Direct Sum Decomposability of Polynomials

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Outline

1. Problem 1: Direct sum decomposability
2. Problem 2: Bounds for Waring rank
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Direct sum decompositions

Can $F = F(x_1, \ldots, x_n)$ be written as a sum of polynomials in separate variables? We allow a linear change of coordinates:

$$F = G(t_1, \ldots, t_k) + H(t_{k+1}, \ldots, t_n)$$

where the $t_i$ are linearly independent linear forms.

**Example**

$xy \neq G(x) + H(y)$, but $xy = \frac{1}{4}(x + y)^2 - \frac{1}{4}(x - y)^2$.

Let $\det_n = \det((x_{ij})_{1 \leq i, j \leq n})$, the $n \times n$ generic determinant.

**Problem**

$\det_2 = ad - bc$ is decomposable. Is $\det_n$ decomposable for $n > 2$?
Apolarity

- \( S = \mathbb{C}[x_1, \ldots, x_n], \quad T = \mathbb{C}[\alpha_1, \ldots, \alpha_n] \) dual ring: \( \alpha_i \) acts as \( \partial/\partial x_i \).
- For \( F \in S, \ F^\perp \subset T \) ideal.

Example

\( \det_n^\perp \) is generated by:

- \( \alpha_{i,j}^2 \)
- \( \alpha_{i,j_1} \alpha_{i,j_2} \) (two entries from same row)
- \( \alpha_{i_1,j} \alpha_{i_2,j} \) (two entries from same column)
- permanents of 2 \( \times \) 2 submatrices:
  \[
  \text{per} \begin{pmatrix} \alpha_{i,j} & \alpha_{i,l} \\ \alpha_{k,j} & \alpha_{k,l} \end{pmatrix} = \alpha_{i,j} \alpha_{k,l} + \alpha_{i,l} \alpha_{k,j}.
  \]

[Shafiei, 2012]
Ranestad-Schreyer bound for Waring rank

Waring rank: \( r(F) = \) least \( r \) such that \( F = \ell_1^d + \cdots + \ell_r^d \).

**Example**

\( \det_n = \) sum of \( n! \) terms of the form \( x_1 \cdots x_n \). Each term has rank \( r(x_1 \cdots x_n) = 2^{n-1} \). So \( r(\det_n) \leq 2^{n-1} n! \).

Lower bounds for rank:

- **Sylvester:** \( r(F) \geq \max\{\dim(T/F^\perp)_a\} \)
- **Ranestad-Schreyer:** \( r(F) \geq \frac{1}{\delta} \sum_a \dim(T/F^\perp)_a \), where \( \delta = \) maximum degree of a generator of \( F^\perp \).

**Example**

Sylvester’s bound gives \( r(\det_n) \geq \left( \left\lfloor \frac{n}{2} \right\rfloor \right)^2 \left( r(\det_3) \geq 9 \right) \).

The Ranestad-Schreyer bound gives \( r(\det_n) \geq \frac{1}{2} \left( \binom{2n}{n} \right) \left( r(\det_3) \geq 10 \right) \).
Apolar generating degree

What is the generating degree of $F^\perp$?

Say $\deg F = d$. Then $F^\perp$ contains all differential operators of degree $> d$, so no generators of degree $d + 2$ or higher (but maybe $d + 1$).

**Problem**

Give conditions for $F^\perp$ to have high-degree or low-degree generators.

**Theorem (Casnati–Notari)**

$F^\perp$ has a minimal generator of degree $d + 1$ if and only if $r(F) = 1$, $F = x^d$. 
Theorem

If $F$ is decomposable as a direct sum then $F^\perp$ has a minimal generator of degree $d$.

Corollary

For $n > 2$, $\det_n$ is not decomposable.

Proof.

Say $F = G(X) - H(Y)$ is a direct sum decomposition.

- For $0 \leq a < d$, $F^\perp_a = G^\perp_a \cap H^\perp_a$: If $DG = DH = 0 \in \mathbb{C}[X] \cap \mathbb{C}[Y]$ then $\deg DG = \deg DH = 0$ so $\deg D = d$.
- Let $\Delta_X(G) = 1$, $\Delta_Y(H) = 1$, $\Delta = \Delta_X + \Delta_Y$. Then $F^\perp_d = (G^\perp \cap H^\perp)_d + \langle \Delta \rangle$.

So $F^\perp = (G^\perp \cap H^\perp) + \Delta$ and $\Delta$ is a minimal generator.
Converse

The converse fails.

Example

\[ F = xy^{d-1} \] has \( F^\perp = \langle \alpha^2, \beta^d \rangle \).

But \( F \) is indecomposable because \( \ell_1^d - \ell_2^d \) has distinct factors.

However \( xy^{d-1} \) is a limit of direct sums:

\[
xy^{d-1} = \lim_{t \to 0} \frac{1}{dt} \left((y + tx)^d - y^d \right).
\]

Theorem

If \( F^\perp \) has a minimal generator of degree \( d \) then \( F \) is a limit of direct sums.
Converse again

Theorem

If $F \perp$ has a minimal generator of degree $d$ then $F$ is a limit of direct sums.

Once again the converse fails!

Example

- $x^d - ty^d \rightarrow x^d$, but $(x^d) \perp = \langle \alpha^{d+1}, \beta \rangle$.
- $xyz - tw^3 \rightarrow xyz$, but $(xyz) \perp = \langle \alpha^2, \beta^2, \gamma^2, \delta \rangle$.

Theorem

Let $n \geq 2$, $d \geq 3$. If $F$ is a limit of direct sums and $F$ cannot be written using fewer variables, then $F \perp$ has a minimal generator of degree $d$. 
Chart of inclusions

\[
\text{DirSum} \cup \text{DirSum} \cap \text{Con} \subsetneq \text{ApoMax} \cup \text{ApoMax} \cap \text{Con} = \text{DirSum} \cap \text{Con}
\]

- **DirSum**: decomposable as a direct sum
- **ApoMax**: $F^\perp$ has a minimal generator of degree $d$
- **Con**: concise, i.e., cannot be written using fewer variables
An idea that doesn’t work

Suppose $F_t \to F_0 = F$ and the $F_t$ are direct sums for $t \neq 0$. Does semicontinuity of graded Betti numbers show that $F^\perp$ has a generator of degree $d$?

No, because $F^\perp$ is not necessarily the flat limit of the $F_t^\perp$.

Example

- $(x^d - ty^d)^\perp = \langle \alpha \beta, t\alpha^d + \beta^d \rangle$
- $(x^d)^\perp = \langle \alpha^{d+1}, \beta \rangle$

So $(x^d - ty^d)^\perp \not\leadsto (x^d)^\perp$. 
Well, it works sometimes

**Theorem**

- For $d = 3$, if $F_t \to F$ and $F$ is concise then $F_t^\perp \to F^\perp$ is always flat.
- For $n = 3$, if $F^\perp$ has a minimal generator of degree $d$ then there exists some family $F_t \to F$, $F_t$ direct sums, such that $F_t^\perp \to F^\perp$ is flat.

**But**

- There exists $F$ such that $F^\perp$ has a minimal generator of degree $d$, so $F$ is a limit of direct sums; but for every family of direct sums $F_t \to F$, $F_t^\perp \to F^\perp$ is not flat.

The last item is a consequence of the existence of non-smoothable Gorenstein schemes.

This forces trickier proofs for the previous theorems.
An idea that does work

**Theorem**

If $F^\perp$ has a minimal generator of degree $d$ then $F$ is a limit of direct sums.

**Proof.**

Suppose $F^\perp$ has a minimal generator of degree $d$.

- By Gorenstein duality, $F^\perp$ has a high degree syzygy.
- By Koszul homology, $F^\perp$ contains quadratic generators: the $2 \times 2$ minors of a matrix $L$ of linear forms.
- Jordan normal form of $L$ either gives
  - a direct sum decomposition of $F$,
  - or (if $L$ is nilpotent) a limit of direct sums.
Theorem

Let $n \geq 2$, $d \geq 3$. If $F$ is a limit of direct sums and $F$ cannot be written using fewer variables, then $F^\perp$ has a minimal generator of degree $d$.

Proof.

Suppose $F$ is a concise limit of direct sums, $F_t \to F$.

- Let $J = \lim F_t^\perp$. $J \subseteq F^\perp$.
- Each $F_t^\perp$ has a minimal generator of degree $d$
- By Gorenstein duality $\beta_{n-1,n}(F_t^\perp) > 0$
- By semicontinuity of graded Betti numbers, $\beta_{n-1,n}(J) > 0$
- This syzygy lies in the minimal-degree strand by conciseness (no linear generators).
- So $F^\perp$ has the same high degree syzygy
- Hence $F^\perp$ also has a minimal generator of degree $d$. 
Low-degree generators

**Theorem**

If $F$ is a homogeneous form of degree $d$ in $n$ variables and $\delta$ is the generating degree of $F^\perp$ then $d \leq (\delta - 1)n$.

If $F^\perp$ is generated by quadrics then $d \leq n$.

**Question**

What are the forms $F$ such that $d = n$ and $F^\perp$ is generated by quadrics?

$F = x_1 \cdots x_n$ has $F^\perp = \langle \alpha_1^2, \ldots, \alpha_n^2 \rangle$. 