\geq 0} \]
- open mapping theorem + generating Lemma
  \[ \text{element in } L = \text{product of } 2n \text{ elements in } D_{\leq 0} \]
bidiagonal decomposition

- \( P_0 \): all principal minors nonzero
  \[ \implies P_0 = LU, \ L \in L, \ U \in U \]

- \( P_0 \) = product of 4\(n\) bidiagonal matrices
  \[ \implies \text{generic matrix} = \text{product of 4}n\ \text{bidiagonal matrices} \]

- open mapping theorem + generating Lemma
  \[ \implies \text{invertible matrix} = \text{product of 8}n\ \text{bidiagonal matrices} \]

- Gaussian elimination
  \[ \implies \text{any matrix} = \text{product of 16}n\ \text{bidiagonal matrices} \]

This proves

**Theorem**

*Every \( n \times n \) matrix is a product of 16 bidiagonal matrices.*
**Question**

- know: a matrix = product of $16n$ tridiagonal matrices
- expected number of factors: $\left\lfloor \frac{n^2}{3n-2} \right\rfloor + 1 \approx \left\lfloor \frac{n}{3} \right\rfloor + 1$
- questions:
  1. better decomposition?
  2. least number of factors needed = expected number?

answers:
  1. yes
  2. no
Motivation and general problems

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Bidiagonal decomposition and Tridiagonal decomposition

definition

- $D_k$: space of $n \times n$ matrices with $a_{ij} = 0$ if $|i - j| > k$, $k = 1, 2, \ldots, n - 1$
- $D_1^{\times r} = D_1 \times \cdots \times D_1$
  \text{r copies}
- $\phi: D_1^{\times r} \to M_n$ defined by matrix multiplication
bidiagonal decomposition

- $A \in D_1, B \in D_k \implies AB \in D_{k+1} \implies r \geq n - 1$ if $\phi$ dominant
- Gaussian elimination $\implies$ a matrix $= LDU$, $L$ lower triangular, $D$ diagonal, $P$ permutation and $U$ upper triangular
- element in $L = \text{product of } 2n$ lower triangular $\implies$ element in $L = 2n$ triangular
- (M.D Samson and M. F Ezerman) permutation matrix $= \text{product of } 2n - 1$ tridiagonal matrices

this proves

**Theorem**

*If $\phi$ is surjective, then $n - 1 \leq r \leq 6n$.***
Important Implication of tridiagonal decomposition

solving linear systems

- Thomas algorithm: $O(n)$ operations for tridiagonal linear systems
- K. Ye and L.H Lim: $O(n^2)$ operations for general linear systems
Open questions

- smallest number of factors needed to for Toeplitz decomposition?
  conjecture: $\left\lfloor \frac{n}{2} \right\rfloor + 1$

- same questions for Hankel, tridiagonal, bidiagonal decompositions

- explicit algorithms for these decompositions?
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Thank You!