Decomposition of Tensors of Small Rank

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Let $V_1, \ldots, V_k$ be complex vector spaces. A decomposition of $f \in V_1 \otimes \ldots \otimes V_k$ is

$$f = \sum_{i=1}^{r} c_i v_{i,1} \otimes \ldots \otimes v_{i,k} \quad \text{with} \quad c_i \in \mathbb{C}, \quad v_{i,j} \in V_j$$

**Definition**

$\text{rk}(f)$ is the minimum number of summands in a decomposition of $f$. A minimal decomposition has $\text{rk}(f)$ summands and it is called CANDECOMP or PARAFAC.

We may assume $c_i = 1$, although in practice it is more convenient to determine $v_{i,k}$ up to scalars, and then solve for $c_i$. 

Symmetric Tensor Decomposition (Waring)

In the case $V_1 = \ldots = V_k = V$ we may consider symmetric tensors $f \in S^d V$. A Waring decomposition of $f \in S^d V$ is

$$f = \sum_{i=1}^{r} c_i (l_i)^d \quad \text{with} \; l_i \in V$$

with minimal $r$.

Example: $7x^3 - 30x^2y + 42xy^2 - 19y^3 = (-x + 2y)^3 + (2x - 3y)^3$

$rk \left( 7x^3 - 30x^2y + 42xy^2 - 19y^3 \right) = 2$
Applications

- Signal Processing, [Comon], ...
- Phylogenetics, [Allman-Rhodes], [Chang], ...
- ....
Recent Updates and Questions

- There are MATLAB packages computing a decomposition, ([Sidiropoulos - Bro],[Kolda et al.])..., working very well when the rank is small, efficiently storing very large tensors, but when the rank is high they often compute a not minimal decomposition.

- Very interesting new techniques, by extending the moment matrices, by [Bernardi-Brachat-Comon-Mourrain-Tsigaridas]

- **Main Questions** Is there only one minimal decomposition? How to compute it? The problem of efficiently computing a minimal decomposition in any rank is still open.
Weak defectivity goes back to classical papers by Terracini. Also studied extensively by Chiantini and Ciliberto. We realized recently that it allows computer experiments regarding the tensor decomposition.

Given $t = \sum_{i=1}^{r} x_i$, the uniqueness of the decomposition of $t$ is implied by the equality

$$\{x \in X | T_xX \subset T_{t\sigma_r(X)} \} = \{x_1, \ldots, x_r \}$$

The left hand side is called the contact locus and always contains the right hand side by Terracini’s Lemma. When equality fails for some $t$, we say $X$ is weakly defective.

Not weakly defective $\implies$ unique decomposition.
The symmetric case: uniqueness in the subgeneric case

Theorem (Sylvester[1851], Chiantini-Ciliberto, Mella, Ballico, [2002-2005])

The general \( f \in S^d \mathbb{C}^{n+1} \) of rank \( s \) smaller than the generic rank has a unique Waring decomposition, with the only exceptions

- rank \( s = \left( \frac{n+2}{2} \right) - 1 \) in \( S^4 \mathbb{C}^{n+1}, 2 \leq n \leq 4 \): infinitely many decompositions
- rank 7 in \( S^3 \mathbb{C}^5 \): infinitely many decompositions
- rank 9 in \( S^6 \mathbb{C}^3 \): exactly two decompositions
- rank 8 in \( S^4 \mathbb{C}^4 \): exactly two decompositions

The cases listed in red are the defective cases.
The cases listed in blue are the weakly defective cases.
The case of $V_1 \otimes V_2 \otimes V_3$

**Defective and Weakly Defective examples**

Only known examples when the decomposition of the general $f \in V_1 \otimes V_2 \otimes V_3 \ (\dim V_i = n_i + 1)$ of subgeneric rank $s$ is NOT UNIQUE are

- unbalanced case, $n_3 \geq n_1 n_2 + 2$,
  \[n_1 n_2 + 2 \leq s \leq \min(n_3 + 1, (n_1 + 1)(n_2 + 1))\]
  [Catalisano-Geramita-Gimigliano]

- $k = 3$, $(n_1, n_2, n_3) = (2, m, m)$ with $m$ even [Strassen],

- $k = 3$, $(n_1, n_2, n_3) = (2, 3, 3)$, sporadic case [Abo-O-Peterson]

- unbalanced case, rank $s = n_1 n_2 + 1$, $n_3 \geq n_1 n_2 + 1$

- rank 6 \( (n_1, n_2, n_3) = (3, 3, 3) \): two decompositions

- rank 8 \( (n_1, n_2, n_3) = (2, 5, 5) \), sporadic case [Chiantini-O]: maybe six decompositions
The exceptions to uniqueness listed in the previous slide are the only ones in the cases

- unbalanced
- \( n_i \leq 6 \)
- \( s \leq 6 \)
**Main Theorem**

- **There is a unique decomposition for tensor of rank** $s$ in $\mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1}$
  - if $s \leq \frac{3n+1}{2}$ [Kruskal, 1977] (it may be applied to tensors satisfying the Kruskal’s condition)
  - if $s \leq \frac{(n+2)^2}{16}$ [Chiantini-O., 2011] (it holds for general tensors of given rank, application to specific tensors requires weak defectivity)

- **Decomposition is unique for general tensor of rank** $s$ in $\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2$ ($k$ times) unless $k = 4$ or $5$ (two decompositions). The result is proved for all possible values of $s$ except one. [Bocci-Chiantini-O., 2011]

Proof uses a generalization of the inductive technique in [AOP] to the weak defectivity setting.
Comparison with Kruskal bound, for general tensors of given rank

<table>
<thead>
<tr>
<th>$a$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>gen.rank ($a \neq 3$)</td>
<td>$\left\lceil \frac{a^3}{3a-2} \right\rceil$</td>
<td>2</td>
<td>4</td>
<td>7</td>
<td>10</td>
<td>14</td>
<td>19</td>
<td>24</td>
<td>30</td>
</tr>
<tr>
<td>Kruskal bound</td>
<td>$\left\lfloor \frac{3a-2}{2} \right\rfloor$</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>6</td>
<td>8</td>
<td>9</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>Chiantini-O</td>
<td>$k(a)$</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>9</td>
<td>13</td>
<td>18</td>
<td>22</td>
<td>27</td>
</tr>
</tbody>
</table>
Let $E$ be a vector bundle on $X$, embedded by the very ample line bundle $L$ in $\mathbb{P}(H^0(L)^\vee)$. Consider the natural morphism

$$H^0(E) \otimes H^0(L)^\vee \xrightarrow{A} H^0(E^\vee \otimes L)^\vee$$

which induces $\forall f \in H^0(L)^\vee$ the linear map

$$A_f : H^0(E) \to H^0(E^\vee \otimes L)^\vee$$
Theorem (Landsberg-O)

Let \( Z = \{x_1, \ldots, x_k \} \subset X \) such that \( H^0(E^\vee \otimes L) \to H^0(E^\vee \otimes L|_Z) \) is surjective.

Let \( f = \sum_{i=1}^k x_i \in H^0(L)^\vee \). Then

\[
Z \subseteq \text{base locus of } \ker A_f
\]

\[
\text{rk } A_f = \text{rk } E \cdot \text{rk } f
\]
When \( E = \mathcal{O}(a) \) is a line bundle we get the classical apolarity of XIX century. If \( f \in S^d V \) the map \( A_f \) becomes the catalecticant map

\[
C_f : S^a \mathbb{C}^{n+1} \to S^{d-a} \mathbb{C}^{n+1} \quad [\text{Sylvester, Iarrobino-Kanev}]
\]

It is convenient to set \( a = \lceil \frac{d}{2} \rceil \). The base locus of \( \ker C_f \) is cut out by polynomials of degree \( a \).
Explicit construction of the minors of $A_f$ from a presentation of $E$

We have a presentation of $E$

$$\cdots \longrightarrow L_2 \longrightarrow L_1 \longrightarrow L_0 \longrightarrow L_{-1} \longrightarrow \cdots$$

We get that $A_f$ factors through the map $P_f$ obtained by differentiating with respect to $p_E$ the catalecticant matrix. Minors of $P_f$ and minors of $A_f$ coincide.

The presentation of $E = Q(m)$ on $\mathbb{P}^2$ is

$$\begin{bmatrix} x_2 & -x_1 \\ -x_2 & x_0 \\ x_1 & -x_0 \end{bmatrix}$$
Consider $X = \mathbb{P}^2$, $L = \mathcal{O}(2\delta + 1)$, $f \in S^{2\delta + 1} \mathbb{C}^3$.
Set $E = Q(\delta)$ where $Q$ is the quotient bundle (rank two) on $\mathbb{P}^2$.

$$A_f : H^0(E) = H^0(Q(\delta)) \to H^0(Q(\delta))^\vee = H^0(E^\vee \otimes L)^\vee$$

is now skew-symmetric.
Block structure of $P_f$ for odd degree plane curves

$P_f$ is represented by the following $3\left(\frac{\delta+2}{2}\right) \times 3\left(\frac{\delta+2}{2}\right)$ matrix, where each depicted block is the $\left(\frac{\delta+2}{2}\right) \times \left(\frac{\delta+2}{2}\right)$ catalecticant of $f_i = \frac{\partial f}{\partial x_i}$.

$$
\begin{bmatrix}
0 & C_{f_2} & -C_{f_1} \\
-C_{f_2} & 0 & C_{f_0} \\
C_{f_1} & -C_{f_0} & 0
\end{bmatrix}
$$

If $(g_0, g_1, g_2)$ is in the kernel and the rank of $f$ is $\leq \left(\frac{\delta+2}{2}\right)$ then the base locus of the 2-minors of

$$
\begin{bmatrix}
x_0 & x_1 & x_2 \\
g_0 & g_1 & g_2
\end{bmatrix}
$$

give the Waring decomposition of $f$. 
Theorem (Sylvester)

Given a general \( f \in S^3 \mathbb{C}^4 \), there exist unique \( l_1, \ldots, l_5 \) such that
\[
f = \sum_{i=1}^{5} l_i^3.
\]

Let \( E = \wedge^2 Q(1) \). We have \( A_f : H^0(\wedge^2 Q(1)) \to H^0(Q(1)) \vee \).

Algorithm (Oeding-O)

The sections in \( \ker A_f \) vanish on \( \{ l_1, \ldots, l_5 \} \).
Note that \( c_3(\wedge^2 Q(1)) = 5 \).
Let $f = \sum_{x_i \in Z} x_i$ be general.

[Landsberg-O., [2010]]

If

$$H^0(I_Z \otimes E) \otimes H^0(I_Z \otimes E^\vee \otimes L) \to H^0(I_{Z^2} \otimes L)$$

is surjective, then the locus $\{ f | \text{rk } A_f \leq k \cdot \text{rk } E \}$ contains $\sigma_k(X)$ as irreducible component.
Let $d = 2\delta + 1$, $a = \lfloor \frac{n}{2} \rfloor$. Let $Z \subset \mathbb{P}^n$ of length $k \leq \binom{\delta + n}{n}$. Then the map

$$H^0(I_Z \otimes \wedge^a Q(\delta)) \otimes H^0(I_Z \otimes \wedge^{n-a} Q(\delta)) \to H^0(I_{Z^2}(d))$$

is surjective. Then we may apply the Infinitesimal Criterion and we get local equations for $\sigma_k(v_d(\mathbb{P}^n))$. 

Theorem
Thanks !!
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